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ON THE FLOW OF A COMPRESSIBLE FLUID BY THE HODOGRAPH METHOD

II—FUNDAMENTAL SET OF PARTICULAR FLOW SOLUTIONS OF THE CHAPLYGIN DIFFERENTIAL EQUATION

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SUMMARY

The differential equation of Chaplygin's jet problem is utilized to give a systematic development of particular solutions of the hodograph flow equations, which extends the treatment of Chaplygin into the supersonic range and completes the set of particular solutions.

The particular solutions serve to place on a reasonable basis the use of velocity correction formulas for the comparison of incompressible and compressible flows. It is shown that the geometric-mean type of velocity correction formula introduced in part I has significance as an over-all type of approximation in the subsonic range.

A brief review of general conditions limiting the potential flow of an adiabatic compressible fluid is given and application is made to the particular solutions, yielding conditions for the existence of singular loci in the supersonic range.

The combining of particular solutions in accordance with prescribed boundary flow conditions is not treated in the present paper.

INTRODUCTION

This paper presents a theoretical investigation that may be regarded as a continuation of studies initiated in part I (reference 1). In part I an attempt was made to unify the results of Chaplygin, von Kármán and Tsien, Temple and Yarwood, and Prandtl and Glauert insofar as their results were concerned with velocity and pressure correction factors for the correspondence of incompressible and compressible flows. In addition, two new velocity correction formulas were introduced that appeared to have a somewhat wider range of applicability than the formulas of the aforementioned authors. Most of the results of part I were obtained with the use of two particular solutions of the hodograph equations. These two basic solutions correspond to a vortex and a source in a compressible fluid.

It was mentioned in part I that, in order to treat the exact boundary problem of uniform flow of a compressible fluid past a prescribed body, a general set of particular solutions of the hodograph equations had to be obtained. Such a study is given in the present paper, which incidentally helps to clarify the nature of the velocity correction factors of part I—in particular, the one referred to as the "geometric-mean" type of approximation. In addition, many interesting types of flows are disclosed from a physical interpretation of the particular solutions. A few such solutions have already been obtained and discussed by Ringleb (reference 2).

Several mathematical approaches exist by means of which particular integrals of the hodograph equations may be obtained. Two such approaches, mentioned in part I, may be attributed to Chaplygin (reference 3) and Bers and Gelbart (reference 4) and are analogous to an exponential and to a power-series approach, respectively. Another method of defining particular integrals is the integral-operator method of Bergman (reference 5). In the present paper the differential equation, first used by Chaplygin in his treatment of jets (reference 3), provides the basis for the definition of a complete set of particular solutions.

The scope of the present paper is limited chiefly to a systematic study of the fundamental solutions and to the physical interpretation of some of the particular flows represented by them. The combining of particular solutions to represent uniform flow past a prescribed body is not treated herein. It is believed, however, that the present study may serve as a basis for further development and clarification of this important problem.

SYMBOLS

x, y	rectangular coordinates in plane of flow
q	magnitude of fluid velocity
θ	angle included by velocity vector and positive direction of x-axis
ρ	density of fluid
p	pressure in fluid
a	velocity of sound in fluid
M	Mach number (q/a)
ρ_0, p_0, a_0	quantities referred to stagnation point $q=0$
ϕ	velocity potential
ψ	stream function
γ	ratio of specific heats (approx. 1.4 for air)
$\beta=\dfrac{1}{\gamma-1}$	(approx. 5/2 for air)
$2\beta a_0^2$	maximum fluid velocity (corresponding to $p=\rho=a=0$)
τ	dimensionless speed variable $\left(\tau=\dfrac{q^2}{2\beta a_0^2}=\dfrac{M^2}{2\beta+M^2}\right)$
τ_s	sonic value of τ $\left(\tau_s=\dfrac{1}{2\beta+1}\text{; approx. 1/6 for air}\right)$

For $\beta>0$ (or $1<\gamma<\infty$), the range of τ is $0\leqq\tau\leqq1$.

GENERAL PARTICULAR SOLUTIONS OF THE HODOGRAPH EQUATIONS

HODOGRAPH EQUATIONS

The linear equations in the hodograph variables θ and q, which relate the velocity potential ϕ and the stream function ψ for the steady two-dimensional flow of a nonviscous compressible fluid, are

$$\left.\begin{aligned} \frac{\partial\phi}{\partial\theta}&=\lambda_1(q)\frac{\partial\psi}{\partial q} \\ \frac{\partial\phi}{\partial q}&=-\lambda_2(q)\frac{\partial\psi}{\partial\theta} \end{aligned}\right\} \tag{1}$$

in which, for the adiabatic equation of state between pressure and density,

$$\lambda_1(q)=\frac{\rho_0}{\rho}q=\frac{q}{(1-\tau)^\beta}$$

and

$$\lambda_2(q)=-q\frac{d}{dq}\left(\frac{\rho_0}{\rho q}\right)=\frac{\rho_0}{\rho q}(1-M^2)=\frac{1-(2\beta+1)\tau}{q(1-\tau)^{\beta+1}}$$

(See equations (21) and (25) of reference 1.)

In the incompressible case $\tau\rightarrow 0$, equations (1) can be expressed in the Cauchy-Riemann form. Particular solutions $\Omega=\phi+i\psi$ can be expressed in this case as any analytic function of the complex variable

$$w=\theta+i\log q \tag{2}$$

or as any analytic function of the related exponential function

$$e^{-iw}=qe^{-i\theta} \tag{3}$$

Thus, an infinite set of particular integrals of equations (1), in the incompressible case, referred to herein as "the powers set," is w^k. When k is a positive integer, the particular solutions vanish at the origin ($\theta=0$, $\log q=0$) and, when k is a negative integer, the particular solutions are infinite at the origin. In the case of nonintegral values of k, the origin is a branch point of the functions w^k.

Another infinite set of particular integrals of equations (1) in the incompressible case, referred to herein as "the exponential set," is

$$(e^{-iw})^k=q^ke^{-ik\theta}$$

where, again, k can take on any value—integral, nonintegral, positive, or negative.

In the compressible case, the particular solutions corresponding to the powers set w^k (that is, the particular solutions which reduce to w^k in the incompressible case $\tau\rightarrow 0$) depend on whether the coefficient of w^k is real or imaginary—a consequence of the fact that, in the compressible case, ϕ and ψ do not satisfy the same differential equation. For example, for $k=1$, the two functions corresponding to w and iw, which have been developed in part I, are

$$W=\theta+iL$$

and

$$i\tilde{W}=i(\theta+i\tilde{L})$$

where

$$L=\log q+f(\tau)$$

and

$$\tilde{L}=\log q+g(\tau)$$

and $f(\tau)$ and $g(\tau)$ each vanish for $\tau=0$. (See equations (26) and (27) of reference 1.)

The development of other functions corresponding to the power set w^k, for positive integral values of k, follows according to the method of Bers and Gelbart. (See expression (22) of reference 1.) Since the present paper is chiefly concerned with the functions corresponding to the exponential set e^{-ikw}, the powers set is not further discussed.

CHAPLYGIN DIFFERENTIAL EQUATION

THE FUNCTIONS P_k AND Q_k

Corresponding to the exponential sets in the incompressible case

$$e^{-ikw}=q^k\cos k\theta-iq^k\sin k\theta$$

and

$$ie^{-ikw}=q^k\sin k\theta+iq^k\cos k\theta$$

there appear in the compressible case functions designated, respectively,

$$P_k(q)\cos k\theta-iQ_k(q)\sin k\theta$$

and

$$P_k(q)\sin k\theta+iQ_k(q)\cos k\theta$$

where the functions $P_k(q)$ and $Q_k(q)$ satisfy second-order differential equations. These equations are easily obtained by substituting in equations (1) the product-type solutions

$$\left.\begin{aligned} \phi_k&=P_k(q)\begin{matrix}\cos\\ \sin\end{matrix}(k\theta) \\ \psi_k&=Q_k(q)\begin{matrix}\sin\\ \cos\end{matrix}(-k\theta) \end{aligned}\right\} \tag{4}$$

In view of equations (1) it is observed that

$$\left.\begin{aligned} kP_k(q)&=\frac{\rho_0}{\rho}q\frac{dQ_k(q)}{dq} \\ \frac{dP_k(q)}{dq}&=-kq\frac{d}{dq}\left(\frac{\rho_0}{\rho q}\right)Q_k(q) \end{aligned}\right\}. \tag{5}$$

The functions $Q_k(q)$ satisfy the second-order differential equation

$$q^2\frac{d^2Q_k}{dq^2}+(1+M^2)q\frac{dQ_k}{dq}-k^2(1-M^2)Q_k=0 \tag{6}$$

The functions $P_k(q)$ can be obtained from $Q_k(q)$ by means of the first of equations (5). Equation (6) may be reduced to a standard type by introducing τ as the independent variable. Put

$$Q_k(q)=q^kY_k(\tau) \tag{7}$$

where clearly $Y_k(\tau)\rightarrow 1$ as $\tau\rightarrow 0$ (incompressible case). With the use of the symbolic relations

$$q\frac{d}{dq}=2\tau\frac{d}{d\tau}$$

$$q^2\frac{d^2}{dq^2}=4\tau^2\frac{d^2}{d\tau^2}+2\tau\frac{d}{d\tau}$$

and the relation

$$M^2=\frac{2\beta\tau}{1-\tau}$$

the desired differential equation is

$$\tau(1-\tau)\frac{d^2Y_k}{d\tau^2}+[(k+1)-(k+1-\beta)\tau]\frac{dY_k}{d\tau}+\frac{1}{2}\beta k(k+1)Y_k=0 \tag{8}$$

Equation (8), which is of the hypergeometric type, was first introduced by Chaplygin in his memoir on gas jets (reference 3).

THE FUNCTIONS Y_k AND Y_{-k}

Chaplygin treated the subsonic flow of a compressible fluid through jets with straight-line boundaries. For such problems the hodograph variables θ and q are natural variables in the sense that the solid and fluid boundaries are described by θ=Constant and q=Constant, respectively, and only the particular solutions of equation (8) with positive characteristic index k are needed. In the present paper a complete ordered set of particular solutions of equation (8) is obtained, which extends the results of Chaplygin into the supersonic range and to negative values of the index k. Two types of solutions of equation (8) for nonintegral values of k are

$$Y_k(\tau)=F(a_k, b_k, k+1; \tau) \tag{9}$$

and

$$\overline{Y}_k(\tau)=\tau^{-k}F(a_k-k, b_k-k, 1-k; \tau) \tag{10}$$

where

$$a_k+b_k=k-\beta$$

$$a_kb_k=-\frac{k}{2}(k+1)\beta$$

and

$$F(a, b, c; \tau)=1+\frac{ab}{c}\tau+\frac{a(a+1)b(b+1)}{2!\,c(c+1)}\tau^2+\ldots$$

It is now shown that only one of the solutions need be used. For positive values of k, the requirement that $Y_k(0)=1$ excludes the use of equation (10). For negative values of the index, the solution $\overline{Q}_{-k}(q)=q^{-k}\overline{Y}_{-k}(\tau)$ obtained with the aid of equation (10) is, except for a constant factor, equivalent to the solution $Q_k(q)=q^kY_k(\tau)$ obtained with the aid of equation (9). Thus

$$\begin{aligned}\overline{Q}_{-k}(q)&=q^{-k}\overline{Y}_{-k}(\tau)\\ &=q^{-k}\tau^kF(a_{-k}+k, b_{-k}+k, k+1; \tau)\\ &=q^{-k}\tau^kF(a_k, b_k, k+1; \tau)\\ &=\left(\frac{1}{2\beta a_0^2}\right)^kQ_k(q)\end{aligned}$$

Hence, only the solutions given by equation (9) are needed for the determination of $Q_k(q)$ and $Q_{-k}(q)$.

Then

$$\begin{aligned}Q_k(q)&=q^kY_k(\tau)\\ &=q^kF(a_k, b_k, k+1; \tau)\end{aligned} \tag{11}$$

and

$$\begin{aligned}Q_{-k}(q)&=q^{-k}Y_{-k}(\tau)\\ &=q^{-k}F(a_{-k}, b_{-k}, -k+1; \tau)\\ &=q^{-k}F(a_k-k, b_k-k, -k+1; \tau)\end{aligned} \tag{12}$$

Observe that both types of hypergeometric functions appearing in equations (9) and (10) are utilized in the expressions for $Q_k(q)$ and $Q_{-k}(q)$.

The foregoing discussion has been limited to nonintegral values of the index, positive or negative. When the index is integral and positive, equations (9) and (11) remain valid. When the index is integral and negative, however, equation (12) does not in general lead to a meaningful solution and consequently another independent solution is to be sought. The desired solution for $Y_{-k}(\tau)$ in such cases contains a logarithmic term and again is subject to the condition that it reduce to unity for $\tau=0$ (incompressible case). The expression for $Q_{-k}(q)$ is then given by

$$Q_{-k}(q)=q^{-k}Y_{-k}(\tau) \tag{13}$$

where

$$\begin{aligned}Y_{-k}(\tau)=1&-\frac{(a_k-k)(b_k-k)}{1!(k-1)}\tau+\frac{(a_k-k)(a_k-k+1)(b_k-k)(b_k-k+1)}{2!(k-1)(k-2)}\tau^2\\ &-\frac{(a_k-k)(a_k-k+1)(a_k-k+2)(b_k-k)(b_k-k+1)(b_k-k+2)}{3!(k-1)(k-2)(k-3)}\tau^3+\ldots\\ &+(-1)^{k-1}\frac{(a_k-k)(a_k-k+1)\ldots(a_k-2)(b_k-k)(b_k-k+1)\ldots(b_k-2)}{(k-1)!(k-1)!}\tau^{k-1}\\ &+c\left[\tau^kF(a_k, b_k, k+1; \tau)\log\tau+\frac{a_kb_k}{1!(k+1)}\left(\frac{1}{a_k}+\frac{1}{b_k}-\frac{1}{1}-\frac{1}{k+1}\right)\tau^{k+1}\right.\\ &\left.+\frac{a_k(a_k+1)b_k(b_k+1)}{2!(k+1)(k+2)}\left(\frac{1}{a_k}+\frac{1}{a_k+1}+\frac{1}{b_k}+\frac{1}{b_k+1}-\frac{1}{1}-\frac{1}{2}-\frac{1}{k+1}-\frac{1}{k+2}\right)\tau^{k+2}+\ldots\right]\end{aligned}$$

and

$$c=(-1)^{k+1}\frac{(a_k-1)(a_k-2)\ldots(a_k-k)(b_k-1)(b_k-2)\ldots(b_k-k)}{k!(k-1)!}$$

Note that, if a_k or b_k takes on any of the values 1, 2, . . . k, the constant c equals zero and the function $Y_{-k}(\tau)$ becomes a polynomial. It should be pointed out, however, that equation (13) is to be utilized only if equation (12) does not yield a relevant and finite result. This statement is illustrated in some of the following special examples.

Case of $\gamma=-1$:

Consider as an example the von Kármán-Tsien treatment of compressible flow (reference 6) in which the adiabatic index $\gamma=-1$ or $\beta=-\frac{1}{2}$. Then

$$a_k=\frac{k+1}{2}$$

$$b_k=\frac{k}{2}$$

For a negative integral index, equation (13) may appear to be applicable, in which case the expression for $Y_{-k}(\tau)$ would be a polynomial of degree $k-1$. An examination of equation (12) shows, however, that for this case no infinities arise and that, when the index is negative, integral, or nonintegral,

$$Y_{-k}(\tau)=F\left(\frac{1-k}{2}, -\frac{k}{2}, 1-k; \tau\right)$$

The hypergeometric series represented by $Y_{-k}(\tau)$ converges for values $0\leqq|\tau|<1$. For the present case of $\gamma=-1$ or $\beta=-\frac{1}{2}$, values of τ corresponding to positive values of M lie outside the range of convergence. A closed expression for $Y_{-k}(\tau)$ can be found, however, for this case which, by analytic continuation, is therefore valid for all values of τ. Thus

$$\begin{aligned}Y_{-k}(\tau)&=F\left(\frac{1-k}{2}, -\frac{k}{2}, 1-k; \tau\right)\\&=\left[\frac{1+(1-\tau)^{1/2}}{2}\right]^k\end{aligned}$$

Similarly, from equation (9), when the index is positive,

$$\begin{aligned}Y_k(\tau)&=F\left(\frac{1+k}{2}, \frac{k}{2}, 1+k; \tau\right)\\&=\left[\frac{1+(1-\tau)^{1/2}}{2}\right]^{-k}\end{aligned}$$

Observe that

$$\begin{aligned}Q_k(q)&=\frac{1}{Q_{-k}(q)}\\&=\left[q\frac{2}{1+(1-\tau)^{1/2}}\right]^k\end{aligned} \tag{14}$$

This identity for the von Kármán-Tsien case corresponds to the identity $q^k=\frac{1}{q^{-k}}$ for the incompressible case.

