

REPORT 980

GENERAL ALGEBRAIC METHOD APPLIED TO CONTROL ANALYSIS OF COMPLEX ENGINE TYPES

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SUMMARY

A general algebraic method of attack on the problem of controlling gas-turbine engines having any number of independent variables was utilized employing operational functions to describe the assumed linear characteristics for the engine, the control, and the other units in the system. Matrices were used to describe the various units of the system, to form a combined system showing all effects, and to form a single condensed matrix showing the principal effects. This method directly led to the conditions on the control system for noninteraction so that any setting disturbance would affect only its corresponding controlled variable. The response-action characteristics were expressed in terms of the control system and the engine characteristics. The ideal control-system characteristics were explicitly determined in terms of any desired response action.

INTRODUCTION

The current development of gas-turbine engines indicates a future trend toward a wide variety of engine types. As new engines are developed or built by combining basic components of existing engines, the control problem presented by each engine type will be different. Control systems satisfactory for one engine type could not be expected to operate for another engine type. Even engines of the same type but of different size or operated under radically different conditions may have different engine characteristics that require special control characteristics. The current fluid and indeterminate state of gas-turbine development indicates a need for a general control analysis that could be applied to any of the new engine types as they are developed.

The control problem for engines differs from that for many plants, servomechanisms, or electric networks, because the control of more than one variable is desired. At least those variables that may cause failure should be controlled. For gas-turbine engines, failure may be caused by excessive temperature, speed, or torque. Exceeding specified burner operating limits may cause burner blow-out. The problem of controlling more than one variable is therefore basic for gas turbines.

In control design, limitations may be placed on such characteristics as response time, maximum undershoot or overshoot, dead-band, or steady-state error. This problem is the subject of much of the control literature to date and various methods of exact, approximate, and graphical means of solution have been developed (reference 1).

Controlling more than one variable generally introduces an interaction among the controlled variables. Making a new setting for one variable may cause, during the transient state, changes in the other controlled variables. If these other variables are operating at or near a maximum point, this interaction may cause excessive values and possible damage to the engine. It would therefore be desirable that each new setting of a controlled variable affect only the variable being set, thus giving separate noninteracting control of all the variables being controlled.

A general algebraic approach to control problems of multi-variable engines, which was developed at the NACA Lewis laboratory during 1949, is presented herein. This method leads to the conditions on the control characteristics that eliminate interaction between control variables and to an explicit solution for control characteristics that yields any desired response action.

SYMBOLS

The following symbols are used in this report:

a, b, c, e	constants
C or $C(p)$	control function to which engine-dependent-variable errors are applied
C' or $C'(p)$	control function to which engine-independent-variable errors are applied
E or $E(p)$	engine-characteristic function
E^*	square matrix using first i columns of engine matrix
$ E^* $	determinant of E^* matrix
$ E^*_{jk} $	cofactor of E^*_{jk} element of E^* matrix
$F(p)$	general operational function
i	number of engine-dependent variables being controlled
L or $L(p)$	instrument-characteristic function
N	engine speed
\bar{N}	measured value of engine speed
n	number of engine-independent variables
p	operational symbol
R or $R(p)$	response function for controlled engine-dependent variables
R' or $R'(p)$	response function for controlled engine-independent variables
S or $S(p)$	servo-characteristic function
T	engine temperature
\bar{T}	measured value of engine temperature

- V uncontrolled transient disturbance in engine-independent variable
- w engine fuel flow
- \bar{w} measured value of engine fuel flow
- \underline{w} signal to engine fuel-flow servo
- X setting of engine-independent variable
- x engine-independent variable
- \bar{x} measured value of engine-independent variable
- \underline{x} signal to engine-independent variable servo
- Y setting of engine-dependent variable
- y engine-dependent variable
- \bar{y} measured value of engine-dependent variable
- β propeller-blade angle
- $\underline{\beta}$ signal to propeller-blade-angle servo
- γ engine-interaction factor ($\gamma = \frac{be}{c}$)
- τ engine time constant
- Subscripts: j, k, r, t, v, μ indices
- s setting of specific engine variable

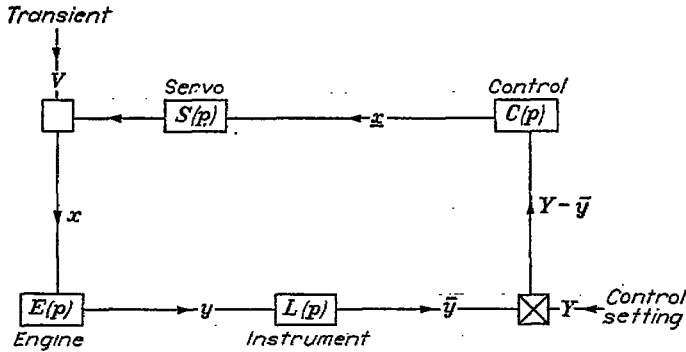


FIGURE 1.—Basic controlled-engine configuration.

ANALYSIS

The system considered herein is essentially that shown in figure 1. This block diagram describes a typical engine and control system. The general system configuration chosen is the one employing negative feedback in which error in the controlled variable is fed back into the control. The engine is represented as a box with a characteristic function E , the input of which is the engine-independent variable x and the output is the engine-dependent variable y . The characteristics of the instrument measuring the value of the controlled variable are denoted as L for which the input is the variable itself and the output is the measured value \bar{y} . The error is the difference between the setting Y and the measured value of the variable \bar{y} . The characteristics of the control are denoted as a function C for which the input is the error and the output is a signal \underline{x} to a servo unit, which is the device that actually changes the engine-independent variable and completes the loop. The servo characteristic is denoted as the function S and may be considered as a part of the control or as a separate entity, as shown in figure 1.

In addition to the intentional disturbance made on the control setting, inadvertent transient disturbances may appear in any part of the system. One possible source of such transient disturbances has been included as an uncontrolled change in engine-independent variable. The variable x is thus the sum of a controlled part and of an uncontrolled part V .

The characteristic functions used to describe the various units of the system are operational functions or Laplace transforms. The dynamic characteristics of gas-turbine engines, controls, and servos can be approximated by such operational functions at least for the small deviations of variables in which the linear assumption is valid. These operational functions can be used without reference to the initial conditions of operation before disturbances occur, if the initial condition is a steady-state condition and the values of all variables are their deviations from the initial steady-state condition. In general, the unit can be described by a general operational or transfer function as follows:

$$y = F(p) \cdot x \tag{1}$$

The block diagram for this unit can be shown as a box containing the function $F(p)$ for which the input is x and the output is y .

One method of deriving the operational function for a unit that approximately follows a linear differential equation is as follows for the equation given by

$$\dot{y} + ay + by = cx + ex \tag{2}$$

where the dots above symbols indicate derivatives with respect to time. This equation can be algebraically represented in terms of the differential operator or in terms of the Laplace transform variable (reference 2). If the differential operator p is substituted for d/dt , equation (2) can be written

$$p^2y + apy + by = cpx + ex \tag{3}$$

If this equation is solved for y , an explicit expression for the output can then be obtained as a function of p operating on the input x as follows:

$$y = \frac{cp + e}{p^2 + ap + b} \cdot x = F(p) \cdot x \tag{4}$$

For each unit of the system, an equation similar to equation (4) can be written in terms of the input, the transfer function, and the output. A set of simultaneous algebraic equations is then obtained that can be combined or manipulated in any desired manner. The resulting transfer functions can then be interpreted from the domain of the variable p to the real time domain as time responses of a variable to some specified disturbance.