Case of $k=1$:

For $k=1$,

$$a_1=1 \qquad b_1=-\beta \qquad c_1=2$$

Then, for the positive index,

$$\begin{aligned}Q_1(q)&=qY_1(\tau)\\&=qF(1, -\beta, 2; \tau)\\&=q\frac{1-(1-\tau)^{\beta+1}}{(\beta+1)\tau}\end{aligned} \tag{15}$$

For the negative integral index, it may appear at first glance that equation (13) is needed; however, equation (12) does yield a relevant and finite result and accordingly is the equation to be used. Thus

$$\begin{aligned}\lim_{k\to 1} F(a_k-k, b_k-k, 1-k; \tau)&=1+\frac{\beta}{2}\tau-\frac{\beta^2}{2\times 2!}\tau^2+\frac{\beta^2(\beta-1)}{2\times 3!}\tau^3\\&\quad-\frac{\beta^2(\beta-1)(\beta-2)}{2\times 4!}\tau^4+\ldots\\&=1+\frac{1}{2}\frac{\beta}{\beta+1}[1-(1-\tau)^{\beta+1}]\end{aligned}$$

and therefore

$$Q_{-1}(\tau)=q^{-1}\left\{1+\frac{1}{2}\frac{\beta}{\beta+1}[1-(1-\tau)^{\beta+1}]\right\} \tag{16}$$

Case of $k=0$:

The exceptional case of $k=0$ is directly treated by means of equation (8). The differential equation for $Y_0(\tau)$ or $Q_0(\tau)$ then is

$$\frac{d}{d\tau}\left[\frac{\tau}{(1-\tau)^\beta}\frac{dQ_0}{d\tau}\right]=0$$

The general solution of this equation can be written as

$$Q_0(q)=2C_1\log q+C_1\int_0^\tau [(1-\tau)^\beta-1]\frac{d\tau}{\tau}+C_2$$

where C_1 and C_2 are arbitrary constants of integration. The constants C_1 and C_2 are determined by the imposed condition that the expression for $Q_0(q)$ reduce in the incompressible case simply to log q. Then

$$C_1=\frac{1}{2}$$

$$C_2=0$$

and therefore

$$Q_0(q)=\log q+\frac{1}{2}\int_0^\tau [(1-\tau)^\beta-1]\frac{d\tau}{\tau} \tag{17}$$

In a similar manner, from the differential equation for P_0,

$$\frac{d}{d\tau}\left[\frac{\tau(1-\tau)^{\beta+1}}{1-(2\beta+1)\tau}\frac{dP_0}{d\tau}\right]=0$$

the expression for P_0 is obtained as

$$P_0(q)=\log q+\frac{1}{2}\int_0^\tau\left[\frac{1-(2\beta+1)\tau}{(1-\tau)^{\beta+1}}-1\right]\frac{d\tau}{\tau} \tag{18}$$

It is remarked that the functions $Q_0(q)$ and $P_0(q)$ are identical with the elementary functions $L(q)$ and $\tilde{L}(q)$, respectively, introduced in part I (reference 1) and are associated with a vortex and a source type of flow.

THE FUNCTIONS R_k AND S_k

A linear homogeneous differential equation of order n can, in general, be reduced to a differential equation of order $n-1$ by means of an exponential-type substitution for the dependent variable. Chaplygin made use of such a substitution to reduce the second-order differential equations satisfied by P_k and Q_k to first-order equations of the Riccati form, in order to study properties of the functions P_k and Q_k in the subsonic range for only positive values of k. In the present analysis the Riccati equations are also found useful in order to extend the study of the functions P_k and Q_k to the supersonic range for both positive and negative values of the index k.

The second-order differential equations for P_k and Q_k, with τ as the independent variable, are

$$\frac{d}{d\tau}\left[\frac{\tau(1-\tau)^{\beta+1}}{1-(2\beta+1)\tau}\frac{dP_k}{d\tau}\right]-\frac{k^2}{4}\frac{(1-\tau)^\beta}{\tau}P_k=0$$

and

$$\frac{d}{d\tau}\left[\frac{\tau}{(1-\tau)^\beta}\frac{dQ_k}{d\tau}\right]-\frac{k^2}{4}\frac{1-(2\beta+1)\tau}{\tau(1-\tau)^{\beta+1}}Q_k=0$$

The corresponding first-order Riccati equations are obtained by substituting for P_k and Q_k new dependent variables R_k and S_k, respectively, as follows:

$$P_k=e^{\int\frac{k}{2\tau}R_k d\tau}$$

or

$$R_k=\frac{2\tau}{k}\frac{1}{P_k}\frac{dP_k}{d\tau}$$

$$=\frac{2\tau}{k}\frac{d}{d\tau}\log P_k \tag{19}$$

and

$$Q_k=e^{\int\frac{k}{2\tau}S_k d\tau}$$

or

$$S_k=\frac{2\tau}{k}\frac{1}{Q_k}\frac{dQ_k}{d\tau}$$

$$=\frac{2\tau}{k}\frac{d}{d\tau}\log Q_k \tag{20}$$

The equations satisfied by $R_k(\tau)$ and $S_k(\tau)$ are

$$\frac{dR_k}{d\tau}+\frac{1+(2\beta+1)\tau}{1-(2\beta+1)\tau}\frac{\beta}{1-\tau}R_k+\frac{k}{2\tau}\left[R_k^2-\frac{1-(2\beta+1)\tau}{1-\tau}\right]=0 \tag{21}$$

and

$$\frac{dS_k}{d\tau}+\frac{\beta}{1-\tau}S_k+\frac{k}{2\tau}\left[S_k^2-\frac{1-(2\beta+1)\tau}{1-\tau}\right]=0 \tag{22}$$

Initial conditions for $R_k(\tau)$ and $S_k(\tau)$ are found by examination of the incompressible case $\tau\rightarrow 0$. In this case $P_k=Q_k=q^k$ and, since $2\tau\frac{d}{d\tau}=q\frac{d}{dq}$, it follows from equations (19) and (20) that

$$R_k(0)=S_k(0)=1$$

The following important relation exists between the functions $R_k(\tau)$ and $S_k(\tau)$:

$$R_k(\tau)S_k(\tau)=\frac{1-(2\beta+1)\tau}{1-\tau}$$

$$=1-M^2 \tag{23}$$

Equation (23) can be verified directly from the hodograph equations (1). It may be noted at this point that this result is of significance in connection with the geometric-mean type of velocity correction factor introduced in part I and is discussed more fully in a later section.

Before the functions $R_k(\tau)$ and $S_k(\tau)$ are treated, certain general observations can be made regarding the functions $P_k(\tau)$, $Q_k(\tau)$, $R_k(\tau)$, and $S_k(\tau)$. Chaplygin, who limited his investigations to the subsonic range and to positive values of the index k, has shown that Q_k and consequently the other functions possess no roots for any value of the independent variable in the subsonic range, with $M=0$ excluded. In the supersonic range $M>1$, $P_k(\tau)$ and $Q_k(\tau)$ in general possess zeros. Certain relations obtained by means of equations (19), (20), and (23) between P_k, Q_k, R_k, and S_k at the zeros of P_k and Q_k are summarized as follows:

P_k	Q_k	$\frac{dP_k}{d\tau}$	$\frac{dQ_k}{d\tau}$	R_k	S_k
0	Max or min	----	0	∞	0
Max or min	0	0	----	0	∞

It is remarked that the number of zeros of Q_k, as a function of the index k, can be found from an expression developed by Klein and Hurwitz (reference 7) in connection with the zeros of the hypergeometric function. In general, the number of zeros increases with the magnitude of the index k and is infinite for $k=\pm\infty$.

A further observation of interest can be made in connection with equation (23). Chaplygin has shown that, for positive finite values of k (and the same is true for negative finite values of k), the functions $S_k(\tau)$ are not zero for the sonic value $\tau=\tau_s$ or $M=1$. From equation (23) then, it follows that the functions $R_k(\tau)=0$ for $M=1$.

In view of the relation between the functions R_k and S_k given by equation (23), only S_k need be discussed. The Riccati equation (22) may be used to discuss certain properties of the function S_k but in general, for numerical evaluation, the original definition (equation (20)) in terms of the function Q_k may be used directly:

$$S_k=\frac{2\tau}{k}\frac{1}{Q_k}\frac{dQ_k}{d\tau}$$

or

$$S_k=1+\frac{2\tau}{Y_k}\frac{dY_k}{d\tau}$$

In general, the functions S_k are expressible in infinite series. For several values of k, however, S_k can be expressed in closed forms. For $k=0$ and $k=\pm\infty$, S_k may be obtained by a limiting process from equation (20); however, for these special cases the Riccati equation (equation (22)) yields the results directly. Thus

$$S_0=(1-\tau)^\beta \tag{24}$$

and

$$S_{\pm\infty}=\left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{1/2} =(1-M^2)^{1/2} \tag{25}$$

The cases $k=1$ and $k=-1$ may also be expressed in closed form. With the aid of the equations (15) and (16) for Q_1 and Q_{-1}, equation (20) yields

$$S_1=1-2\frac{1-(1+\beta\tau)(1-\tau)}{1-(1-\tau)^{\beta+1}} \tag{26}$$

and

$$S_{-1}=1-\frac{\beta\tau(1-\tau)^\beta}{1+\frac{1}{2}\frac{\beta}{\beta+1}[1-(1-\tau)^{\beta+1}]} \tag{27}$$

In order to illustrate the behavior of some of the functions thus far introduced, a number of tables and figures are given. All the calculations have been performed with the adiabatic index $\gamma=1.4$. Table 1 gives values of Y_k as a function of M or τ for several positive and negative values of the index k. Figure 1 shows the functions Y_k plotted against M. Values of the functions S_k and R_k are given in tables 2 and 3 and are plotted against M in figures 2 and 3.

THE FUNCTIONS $f_k(\tau)$ AND $g_k(\tau)$

In the incompressible case, the sets of functions Q_k and P_k can be reduced to a single function $\log q$ by means of a simple operator $\frac{1}{k}\log$. Thus

$$Q_k=P_k=q^k$$

and

$$\frac{1}{k}\log q^k=\log q$$

This same operation applied to the functions Q_k and P_k in the compressible case serves to define two useful sets of functions $\log q+f_k(\tau)$ and $\log q+g_k(\tau)$, respectively. Thus

$$\frac{1}{k}\log Q_k=\log q+f_k(\tau) \tag{28}$$

and

$$\frac{1}{k}\log P_k=\log q+g_k(\tau) \tag{29}$$

From equation (7), namely,

$$Q_k=q^kY_k(\tau)$$

it follows that

$$f_k(\tau)=\frac{1}{k}\log Y_k(\tau) \tag{30}$$

From equation (5) for P_k and equation (20), which defines S_k,

$$P_k=\frac{1}{(1-\tau)^\beta}Q_kS_k$$

It follows that

$$g_k(\tau)=\frac{1}{k}\log\frac{Y_k(\tau)S_k(\tau)}{(1-\tau)^\beta} =f_k(\tau)+\frac{1}{k}\log\frac{S_k}{(1-\tau)^\beta} \tag{31}$$

For example, for $k=1$ and $k=-1$ and with the use of equations (15) and (16),

$$f_1(\tau)=\log\frac{1-(1-\tau)^{\beta+1}}{(\beta+1)\tau} \tag{32}$$

$$g_1(\tau)=\log\frac{(1-\tau)^\beta[1+(2\beta+1)\tau]-1}{(\beta+1)\tau(1-\tau)^\beta} \tag{33}$$

$$f_{-1}(\tau)=-\log\left\{1+\frac{1}{2}\frac{\beta}{\beta+1}[1-(1-\tau)^{\beta+1}]\right\} \tag{34}$$

$$g_{-1}(\tau)=-\log\frac{(3\beta+1)-\beta(1+\tau)(1-\tau)^\beta}{2(\beta+1)(1-\tau)^\beta} \tag{35}$$

For $k=0$ and $k=\pm\infty$, equations (30) and (31) require a limiting process for their evaluation. Alternate forms for $f_k(\tau)$ and $g_k(\tau)$ may be obtained, however, by means of equations (19) and (20) defining $R_k(\tau)$ and $S_k(\tau)$, which yield the results for $k=0$ and $k=\pm\infty$ directly. Thus

$$f_k(\tau)=\frac{1}{2}\int_0^\tau[S_k(\tau)-1]\frac{d\tau}{\tau} \tag{36}$$

and

$$g_k(\tau)=\frac{1}{2}\int_0^\tau[R_k(\tau)-1]\frac{d\tau}{\tau} \tag{37}$$

where $R_k(\tau)$ and $S_k(\tau)$ are related according to equation (23). Then

$$f_0(\tau)=\frac{1}{2}\int_0^\tau[(1-\tau)^\beta-1]\frac{d\tau}{\tau} \tag{38}$$

$$g_0(\tau)=\frac{1}{2}\int_0^\tau\left[\frac{1-(2\beta+1)\tau}{(1-\tau)^{\beta+1}}-1\right]\frac{d\tau}{\tau} \tag{39}$$

and

$$f_{\pm\infty}(\tau)=g_{\pm\infty}(\tau)=\frac{1}{2}\int_0^\tau\left\{\left[\frac{1-(2\beta+1)\tau}{1-\tau}\right]^{1/2}-1\right\}\frac{d\tau}{\tau} \tag{40}$$

It is worthy of special notice that the functions $f_0(\tau)$, $g_0(\tau)$, and $f_{\pm\infty}(\tau)$ are identical with the functions $f(\tau)$, $g(\tau)$, and $h(\tau)$, respectively, which formed the basis of part I (reference 1). In addition, the expressions $\log q+f_0(\tau)$, $\log q+g_0(\tau)$, and $\log q+f_{\pm\infty}(\tau)$ are identical with the functions L, $\bar{L}$ and H, respectively, which were introduced in part I.

A number of functions f_k and g_k have been calculated, with $\gamma=1.4$, for several positive and negative values of the index k, and the values are given in tables 4 and 5 and plotted in figures 4 and 5.

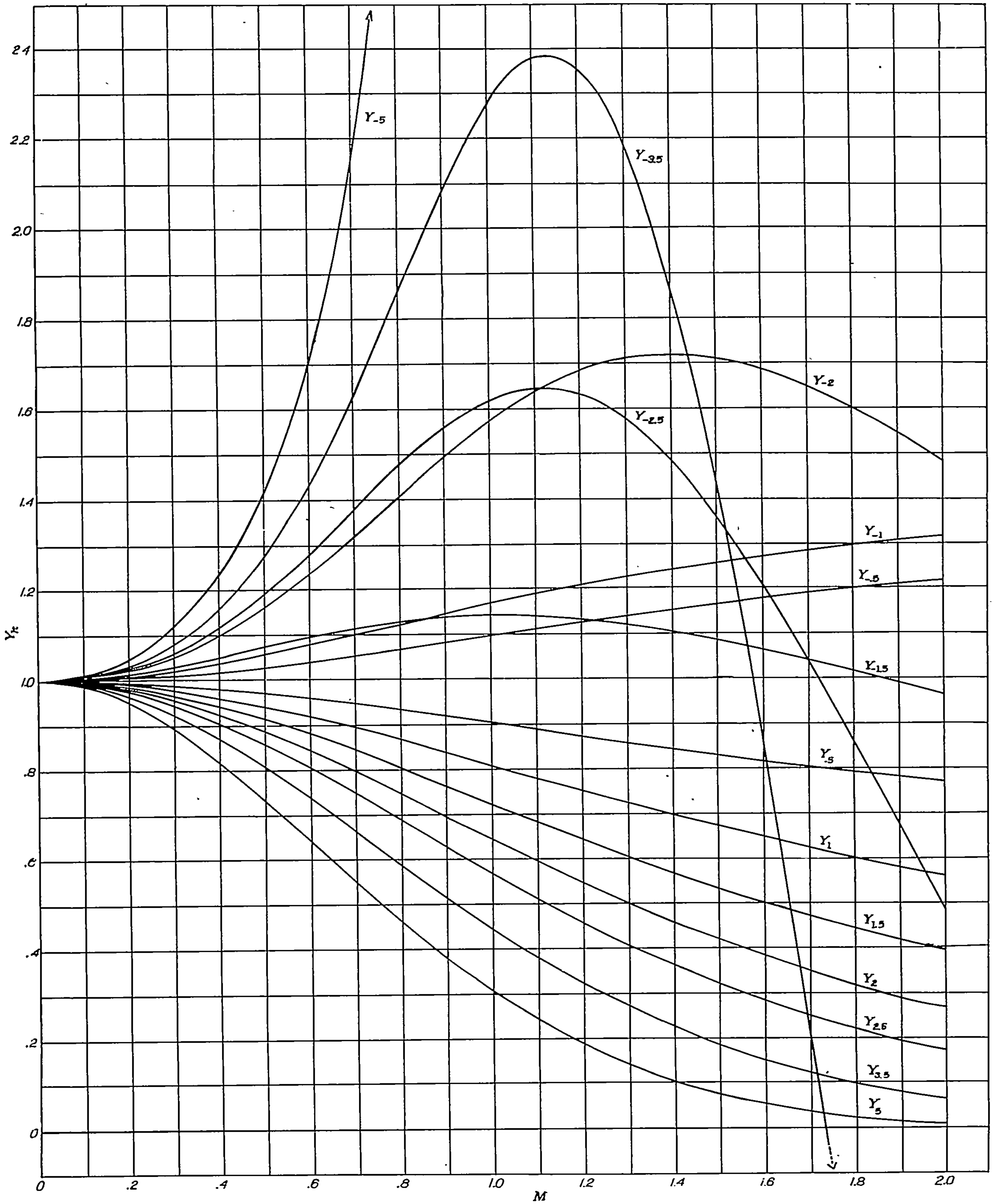

FIGURE 1.—The functions Y_k against M for several values of the index k.

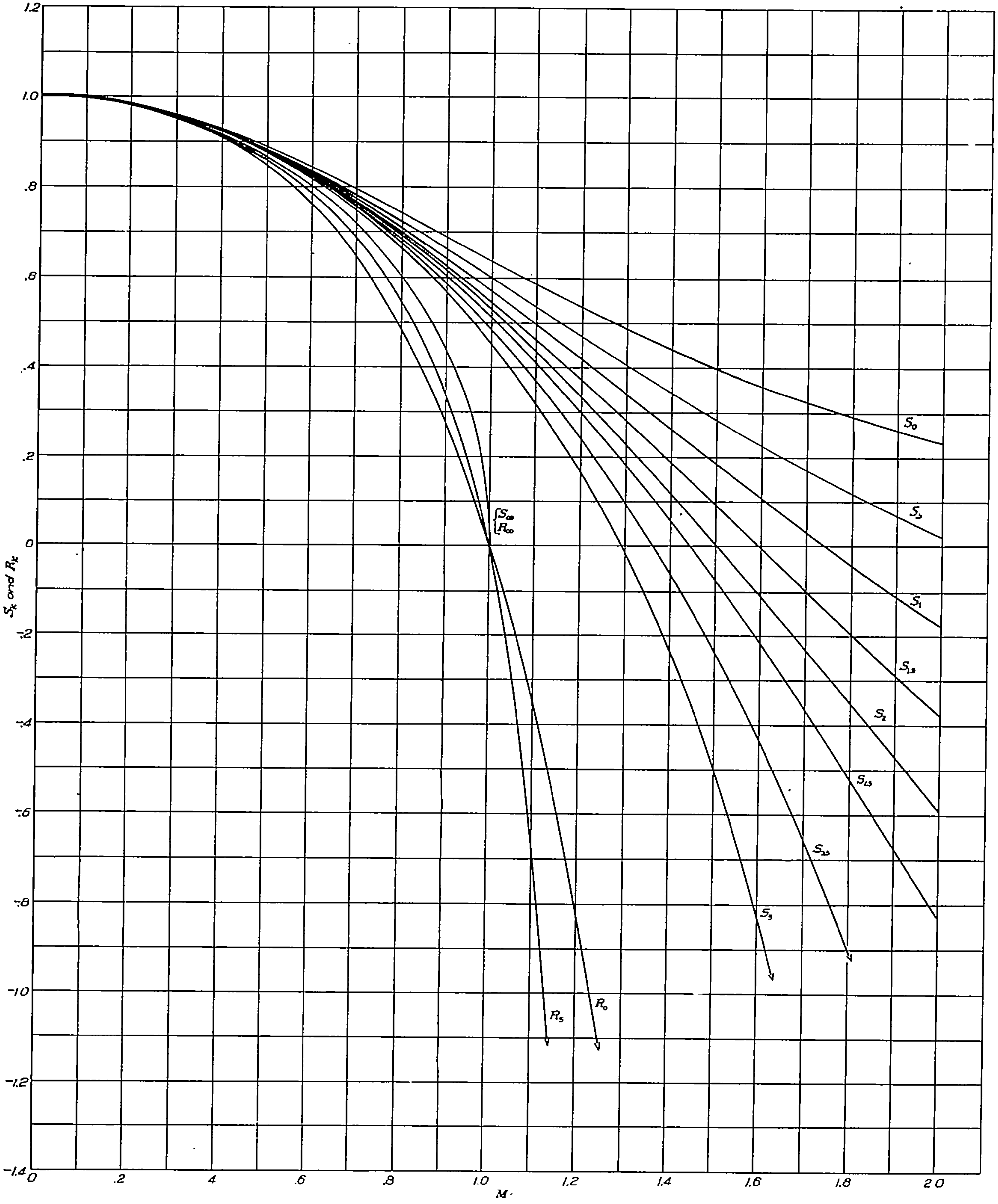

FIGURE 2.—The functions S_k and R_k against M for several positive values of the index k.

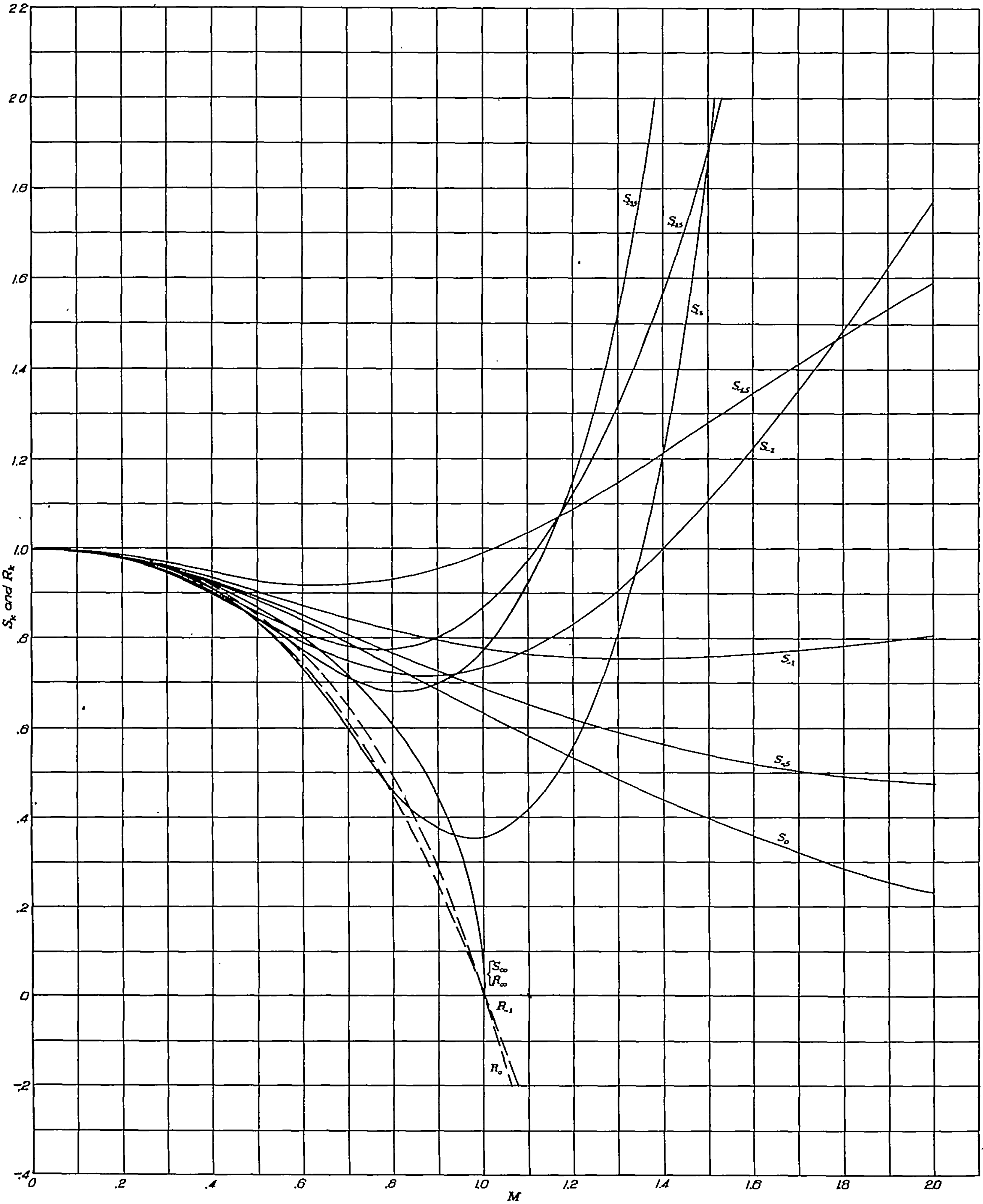

FIGURE 3.—The functions S_k and R_k against M for several negative values of the index k.

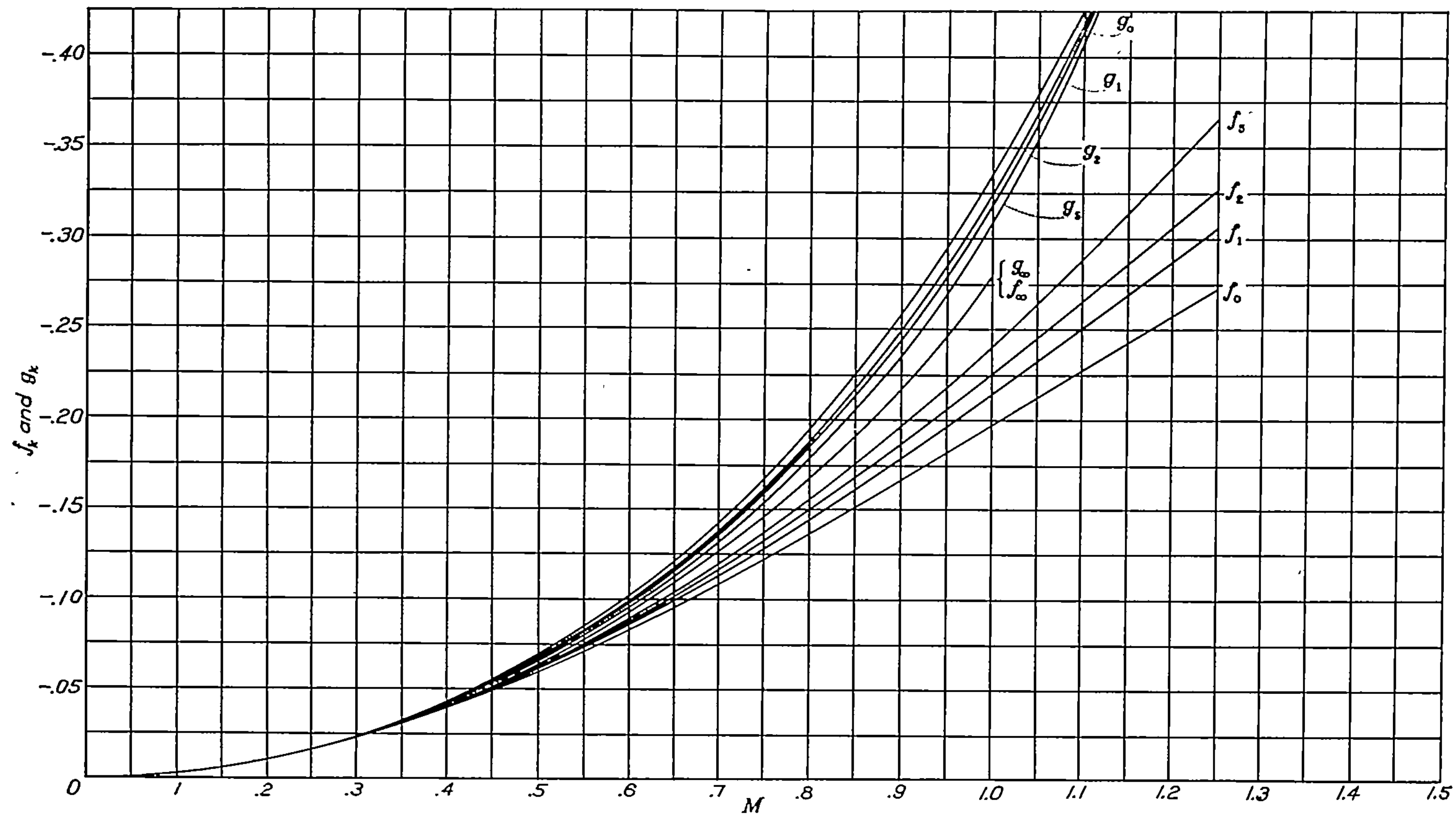

FIGURE 4.—The functions f_k and g_k against M for several positive values of the index k.

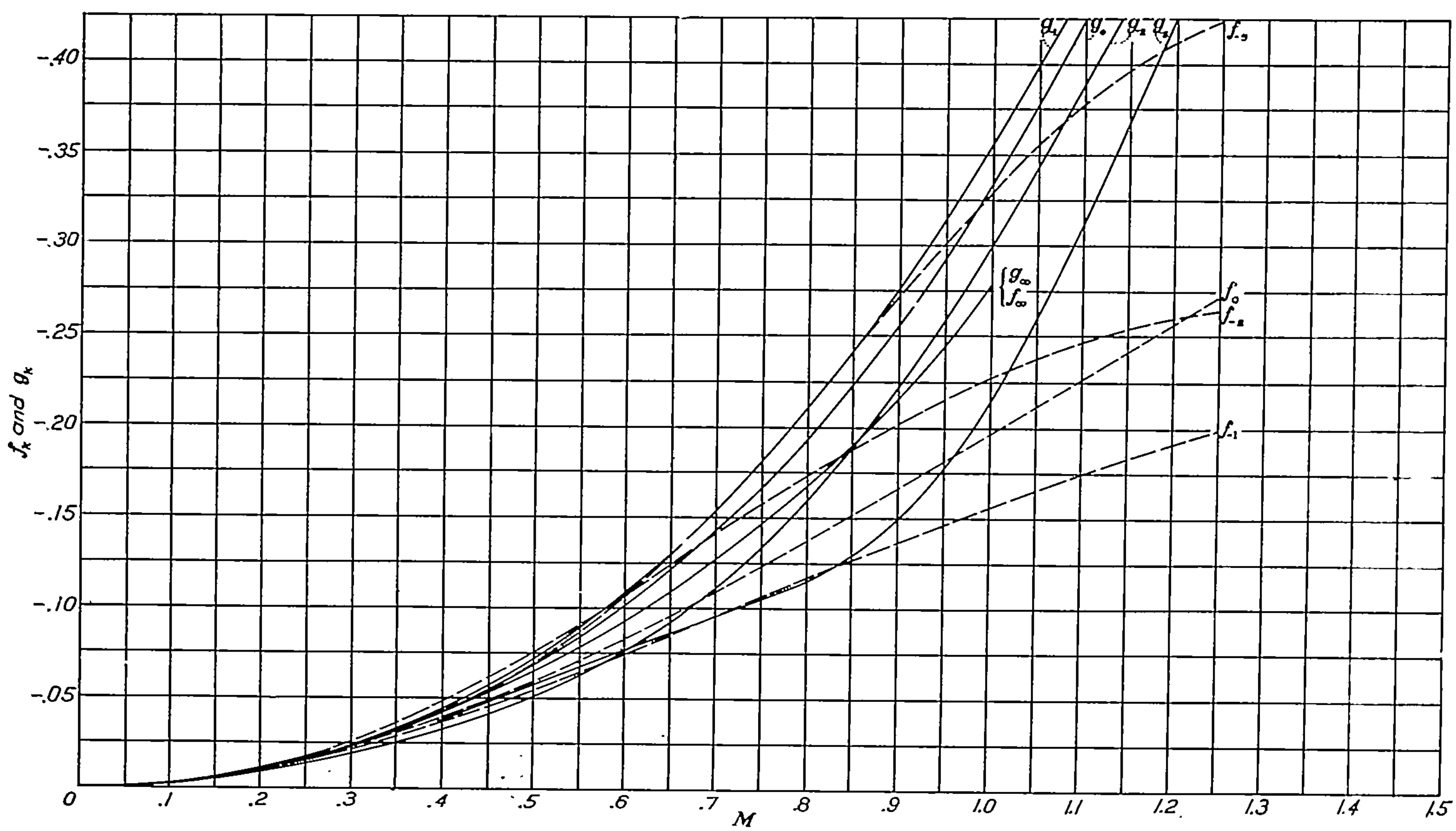

FIGURE 5.—The functions f_k and g_k against M for several negative values of the index k.

The opportunity is taken here to note that, for the von Kármán-Tsien case $\left(\gamma=-1 \text{ or } \beta=-\frac{1}{2}\right)$,

$$f_k=g_k=-\log\frac{1+(1-\tau)^{1/2}}{2}$$

$$=\log\frac{2\sqrt{1-M^2}}{1+\sqrt{1-M^2}}$$

and that the sets of functions P_k and Q_k, as in the incompressible case, are reduced to a single function by the operator $\frac{1}{k}\log$; namely (compare equation (14)),

$$\log q+\log\frac{2\sqrt{1-M^2}}{1+\sqrt{1-M^2}}$$

In fact, the complex flow potential $\phi+i\psi$ can be expressed as an analytic function of a single complex variable $\theta+i\log q\frac{2\sqrt{1-M^2}}{1+\sqrt{1-M^2}}$. Tsien has made use of this complex variable in his hodograph treatment of the compressible flow past an elliptic cylinder (reference 8).

VELOCITY CORRECTION FACTOR

The solution of the problem of an exact correspondence between the flow past a prescribed body in an incompressible fluid and the flow past the same body in a compressible fluid is a difficult matter. This problem can be solved exactly for certain types of flow patterns (not past closed shapes), such as flows inside or outside angles or channels, and for certain flow singularities such as a vortex, source, and doublet—types of flow which can be associated with the particular solutions Q_k. Some of these types of flow are illustrated by examples in the following section. Combining particular solutions to represent uniform flow past a prescribed body is a complicated process, since the treatment of infinite series in the functions Q_k for both positive and negative values of k is involved. Furthermore, the process of returning to the physical-plane variables from the hodograph-plane variables hinges on nonelementary parts of differential geometry. Certain types of jet problems can be properly treated in the subsonic range by series in Q_k with k positive, as was shown by Chaplygin (reference 3). Thus, it appears that much work remains to be done in order to render feasible exact and practical solutions for uniform flow past prescribed bodies in a compressible fluid. Because of the difficulty and complexity of the general problem of flow in a compressible fluid, attempts have been made by a number of investigators to obtain results by means of velocity correction formulas that serve to place in correspondence velocities in an incompressible and in a compressible fluid.

In part I the velocity correction factor was discussed with particular reference to the two functions L and $\tilde{L}$ (Q_0 and P_0 of the present paper) associated with a vortex and source type of flow, respectively. The main justification for the results of part I was the yielding and the unifying of the results of Chaplygin, von Kármán and Tsien, Temple and Yarwood, and Prandtl and Glauert. The knowledge of the infinite set of functions P_k and Q_k discussed in the present paper can now serve to establish further on a reasonable basis the concept of a velocity correction formula.