For the system of figure 1, the following equations can be written:

$$\left. \begin{aligned}
 &\text{Engine characteristic:} \\
 &\quad y = E \cdot x \\
 &\text{Control characteristic:} \\
 &\quad \underline{x} = C \cdot (Y - \bar{y}) \\
 &\text{Instrument characteristic:} \\
 &\quad \bar{y} = L \cdot y \\
 &\text{Servo and transient disturbance:} \\
 &\quad x = S \cdot \underline{x} + V
 \end{aligned} \right\} (5)$$

Equations (5) can be combined for the effect of the disturbances Y and V on the controlled variable y as follows:

$$y = \frac{ESC}{ESCL+1} \cdot Y + \frac{E}{ESCL+1} \cdot V \quad (6)$$

The previous procedure is, in general, followed herein. Equations, such as equations (5), are written for each unit in the system and are then combined, as in equation (6), to obtain the effects of the disturbances.

General representation of engine.—If the engine has more than one independent variable, there are additional degrees of freedom in control. The engine-characteristic function must be expanded to show how each dependent variable of interest is affected by all the engine inputs as follows:

$$\left. \begin{aligned}
 y_1 &= E_{11} \cdot x_1 + E_{12} \cdot x_2 + \dots + E_{1n} \cdot x_n \\
 y_2 &= E_{21} \cdot x_1 + E_{22} \cdot x_2 + \dots + E_{2n} \cdot x_n \\
 &\vdots \\
 y_i &= E_{i1} \cdot x_1 + E_{i2} \cdot x_2 + \dots + E_{in} \cdot x_n
 \end{aligned} \right\} (7)$$

In equations (7) each dependent variable y is expressed as a linear sum of effects due to each independent variable x . For n independent variables, there are n terms in each equation. There is an equation for each dependent variable of interest. For i dependent variables, where $i \leq n$, there are i equations.

The number of degrees of freedom of the entire system cannot be greater than the number of degrees of freedom of the engine alone. For continuously acting controls (excluding such controls as limiters), the total number of engine variables being controlled should not be greater than the number of engine-independent variables. Equations (7) can be written in the following index form:

$$y_j = \sum_{k=1}^n E_{jk} \cdot x_k \quad (8)$$

where

$$j = 1, 2, \dots, i$$

The functions E_{jk} represent engine characteristics whereby each input separately affects each output.

Equation (7) or equation (8) can be visualized in the matrix form of figure 2. The engine functions are set up in a rectangular array where E_{jk} represents the element in the j th row and the k th column. The x inputs enter the matrix in the columns and the y outputs are attached to the rows. Each input multiplies all the elements in its column and each output is the linear sum of the resulting products in its row. Thus the effects shown by equations (7) and (8) can be pictorially traced in figure 2. For later use, a square matrix obtained by using only the first i columns in figure 2 is denoted by E^* .

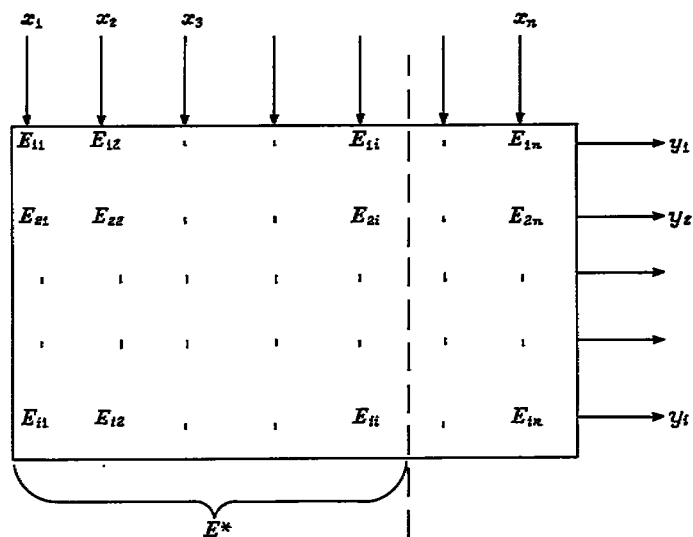


FIGURE 2.—General representation of complex engine types.

General representation of control.—The control system is generalized to this extent: (1) Inasmuch as the engine has n independent variables, there are n degrees of freedom for the entire system. If only a limited number i of engine-dependent variables are to be controlled, the difference $n-i$ of the engine-independent variables can be controlled. (2) The control system employs negative feedback where only errors are applied to the control. (3) Each error is to affect every engine-independent variable.

The control equations are written where each control output \underline{x} is expressed as a linear sum of effects due to the errors in the controlled variables as follows:

$$\left. \begin{aligned}
 \underline{x}_1 &= C_{11} \cdot (Y_1 - \bar{y}_1) + C_{12} \cdot (Y_2 - \bar{y}_2) + \dots + C_{1i} \cdot (Y_i - \bar{y}_i) + \\
 &\quad \cdot C'_{1(i+1)} \cdot (X_{i+1} - \bar{x}_{i+1}) + \dots + C'_{1n} \cdot (X_n - \bar{x}_n) \\
 &\vdots \\
 \underline{x}_n &= C_{n1} \cdot (Y_1 - \bar{y}_1) + C_{n2} \cdot (Y_2 - \bar{y}_2) + \dots + C_{ni} \cdot (Y_i - \bar{y}_i) + \\
 &\quad \cdot C'_{n(i+1)} \cdot (X_{i+1} - \bar{x}_{i+1}) + \dots + C'_{nn} \cdot (X_n - \bar{x}_n)
 \end{aligned} \right\} (9)$$

There are a total of n controlled quantities, n errors, and n terms in each equation. There are also n engine-independent variables and therefore n equations. Those independent variables to be controlled are given subscript numbers from $i+1$ to n and those controls to which the corresponding errors are applied are denoted by C' .

The control characteristics of equations (9) can be written in the following equation in index form:

$$\underline{x}_k = \sum_{v=1}^i C_{kv} \cdot (Y_v - \bar{y}_v) + \sum_{\mu=i+1}^n C'_{k\mu} \cdot (X_\mu - \bar{x}_\mu) \quad (10)$$

where

$$k=1, 2, \dots, n$$

Equations (9) and (10) can be visualized in the matrix form of figure 3. The control functions are set up in a square arrangement where C_{kv} represents the element in the k th row and v th column. The error inputs, $Y - \bar{y}$ and $X - \bar{x}$, enter the matrix in the columns and the \underline{x} outputs are

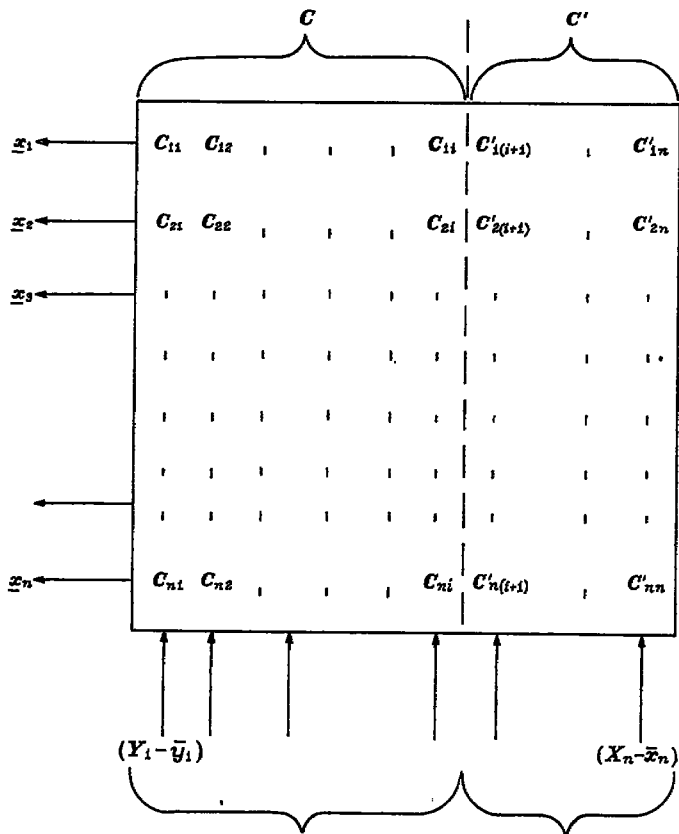


FIGURE 3.—General representation of control system.

attached to the rows. Each input multiplies all the elements in its column and each output is the linear sum of the resulting products in its row. Thus, the effects shown by equations (9) and (10) can be pictorially traced in figure 3.

Representation for instrument and servo functions.—The characteristics of the instrument measuring the value of the controlled variable have been denoted as L in equations (5); the input of this instrument is the variable itself and the

output, the measured value. These characteristics can be written as

$$\left. \begin{aligned} \bar{y}_v &= L_{vv} \cdot y_v \\ \text{where } v &= 1, 2, \dots, i \end{aligned} \right\} \quad (11)$$

$$\left. \begin{aligned} \bar{x}_\mu &= L_{\mu\mu} \cdot x_\mu \\ \text{where } \mu &= i+1, \dots, n \end{aligned} \right\}$$

The general assumption is made that each engine-independent variable would require a servo, which would change the variable in accordance with its corresponding signal \underline{x} . A source of transient disturbance is included that may vary any of the engine-independent variables outside of the controlled variation in them. Thus, the expressions for the engine-independent variables can be given by

$$x_k = S_{kk} \cdot \underline{x}_k + V_k \quad (12)$$

where

$$k=1, 2, \dots, n$$

Each variable x is thus the sum of a controlled part and an uncontrolled part.

Analysis of complete system.—The determining characteristic equations for all the units in the system are repeated.

Engine:

$$y_j = \sum_{k=1}^n E_{jk} \cdot x_k \quad (8)$$

where

$$j=1, 2, \dots, i$$

Control:

$$\underline{x}_k = \sum_{v=1}^i C_{kv} \cdot (Y_v - \bar{y}_v) + \sum_{\mu=i+1}^n C'_{k\mu} \cdot (X_\mu - \bar{x}_\mu) \quad (10)$$

where

$$k=1, 2, \dots, n$$

Instrument:

$$\left. \begin{aligned} \bar{y}_v &= L_{vv} \cdot y_v \\ \text{where } v &= 1, 2, \dots, i \end{aligned} \right\} \quad (11)$$

where

$$\left. \begin{aligned} \bar{x}_\mu &= L_{\mu\mu} \cdot x_\mu \\ \text{where } \mu &= i+1, \dots, n \end{aligned} \right\}$$

Servo and transient:

$$x_k = S_{kk} \cdot \underline{x}_k + V_k \quad (12)$$

where

$$k=1, 2, \dots, n$$

These equations can be visualized in the matrix form of figure 4. Figure 4 represents, by block diagram, the general system considered herein. This figure specifically is for an engine having five independent variables for which three engine-dependent and two engine-independent variables are controlled. The engine matrix is shown for the five inputs and the three outputs to be controlled. The control matrix

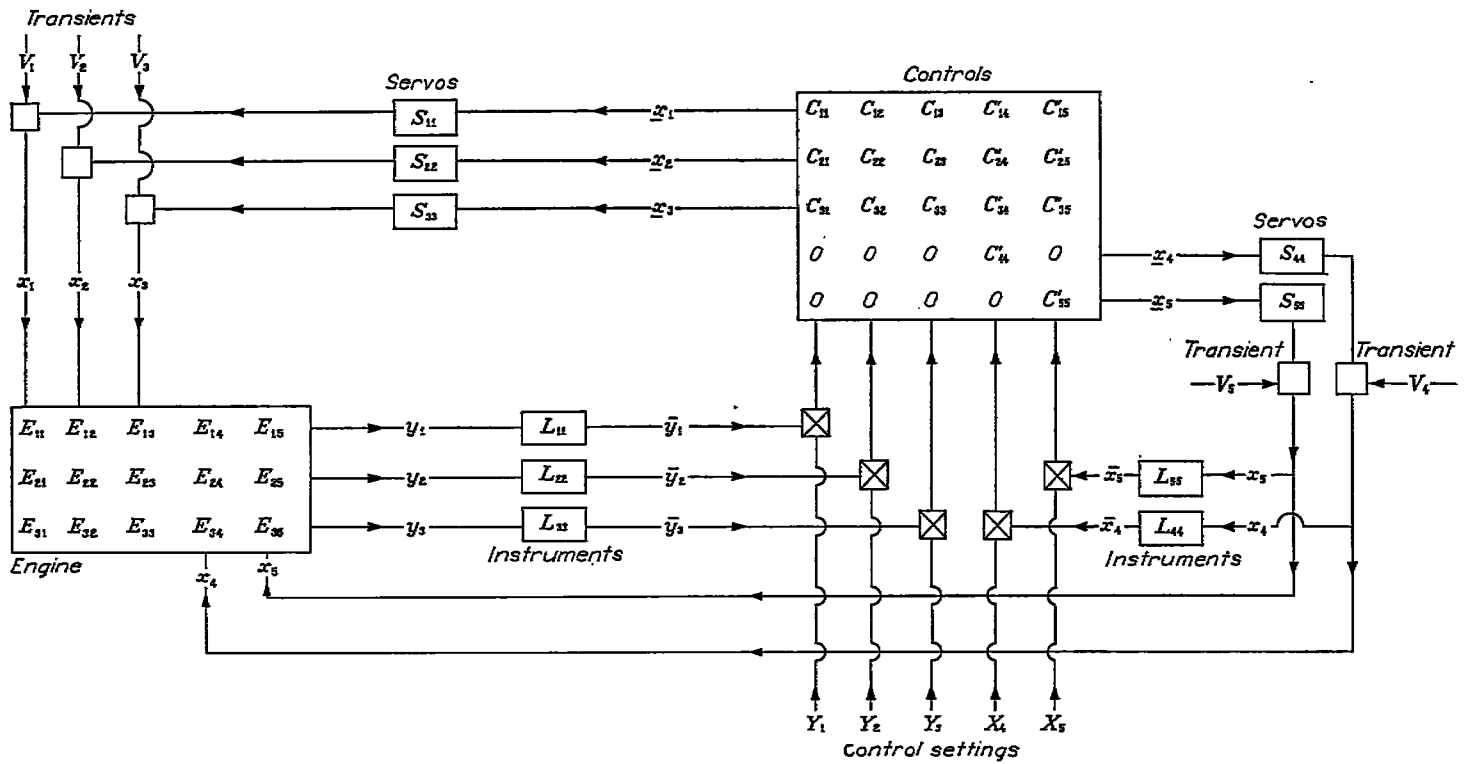


FIGURE 4.—Combined general controlled-engine-system configuration.

is shown for the three y error inputs, the two x error inputs, and the five outputs to the engine servos. The measuring devices, the servos, and the transients are also included. The entire system is enclosed except for the outside disturbances, which can be imposed on the system. These are the five settings of controlled variables Y and X and the five transient disturbances that have been included.