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In order that a single velocity correction factor be feasible, even for a flow associated with a particular solution, it is necessary that $P_k\approx Q_k$. Consider, for example, the functions Q_k and P_k insofar as the first power of the variable τ is concerned. It can be shown easily that

$$\frac{1}{k}\log Q_k=\log q+f_k(\tau)$$

$$\approx\log q-\frac{1}{2}\beta\tau$$

and

$$\frac{1}{k}\log P_k=\log q+g_k(\tau)$$

$$\approx\log q-\frac{1}{2}\beta\tau$$

Thus, to the first power of τ and independent of k,

$$f_k(\tau)=g_k(\tau)$$

$$\approx-\frac{1}{2}\beta\tau$$

Then

$$P_k\approx Q_k$$

$$\approx\left(qe^{-\frac{1}{2}\beta\tau}\right)^k$$

The nature of the correspondence between the incompressible flow and the compressible flow is such that

$$\left.\begin{aligned}\phi_i&=\phi_c\\ \psi_i&=\psi_c\end{aligned}\right\}\tag{41}$$

Without going into any details here of the field point correspondence or of the boundary distortion, the velocities in the incompressible and compressible cases may be placed in correspondence as follows:

$$(\log q)_i=\left(\log q-\frac{1}{2}\beta\tau\right)_c$$

or

$$q_i=q_ce^{-\frac{1}{2}\beta\tau}\tag{42}$$

This result implies that the complex variable

$$\theta+i\left(\log q-\frac{1}{2}\beta\tau\right)$$

in the compressible case corresponds to the complex variable $\theta+i\log q$ in the incompressible case. Equation (42) represents the approximation of Temple and Yarwood discussed in part I.

Consider now the functions Q_k and P_k insofar as large values of the index k are concerned. It is recalled that, as the index $k\rightarrow\pm\infty$,

$$R_k\rightarrow S_k$$

$$\rightarrow(1-M^2)^{1/2}$$

and that

$$\frac{1}{k}\log Q_k \to \frac{1}{k}\log P_k$$
$$\to \log q + h(\tau)$$

where

$$h(\tau) = f_{-\infty}(\tau) = g_{-\infty}(\tau)$$

Then, as $k \to \pm\infty$,

$$P_k \approx Q_k$$
$$\approx [qe^{h(\tau)}]^k$$

The function $h(\tau)$ is expressed in integral form in equation (40) and has been evaluated and tabulated in part I. (See also table 4 and fig. 4 of the present paper.) The correspondence of velocities in the incompressible and the compressible case is given by

$$q_i = q_c e^{h(\tau)} \tag{43}$$

Equation (43) constitutes the geometric-mean velocity correction formula introduced in part I and is limited to the subsonic range $0 \leqq M \leqq 1$. It is observed that, for positive values of k, $h(\tau)$ lies between $f_k(\tau)$ and $g_k(\tau)$ in magnitude. Moreover, the deviation of $e^{h(\tau)}$ from $e^{f_k(\tau)}$ and $e^{g_k(\tau)}$ is quite small in the subsonic range. (See table 6.)

The foregoing remarks, together with the fact that the geometric-mean type of approximation contains the results of Chaplygin, von Kármán and Tsien, Temple and Yarwood, and in the limiting case of small disturbances to the main flow the exact Prandtl-Glauert rule, lead to the suggestion that it may be adopted as an over-all type of approximation in the subsonic range.

FLOW PATTERNS CORRESPONDING TO THE PARTICULAR SOLUTIONS

Before the flow patterns corresponding to the particular solutions ϕ_k and ψ_k given by equations (4) for the compressible case are discussed, it is instructive to examine the incompressible case. Consider the complex velocity potential

$$\Omega = \phi + i\psi$$
$$= Uz^n \tag{44}$$

where U and n are constants and $z = x + iy$. It is well known that, if $n = \frac{\pi}{\alpha}$ where α is an angle between 0 and 2π, equation (44) represents the flow in a sharp angle. For example, the flow inside a right angle is obtained with $n=2$ and the flow outside a right angle is obtained with $n=\frac{2}{3}$. Again, the value $n=1$ or $\alpha=\pi$ corresponds to a uniform flow and the value $n=\frac{1}{2}$ or $\alpha=2\pi$ corresponds to the flow around a semi-infinite line. Clearly, all the angle flows are obtained with values of n between ½ and ∞. Other types of flows are given by other values of n. For example, $n=-1$ corresponds to a doublet and the remaining negative integers are associated with singularities of higher order than the doublet. In addition to the flows described by the powers z^n, there are the two fundamental flows, the source and the vortex, associated with the function $\log z$. If, now, it is desired to obtain generalizations for the compressible case of the foregoing particular flows, the procedure is first to express ϕ or ψ for the incompressible flow as a function of the hodograph variables q and θ and then to replace q^k by P_k or Q_k, respectively. Several examples will best illustrate this procedure:

(1) Consider the compressible generalization of the angle flows. By means of the relation

$$\frac{d\Omega}{dz} = e^{-iw}$$

where $w = \theta + i\log q$, the hodograph complex variable w is introduced as independent variable in place of z. From equation (44)

$$\frac{d\Omega}{dz} = nUz^{n-1}$$
$$= e^{-iw}$$

Hence

$$z = \frac{1}{(Un)^{\frac{1}{n-1}}}(e^{-iw})^{\frac{1}{n-1}}$$

and

$$\Omega = \frac{U}{(Un)^{\frac{n}{n-1}}}(e^{-iw})^{\frac{n}{n-1}}$$

Then

$$\psi = -\frac{U}{(Un)^{\frac{n}{n-1}}} q^{\frac{n}{n-1}} \sin\frac{n}{n-1}\theta$$

If $\frac{n}{n-1}$ is replaced by k, the compressible generalization of the angle flows is given by

$$\psi_k = -\frac{U}{\left(U\frac{k}{k-1}\right)^k} Q_k \sin k\theta \tag{45}$$

The inside angle flows are given by values of k in the range $1 < k < \infty$ and the outside angle flows, by values of k in the range $1 \leqq -k < \infty$. For example, $k=2$ for the flow inside a right angle, and $k=-2$ for the flow outside a right angle. Other types of flow are given by values of k in the range $-1 < k \leqq 1$.

The case $k=1$ or $n=\pm\infty$ is exceptional and, in fact, corresponds to the incompressible flow

$$\Omega = e^{cz} \tag{46}$$

where c is a constant.

(2) Consider the compressible generalization of the doublet. The complex velocity potential for the incompressible doublet at the origin is

$$\Omega = \frac{1}{z}$$

The reflected-velocity vector is

$$\frac{d\Omega}{dz} = -\frac{1}{z^2}$$
$$= e^{-iw}$$

Hence

$$z=ie^{\frac{1}{2}iw}$$

and

$$\Omega=-ie^{-\frac{1}{2}iw}$$

The stream function for the incompressible doublet is then given by

$$\psi=-q^{\frac{1}{2}}\cos\frac{1}{2}\theta$$

The compressible generalization of the doublet is therefore

$$\psi_{\frac{1}{2}}=-Q_{\frac{1}{2}}\cos\frac{1}{2}\theta$$

(3) Consider the compressible generalization of the source. The complex velocity potential for a unit source at the origin is

$$\Omega=\log z$$

The reflected-velocity vector is

$$\frac{d\Omega}{dz}=\frac{1}{z}$$

$$=e^{-iw}$$

Hence

$$z=e^{iw}$$

and

$$\Omega=iw$$

The velocity potential for the incompressible source is

$$\phi=-\log q$$

The compressible generalization of the source is then given by

$$\phi_0=-P_0$$

(4) Consider the compressible generalization of a point vortex. The complex velocity potential for a vortex of unit strength at the origin is

$$\Omega=-i\log z$$

The reflected-velocity vector is

$$\frac{d\Omega}{dz}=-\frac{i}{z}$$

$$=e^{-iw}$$

Hence

$$z=-ie^{iw}$$

and, except for an additive constant,

$$\Omega=w$$

The stream function for the incompressible vortex is

$$\psi=\log q$$

The compressible generalization of the vortex is then given by

$$\psi_0=Q_0$$

TRANSFORMATION FROM THE HODOGRAPH TO THE PHYSICAL VARIABLES

Given the velocity potential ϕ and the stream function ψ in terms of the hodograph variables θ and q, it is possible to express the coordinates x and y of the physical plane in terms of θ and q.

From the basic flow equations

$$\frac{\partial\phi}{\partial x}=\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial y}$$

$$\frac{\partial\phi}{\partial y}=-\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial x}$$

it follows (see equation (6) of reference 1) that

$$dz=\frac{1}{q}e^{i\theta}\left(d\phi+i\frac{\rho_0}{\rho}d\psi\right)$$

The real and imaginary parts of this equation yield

$$\left.\begin{aligned}dx&=\frac{1}{q}\left[\left(\frac{\partial\phi}{\partial q}\cos\theta-\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial q}\sin\theta\right)dq\right.\\&\qquad\left.+\left(\frac{\partial\phi}{\partial\theta}\cos\theta-\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial\theta}\sin\theta\right)d\theta\right]\\dy&=\frac{1}{q}\left[\left(\frac{\partial\phi}{\partial q}\sin\theta+\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial q}\cos\theta\right)dq\right.\\&\qquad\left.+\left(\frac{\partial\phi}{\partial\theta}\sin\theta+\frac{\rho_0}{\rho}\frac{\partial\psi}{\partial\theta}\cos\theta\right)d\theta\right]\end{aligned}\right\}\tag{47}$$

Equations (47) relate the differential line elements in the physical xy-plane and the hodograph θq-plane. When expressions for ϕ and ψ as functions of θ and q are known for a given flow, the integrals of equations (47) are the equations of transformation of the θ,q coordinates to the x,y coordinates. It may be remarked that the hodograph flow equations (1) are the integrability conditions for the differential equations (47). The right-hand sides of equations (47) are therefore perfect differentials.

Consider one set of particular solutions from equations (4)

$$\phi=P_k(q)\cos k\theta$$

$$\psi=-Q_k(q)\sin k\theta$$

where $k=\pm1$ and $k=0$ are excluded. By the use of equations (5), it can easily be verified that

$$\left.\begin{aligned}x_k&=\frac{k}{2}\left[\left(\frac{P_k}{q}-\frac{\rho_0}{\rho q}Q_k\right)\frac{\cos(k+1)\theta}{k+1}\right.\\&\qquad\left.+\left(\frac{P_k}{q}+\frac{\rho_0}{\rho q}Q_k\right)\frac{\cos(k-1)\theta}{k-1}\right]+\text{Constant}\\y_k&=\frac{k}{2}\left[\left(\frac{P_k}{q}-\frac{\rho_0}{\rho q}Q_k\right)\frac{\sin(k+1)\theta}{k+1}\right.\\&\qquad\left.-\left(\frac{P_k}{q}+\frac{\rho_0}{\rho q}Q_k\right)\frac{\sin(k-1)\theta}{k-1}\right]+\text{Constant}\end{aligned}\right\}\tag{48}$$

The equations of transformation corresponding to the other set of particular solutions from equations (4) are obtained by replacing in equations (48) the cosine by the sine and the sine by the negative cosine.

The excluded cases $k=0$ and $k=\pm 1$ are now treated. For $k=0$, one set of particular solutions corresponds to a source and is

$$\phi_0=-P_0$$

$$\psi_0=\theta$$

Equations (47) then yield

$$x=\frac{\rho_0}{\rho q}\cos\theta$$

$$y=\frac{\rho_0}{\rho q}\sin\theta$$

The other set of particular solutions corresponds to a vortex and is

$$\phi_0=\theta$$

$$\psi_0=Q_0$$

Equations (47) then yield

$$x=\frac{1}{q}\sin\theta$$

$$y=-\frac{1}{q}\cos\theta$$

For $k=1$ with

$$\phi_1=P_1\cos\theta$$

$$\psi_1=-Q_1\sin\theta$$

equations (47) yield

$$x_1=\frac{1}{4}\left(\frac{P_1}{q}-\frac{\rho_0}{\rho q}Q_1\right)\cos 2\theta+\frac{1}{2}\int\left(\frac{1}{q}\frac{dP_1}{dq}+\frac{\rho_0}{\rho q}\frac{dQ_1}{dq}\right)dq$$

$$y_1=\frac{1}{4}\left(\frac{P_1}{q}-\frac{\rho_0}{\rho q}Q_1\right)\sin 2\theta-\frac{1}{2}\left(\frac{P_1}{q}+\frac{\rho_0 Q_1}{\rho q}\right)\theta$$

With the use of equations (15) and (5),

$$\left.\begin{aligned}x_1&=\frac{1}{2}\left[1-\frac{1-(1-\tau)^{\beta+1}}{(\beta+1)\tau(1-\tau)^{\beta}}\right]\cos 2\theta+\log q\\&\quad-\frac{\beta}{\beta+1}g(\tau)-\frac{1}{2}\frac{2\beta+1}{\beta+1}\left[\frac{1}{(1-\tau)^{\beta}}-1\right]\\y_1&=\frac{1}{2}\left[1-\frac{1-(1-\tau)^{\beta+1}}{(\beta+1)\tau(1-\tau)^{\beta}}\right]\sin 2\theta-\theta\end{aligned}\right\}\quad(49)$$

where $g(\tau)=g_0(\tau)$ by equation (39) and is evaluated in part I (reference 1).

For $k=1$ with

$$\phi_1=P_1\sin\theta$$

$$\psi_1=Q_1\cos\theta$$

$$\left.\begin{aligned}x_1&=\frac{1}{2}\left[1-\frac{1-(1-\tau)^{\beta+1}}{(\beta+1)\tau(1-\tau)^{\beta}}\right]\sin 2\theta+\theta\\y_1&=\frac{1}{2}\left[1-\frac{1-(1-\tau)^{\beta+1}}{(\beta+1)\tau(1-\tau)^{\beta}}\right]\cos 2\theta+\log q\\&\quad-\frac{\beta}{\beta+1}g(\tau)-\frac{1}{2}\frac{2\beta+1}{\beta+1}\left[\frac{1}{(1-\tau)^{\beta}}-1\right]\end{aligned}\right\}\quad(50)$$

For $k=-1$ with

$$\phi_{-1}=P_{-1}\cos\theta$$

$$\psi_{-1}=Q_{-1}\sin\theta$$

equations (47) yield

$$x_{-1}=\frac{1}{4}\left(\frac{P_{-1}}{q}+\frac{\rho_0 Q_{-1}}{\rho q}\right)\cos 2\theta+\frac{1}{2}\int\left(\frac{1}{q}\frac{dP_{-1}}{dq}-\frac{\rho_0}{\rho q}\frac{dQ_{-1}}{dq}\right)dq$$

$$y_{-1}=\frac{1}{4}\left(\frac{P_{-1}}{q}+\frac{\rho_0 Q_{-1}}{\rho q}\right)\sin 2\theta-\frac{1}{2}\left(\frac{P_{-1}}{q}-\frac{\rho_0}{\rho q}Q_{-1}\right)\theta$$

With the use of equations (16) and (5),

$$\left.\begin{aligned}x_{-1}&=\frac{1}{4q^2}\left[\frac{3\beta+2}{\beta+1}\frac{1}{(1-\tau)^{\beta}}-\frac{\beta}{\beta+1}(1+\beta\tau)\right]\cos 2\theta\\&\quad+\frac{1}{4a_0^2}\log q+\frac{3\beta+2}{4(\beta+1)a_0^2}g(\tau)\\&\quad+\frac{3\beta+2}{8(\beta+1)}\frac{2\beta+1}{\beta a_0^2}\left[\frac{1}{(1-\tau)^{\beta}}-1\right]\\y_{-1}&=\frac{1}{4q^2}\left[\frac{3\beta+2}{\beta+1}\frac{1}{(1-\tau)^{\beta}}-\frac{\beta}{\beta+1}(1+\beta\tau)\right]\sin 2\theta+\frac{1}{4a_0^2}\theta\end{aligned}\right\}\quad(51)$$

For $k=-1$ with

$$\phi_{-1}=-P_{-1}\sin\theta$$

$$\psi_{-1}=Q_{-1}\cos\theta$$

$$\left.\begin{aligned}x_{-1}&=\frac{1}{4q^2}\left[\frac{3\beta+2}{\beta+1}\frac{1}{(1-\tau)^{\beta}}-\frac{\beta}{\beta+1}(1+\beta\tau)\right]\sin 2\theta\\&\quad-\frac{1}{4a_0^2}\theta\\y_{-1}&=-\frac{1}{4q^2}\left[\frac{3\beta+2}{\beta+1}\frac{1}{(1-\tau)^{\beta}}-\frac{\beta}{\beta+1}(1+\beta\tau)\right]\cos 2\theta\\&\quad+\frac{1}{4a_0^2}\log q+\frac{3\beta+2}{4(\beta+1)a_0^2}g(\tau)\\&\quad+\frac{3\beta+2}{8(\beta+1)}\frac{2\beta+1}{\beta a_0^2}\left[\frac{1}{(1-\tau)^{\beta}}-1\right]\end{aligned}\right\}\quad(52)$$

Ringleb (reference 2) gives an example of the flow of a compressible fluid around a semi-infinite line. An examination of Ringleb's stream function $\psi=\frac{1}{q}\sin\theta$ shows that it is a linear combination of ψ_1 and ψ_{-1}; that is,

$$\psi=\left(-\frac{1}{4a_0^2}Q_1+Q_{-1}\right)\sin\theta$$

In fact, all the external angle flows ($1\leq -k<\infty$) are nonunique; for, in view of the discussion preceding equation (11), a general form of ψ_{-k} is

$$\psi_{-k}=q^{-k}[A\tau^{k}Y_k(\tau)+Y_{-k}(\tau)]\sin k\theta$$

where A is an arbitrary constant.

OBSERVATIONS ON LIMIT LINES

In the present section there are reviewed briefly certain conditions, discussed by Tollmien (reference 9) and Ringleb (reference 10), with regard to possible limitations on the potential flow of an adiabatic compressible fluid.