All the properties of the system can be obtained from this block-diagram representation. The effect of any outside disturbance or any internal disturbance can be easily traced on figure 4. For specific effects, the equations of each unit of the system, equations (13), are most productive of results.

When equations (13) are combined to give the effects on the controlled variables of the outside disturbances, the following equations are obtained:

$$y_j = \sum_{k=1}^n \sum_{v=1}^i E_{jk} S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{k=1}^n \sum_{\mu=i+1}^n E_{jk} S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} x_\mu) + \sum_{k=1}^n E_{jk} V_k \quad (14)$$

where

$$j=1, 2, \dots, i$$

$$x_k = \sum_{v=1}^i S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{\mu=i+1}^n S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} x_\mu) + V_k \quad (15)$$

where

$$k=1, 2, \dots, n$$

Equation (15) is valid for the effects on any ($k=1, 2, \dots, n$) engine-independent variable. For the effects on the controlled engine-independent variables, equation (15) is used for $k=i+1$ to n . A more condensed form of the matrix representation of the system, shown in figure 5, is indicated by

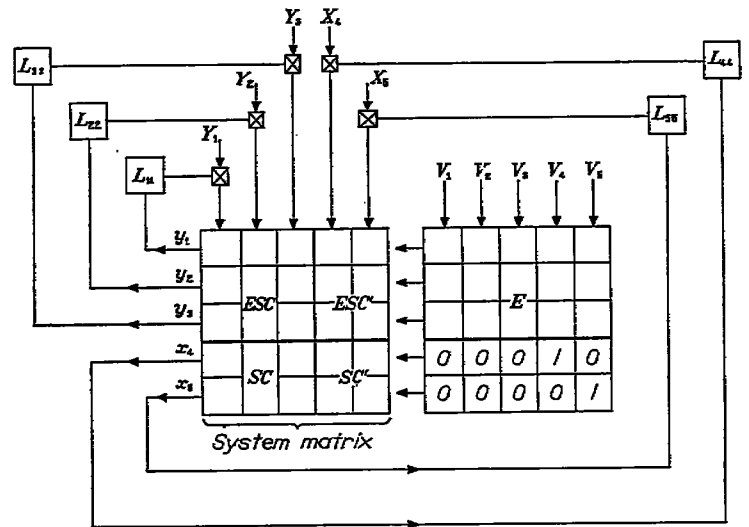


FIGURE 5.—Condensed matrix form of complete controlled system.

equations (14) and (15). A single-system matrix is used in which the inputs are the errors in controlled variables and the outputs are the controlled variables. The ESC matrix is a matrix in which the element in the j th row and v th column is $\sum_{k=1}^n E_{jk} S_{kk} C_{kv}$. Similarly, an ESC' matrix is a matrix in which the element in the j th row and μ th column is $\sum_{k=1}^n E_{jk} S_{kk} C'_{k\mu}$. An SC matrix is defined as $S_{kk} C_{kv}$ and an SC' matrix is defined as $S_{kk} C'_{k\mu}$. An additional matrix is included showing the effects of the transient disturbances.

Noninteraction conditions.—Several types of noninteraction may be specified as follows:

(1) A specific controlled variable, such as y_j , is to be affected only by its own setting Y_j , and is not to be affected by any other setting, which means that in figure 5 the j th

row of the system matrix contains only the diagonal element and the other elements must be zero.

(2) Any specific setting, such as Y_i , is to affect only its corresponding controlled variable y_i , and not any of the other controlled variables. In figure 5, the t th column of the system matrix contains only the diagonal element and the other elements must be zero.

(3) Every controlled variable is to be affected by its corresponding setting only, or every setting is to affect its corresponding controlled variable only. This complete non-interaction means that in figure 5 the system matrix is a diagonal matrix. In general, the effect of the setting on the system is of primary interest and a condition such as (1) is of secondary interest. The interaction effects defined in terms of the effects of the settings, such as (2) and (3), will thus be considered herein.

In order for any setting such as Y_i to affect only y_i , it is sufficient from figure 5 that for all j from 1 to i when $j \neq t$

$$\sum_{k=1}^n E_{jk} S_{kk} C_{kt} = 0 \tag{16}$$

and for all μ from $i+1$ to n

$$S_{\mu\mu} C_{\mu t} = 0 \tag{17}$$

or

$$C_{\mu t} = 0 \tag{17a}$$

Equation (17) states that the $i+1$ to n elements of the t th column of the C matrix (fig. 3) are to be zeroed. Along with this condition, equation (16) becomes

$$\sum_{k=1}^i E_{jk} S_{kk} C_{kt} = 0 \tag{18}$$

for all j from 1 to i where $j \neq t$. Equation (18) expresses $i-1$ homogeneous equations in the i unknown C 's and therefore specifies the ratios between any two elements in the t th column of the C matrix. This result becomes

$$\frac{S_{jj} C_{jt}}{S_{vv} C_{vt}} = \frac{|E^*_{tj}|}{|E^*_{tv}|} \tag{19}$$

as shown in the appendix. The notation E^* refers to the square engine matrix using only the first i columns. The notation $|E^*_{tj}|$ indicates the cofactor of the E_{tj} element of the E^* matrix. The necessary and sufficient conditions for any setting Y_i to affect only y_i are

$$C_{\mu t} = 0 \tag{17a}$$

for all μ from $i+1$ to n and

$$\frac{S_{jj} C_{jt}}{S_{vv} C_{vt}} = \frac{|E^*_{tj}|}{|E^*_{tv}|} \tag{19}$$

as shown in the appendix.

For any setting X_r to affect only x_r , it is sufficient from figure 5 that

$$\sum_{k=1}^n E_{jk} S_{kk} C'_{kr} = 0 \tag{20}$$

for all j from 1 to i ; and for all μ from $i+1$ to n where $\mu \neq r$

$$S_{\mu\mu} C'_{\mu r} = 0 \tag{21}$$

or

$$C'_{\mu r} = 0 \tag{21a}$$

Equation (21) states that the $i+1$ to n elements of the r th column of the C' matrix contain only the diagonal term. With this condition, equation (20) becomes

$$\sum_{k=1}^i (E_{jk} S_{kk} C'_{kr}) + E_{jr} S_{rr} C'_{rr} = 0 \tag{22}$$

for all j from 1 to i . Equation (22) expresses i homogeneous equations in the $i+1$ unknown (C')'s and therefore specifies the ratios between any two elements in the r th column of the C' matrix. This result becomes

$$\frac{S_{jj} C'_{jr}}{S_{vv} C'_{vr}} = \frac{\sum_{k=1}^i |E^*_{kj}| E_{kr}}{\sum_{k=1}^i |E^*_{kv}| E_{kr}} \tag{23}$$

for any j and v from 1 to i or

$$\frac{S_{jj} C'_{jr}}{S_{rr} C'_{rr}} = \frac{-\sum_{k=1}^i |E^*_{kj}| E_{kr}}{|E^*|} \tag{24}$$

as shown in the appendix. The necessary and sufficient condition for any setting X_r to affect only x_r is

$$C'_{\mu r} = 0 \tag{21a}$$

for all μ from $i+1$ to n when $\mu \neq r$ and

$$\frac{S_{jj} C'_{jr}}{S_{rr} C'_{rr}} = \frac{-\sum_{k=1}^i |E^*_{kj}| E_{kr}}{|E^*|} \tag{24}$$

as shown in the appendix.

For complete noninteraction, where every setting is to affect its corresponding variable only, the conditions can be summarized as follows:

$$C_{\mu t} = 0 \tag{17a}$$

where

$$\mu = i+1, \dots, n$$

$$t = 1, 2, \dots, i$$

$$C'_{\mu r} = 0 \tag{21a}$$

where

$$\mu = i+1, \dots, n \quad (\mu \neq r)$$

$$r = i+1, \dots, n$$

$$\frac{S_{jj} C_{jt}}{S_{vv} C_{vt}} = \frac{|E^*_{tj}|}{|E^*_{tv}|} \tag{19}$$

where

$$j, t, v = 1, 2, \dots, i$$

$$\frac{S_{jj}C'_{jr}}{S_{rr}C'_{rr}} = \frac{-\sum_{k=1}^i |E^*_{kj}| E_{kr}}{|E^*_{jj}|} \quad (24)$$

where

$$j = 1, 2, \dots, i$$

$$r = i + 1, \dots, n$$

Equations (17a) and (21a) state that the $i+1$ to n rows of the control matrix contain only diagonal elements. Equation (19) gives the required ratios between any two elements of any column of the C matrix. Equation (24) gives the required ratios of any element in any column of the C' matrix to the diagonal element.

Response equations.—The use of the operational functions to characterize each unit in the system allows algebraic representations of the relations among the variables of the system. With complete noninteraction specified, the system matrix of figure 5 becomes a diagonal matrix and each circuit acts as follows:

$$y_j = \sum_{k=1}^i E_{jk} S_{kk} C_{kj} (Y_j - L_{jj} y_j) + \sum_{k=1}^i E_{jk} V_k \quad (25)$$

and

$$x_r = S_{rr} C'_{rr} (X_r - L_{rr} x_r) + V_r \quad (26)$$

When these equations are solved for the controlled variables as a function of the settings and the transient disturbances, the following equations are obtained:

$$y_j = \frac{\sum_{k=1}^i E_{jk} S_{kk} C_{kj}}{\sum_{k=1}^i (E_{jk} S_{kk} C_{kj} L_{jj}) + 1} \cdot Y_j + \frac{\sum_{k=1}^i E_{jk} V_k}{\sum_{k=1}^i (E_{jk} S_{kk} C_{kj} L_{jj}) + 1} \quad (27)$$

$$x_r = \frac{S_{rr} C'_{rr}}{S_{rr} C'_{rr} L_{rr} + 1} \cdot X_r + \frac{V_r}{S_{rr} C'_{rr} L_{rr} + 1} \quad (28)$$

If the operational response function of y_j to Y_j is defined as R_{jj} and the operational response function of x_r to X_r as R'_{rr} , equations (27) and (28) become

$$y_j = R_{jj} Y_j - \sum_{k=1}^i (R_{jj} L_{jj} - 1) E_{jk} V_k \quad (29)$$

$$x_r = R'_{rr} X_r - (R'_{rr} L_{rr} - 1) V_r \quad (30)$$

where

$$R_{jj} = \frac{\sum_{k=1}^i E_{jk} S_{kk} C_{kj}}{\sum_{k=1}^i (E_{jk} S_{kk} C_{kj} L_{jj}) + 1} \quad (31)$$

and

$$R'_{rr} = \frac{S_{rr} C'_{rr}}{S_{rr} C'_{rr} L_{rr} + 1} \quad (32)$$

The response of any y_j to Y_j insofar as the control matrix is concerned depends only on the elements in the j th column of the control matrix (fig. 3). Inasmuch as for complete noninteraction only the ratios of the elements of that column of the control matrix are specified, the response can be independently set by choosing any one element of that column. If the conditions for noninteraction are used, R_{jj} is a function of any C_{vj}

$$R_{jj} = \frac{|E^*_{jj}| S_{vv} C_{vj}}{|E^*_{jj}| S_{vv} C_{vj} L_{jj} + |E^*_{jj}|} \quad (33)$$

Equations (32) and (33) show the response of any controlled variable to its setting as determined by the control functions. The response of a controlled engine-independent variable does not depend on engine characteristics (equation (32)). The response of a controlled engine-dependent variable depends on the square engine matrix E^* (fig. 2).

An important result is obtained when equations similar to (29) and (30) are written for the errors in controlled quantities as follows:

$$Y_j - \bar{y}_j = (1 - R_{jj} L_{jj}) \left(Y_j - L_{jj} \sum_{k=1}^n E_{jk} V_k \right) \quad (34)$$

$$X_r - \bar{x}_r = (1 - R'_{rr} L_{rr}) (X_r - L_{rr} V_r) \quad (35)$$

The errors in controlled quantities respond in the same manner to both setting and transient disturbances and the control configuration chosen can act both as a control to setting disturbances and a regulator to transient disturbances, at least with the transient disturbances assumed.

Ideal control functions.—The control-design problem is one of fitting a control system to a particular plant or engine in order to control certain variables in some specified manner. The nature of the final system is characterized by the desired response of controlled quantities to outside disturbances. These responses may be exactly specified or given in general qualitative terms. The nature of the engine or plant to be controlled must be known in its static and dynamic states. Any method of explicitly solving for the control functions for any given plant and corresponding desired response action would give an exact answer to this problem.

The use of the operational functions to describe the engine, control, and response functions and the algebraic methods employed herein lead directly to an explicit solution for the ideal control functions. Equations (32) and (33), which give the response actions in terms of the control function, can be solved for the control functions in terms of the response functions as follows:

$$S_{vv} C_{vj} = \frac{R_{jj}}{(1 - R_{jj} L_{jj})} \frac{|E^*_{jj}|}{|E^*_{jj}|} \quad (36)$$

$$S_{rr} C'_{rr} = \frac{R'_{rr}}{1 - R'_{rr} L_{rr}} \quad (37)$$

Choosing any desired response characteristics and solving for the required control functions to achieve this response are therefore theoretically possible. The use of these equations as well as the noninteraction conditions previously given for control design has certain limitations that preclude the use of this method in its most general sense. These limitations arise from the inability to incorporate exactly in a control the engine characteristics that the control is to match. A simple first-order response can be called for in equation (36) in spite of the higher-order transfer functions of engine and servo. This type of response is possible only if the control function can reduce the order of the transfer function by allowing factoring of the numerator and the denominator of equation (33) into exactly the same terms.