Consider the family of streamlines

$$\psi(\theta, q) = \text{Constant}$$

Then along a streamline

$$d\psi = \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial q} dq = 0$$

and, from equations (47), the line elements along a streamline are

$$\left.\begin{aligned} dx &= \frac{\rho_0}{\rho}\left[\frac{1}{q^2}(M^2-1)\left(\frac{\partial \psi}{\partial \theta}\right)^2 - \left(\frac{\partial \psi}{\partial q}\right)^2\right]\frac{\cos\theta}{\partial\psi/\partial\theta} dq \\ dy &= \frac{\rho_0}{\rho}\left[\frac{1}{q^2}(M^2-1)\left(\frac{\partial \psi}{\partial \theta}\right)^2 - \left(\frac{\partial \psi}{\partial q}\right)^2\right]\frac{\sin\theta}{\partial\psi/\partial\theta} dq \end{aligned}\right\} \tag{53}$$

Singular points along a streamline are characterized by the vanishing of the common factor of equations (53):

$$\left(\frac{\partial \psi}{\partial q}\right)^2 - \frac{1}{q^2}(M^2-1)\left(\frac{\partial \psi}{\partial \theta}\right)^2 = 0 \tag{54}$$

(Stagnation points at which $\frac{\partial \psi}{\partial \theta}$ and $\frac{\partial \psi}{\partial q}$ vanish, the vortex for which $\frac{\partial \psi}{\partial \theta}=0$ and the source for which $\frac{\partial \psi}{\partial q}=0$, are excluded from this discussion.) Observe now from equations (47) that the Jacobian of the transformation from the hodograph variables θ and q to the physical-plane variables x and y is given by

$$\begin{aligned} J\left(\frac{x,y}{\theta,q}\right) &= \frac{\rho_0}{\rho q^2}\left(\frac{\partial \phi}{\partial \theta}\frac{\partial \psi}{\partial q} - \frac{\partial \phi}{\partial q}\frac{\partial \psi}{\partial \theta}\right) \\ &= \frac{1}{q}\left(\frac{\rho_0}{\rho}\right)^2\left[\left(\frac{\partial \psi}{\partial q}\right)^2 - \frac{1}{q^2}(M^2-1)\left(\frac{\partial \psi}{\partial \theta}\right)^2\right] \end{aligned} \tag{55}$$

Thus, the vanishing of the Jacobian is equivalent to the condition for the existence of a singular locus for the family of streamlines

$$\psi(\theta,q) = \text{Constant}$$

This singular locus consists of points at which the streamlines undergo an abrupt change of curvature and means, physically, that the acceleration $q\frac{dq}{ds}$ of a fluid particle is infinite at such points.

Both Ringleb and Tollmien have shown that the singular locus for the streamlines is also the envelope of the Mach lines in the plane of flow. The Mach lines are related to the streamlines in such a way that the component of the fluid velocity normal to a Mach line is equal to the local velocity of sound. The Mach lines are identical with the so-called characteristic curves of the second-order partial differential equations for ϕ and ψ and are the integral curves of the ordinary differential equation

$$d\theta^2 - \frac{1}{q^2}(M^2-1)dq^2 = 0$$

or

$$d\theta = \pm\frac{1}{q}\sqrt{M^2-1}\, dq \tag{56}$$

The real solutions of this differential equation interpreted in the physical xy-plane yield the Mach lines for a given flow. The solution of equation (56) is

$$\theta - \theta_0 = \pm\left[\frac{1}{\sqrt{\tau_s}}\tan^{-1}(\sqrt{\tau_s}\sqrt{M^2-1}) - \tan^{-1}\sqrt{M^2-1}\right] \tag{57}$$

where $\tau_s = \frac{\gamma-1}{\gamma+1}$ and where θ_0 assumes the values of θ along the $M=1$ line for a given flow.

It is recalled that the function

$$H = \log q + h(\tau)$$

introduced in part I (reference 1) in connection with the geometric-mean type of velocity correction formula is a solution of the differential equation

$$dH = \pm\frac{\sqrt{1-M^2}}{q} dq$$

in the subsonic range. Observe that a continuation of the function H into the supersonic range is given by equation (56) as

$$d\theta = \pm\frac{\sqrt{M^2-1}}{q} dq$$

In the supersonic range, the function $H=\theta-\theta_0$ can thus be interpreted as the hodograph of the Mach lines for a given flow.

The differential line elements dx and dy for the Mach lines in the physical plane are now given. From equations (47) and (56), the line elements along a Mach line for a given flow are

$$\left.\begin{aligned} dx &= \frac{\rho_0}{\rho q}(\pm\sqrt{M^2-1}\cos\theta - \sin\theta)\left(\frac{\partial \psi}{\partial q} \pm \frac{1}{q}\sqrt{M^2-1}\frac{\partial \psi}{\partial \theta}\right) dq \\ dy &= \frac{\rho_0}{\rho q}(\pm\sqrt{M^2-1}\sin\theta + \cos\theta)\left(\frac{\partial \psi}{\partial q} \pm \frac{1}{q}\sqrt{M^2-1}\frac{\partial \psi}{\partial \theta}\right) dq \end{aligned}\right\} \tag{58}$$

Singular points along a Mach line are characterized by the vanishing of the common factor of equations (58)

$$\frac{\partial \psi}{\partial q} \pm \frac{1}{q}\sqrt{M^2-1}\frac{\partial \psi}{\partial \theta} = 0 \tag{59}$$

Equation (59) represents in the plane of flow two possible singular loci or "limit lines" for the two families of Mach lines associated with the plus and minus signs in equation (56). Clearly, the two singular loci cannot occur simultaneously since the two conditions cannot be satisfied simultaneously. Observe that equation (59) is equivalent to the vanishing of the Jacobian given in equations (55). Thus, the vanishing of the Jacobian is not only the condition for the existence of a singular (cusp) locus for the streamlines but also the condition for the existence of a limit line (envelope) for the Mach lines.

The existence of a singular locus may be looked upon as being equivalent to the vanishing along a curve of the Jacobian $J\left(\frac{x,y}{\theta,q}\right)$ of the transformation from the hodograph-plane variables θ and q to the physical-plane variables x and y. It is remarked that singular solutions exist for which the Jacobian $J\left(\frac{\theta,q}{x,y}\right)$ of the transformation from the physical-plane variables x and y to the hodograph-plane variables θ and q vanishes identically in a region of the physical plane. In this case, as Tollmien pointed out, θ and q are no longer independent variables and the flow cannot be described in the hodograph plane. Examples of these "missed flows" are the solutions of Meyer (reference 11) for supersonic flow inside and outside sharp angles.

It is of special interest to apply the condition for the vanishing of the Jacobian to the particular solutions ϕ_k and ψ_k treated in the early part of this paper. The expression for the Jacobian for a particular solution

$$\phi_k = P_k \cos k\theta$$

$$\psi_k = -Q_k \sin k\theta$$

is, with the use of equations (5), for $k \neq 0$,

$$\begin{aligned} J &= \frac{\rho_0}{\rho q^2}\left(\frac{\partial\phi}{\partial\theta}\frac{\partial\psi}{\partial q} - \frac{\partial\phi}{\partial q}\frac{\partial\psi}{\partial\theta}\right) \\ &= \frac{k^2}{q^3}\left[P_k^2 \sin^2 k\theta + \left(\frac{\rho_0}{\rho}\right)^2 (1-M^2) Q_k^2 \cos^2 k\theta\right] \end{aligned} \tag{60}$$

Clearly, this expression for J is positive in the subsonic range $M<1$. At the sonic value $M=1$, $P_k \neq 0$ (see table following equation (23)) and J is again positive. At the first zero of P_k in the supersonic range $M>1$, $Q_k \neq 0$; hence, J is negative. The values of M, for all the pairs of values θ, M for which the Jacobian J vanishes, therefore lie between $M=1$ and the value of M at the first zero of P_k (or S_k) in the supersonic range.

By means of the relation

$$P_k = \frac{\rho_0}{\rho} Q_k S_k$$

the vanishing of the Jacobian yields

$$\cot k\theta = \mp \frac{S_k}{\sqrt{M^2-1}} \tag{61}$$

Equation (61) is the relation for pairs of values θ, M, which interpreted in the physical xy-plane constitute the limit line for the particular flow ϕ_k, ψ_k. The values of M that satisfy equation (61) accordingly lie between $M=1$ and the value of M at the first zero of S_k in the supersonic range.

This paper is closed with the following remarks on limiting values of M in connection with the use of velocity correction formulas. The limiting local values of M in the case of uniform flow past a prescribed boundary, in general, depend on shape parameters. The use of a velocity correction formula, however, yields a constant limiting value of M that depends only on the particular correction formula used. The geometric-mean correction formula yields the value $M=1$; the approximation of Temple and Yarwood yields $M=1.35$; and the arithmetic-mean correction formula given in part I (reference 1), which is based on a linear combination of a source (limiting value $M=1$) and a vortex (limiting value $M=\infty$) or a spiral flow, yields the value $M=1.15$.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., *September 29, 1944.*

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TABLE 1.—THE FUNCTIONS Y_k FOR SEVERAL VALUES OF THE INDEX k

M	τ	$Y_{0.5}$	Y_1	$Y_{1.5}$	Y_2	$Y_{2.5}$	$Y_{3.5}$	Y_5	$Y_{-0.5}$	Y_{-1}	$Y_{-1.5}$	Y_{-2}	$Y_{-2.5}$	$Y_{-3.5}$	Y_{-5}
0.1	0.00200	0.99875	0.99751	0.99626	0.99502	0.99378	0.99130	0.98759	1.00125	1.00249	1.00371	1.00517	1.00634	1.00883	1.01261
.2	.00794	.99505	.99012	.98520	.98030	.97542	.96572	.95134	1.00494	1.00982	1.01442	1.02207	1.02642	1.03632	1.05178
.3	.01768	.98902	.97809	.96724	.95648	.94582	.92481	.89414	1.01094	1.02162	1.03089	1.05313	1.06271	1.08597	1.12213
.4	.03101	.98085	.96184	.94306	.92456	.90638	.87096	.82027	1.01904	1.03728	1.05128	1.10000	1.11755	1.16425	1.23409
.5	.04762	.97078	.94188	.91353	.88584	.85886	.80709	.73491	1.02897	1.05606	1.07340	1.16253	1.19120	1.27940	1.41003
.55	.05705	.96513	.93071	.89707	.86438	.83268	.77240	.68961	1.03453	1.06637	1.08441	1.19901	1.23423	1.35293	1.53449
.6	.06716	.95911	.91884	.87966	.84176	.80524	.73642	.64344	1.04042	1.07714	1.09502	1.23835	1.28044	1.43746	1.69314
.65	.07792	.95276	.90636	.86141	.81818	.77678	.69954	.59703	1.04660	1.08828	1.10499	1.27993	1.32883	1.53249	1.89526
.7	.08925	.94613	.89336	.84247	.79383	.74755	.66213	.55092	1.05304	1.09967	1.11410	1.32307	1.37819	1.63666	2.15092
.75	.10112	.93925	.87991	.82297	.76888	.71778	.62455	.50563	1.05969	1.11122	1.12216	1.36702	1.42716	1.74773	2.47001
.8	.11348	.93216	.86609	.80303	.74352	.68772	.58723	.46159	1.06651	1.12285	1.12901	1.41099	1.47429	1.86253	2.86074
.825	.11982	.92855	.85907	.79294	.73074	.67264	.56857	.44016	1.06997	1.12866	1.13195	1.43274	1.49670	1.92011	3.08469
.85	.12626	.92491	.85198	.78278	.71791	.65756	.55015	.41917	1.07347	1.13446	1.13454	1.45421	1.51810	1.97707	3.32803
.875	.13279	.92122	.84485	.77257	.70506	.64251	.53192	.39867	1.07698	1.14024	1.13678	1.47529	1.53830	2.03280	3.59058
.9	.13941	.91751	.83766	.76232	.69220	.62751	.51389	.37868	1.08052	1.14598	1.13866	1.49591	1.55714	2.08667	3.87173
.925	.14612	.91378	.83045	.75205	.67936	.61257	.49610	.35923	1.08407	1.15168	1.14017	1.51597	1.57444	2.13801	4.17044
.95	.15290	.91002	.82320	.74176	.66654	.59773	.47856	.34034	1.08763	1.15734	1.14132	1.53540	1.59006	2.18615	4.48517
.96	.15563	.90851	.82030	.73765	.66143	.59182	.47163	.33294	1.08905	1.15958	1.14167	1.54297	1.59580	2.20436	4.61510
.98	.16113	.90549	.81448	.72942	.65122	.58006	.45790	.31844	1.09191	1.16405	1.14220	1.55776	1.60636	2.23874	4.88108
1.00	.16667	.90246	.80866	.72121	.64106	.56838	.44436	.30432	1.09477	1.16847	1.14249	1.57203	1.61565	2.27013	5.15401
1.02	.17224	.89942	.80283	.71301	.63094	.55680	.43103	.29068	1.09763	1.17285	1.14254	1.58576	1.62360	2.29819	5.43233
1.04	.17785	.89638	.79701	.70483	.62088	.54531	.41792	.27724	1.10048	1.17718	1.14236	1.59891	1.63016	2.32258	5.71424
1.06	.18349	.89334	.79119	.69667	.61088	.53394	.40502	.26430	1.10333	1.18147	1.14194	1.61145	1.63528	2.34300	5.99777
1.08	.18915	.89029	.78538	.68855	.60094	.52267	.39235	.25175	1.10618	1.18570	1.14129	1.62337	1.63891	2.35914	6.28077
1.10	.19485	.88725	.77958	.68046	.59108	.51153	.37992	.23961	1.10902	1.18987	1.14042	1.63462	1.64102	2.37071	6.56092
1.12	.20056	.88421	.77379	.67241	.58129	.50052	.36773	.22786	1.11185	1.19399	1.13933	1.64520	1.64157	2.37746	6.83579
1.15	.20917	.87966	.76515	.66042	.56677	.48424	.34990	.21098	1.11608	1.20006	1.13728	1.65975	1.63943	2.37799	7.23249
1.18	.21782	.87512	.75655	.64854	.55245	.46828	.33263	.19499	1.12028	1.20599	1.13475	1.67269	1.63367	2.36639	7.60241
1.20	.22360	.87211	.75085	.64069	.54303	.45782	.32143	.18481	1.12306	1.20987	1.13282	1.68040	1.62782	2.35161	7.82954
1.25	.23810	.86462	.73673	.62132	.51991	.43234	.29456	.16102	1.12995	1.21927	1.12712	1.69640	1.60612	2.28904	8.30880
1.30	.25262	.85721	.72282	.60238	.49748	.40784	.26929	.13953	1.13673	1.22825	1.12030	1.70770	1.57447	2.18883	8.62759
1.35	.26713	.84990	.70915	.58388	.47576	.38436	.24563	.12022	1.14339	1.23680	1.11246	1.71430	1.53323	2.05067	8.74667
1.40	.28161	.84270	.69576	.56587	.45480	.36192	.22354	.10297	1.14990	1.24491	1.10370	1.71632	1.48294	1.87536	8.63279
1.45	.29602	.83563	.68265	.54838	.43461	.34053	.20301	.08763	1.15627	1.25260	1.09414	1.71392	1.42427	1.66476	8.26094
1.50	.31034	.82869	.66985	.53141	.41521	.32019	.18396	.07406	1.16249	1.25986	1.08389	1.70731	1.35802	1.42162	7.61597
1.55	.32455	.82190	.65738	.51500	.39660	.30089	.16635	.06213	1.16854	1.26669	1.07306	1.69677	1.28508	1.14948	6.69350
1.60	.33862	.81526	.64524	.49913	.37879	.28262	.15012	.05168	1.17442	1.27312	1.06176	1.68258	1.20639	.85252	5.50014
1.65	.35254	.80877	.63345	.48383	.36177	.26535	.13520	.04258	1.18014	1.27914	1.05009	1.66508	1.12291	.53530	4.05311
1.70	.36629	.80245	.62200	.46909	.34552	.24905	.12151	.03469	1.18568	1.28479	1.03814	1.64459	1.03562	.20267	2.37916
1.75	.37984	.79630	.61091	.45491	.33004	.23370	.10897	.02789	1.19105	1.29006	1.02599	1.62146	.94546	−.14043	.51304
1.80	.39320	.79031	.60017	.44128	.31530	.21926	.09753	.02206	1.19625	1.29499	1.01373	1.59603	.85334	−.48912	−1.50416
1.85	.40635	.78449	.58978	.42819	.30129	.20569	.08710	.01709	1.20128	1.29957	1.00144	1.56864	.76012	−.83871	−3.62711
1.90	.41928	.77885	.57974	.41564	.28798	.19296	.07761	.01288	1.20614	1.30384	.98916	1.53961	.66659	−1.18479	−5.80829
1.95	.43198	.77337	.57005	.40361	.27535	.18102	.06900	.00934	1.21083	1.30781	.97697	1.50924	.57349	−1.52335	−7.99978
2.00	.44444	.76805	.56070	.39209	.26337	.16984	.06119	.00637	1.21535	1.31150	.96490	1.47782	.48146	−1.85078	−10.15503

TABLE 2.—THE FUNCTIONS S_k FOR SEVERAL VALUES OF THE INDEX k

M	τ	S_0	$S_{0.5}$	S_1	$S_{1.5}$	S_2	$S_{2.5}$	$S_{3.5}$	S_5	S_∞
0.1	0.00200	0.99502	0.99501	0.99501	0.99500	0.99500	0.99500	0.99500	0.99500	0.99499
.2	.00794	.98028	.98018	.98012	.98007	.98004	.98001	.97997	.97994	.97980
.3	.01768	.95638	.95592	.95560	.95537	.95520	.95507	.95487	.95468	.95394
.4	.03101	.92427	.92286	.92189	.92118	.92064	.92022	.91958	.91896	.91652
.5	.04762	.88517	.88187	.87957	.87788	.87657	.87553	.87397	.87242	.86603
.55	.05705	.86342	.85871	.85541	.85295	.85105	.84953	.84723	.84492	.83516
.6	.06716	.84045	.83396	.82938	.82594	.82326	.82110	.81784	.81451	.80000
.65	.07792	.81644	.80777	.80158	.79692	.79325	.79028	.78575	.78109	.75993
.7	.08925	.79158	.78028	.77215	.76596	.76105	.75706	.75092	.74454	.71414
.75	.10112	.76604	.75164	.74118	.73315	.72674	.72148	.71832	.70473	.66144
.8	.11348	.73999	.72202	.70881	.69858	.69035	.68355	.67295	.66153	.60000
.825	.11982	.72683	.70688	.69214	.68067	.67140	.66371	.65161	.63860	.56513
.85	.12626	.71361	.69155	.67517	.66236	.65196	.64330	.62959	.61475	.52678
.875	.13279	.70034	.67605	.65791	.64365	.63203	.62231	.60684	.58995	.48412
.9	.13941	.68704	.66039	.64038	.62457	.61162	.60075	.58334	.56419	.43589
.925	.14612	.67374	.64459	.62259	.60512	.59074	.57862	.55910	.53742	.37997
.95	.15290	.66044	.62868	.60457	.58532	.56941	.55593	.53409	.50962	.31225
.96	.15563	.65513	.62229	.59730	.57730	.56075	.54669	.52387	.49820	.28000
.98	.16113	.64452	.60945	.58265	.56111	.54321	.52795	.50306	.47484	.19900
1.00	.16667	.63394	.59657	.56787	.54472	.52539	.50886	.48174	.45076	0
1.02	.17224	.62339	.58364	.55298	.52812	.50729	.48941	.45992	.42594	-------
1.04	.17785	.61289	.57068	.53796	.51134	.48893	.46962	.43758	.40036	-------
1.06	.18349	.60243	.55768	.52285	.49436	.47030	.44947	.41473	.37399	-------
1.08	.18915	.59203	.54468	.50763	.47721	.45140	.42897	.39135	.34681	-------
1.10	.19485	.58170	.53165	.49233	.45990	.43225	.40813	.36743	.31879	-------
1.12	.20056	.57143	.51863	.47695	.44241	.41285	.38695	.34297	.28989	-------
1.15	.20917	.55616	.49910	.45374	.41590	.38330	.35454	.30526	.24482	-------
1.18	.21782	.54108	.47961	.43040	.38906	.35321	.32138	.26628	.19759	-------
1.20	.22360	.53114	.46665	.41477	.37100	.33286	.29885	.23958	.16482	-------
1.25	.23810	.50670	.43438	.37554	.32531	.28104	.24110	.17025	.07809	-------
1.30	.25262	.48290	.40241	.33616	.27895	.22793	.18133	.09712	−.01622	-------
1.35	.26713	.45980	.37082	.29675	.23203	.17362	.11960	.02001	−.11915	-------
1.40	.28161	.43742	.33969	.25740	.18467	.11821	.05595	−.06128	−.23203	-------
1.45	.29602	.41581	.30909	.21823	.13698	.06181	−.00956	−.14700	−.35657	-------
1.50	.31034	.39498	.27908	.17932	.08907	.00451	−.07688	−.23739	−.49497	-------
1.55	.32455	.37495	.24971	.14076	.04105	−.05357	−.14595	−.33276	−.65017	-------
1.60	.33862	.35573	.22103	.10588	−.00698	−.11233	−.21670	−.43347	−.82614	-------
1.65	.35254	.33731	.19308	.06500	−.05491	−.17166	−.28906	−.53992	−1.02840	-------
1.70	.36629	.31969	.16589	.02795	−.10264	−.23145	−.36296	−.65256	−1.26488	---- --
1.75	.37984	.30287	.13948	−.00847	−.15009	−.29150	−.43834	−.77196	−1.54739	-------
1.80	.39320	.28682	.11387	−.04421	−.19716	−.35197	−.51511	−.89874	−1.89427	-------
1.85	.40635	.27153	.08907	−.07921	−.24376	−.41249	−.59320	−1.03367	−2.33576	-------
1.90	.41928	.25699	.06509	−.11343	−.28982	−.47303	−.67251	−1.17762	−2.92543	-------
1.95	.43198	.24317	.04194	−.14685	−.33527	−.53348	−.75295	−1.33167	−3.76819	-------
2.00	.44444	.23005	.01960	−.17942	−.38003	−.59375	−.83445	−1.49711	−5.10125	-------

TABLE 2.—THE FUNCTIONS S_k FOR SEVERAL VALUES OF THE INDEX k—Concluded

M	τ	S_0	$S_{-0.5}$	S_{-1}	$S_{-1.5}$	S_{-2}	$S_{-2.5}$	$S_{-3.5}$	S_{-5}	S_∞
0.1	0.00200	0.99502	0.99503	0.99505	0.99511	0.99470	0.99487	0.99494	0.99497	0.99499
.2	.00794	.98028	.98043	.98074	.98165	.97674	.97823	.97902	.97944	.97980
.3	.01768	.95638	.95714	.95862	.96292	.94371	.94794	.94966	.95172	.95394
.4	.03101	.92427	.92656	.93093	.94336	.89696	.90476	.90341	.90716	.91652
.5	.04762	.88517	.89040	.90022	.92750	.84251	.85489	.84049	.83602	.86603
.55	.05705	.86342	.87082	.88452	.92218	.81515	.83074	.80524	.78695	.83516
.6	.06716	.84045	.85053	.86899	.91911	.78929	.80916	.77005	.72839	.80000
.65	.07792	.81644	.82977	.85386	.91858	.76602	.79166	.73743	.66200	.75993
.7	.08925	.79158	.80874	.83938	.92077	.74630	.77959	.70999	.59148	.71414
.75	.10112	.76604	.78764	.82572	.92583	.73089	.77411	.69027	.52200	.66144
.8	.11348	.73999	.76666	.81304	.93381	.72037	.77615	.68050	.45926	.60000
.825	.11982	.72683	.75625	.80710	.93890	.71707	.78022	.67994	.43205	.56513
.85	.12626	.71361	.74594	.80145	.94470	.71513	.78643	.68256	.40833	.52678
.875	.13279	.70034	.73573	.79610	.95122	.71456	.79485	.68855	.38856	.48412
.9	.13941	.68704	.72564	.79104	.95844	.71539	.80553	.69808	.37302	.43589
.925	.14612	.67374	.71568	.78630	.96635	.71763	.81852	.71132	.36202	.37997
.95	.15290	.66044	.70586	.78186	.97493	.72127	.83387	.72846	.35570	.31225
.96	.15563	.65513	.70198	.78018	.97855	.72311	.84067	.73644	.35452	.28000
.98	.16113	.64452	.69430	.77696	.98609	.72747	.85545	.75441	.35449	.19900
1.00	.16667	.63394	.68673	.77394	.99402	.73272	.87180	.77513	.35761	0
1.02	.17224	.62339	.67927	.77113	1.00235	.73884	.88974	.79870	.36389	-------
1.04	.17785	.61289	.67193	.76851	1.01106	.74582	.90928	.82526	.37338	-------
1.06	.18349	.60243	.66472	.76610	1.02012	.75366	.93046	.85493	.38608	-------
1.08	.18915	.59203	.65763	.76388	1.02954	.76234	.95329	.88787	.40203	-------
1.10	.19485	.58170	.65068	.76186	1.03930	.77186	.97780	.92424	.42126	-------
1.12	.20056	.57143	.64386	.76003	1.04938	.78218	1.00401	.96425	.44384	-------
1.15	.20917	.55616	.63388	.75765	1.06507	.79915	1.04659	1.03159	.48414	-------
1.18	.21782	.54108	.62421	.75568	1.08141	.81787	1.09319	1.10856	.53245	-------
1.20	.22360	.53114	.61794	.75459	1.09264	.83128	1.12656	1.16579	.56933	-------
1.25	.23810	.50670	.60290	.75263	1.12176	.86797	1.21833	1.33275	.67937	-------
1.30	.25262	.48290	.58875	.75170	1.15220	.90893	1.32274	1.54142	.81850	-------
1.35	.26713	.45980	.57549	.75173	1.18372	.95389	1.44086	1.80566	.99367	-------
1.40	.28161	.43742	.56312	.75264	1.21609	1.00255	1.57409	2.14809	1.21624	-------
1.45	.29602	.41581	.55161	.75433	1.24906	1.05464	1.72424	2.60776	1.50541	-------
1.50	.31034	.39498	.54095	.75676	1.28242	1.10986	1.89364	3.25805	1.89549	-------
1.55	.32455	.37495	.53111	.75982	1.31595	1.16794	2.08535	4.25443	2.45352	-------
1.60	.33862	.35573	.52204	.76346	1.34945	1.22861	2.30340	5.99252	3.32923	-------
1.65	.35254	.33731	.51373	.76759	1.38273	1.29160	2.55316	9.86167	4.93773	-------
1.70	.36629	.31969	.50612	.77214	1.41560	1.35665	2.84192	26.61701	9.01193	-------
1.75	.37984	.30287	.49919	.77706	1.44790	1.42349	3.17976	−38.80980	43.87853	-------
1.80	.39320	.28682	.49289	.78228	1.47949	1.49187	3.58098	−11.12607	−15.39152	-------
1.85	.40635	.27153	.48719	.78772	1.51022	1.56155	4.06658	−6.40028	−6.42359	-------
1.90	.41928	.25699	.48205	.79340	1.53998	1.63226	4.66846	−4.41184	−3.94403	-------
1.95	.43198	.24317	.47743	.79920	1.56866	1.70378	5.43767	−3.29536	−2.74361	-------
2.00	.44444	.23005	.47329	.80510	1.59617	1.77585	6.46116	−2.56565	−2.00955	-------

TABLE 3.—THE FUNCTIONS R_k FOR SEVERAL VALUES OF THE INDEX k

M	τ	R_0	$R_{0.5}$	R_1	$R_{1.5}$	R_2	$R_{2.5}$	$R_{3.5}$	R_5	R_∞
0.1	0.00200	0.99496	0.99496	0.99497	0.99497	0.99497	0.99497	0.99498	0.99498	0.99499
.2	.00794	.97932	.97941	.97947	.97952	.97955	.97958	.97962	.97966	.97980
.3	.01768	.95150	.95197	.95228	.95251	.95268	.95281	.95301	.95320	.95394
.4	.03101	.90882	.91022	.91117	.91187	.91241	.91283	.91346	.91408	.91652
.5	.04762	.84729	.85047	.85269	.85433	.85561	.85662	.85815	.85968	.86603
.55	.05705	.80783	.81227	.81540	.81775	.81958	.82104	.82327	.82552	.83516
.6	.06716	.76149	.76742	.77166	.77487	.77740	.77944	.78255	.78575	.80000
.65	.07792	.70734	.71493	.72045	.72467	.72802	.73076	.73497	.73935	.75993
.7	.08925	.64428	.65361	.66050	.66584	.67012	.67366	.67917	.68499	.71414
.75	.10112	.57112	.58206	.59027	.59674	.60201	.60639	.61333	.62080	.66144
.8	.11348	.48649	.49860	.50789	.51533	.52147	.52666	.53496	.54419	.60000
.825	.11982	.43941	.45181	.46143	.46921	.47568	.48119	.49014	.50012	.56513
.85	.12626	.38887	.40127	.41101	.41896	.42564	.43137	.44076	.45140	.52678
.875	.13279	.33466	.34668	.35624	.36413	.37083	.37662	.38622	.39728	.48412
.9	.13941	.27655	.28771	.29670	.30421	.31065	.31627	.32571	.33677	.43589
.925	.14612	.21429	.22398	.23189	.23859	.24440	.24952	.25823	.26864	.37997
.95	.15290	.14763	.15509	.16127	.16658	.17123	.17538	.18255	.19132	.31225
.96	.15563	.11967	.12599	.13126	.13580	.13981	.14341	.14966	.15737	.28000
.98	.16113	.06144	.06498	.06797	.07057	.07290	.07501	.07872	.08340	.19900
1.00	.16667	0	0	0	0	0	0	0	0	0
1.02	.17224	−.06481	−.06922	−.07306	−.07650	−.07964	−.08255	−.08784	−.09485	-------
1.04	.17785	−.13314	−.14299	−.15168	−.15958	−.16690	−.17376	−.18648	−.20382	-------
1.06	.18349	−.20517	−.22163	−.23640	−.25002	−.26281	−.27499	−.29803	−.33049	-------
1.08	.18915	−.28107	−.30550	−.32780	−.34869	−.36863	−.38790	−.42520	−.47980	-------
1.10	.19485	−.36101	−.39499	−.42654	−.45663	−.48583	−.51454	−.57154	−.65874	-------
1.12	.20056	−.44520	−.49052	−.53339	−.57503	−.61620	−.65745	−.74175	−.87758	-------
1.15	.20917	−.57986	−.64616	−.71076	−.77543	−.84139	−.90962	−1.05649	−1.31727	-------
1.18	.21782	−.72521	−.81816	−.91172	−1.00858	−1.11096	−1.22100	−1.47365	−1.98596	-------
1.20	.22360	−.82840	−.94290	−1.06083	−1.18598	−1.32187	−1.47231	−1.83658	−2.66958	-------
1.25	.23810	−1.11012	−1.29494	−1.49784	−1.72910	−2.00149	−2.33308	−3.30395	−7.20339	-------
1.30	.25262	−1.42886	−1.71465	−2.05258	−2.47353	−3.02728	−3.80519	−7.10444	42.54915	-------
1.35	.26713	−1.78883	−2.21804	−2.77171	−3.54472	−4.73743	−6.87721	−41.10040	6.90320	-------
1.40	.28161	−2.19467	−2.82609	−3.72953	−5.19836	−8.12116	−17.15880	15.66493	4.13736	-------
1.45	.29602	−2.65144	−3.56693	−5.05195	−8.04849	−17.83792	115.26579	7.50025	3.09196	-------
1.50	.31034	−3.16468	−4.47904	−6.97078	−14.03370	−277.06111	16.25833	5.26566	2.52540	-------
1.55	.32455	−3.74045	−5.61652	−9.96383	−34.16602	26.18116	9.60956	4.21470	2.15713	-------
1.60	.33862	−4.38535	−7.05782	−14.73354	223.57482	13.88787	7.19905	3.59884	1.88831	-------
1.65	.35254	−5.10657	−8.92117	−26.49894	31.37072	10.03451	5.95902	3.19031	1.67494	-------
1.70	.36629	−5.91193	−11.39331	−67.62261	18.41330	8.16600	5.20712	2.89627	1.49421	-------
1.75	.37984	−6.80994	−14.78750	243.44556	13.74179	7.07332	4.70523	2.67177	1.33289	-------
1.80	.39320	−7.80982	−19.67235	50.67121	11.36148	6.36416	4.34856	2.49237	1.18251	-------
1.85	.40635	−8.92157	−27.19842	30.58443	9.93797	5.87292	4.08381	2.34360	1.03714	-------
1.90	.41928	−10.15601	−40.09831	23.00938	9.00545	5.51767	3.88101	2.21633	.89218	-------
1.95	.43198	−11.52488	−66.82960	19.08453	8.35894	5.25323	3.72201	2.10449	.74372	-------
2.00	.44444	−13.04075	−153.05279	16.72046	7.89411	5.05263	3.59520	2.00386	.58809	-------