The control functions found from equations (36) and (37) for responses that are stable would be expected to give satisfactory results for small changes in characteristics and in the engine characteristics the control function is to match. Special cases of engine characteristics may require that an unstabilizing term in the engine functions be factored out by the control. In this case, variations in engine or control characteristics would unstabilize the system. In such cases, equations (36) and (37) should be used with the proviso that the control does not decrease the order of the response function and the response functions must be chosen accordingly.

The algebraic methods used in dealing with operational functions can be further generalized in terms of equivalent relations when the engine, the control, and the other unit characteristics are expressed in other forms. For instance, if the frequency-response characteristics are graphically known from tests or analysis, all the results described herein can be graphically performed in terms of the frequency-response characteristics in place of the operational functions.

Summary of basic theory and results.—For an engine having n engine-independent variables and for which i engine-dependent variables are to be set and controlled ($i \leq n$) and for which $n-i$ engine-independent variables are to be set and controlled, engine, control system, instrument characteristics, servo, and transient disturbances are

Engine:

$$y_j = \sum_{k=1}^n E_{jk} \cdot x_k \quad (8)$$

where

$$j = 1, 2, \dots, i$$

Control:

$$x_k = \sum_{v=1}^i C_{kv} \cdot (Y_v - \bar{y}_v) + \sum_{\mu=i+1}^n C'_{k\mu} \cdot (X_\mu - \bar{x}_\mu) \quad (10)$$

where

$$k = 1, 2, \dots, n$$

Instruments:

$$\left. \begin{aligned} \bar{y}_v &= L_{vv} \cdot y_v \\ \bar{x}_\mu &= L_{\mu\mu} \cdot x_\mu \end{aligned} \right\} \quad (11)$$

where

$$v = 1, 2, \dots, i$$

$$\mu = i+1, \dots, n$$

Servo and transient disturbance:

$$x_k = S_{kk} \cdot \underline{x}_k + V_k \quad (12)$$

where

$$k = 1, 2, \dots, n$$

The necessary and sufficient conditions for any setting Y_t to affect only y_t are

$$C_{\mu t} = 0 \quad (17a)$$

for all μ from $i+1$ to n , and

$$\frac{S_{jt} C_{jt}}{S_{vv} C_{vt}} = \frac{|E^*_{tj}|}{|E^*_{tv}|} \quad (19)$$

The necessary and sufficient conditions for any setting X_r to affect only x_r are

$$C'_{\mu r} = 0 \quad (21a)$$

for all μ from $i+1$ to n when $\mu \neq r$

$$\frac{S_{jt} C'_{jr}}{S_{rr} C'_{rr}} = \frac{-\sum_{k=1}^i |E^*_{kj}| E_{kr}}{|E^*_{rj}|} \quad (24)$$

For complete noninteraction, equations (17) and (19) must be true for all values of t from 1 to i and equations (21) and (24) must be true for all values of r from $i+1$ to n .

With complete noninteraction specified, the response of the controlled variables and their measured error signals to setting and transient disturbances are

$$y_j = R_{jj} \cdot Y_j - \sum_{k=1}^n (R_{jj} L_{jj} - 1) E_{jk} V_k \quad (29)$$

$$x_r = R'_{rr} \cdot X_r - (R'_{rr} L_{rr} - 1) V_r \quad (30)$$

and

$$Y_j - \bar{y}_j = (1 - R_{jj} L_{jj}) \left(Y_j - L_{jj} \sum_{k=1}^n E_{jk} V_k \right) \quad (34)$$

$$X_r - \bar{x}_r = (1 - R'_{rr} L_{rr}) (X_r - L_{rr} V_r) \quad (35)$$

where

$$R_{jj} = \frac{\sum_{k=1}^i E_{jk} S_{kk} C_{kj}}{\sum_{k=1}^i (E_{jk} S_{kk} C_{kj} L_{jj}) + 1} = \frac{|E^*_{jj}| S_{vv} C_{vj}}{|E^*_{jj}| S_{vv} C_{vj} L_{jj} + |E^*_{jj}|} \quad (31) \text{ and } (33)$$

and

$$R'_{rr} = \frac{S_{rr} C'_{rr}}{S_{rr} C'_{rr} L_{rr} + 1} \quad (32)$$

With complete noninteraction specified, the explicit expressions for the ideal control functions in terms of the desired response functions are

$$S_{jj}C'_{jj} = \frac{R_{jj}}{(1 - R_{jj}L_{jj})} \frac{|E^*_{jj}|}{|E^*|} \quad (36)$$

$$S_{rr}C'_{rr} = \frac{R'_{rr}}{1 - R'_{rr}L_{rr}} \quad (37)$$

SPECIFIC EXAMPLE

If the turbine-propeller engine, which has the two independent variables fuel flow and blade angle, is considered, the defining differential equations obtained by linearizing the general functional forms of engine torque as a function of speed and temperature, propeller torque as a function of speed and blade angle, and temperature as a function of speed and fuel flow are

$$N + \tau N = -a\beta + bw_f \quad (38)$$

$$T = cw_f - eN \quad (39)$$

When these equations are placed in operational form and speed and temperature are solved for in terms of fuel flow and blade angle,

$$N = \frac{-a}{1 + \tau p} \beta + \frac{b}{1 + \tau p} w_f \quad (40)$$

$$T = \frac{ae}{1 + \tau p} \beta + \frac{c(1 - \gamma + \tau p)}{1 + \tau p} w_f \quad (41)$$

where

$$\gamma = \frac{be}{c}$$

CASE I

Controlling N and w_f .—For the case where $n=2$ and $i=1$, the engine matrix of figure 2 is obtained from equation (40) and

$$E_{11} = \frac{-a}{1 + \tau p} \quad E_{12} = \frac{b}{1 + \tau p}$$

$$x_1 = \beta \quad x_2 = w_f$$

$$y_1 = N$$

The general control system is

$$\underline{\beta} = C_{11}(N_s - \bar{N}) + C'_{12}(w_{f,s} - \bar{w}_f)$$

$$\underline{w}_f = C_{21}(N_s - \bar{N}) + C'_{22}(w_{f,s} - \bar{w}_f)$$

From equation (17a), the conditions for complete noninteraction give

$$C_{21} = 0$$

and from equation (24) for complete noninteraction,

$$\frac{S_{11}C'_{12}}{S_{22}C'_{22}} = \frac{-|E^*_{11}|E_{12}}{|E^*|} = -\frac{E_{12}}{E_{11}} \frac{b}{a}$$

where the determinant of a single-element matrix written as $|E^*| = |E^*_{11}|E_{11}$ allows the use of the equations employed herein for this case.