TABLE 3.—THE FUNCTIONS R_k FOR SEVERAL VALUES OF THE INDEX k—Concluded

M	τ	R_0	$R_{-0.5}$	R_{-1}	$R_{-1.5}$	R_{-2}	$R_{-2.5}$	$R_{-3.5}$	R_{-5}	R_∞
0.1	0.00200	0.99496	0.99495	0.99493	0.99487	0.99528	0.99510	0.99503	0.99501	0.99499
.2	.00794	.97932	.97916	.97885	.97795	.98287	.98137	.98057	.98016	.97980
.3	.01768	.95150	.95075	.94928	.94504	.96428	.95998	.95824	.95616	.95394
.4	.03101	.90882	.90658	.90233	.89043	.93649	.92842	.92081	.92597	.91652
.5	.04762	.84729	.84232	.83313	.80662	.89020	.87731	.89234	.89711	.86603
.55	.05705	.80783	.80097	.78856	.75636	.85567	.83961	.86620	.88633	.83516
.6	.06716	.76149	.75247	.73649	.69632	.81086	.79094	.83111	.87865	.80000
.65	.07792	.70734	.69598	.67634	.62869	.75389	.72948	.78313	.87235	.75993
.7	.08925	.64428	.63061	.60759	.55388	.68337	.65419	.71832	.86225	.71414
.75	.10112	.57112	.55546	.52984	.47255	.59859	.56516	.63381	.83812	.66144
.8	.11348	.48649	.46957	.44278	.38552	.49975	.46383	.52903	.78387	.60000
.825	.11982	.43941	.42231	.39570	.34016	.44539	.40934	.46971	.73921	.56513
.85	.12626	.38887	.37201	.34625	.29374	.38804	.35286	.40656	.67959	.52678
.875	.13279	.33466	.31856	.29441	.24639	.32800	.29487	.34039	.60321	.48412
.9	.13941	.27655	.26184	.24019	.19824	.26559	.23587	.27218	.50935	.43589
.925	.14612	.21429	.20173	.18361	.14940	.20118	.17639	.20297	.39881	.37997
.95	.15290	.14763	.13813	.12470	.10001	.13518	.11693	.13384	.27411	.31225
.96	.15563	.11967	.11168	.10049	.08012	.10842	.09326	.10646	.22114	.28000
.98	.16113	.06144	.05704	.05097	.04016	.05444	.04629	.05249	.11171	.19900
1.00	.16667	0	0	0	0	0	0	0	0	0
1.02	.17224	−.06481	−.05948	−.05239	−.04031	−.05468	−.04541	−.05058	−.11102	-------
1.04	.17785	−.13314	−.12144	−.10618	−.08071	−.10941	−.08974	−.09888	−.21855	-------
1.06	.18349	−.20517	−.18594	−.16134	−.12116	−.16400	−.13284	−.14457	−.32014	-------
1.08	.18915	−.28107	−.25303	−.21783	−.16163	−.21827	−.17455	−.18742	−.41390	-------
1.10	.19485	−.36101	−.32274	−.27564	−.20206	−.27207	−.21477	−.22721	−.49850	-------
1.12	.20056	−.44520	−.39512	−.33472	−.24243	−.32525	−.25338	−.26383	−.57318	-------
1.15	.20917	−.57986	−.50877	−.42566	−.30280	−.40355	−.30814	−.31262	−.66613	-------
1.18	.21782	−.72521	−.62863	−.51927	−.36286	−.47978	−.35895	−.35397	−.73698	-------
1.20	.22360	−.82840	−.71204	−.58310	−.40269	−.52930	−.39057	−.37743	−.77284	-------
1.25	.23810	−1.11012	−.93300	−.74738	−.50144	−.64807	−.46170	−.42206	−.82797	-------
1.30	.25262	−1.42886	−1.17198	−.91792	−.59885	−.75913	−.52165	−.44764	−.84300	-------
1.35	.26713	−1.78883	−1.42922	−1.09415	−.69484	−.86226	−.57084	−.45551	−.82774	-------
1.40	.28161	−2.19467	−1.70479	−1.27552	−.78942	−.95755	−.60968	−.44691	−.78932	-------
1.45	.29602	−2.65144	−1.99868	−1.46156	−.88266	−1.04538	−.63941	−.42278	−.73236	-------
1.50	.31034	−3.16468	−2.31074	−1.65179	−.97472	−1.12627	−.66011	−.38356	−.65946	-------
1.55	.32455	−3.74045	−2.64071	−1.84583	−1.06577	−1.20083	−.67255	−.32966	−.57163	-------
1.60	.33862	−4.38535	−2.98826	−2.04334	−1.15672	−1.26972	−.67726	−.26032	−.46858	-------
1.65	.35254	−5.10657	−3.35295	−2.24405	−1.24573	−1.33361	−.67465	−.17467	−.34884	-------
1.70	.36629	−5.91193	−3.73428	−2.44773	−1.33512	−1.39314	−.66504	−.07101	−.20972	-------
1.75	.37984	−6.80994	−4.13169	−2.65423	−1.42447	−1.44890	−.64863	.05314	−.04700	-------
1.80	.39320	−7.80982	−4.54459	−2.86343	−1.51404	−1.50147	−.62553	.20133	.14553	-------
1.85	.40635	−8.92157	−4.97236	−3.07533	−1.60407	−1.55135	−.59571	.37850	.37713	-------
1.90	.41928	−10.15601	−5.41436	−3.28965	−1.69483	−1.59901	−.55907	.59159	.66176	-------
1.95	.43198	−11.52488	−5.86997	−3.50664	−1.78656	−1.64488	−.51539	.85044	1.02146	-------
2.00	.44444	−13.04075	−6.33855	−3.72624	−1.87950	−1.68933	−.46431	1.16930	1.49287	-------

TABLE 4.—THE FUNCTIONS f_k FOR SEVERAL VALUES OF THE INDEX k

M	τ	f_0	$f_{0.5}$	f_1	$f_{1.5}$	f_2	$f_{2.5}$	$f_{3.5}$	f_5	f_∞
0.1	0.00200	−0.00250	−0.00250	−0.00250	−0.00250	−0.00250	−0.00250	−0.00250	−0.00250	−0.00250
.2	.00794	−.00989	−.00991	−.00993	−.00994	−.00995	−.00996	−.00997	−.00998	−.01001
.3	.01768	−.02196	−.02207	−.02215	−.02221	−.02225	−.02228	−.02233	−.02238	−.02256
.4	.03101	−.03831	−.03867	−.03891	−.03908	−.03922	−.03932	−.03947	−.03962	−.04020
.5	.04762	−.05847	−.05930	−.05987	−.06029	−.06061	−.06086	−.06123	−.06160	−.06306
.55	.05705	−.06979	−.07099	−.07181	−.07241	−.07287	−.07324	−.07379	−.07433	−.07650
.6	.06716	−.08186	−.08350	−.08464	−.08548	−.08613	−.08665	−.08742	−.08818	−.09133
.65	.07792	−.09457	−.09678	−.09832	−.09946	−.10034	−.10104	−.10210	−.10316	−.10768
.7	.08925	−.10787	−.11075	−.11277	−.11428	−.11545	−.11638	−.11780	−.11923	−.12541
.75	.10112	−.12167	−.12534	−.12794	−.12989	−.13141	−.13263	−.13449	−.13639	−.14484
.8	.11348	−.13588	−.14049	−.14377	−.14624	−.14818	−.14975	−.15210	−.15462	−.16605
.825	.11982	−.14313	−.14825	−.15191	−.15467	−.15685	−.15862	−.16133	−.16412	−.17740
.85	.12626	−.15045	−.15613	−.16019	−.16327	−.16571	−.16769	−.17073	−.17389	−.18927
.875	.13279	−.15785	−.16410	−.16860	−.17202	−.17474	−.17695	−.18036	−.18392	−.20173
.9	.13941	−.16530	−.17218	−.17714	−.18093	−.18394	−.18640	−.19021	−.19421	−.21487
.925	.14612	−.17281	−.18034	−.18579	−.18997	−.19330	−.19604	−.20028	−.20476	−.22876
.95	.15290	−.18036	−.18858	−.19455	−.19915	−.20283	−.20585	−.21056	−.21556	−.24353
.96	.15563	−.18339	−.19189	−.19809	−.20286	−.20668	−.20982	−.21473	−.21996	−.24976
.98	.16113	−.18946	−.19856	−.20520	−.21033	−.21445	−.21785	−.22318	−.22887	−.26292
1.00	.16667	−.19556	−.20526	−.21238	−.21789	−.22232	−.22598	−.23175	−.23794	−.27757
1.02	.17224	−.20166	−.21201	−.21961	−.22551	−.23027	−.23422	−.24045	−.24717	--------
1.04	.17785	−.20778	−.21878	−.22689	−.23320	−.23831	−.24256	−.24928	−.25657	--------
1.06	.18349	−.21391	−.22558	−.23422	−.24096	−.24643	−.25099	−.25823	−.26614	--------
1.08	.18915	−.22003	−.23241	−.24159	−.24878	−.25463	−.25952	−.26731	−.27586	--------
1.10	.19485	−.22616	−.23925	−.24900	−.25666	−.26290	−.26814	−.27651	−.28575	--------
1.12	.20056	−.23228	−.24612	−.25645	−.26459	−.27125	−.27685	−.28583	−.29581	--------
1.15	.20917	−.24144	−.25644	−.26769	−.27659	−.28390	−.29007	−.30003	−.31120	--------
1.18	.21782	−.25059	−.26678	−.27898	−.28869	−.29669	−.30348	−.31450	−.32697	--------
1.20	.22360	−.25665	−.27368	−.28655	−.29681	−.30530	−.31251	−.32428	−.33769	--------
1.25	.23810	−.27177	−.29093	−.30553	−.31727	−.32705	−.33542	−.34922	−.36524	--------
1.30	.25262	−.28673	−.30814	−.32459	−.33792	−.34910	−.35875	−.37485	−.39389	--------
1.35	.26713	−.30148	−.32527	−.34368	−.35871	−.37142	−.38247	−.40112	−.42368	--------
1.40	.28161	−.31604	−.34229	−.36276	−.37959	−.39395	−.40653	−.42804	−.45467	--------
1.45	.29602	−.33034	−.35915	−.38177	−.40053	−.41665	−.43090	−.45558	−.48693	--------
1.50	.31034	−.34438	−.37582	−.40070	−.42148	−.43949	−.45554	−.48372	−.52057	--------
1.55	.32455	−.35815	−.39228	−.41950	−.44240	−.46241	−.48040	−.51247	−.55572	--------
1.60	.33862	−.37163	−.40851	−.43813	−.46325	−.48539	−.50546	−.54180	−.59255	--------
1.65	.35254	−.38478	−.42448	−.45658	−.48401	−.50838	−.53068	−.57172	−.63129	--------
1.70	.36629	−.39763	−.44017	−.47481	−.50464	−.53135	−.55603	−.60223	−.67226	--------
1.75	.37984	−.41014	−.45556	−.49281	−.52511	−.55427	−.58148	−.63332	−.71589	--------
1.80	.39320	−.42233	−.47066	−.51055	−.54539	−.57711	−.60700	−.66502	−.76278	--------
1.85	.40635	−.43418	−.48543	−.52800	−.56545	−.59984	−.63255	−.69734	−.81382	--------
1.90	.41928	−.44573	−.49989	−.54517	−.58529	−.62243	−.65811	−.73029	−.87038	--------
1.95	.43198	−.45690	−.51401	−.56203	−.60487	−.64485	−.68366	−.76391	−.93478	--------
2.00	.44444	−.46775	−.52779	−.57857	−.62418	−.66709	−.70916	−.79823	−1.01119	--------

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TABLE 4.—THE FUNCTIONS f_k FOR SEVERAL VALUES OF THE INDEX k—Concluded

M	τ	f_0	$f_{-0.5}$	f_{-1}	$f_{-1.5}$	f_{-2}	$f_{-2.5}$	$f_{-3.5}$	f_{-5}	f_∞
0.1	0.00200	−0.00250	−0.00250	−0.00249	−0.00247	−0.00258	−0.00253	−0.00251	−0.00251	−0.00250
.2	.00794	−.00989	−.00986	−.00977	−.00954	−.01091	−.01043	−.01019	−.01010	−.01001
.3	.01768	−.02196	−.02176	−.02139	−.02028	−.02588	−.02433	−.02356	−.02305	−.02256
.4	.03101	−.03831	−.03772	−.03660	−.03334	−.04766	−.04446	−.04345	−.04207	−.04020
.5	.04762	−.05847	−.05712	−.05455	−.04722	−.07530	−.06998	−.07040	−.06872	−.06306
.55	.05705	−.06979	−.06789	−.06426	−.05402	−.09075	−.08418	−.08636	−.08564	−.07650
.6	.06716	−.08186	−.07925	−.07431	−.06052	−.10689	−.09888	−.10368	−.10532	−.09133
.65	.07792	−.09457	−.09109	−.08459	−.06656	−.12340	−.11372	−.12197	−.12787	−.10768
.7	.08925	−.10787	−.10336	−.09501	−.07203	−.13998	−.12831	−.14076	−.15318	−.12541
.75	.10112	−.12167	−.11595	−.10546	−.07684	−.15632	−.14227	−.15952	−.18084	−.14484
.8	.11348	−.13588	−.12878	−.11587	−.08089	−.17215	−.15527	−.17770	−.21022	−.16605
.825	.11982	−.14313	−.13526	−.12103	−.08263	−.17979	−.16131	−.18639	−.22529	−.17740
.85	.12626	−.15045	−.14179	−.12616	−.08415	−.18723	−.16698	−.19475	−.24048	−.18927
.875	.13279	−.15785	−.14832	−.13123	−.08547	−.19443	−.17227	−.20269	−.25566	−.20173
.9	.13941	−.16530	−.15488	−.13626	−.08657	−.20137	−.17714	−.21016	−.27074	−.21487
.925	.14612	−.17281	−.16144	−.14122	−.08745	−.20803	−.18156	−.21711	−.28560	−.22876
.95	.15290	−.18036	−.16800	−.14612	−.08812	−.21440	−.18551	−.22347	−.30016	−.24353
.96	.15563	−.18339	−.17061	−.14806	−.08833	−.21686	−.18695	−.22584	−.30587	−.24976
.98	.16113	−.18946	−.17586	−.15190	−.08864	−.22162	−.18959	−.23026	−.31707	−.26292
1.00	.16667	−.19556	−.18109	−.15570	−.08881	−.22619	−.19189	−.23424	−.32796	−.27757
1.02	.17224	−.20166	−.18631	−.15944	−.08884	−.23053	−.19386	−.23775	−.33847	--------
1.04	.17785	−.20778	−.19149	−.16312	−.08873	−.23466	−.19547	−.24077	−.34859	--------
1.06	.18349	−.21391	−.19667	−.16676	−.08849	−.23857	−.19673	−.24327	−.35828	--------
1.08	.18915	−.22003	−.20183	−.17033	−.08811	−.24225	−.19761	−.24523	−.36750	--------
1.10	.19485	−.22616	−.20695	−.17385	−.08760	−.24571	−.19813	−.24663	−.37623	--------
1.12	.20056	−.23228	−.21205	−.17730	−.08696	−.24893	−.19826	−.24744	−.38443	--------
1.15	.20917	−.24144	−.21965	−.18237	−.08576	−.25333	−.19774	−.24750	−.39572	--------
1.18	.21782	−.25059	−.22716	−.18730	−.08427	−.25722	−.19633	−.24610	−.40569	--------
1.20	.22360	−.25665	−.23211	−.19051	−.08314	−.25952	−.19490	−.24431	−.41158	--------
1.25	.23810	−.27177	−.24435	−.19825	−.07978	−.26425	−.18953	−.23661	−.42346	--------
1.30	.25262	−.28673	−.25631	−.20559	−.07573	−.26757	−.18157	−.22382	−.43099	--------
1.35	.26713	−.30148	−.26800	−.21253	−.07105	−.26950	−.07095	−.20519	−.43373	--------
1.40	.28161	−.31604	−.27935	−.21907	−.06578	−.27009	−.15761	−.17966	−.43111	
1.45	.29602	−.33034	−.29040	−.22522	−.05998	−.26939	−.14146	−.14562	−.42231	--------
1.50	.31034	−.34438	−.30113	−.23100	−.05370	−.26746	−.12241	−.10051	−.40605	--------
1.55	.32455	−.35815	−.31151	−.23641	−.04701	−.26436	−.10033	−.03980	−.38023	--------
1.60	.33862	−.37163	−.32155	−.24147	−.03995	−.26016	−.07505	.04559	−.34095	--------
1.65	.35254	−.38478	−.33127	−.24619	−.03258	−.25494	−.04637	.17855	−.27990	--------
1.70	.36629	−.39763	−.34063	−.25060	−.02495	−.24875	−.01400	.45605	−.17335	--------
1.75	.37984	−.41014	−.34967	−.25469	−.01711	−.24166	.02244	--------	.13348	--------
1.80	.39320	−.42233	−.35838	−.25850	−.00909	−.23376	.06344	--------	--------	--------
1.85	.40635	−.43418	−.36678	−.26203	−.00096	−.22510	.10971	--------	--------	--------
1.90	.41928	−.44573	−.37485	−.26531	.00727	−.21576	.16223	--------	--------	--------
1.95	.43198	−.45690	−.38261	−.26835	.01554	−.20580	.22240	--------	--------	--------
2.00	.44444	−.46775	−.39006	−.27117	.02382	−.19528	.29237	--------	--------	--------

TABLE 5.—THE FUNCTIONS g_k FOR SEVERAL VALUES OF THE INDEX k

M	τ	g_0	$g_{0.5}$	g_1	$g_{1.5}$	g_2	$g_{2.5}$	$g_{3.5}$	g_5	g_∞
0.1	0.00200	−0.00251	−0.00251	−0.00251	−0.00250	−0.00250	−0.00250	−0.00250	−0.00250	−0.00250
.2	.00794	−.01013	−.01010	−.01009	−.01008	−.01007	−.01006	−.01006	−.01005	−.01001
.3	.01768	−.02316	−.02304	−.02296	−.02291	−.02287	−.02283	−.02278	−.02274	−.02256
.4	.03101	−.04208	−.04174	−.04149	−.04132	−.04119	−.04108	−.04093	−.04078	−.04020
.5	.04762	−.06760	−.06678	−.06622	−.06581	−.06549	−.06524	−.06487	−.06451	−.06306
.55	.05705	−.08309	−.08194	−.08113	−.08054	−.08009	−.07973	−.07919	−.07866	−.07650
.6	.06716	−.10059	−.09901	−.09791	−.09709	−.09646	−.09596	−.09521	−.09445	−.09133
.65	.07792	−.12023	−.11814	−.11668	−.11560	−.11475	−.11407	−.11305	−.11201	−.10768
.7	.08925	−.14216	−.13951	−.13763	−.13621	−.13511	−.13421	−.13287	−.13149	−.12541
.75	.10112	−.16657	−.16327	−.16093	−.15915	−.15774	−.15660	−.15486	−.15307	−.14484
.8	.11348	−.19363	−.18967	−.18681	−.18464	−.18290	−.18148	−.17924	−.17703	−.16605
.825	.11982	−.20822	−.20392	−.20081	−.19841	−.19652	−.19496	−.19255	−.19001	−.17740
.85	.12626	−.22354	−.21893	−.21556	−.21296	−.21088	−.20918	−.20652	−.20372	−.18927
.875	.13279	−.23964	−.23471	−.23110	−.22829	−.22605	−.22421	−.22130	−.21823	−.20173
.9	.13941	−.25652	−.25131	−.24747	−.24449	−.24208	−.24009	−.23695	−.23362	−.21487
.925	.14612	−.27423	−.26878	−.26474	−.26158	−.25903	−.25692	−.25357	−.24997	−.22876
.95	.15290	−.29280	−.28715	−.28295	−.27966	−.27698	−.27476	−.27123	−.26741	−.24353
.96	.15563	−.30047	−.29475	−.29051	−.28717	−.28446	−.28220	−.27861	−.27472	−.24976
.98	.16113	−.31624	−.31045	−.30612	−.30272	−.29996	−.29765	−.29398	−.28997	−.26292
1.00	.16667	−.33261	−.32677	−.32243	−.31901	−.31622	−.31389	−.31019	−.30614	−.27757
1.02	.17224	−.34958	−.34380	−.33947	−.33608	−.33331	−.33101	−.32735	−.32335	--------
1.04	.17785	−.36718	−.36150	−.35728	−.35397	−.35129	−.34907	−.34554	−.34174	--------
1.06	.18349	−.38542	−.37995	−.37590	−.37277	−.37024	−.36816	−.36491	−.36148	--------
1.08	.18915	−.40432	−.39915	−.39539	−.39251	−.39023	−.38838	−.38559	−.38282	--------
1.10	.19485	−.42390	−.41915	−.41579	−.41329	−.41137	−.40988	−.40777	−.40242	--------
1.12	.20056	−.44417	−.44001	−.43718	−.43518	−.43377	−.43278	−.43168	−.43153	--------
1.15	.20917	−.47594	−.47293	−.47122	−.47033	−.47003	−.47016	−.47143	−.47530	--------
1.18	.21782	−.50938	−.50798	−.50785	−.50858	−.50995	−.51187	−.51708	−.52845	--------
1.20	.22360	−.53263	−.53260	−.53385	−.53602	−.53895	−.54255	−.55175	−.57172	--------
1.25	.23810	−.59432	−.59892	−.60509	−.61269	−.62176	−.63251	−.66083	−.73925	--------
1.30	.25262	−.66139	−.67281	−.68682	−.70377	−.72450	−.75054	−.83309	--------	--------
1.35	.26713	−.73409	−.75539	−.78158	−.81464	−.85838	−.92113	−1.29668	--------	--------
1.40	.28161	−.81292	−.84801	−.89301	−.95446	−1.04817	−1.22912	--------	--------	--------
1.45	.29602	−.89817	−.95235	−1.02644	−1.14078	−1.36976	--------	--------	--------	--------
1.50	.31034	−.99030	−1.07053	−1.19037	−1.41442	−2.67558	--------	--------	--------	--------
1.55	.32455	−1.08984	−1.20528	−1.39926	−1.91708	--------	--------	--------	--------	--------
1.60	.33862	−1.19706	−1.36024	−1.64999	--------	--------	--------	--------	--------	--------
1.65	.35254	−1.31258	−1.54028	−2.10316	--------	--------	--------	--------	--------	--------
1.70	.36629	−1.43687	−1.75228	−2.91178	--------	--------	--------	--------	--------	--------
1.75	.37984	−1.57037	−2.00636	--------	--------	--------	--------	--------	--------	--------
1.80	.39320	−1.71378	−2.31832	--------	--------	--------	--------	--------	--------	--------
1.85	.40635	−1.86763	−2.71480	--------	--------	--------	--------	--------	--------	--------
1.90	.41928	−2.03240	−3.24642	--------	--------	--------	--------	--------	--------	--------
1.95	.43198	−2.20875	−4.02928	--------	--------	--------	--------	--------	--------	--------
2.00	.44444	−2.39742	−5.45320	--------	--------	--------	--------	--------	--------	--------

TABLE 5.—THE FUNCTIONS g_k FOR SEVERAL VALUES OF THE INDEX k—Concluded

M	τ	g_0	$g_{-0.5}$	g_{-1}	$g_{-1.5}$	g_{-2}	$g_{-2.5}$	$g_{-3.5}$	g_{-5}	g_{∞}
0.1	0.00200	−0.00251	−0.00252	−0.00252	−0.00253	−0.00242	−0.00247	−0.00249	−0.00250	−0.00250
.2	.00794	−.01013	−.01018	−.01025	−.01048	−.00911	−.00959	−.00983	−.00992	−.01001
.3	.01768	−.02316	−.02335	−.02373	−.02483	−.01922	−.02078	−.02155	−.02207	−.02256
.4	.03101	−.04208	−.04265	−.04377	−.04697	−.03266	−.03592	−.03693	−.03833	−.04020
.5	.04762	−.06760	−.06891	−.07141	−.07837	−.05060	−.05606	−.05560	−.05730	−.06306
.55	.05705	−.08309	−.08494	−.08840	−.09792	−.06198	−.06875	−.06643	−.06709	−.07650
.6	.06716	−.10059	−.10309	−.10770	−.12016	−.07548	−.08371	−.07869	−.07670	−.09133
.65	.07792	−.12023	−.12347	−.12941	−.14514	−.09153	−.10139	−.09289	−.08593	−.10768
.7	.08925	−.14216	−.14625	−.15364	−.17282	−.11053	−.12221	−.10968	−.09490	−.12541
.75	.10112	−.16657	−.17158	−.18049	−.20314	−.13283	−.14646	−.12976	−.10413	−.14484
.8	.11348	−.19363	−.19958	−.21001	−.23598	−.15871	−.17435	−.15375	−.11481	−.16605
.825	.11982	−.20822	−.21463	−.22579	−.25330	−.17303	−.18966	−.16734	−.12126	−.17740
.85	.12626	−.22354	−.23041	−.24225	−.27118	−.18829	−.20585	−.18205	−.12883	−.18927
.875	.13279	−.23964	−.24691	−.25939	−.28959	−.20448	−.22290	−.19784	−.13783	−.20173
.9	.13941	−.25652	−.26419	−.27721	−.30851	−.22159	−.24078	−.21472	−.14859	−.21487
.925	.14612	−.27423	−.28222	−.29571	−.32792	−.23958	−.25942	−.23262	−.16137	−.22876
.95	.15290	−.29280	−.30103	−.31489	−.34776	−.25844	−.27877	−.25148	−.17639	−.24353
.96	.15563	−.30047	−.30877	−.32275	−.35582	−.26622	−.28670	−.25927	−.18305	−.24976
.98	.16113	−.31624	−.32466	−.33879	−.37213	−.28216	−.30284	−.27524	−.19751	−.26292
1.00	.16667	−.33261	−.34107	−.35524	−.38568	−.29859	−.31934	−.29169	−.21345	−.27757
1.02	.17224	−.34958	−.35800	−.37212	−.40545	−.31548	−.33616	−.30855	−.23081	--------
1.04	.17785	−.36718	−.37545	−.38940	−.42244	−.33281	−.35326	−.32577	−.24947	--------
1.06	.18349	−.38542	−.39345	−.40709	−.43962	−.35055	−.37061	−.34328	−.26929	--------
1.08	.18915	−.40432	−.41201	−.42518	−.45697	−.36867	−.38815	−.36102	−.29009	--------
1.10	.19485	−.42390	−.43109	−.44366	−.47451	−.38713	−.40587	−.37892	−.31169	--------
1.12	.20056	−.44417	−.45073	−.46253	−.49217	−.40590	−.42371	−.39693	−.33390	--------
1.15	.20917	−.47594	−.48124	−.49153	−.51892	−.43458	−.45063	−.42402	−.36798	--------
1.18	.21782	−.50938	−.51299	−.52134	−.54590	−.46378	−.47764	−.45103	−.40247	--------
1.20	.22360	−.53263	−.53485	−.54166	−.56403	−.48348	−.49566	−.46893	−.42547	--------
1.25	.23810	−.59432	−.59200	−.59390	−.60960	−.53337	−.54046	−.51292	−.48211	--------
1.30	.25262	−.66139	−.65267	−.64811	−.65547	−.58380	−.58462	−.55543	−.53653	--------
1.35	.26713	−.73409	−.71688	−.70412	−.70147	−.63439	−.62784	−.59602	−.58785	--------
1.40	.28161	−.81292	−.78453	−.76175	−.74745	−.68479	−.66982	−.63435	−.63564	--------
1.45	.29602	−.89817	−.85564	−.82083	−.79326	−.73475	−.71038	−.67019	−.67963	--------
1.50	.31034	−.99030	−.93010	−.88119	−.83880	−.78403	−.74937	−.70338	−.71973	--------
1.55	.32455	−1.08984	−1.00784	−.94269	−.88402	−.83246	−.78668	−.73377	−.75592	--------
1.60	.33862	−1.19706	−1.08871	−1.00515	−.92881	−.87989	−.82224	−.76129	−.78822	--------
1.65	.35254	−1.31258	−1.17264	−1.06844	−.97312	−.92626	−.85600	−.78585	−.81663	--------
1.70	.36629	−1.43687	−1.25946	−1.13241	−1.01691	−.97146	−.88795	−.80737	−.84114	--------
1.75	.37984	−1.57037	−1.34906	−1.19692	−1.06016	−1.01545	−.91807	--------	−.86170	--------
1.80	.39320	−1.71378	−1.44127	−1.26187	−1.10283	−1.05823	−.94637	--------	-------	--------
1.85	.40635	−1.86763	−1.53594	−1.32709	−1.14492	−1.09978	−.97288	--------	--------	--------
1.90	.41928	−2.03240	−1.63288	−1.39260	−1.18639	−1.14010	−.99759	--------	--------	--------
1.95	.43198	−2.20875	−1.73193	−1.45820	−1.22728	−1.17922	−1.02054	--------	--------	--------
2.00	.44444	−2.39742	−1.83291	−1.52385	−1.26756	−1.21628	−1.04173	--------	--------	--------

TABLE 6.—EXPONENTIALS OF THE FUNCTIONS f_k AND g_k FOR SEVERAL POSITIVE AND NEGATIVE VALUES OF THE INDEX k

M	e^h	e^{f_0}	e^{g_0}	$e^{f_{0.5}}$	$e^{g_{0.5}}$	e^{f_1}	e^{g_1}	$e^{f_{1.5}}$	$e^{g_{1.5}}$	e^{f_2}	e^{g_2}	$e^{f_{2.5}}$	$e^{g_{2.5}}$	$e^{f_{3.5}}$	$e^{g_{3.5}}$	e^{f_5}	e^{g_5}
0.1	0.99750	0.99750	0.99749	0.99751	0.99749	0.99751	0.99750	0.99751	0.99750	0.99751	0.99750	0.99751	0.99750	0.99750	0.99750	0.99750	0.99750
.2	.99004	.99016	.98992	.99014	.98995	.99012	.98996	.99011	.98997	.99010	.98998	.99009	.98999	.99008	.99000	.99007	.99000
.3	.97769	.97828	.97711	.97817	.97722	.97809	.97730	.97804	.97735	.97800	.97739	.97796	.97743	.97792	.97747	.97787	.97752
.4	.96060	.96241	.95879	.96207	.95912	.96184	.95936	.96167	.95952	.96154	.95965	.96144	.95975	.96129	.95990	.96115	.96004
.5	.93889	.94321	.93463	.94242	.93540	.94188	.93593	.94149	.93631	.94119	.93661	.94095	.93684	.94060	.93719	.94026	.93753
.55	.92635	.93259	.92026	.93147	.92133	.93071	.92207	.93015	.92262	.92972	.92303	.92938	.92337	.92887	.92386	.92837	.92436
.6	.91271	.92140	.90430	.91989	.90573	.91884	.90673	.91807	.90747	.91747	.90804	.91700	.90850	.91630	.90918	.91559	.90987
.65	.89792	.90977	.88671	.90776	.88857	.90636	.88987	.90533	.89083	.90453	.89159	.90390	.89220	.90294	.89311	.90198	.89403
.7	.88213	.89775	.86748	.89516	.86978	.89336	.87142	.89201	.87266	.89097	.87362	.89013	.87441	.88887	.87558	.88760	.87679
.75	.86516	.88544	.84656	.88219	.84936	.87991	.85135	.87820	.85287	.87686	.85407	.87579	.85505	.87416	.85654	.87250	.85807
.8	.84700	.87295	.82397	.86893	.82723	.86609	.82960	.86395	.83140	.86228	.83285	.86092	.83403	.85890	.83591	.85674	.83775
.825	.83744	.86664	.81202	.86221	.81553	.85907	.81807	.85669	.82003	.85483	.82159	.85332	.82287	.85102	.82485	.84864	.82695
.85	.82756	.86032	.79968	.85545	.80338	.85198	.80609	.84936	.80819	.84730	.80987	.84562	.81125	.84305	.81341	.84039	.81569
.875	.81732	.85398	.78691	.84866	.79080	.84485	.79366	.84196	.79589	.83968	.79768	.83782	.79915	.83497	.80148	.83200	.80394
.9	.80664	.84764	.77374	.84183	.77778	.83768	.78077	.83450	.78310	.83199	.78499	.82994	.78656	.82678	.78903	.82348	.79167
.925	.79552	.84130	.76016	.83499	.76431	.83045	.76741	.82698	.76983	.82423	.77180	.82198	.77343	.81850	.77603	.81484	.77882
.95	.78386	.83497	.74617	.82814	.75040	.82320	.75356	.81943	.75604	.81642	.75807	.81396	.75975	.81013	.76244	.80609	.76536
.96	.77899	.83245	.74047	.82539	.74472	.82029	.74789	.81639	.75038	.81328	.75242	.81073	.75412	.80676	.75684	.80255	.75978
.98	.76880	.82741	.72888	.81991	.73312	.81448	.73630	.81031	.73881	.80698	.74085	.80425	.74256	.79997	.74529	.79544	.74828
1.00	.75762	.82237	.71705	.81443	.72125	.80866	.72439	.80422	.72687	.80066	.72890	.79773	.73060	.79315	.73331	.78825	.73628

M	e^h	e^{f_0}	e^{g_0}	$e^{f_{-0.5}}$	$e^{g_{-0.5}}$	$e^{f_{-1}}$	$e^{g_{-1}}$	$e^{f_{-1.5}}$	$e^{g_{-1.5}}$	$e^{f_{-2}}$	$e^{g_{-2}}$	$e^{f_{-2.5}}$	$e^{g_{-2.5}}$	$e^{f_{-3.5}}$	$e^{g_{-3.5}}$	$e^{f_{-5}}$	$e^{g_{-5}}$
0.1	0.99750	0.99750	0.99749	0.99750	0.99749	0.99751	0.99749	0.99753	0.99748	0.99742	0.99758	0.99748	0.99752	0.99749	0.99751	0.99750	0.99751
.2	.99004	.99016	.98992	.99019	.98988	.99027	.98981	.99050	.98958	.98914	.99094	.98962	.99045	.98986	.99022	.98995	.99012
.3	.97769	.97828	.97711	.97847	.97692	.97884	.97655	.97592	.97548	.97445	.98097	.97596	.97943	.97671	.97868	.97722	.97817
.4	.96060	.96241	.95879	.96298	.95824	.96406	.95717	.96721	.95412	.95346	.96787	.95652	.96472	.95748	.96375	.95881	.96240
.5	.93889	.94321	.93463	.94448	.93341	.94691	.93108	.95388	.92463	.92746	.95065	.93241	.94548	.93202	.94592	.93359	.94432
.55	.92635	.93259	.92026	.93436	.91856	.93776	.91539	.94741	.90672	.91325	.93990	.91927	.93356	.91726	.93573	.91793	.93511
.6	.91271	.92140	.90430	.92381	.90205	.92838	.89790	.94128	.88678	.89863	.92729	.90585	.91970	.90151	.92433	.90004	.92617
.65	.89792	.90977	.88671	.91293	.88365	.91869	.87861	.93561	.86490	.88391	.91253	.89251	.90358	.88517	.91130	.87997	.91765
.7	.88213	.89775	.86748	.90180	.86394	.90936	.85758	.93050	.84129	.86938	.89536	.87958	.88496	.86870	.89612	.85798	.90947
.75	.86516	.88544	.84656	.89052	.84233	.89991	.83486	.92604	.81616	.85529	.87561	.86738	.86376	.85255	.87831	.83457	.90111
.8	.84700	.87295	.82397	.87916	.81907	.89059	.81057	.92229	.78980	.84186	.85325	.85618	.84060	.83720	.85749	.81041	.89153
.825	.83744	.86664	.81202	.87349	.80684	.88601	.79789	.92069	.77623	.83544	.84111	.85103	.82724	.82995	.84591	.79828	.88581
.85	.82756	.86032	.79668	.86780	.79421	.88148	.78486	.91929	.76248	.82925	.82837	.84621	.81596	.82204	.83356	.78625	.87913
.875	.81732	.85398	.78691	.86215	.78121	.87701	.77152	.91808	.74857	.82331	.81507	.84175	.80019	.81653	.82050	.77440	.87124
.9	.80664	.84764	.77374	.85651	.76783	.87262	.75789	.91707	.73454	.81761	.80125	.83766	.78601	.81045	.80677	.76281	.86192
.925	.79552	.84130	.76016	.85091	.75411	.86830	.74400	.91626	.72042	.81218	.78696	.83397	.77150	.80484	.79245	.75156	.85097
.95	.78386	.83497	.74617	.84535	.74006	.86405	.72987	.91565	.70627	.80703	.77225	.83068	.75671	.79974	.77765	.74070	.83829
.96	.77899	.83245	.74047	.84315	.73435	.86238	.72416	.91546	.70060	.80505	.76627	.82949	.75074	.79785	.77161	.73648	.83272
.98	.76880	.82741	.72888	.83574	.72277	.85907	.71264	.91518	.68926	.80122	.75415	.82730	.73872	.79433	.75939	.72828	.82077
1.00	.75762	.82237	.71705	.83436	.71101	.85582	.70100	.91502	.67795	.79757	.74186	.82539	.72663	.79117	.74700	.72040	.80779