The control-system equations thus become

$$\underline{\beta} = C_{11}(N_s - \bar{N}) + \frac{b}{a} \frac{S_{22}}{S_{11}} C'_{22}(w_{f,s} - \bar{w}_f)$$

$$\underline{w}_f = C'_{22}(w_{f,s} - \bar{w}_f)$$

The responses of speed and fuel flow to their settings become, from equation (33),

$$R_{11} = \frac{|E^*|S_{11}C_{11}}{|E^*|S_{11}C_{11}L_{11} + |E^*_{11}|} = \frac{E_{11}S_{11}C_{11}}{E_{11}S_{11}C_{11}L_{11} + 1}$$

$$= \frac{aS_{11}C_{11}}{aS_{11}C_{11}L_{11} - (1 + \tau p)}$$

and from equation (32),

$$R'_{22} = \frac{S_{22}C'_{22}}{S_{22}C'_{22}L_{22} + 1}$$

The expressions for the ideal control functions then become, from equation (36),

$$S_{11}C_{11} = \frac{R_{11}}{1 - R_{11}L_{11}} \frac{|E^*_{11}|}{|E^*|} = \frac{R_{11}}{(1 - R_{11}L_{11})} \frac{1}{E_{11}} = \frac{-R_{11}}{(1 - R_{11}L_{11})} \frac{(1 + \tau p)}{a}$$

and from equation (37),

$$S_{22}C'_{22} = \frac{R'_{22}}{1 - R'_{22}L_{22}}$$

CASE II

Controlling N and T .—For the case where $n=2$ and $i=2$, the engine matrix is obtained from equations (40) and (41) and

$$E_{11} = \frac{-a}{1 + \tau p} \quad E_{12} = \frac{b}{1 + \tau p}$$

$$E_{21} = \frac{ae}{1 + \tau p} \quad E_{22} = \frac{c(1 - \gamma + \tau p)}{1 + \tau p}$$

$$x_1 = \beta \quad x_2 = w_f$$

$$y_1 = N \quad y_2 = T$$

The general control system is

$$\underline{\beta} = C_{11}(N_s - \bar{N}) + C_{12}(T_s - \bar{T})$$

$$\underline{w}_f = C_{21}(N_s - \bar{N}) + C_{22}(T_s - \bar{T})$$

The conditions for complete noninteraction give, from equation (19),

$$\frac{S_{11}C_{11}}{S_{22}C_{21}} = \frac{|E^*_{11}|}{|E^*_{12}|} = -\frac{E_{22}}{E_{21}} = -\frac{b}{a\gamma}(1-\gamma+\tau p)$$

$$\frac{S_{11}C_{12}}{S_{22}C_{22}} = \frac{|E^*_{21}|}{|E^*_{22}|} = -\frac{E_{12}}{E_{11}} = \frac{b}{a}$$

The control-system equations thus become

$$\underline{\beta} = C_{11}(N_s - \bar{N}) + \frac{b}{a} \frac{S_{22}}{S_{11}} C_{22}(T_s - \bar{T})$$

$$\underline{w}_f = -\frac{a\gamma}{b(1-\gamma+p)} \frac{S_{11}}{S_{22}} C_{11}(N_s - \bar{N}) + C_{22}(T_s - \bar{T})$$

The responses of speed and fuel flow to their settings become

$$R_{11} = \frac{|E^*| S_{11} C_{11}}{|E^*| S_{11} C_{11} L_{11} + |E^*_{11}|} = \frac{S_{11} C_{11}}{S_{11} C_{11} L_{11} - \frac{(1-\gamma+\tau p)}{a}}$$

or

$$R_{11} = \frac{S_{22} C_{21}}{S_{22} C_{21} L_{11} + \frac{\gamma}{b}}$$

$$R_{22} = \frac{|E^*| S_{22} C_{22}}{|E^*| S_{22} C_{22} L_{22} + |E^*_{22}|} = \frac{S_{22} C_{22}}{S_{22} C_{22} L_{22} + \frac{1}{c}}$$

The expressions for the ideal control functions then become

$$S_{11} C_{11} = \frac{R_{11}}{1 - R_{11} L_{11}} \frac{|E^*_{11}|}{|E^*|} = \frac{-R_{11}}{1 - R_{11} L_{11}} \frac{(1-\gamma+\tau p)}{a}$$

$$S_{22} C_{22} = \frac{R_{22}}{1 - R_{22} L_{22}} \frac{|E^*_{22}|}{|E^*|} = \frac{R_{22}}{(1 - R_{22} L_{22})} \frac{1}{c}$$

SUMMARY OF RESULTS

The control of gas-turbine engines was limited to the linear assumptions for the engine, the control system, and the other units in the system. Operational functions were used for these units, the system was generalized to allow for complex engine types having any degree of freedom, and algebraic methods were employed in setting up the characteristics of the engine, the control, the servo, and the measuring instruments. These algebraic methods were employed to combine the various separate units into a system showing how various effects are transmitted throughout the entire system. A condensed matrix form was used to illustrate the principal effects on controlled variables.

By use of either the index or matrix form of the algebraic equations of the system, the following principal conditions were explicitly expressed:

(a) The conditions that must be imposed on the control functions so that any setting disturbance will affect only the variable being set

(b) The conditions that must be imposed on the control functions for complete noninteraction so that every setting disturbance will affect only its corresponding variable

(c) The response functions, which characterize the controlled-system response action, in terms of the control-system functions

(d) The ideal control-system characteristics in terms of any desired response action.

A specific example using the turbine-propeller engine having two degrees of freedom was given to illustrate the use of the previous results.

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APPENDIX

CALCULATION CONDITIONS

CONDITIONS ON C MATRIX

Necessary condition.—The general equations for the response of the controlled variables are given by equations (A1) and (A2) (equations (14) and (15), respectively, in the text) as follows:

$$y_j = \sum_{k=1}^n \sum_{v=1}^i E_{jk} S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{k=1}^n \sum_{\mu=i+1}^n E_{jk} S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} x_\mu) + \sum_{k=1}^n E_{jk} V_k \quad (\text{A1})$$

where

$$j=1, 2, \dots, i$$

$$x_k = \sum_{v=1}^i S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{\mu=i+1}^n S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} x_\mu) + V_k \quad (\text{A2})$$

where

$$k=1, 2, \dots, n$$

If the effects of any setting Y_i are separated, equations (A1) and (A2) become

$$y_j = \sum_{k=1}^n \sum_{\substack{v=1 \\ v \neq i}}^i E_{jk} S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{k=1}^n \sum_{\mu=i+1}^n E_{jk} S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} x_\mu) + \sum_{k=1}^n E_{jk} V_k + \sum_{k=1}^n E_{jk} S_{kk} C_{ki} (Y_i - L_{ii} y_i) \quad (\text{A3})$$

$$x_k = \sum_{\substack{v=1 \\ v \neq i}}^i S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{\mu=i+1}^n S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} x_\mu) + V_k + S_{kk} C_{ki} (Y_i - L_{ii} y_i) \quad (\text{A4})$$

If any setting Y_i is to affect only y_i , then any y_j is not affected by Y_i ($j \neq i$) and any controlled x_k ($k=i+1$ to n) is not affected by Y_i . If equation (A3) is considered for any $j \neq i$, then the only term that can be a function of Y_i is the last one. Because of the arbitrary nature of the instrument function, $Y_i \neq L_{ii} y_i$, so that

$$\sum_{k=1}^n E_{jk} S_{kk} C_{ki} = 0 \quad (\text{A5})$$

where

$$j \neq i$$

If equation (A4) is considered for $k=i+1$ to n , the only term that can be a function of Y_i is the last one; therefore

$$S_{kk} C_{ki} = 0 \quad (\text{A6})$$

$$C_{ki} = 0 \quad (\text{A7})$$

where

$$k=i+1, \dots, n$$

If equation (A7) is used, equation (A5) becomes

$$\sum_{k=1}^i E_{jk} S_{kk} C_{ki} = \sum_{k=1}^i \delta_{jt} E_{tk} S_{kk} C_{ki}$$

where

$$j=1, 2, \dots, i$$

and

$$\delta_{jt} = 0$$

when

$$j \neq t$$

and

$$\delta_{jt} = 1$$

when

$$j=t$$

The solution of this set of i equations in the i unknown $S_{kk} C_{ki}$ gives

$$S_{jj} C_{ji} = \frac{\sum_{p=1}^i \sum_{k=1}^i |E^*_{pj}| \delta_{pt} E_{tk} S_{kk} C_{ki}}{|E^*|}$$

and because

$$\delta_{pt} = 0$$

when

$$p \neq t$$

when

$$\delta_{pt} = 1$$

$$p=t$$

$$S_{jj} C_{ji} = \frac{|E^*_{tj}| \sum_{k=1}^i E_{tk} S_{kk} C_{ki}}{|E^*|} \quad (\text{A8})$$

where

$$j=1, 2, \dots, i$$

Because j is any number ($1, 2, \dots, i$), a similar expression for any other $S_{vv} C_{vt}$ is

$$S_{vv} C_{vt} = \frac{|E^*_{vt}| \sum_{k=1}^i E_{tk} S_{kk} C_{ki}}{|E^*|} \quad (\text{A9})$$

The ratio of equations (A8) and (A9) becomes

$$\frac{S_{jj} C_{ji}}{S_{vv} C_{vt}} = \frac{|E^*_{tj}|}{|E^*_{vt}|} \quad (\text{A10})$$

where

$$j=1, 2, \dots, i$$

$$v=1, 2, \dots, i$$

Equations (A7) and (A10) thus follow necessarily from the noninteraction condition specified.

Sufficient condition.—If equation (A10) holds,

$$\sum_{j=1}^i E_{rj} \frac{S_{jj} C_{jt}}{S_{rr} C_{rt}} = \frac{\sum_{j=1}^i E_{rj} |E^*_{tj}|}{|E^*_{tr}|}$$

where

$$r=1, 2, \dots, i$$

From the theory of determinants, the expression

$$\sum_{j=1}^i E_{rj} |E^*_{tj}| = 0$$

when

$$r \neq t$$

$$\sum_{j=1}^i E_{rj} |E^*_{tj}| = |E^*|$$

when

$$r = t$$

Therefore,

$$\sum_{j=1}^i E_{rj} S_{jj} C_{jt} = 0 \tag{A11}$$

when

$$r \neq t$$

If equation (A7) also holds, then equation (A11) becomes

$$\sum_{j=1}^n E_{rj} S_{jj} C_{jt} = 0 \tag{A12}$$

where

$$r \neq t$$

$$r=1, 2, \dots, i$$

If equations (A12) and (A7) hold, then any setting Y_t will affect only y_t (fig. 5). Equations (A7) and (A10) thus are sufficient for the noninteraction condition specified.

CONDITIONS ON C' MATRIX

Necessary condition.—When the general equations for the response of the controlled variables (equations (A1) and (A2)) are written separating the effects of any setting X_r , these equations become

$$y_j = \sum_{k=1}^n \sum_{v=1}^i E_{jk} S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{k=1}^n \sum_{\substack{\mu=1 \\ \mu \neq r}}^n E_{jk} S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} x_\mu) + \sum_{k=1}^n E_{jk} V_k + \sum_{k=1}^n E_{jk} S_{kk} C'_{kr} (X_r - L_{rr} x_r) \tag{A13}$$

$$x_k = \sum_{v=1}^i S_{kk} C_{kv} (Y_v - L_{vv} y_v) + \sum_{\substack{\mu=1 \\ \mu \neq r}}^n S_{kk} C'_{k\mu} (X_\mu - L_{\mu\mu} x_\mu) + V_k + S_{kk} C'_{kr} (X_r - L_{rr} x_r) \tag{A14}$$

If any setting X_r is to affect only x_r , then any y_j is not affected by X_r and any controlled x_k ($k = i+1$ to n) is not affected by X_r for $k \neq r$. In equation (A13) the only term that can be a function of X_r is the last one. Because of the arbitrary nature of the instrument function, $X_r \neq L_{rr} x_r$ and

$$\sum_{k=1}^n E_{jk} S_{kk} C'_{kr} = 0 \tag{A15}$$

where

$$j=1, 2, \dots, i$$

If equation (A14) is considered for any k ($i+1$ to n) where $k \neq r$, the only term that can be a function of X_r is the last one and therefore

$$S_{kk} C'_{kr} = 0 \tag{A16}$$

and

$$C'_{kr} = 0 \tag{A17}$$

where

$$k \neq r$$

If equation (A17) is used, equation (A15) becomes

$$\sum_{k=1}^i E_{jk} S_{kk} C'_{kr} = -E_{jr} S_{rr} C'_{rr}$$

where

$$j=1, 2, \dots, i$$

The solution of this set of i equations in the i unknown $S_{kk} C'_{kr}$ gives

$$S_{jj} C'_{jr} = \frac{-\sum_{p=1}^i |E^*_{pj}| E_{pr} S_{rr} C'_{rr}}{|E^*|}$$

or

$$\frac{S_{jj} C'_{jr}}{S_{rr} C'_{rr}} = \frac{-\sum_{p=1}^i |E^*_{pj}| E_{pr}}{|E^*|} \tag{A18}$$

where

$$j=1, 2, \dots, i$$

Equations (A17) and (A18) thus follow necessarily from the noninteraction condition specified.

Sufficient condition.—If equation (A18) holds,

$$\sum_{j=1}^i \frac{E_{vj} S_{jj} C'_{jr}}{S_{rr} C'_{rr}} + \frac{\sum_{p=1}^i \sum_{j=1}^i E_{vj} |E^*_{pj}| E_{pr}}{|E^*|} = 0$$

From the theory of determinants applied to the second term

$$\sum_{j=1}^i \frac{E_{vj} S_{jj} C'_{jr}}{S_{rr} C'_{rr}} + E_{vr} = 0$$

and

$$\sum_{j=1}^n E_{vj} S_{jj} C'_{jr} + E_{vr} S_{rr} C'_{rr} = 0$$

If equation (A17) also holds, the previous equation becomes

$$\sum_{j=1}^n E_{vj} S_{jj} C'_{jr} = 0 \tag{A19}$$

If equations (A17) and (A19) hold, then any setting X_r will affect only x_r (fig. 5). Equations (A17) and (A18) thus are sufficient for the specified noninteraction conditions to hold.

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