REPORT 1068

AUTOMATIC CONTROL SYSTEMS SATISFYING CERTAIN GENERAL CRITERIONS ON TRANSIENT BEHAVIOR [1]

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SUMMARY

An analytic method for the design of automatic controls is developed that starts from certain arbitrary criterions on the behavior of the controlled system and gives those physically realizable equations that the control system can follow in order to realize this behavior. The criterions used are developed in the form of certain time integrals.

General results are shown for systems of second order and of any number of degrees of freedom. Detailed examples for several cases in the control of a turbojet engine are presented.

INTRODUCTION

In the past several years, there has been increasing development and interest in automatic control; in the fields of gun direction, guided-missile control, and control of gas-turbine engines, for instance, where very refined and accurate controlled behavior is required, need still exists for further development of the methods of controls analysis and design.

Recent developments in this field have been concerned chiefly with the problem of control analysis both in the realm of linear systems (reference 1) and in the realm of nonlinear systems (reference 2). These analytical works answer questions concerning the behavior of a given system or the effect on its behavior of changing certain constants in the system.

Another problem of equal and, in some cases, greater importance is that of control synthesis. Work on this problem seeks to find the method of design when certain criterions concerning the behavior of a controlled system are given. The method decided upon should give all aspects of the system; for instance, where the system should be linear, what the general configuration should be, and what the precise values of all the constants should be.

This synthesis problem has hardly been broached in literature. The use of analysis as a design procedure offers a partial solution to this problem in that the analysis of a large number of cases may reveal, coincidently, one that satisfies the desired criterions of controlled behavior. Such a method is, at best, long and tedious and almost always would result in compromises because the systems chosen to be analyzed would probably be such that they could never satisfy all the desired criterions.

A method for designing a linear system to satisfy certain special criterions when operating on a random input is developed in reference 3. This method is applicable as an addition to a control system whenever random external disturbances are involved. A partial solution to the synthesis problem is developed in reference 4 in satisfying the criterions of noninteraction for systems with many degrees of freedom.

An analysis made at the NACA Lewis laboratory during the latter part of 1950 and presented herein develops a rational method of control synthesis that starts from any arbitrary but physically realizable criterions and results in the equations for the best system that satisfies these criterions. As is shown, the nature of the criterions, in general, requires minimizing certain time integrals by using the calculus of variations and the methods developed are an application of the calculus of variations to the problem of control synthesis.

A careful scrutiny is first made of the whole problem, followed by a development of general results. These general results are then applied in examples to the design of turbojet-engine control systems. In general, the methods used vary according to the order of the differential equation describing the plant, the process, or the system being controlled and according to the number of degrees of freedom or independent variables. Detailed analyses are presented for application to a system of first order and of one degree of freedom. General results for systems of second order and any number of degrees of freedom are developed in the appendix.

SYMBOLS

The following symbols are used in this report:

a, b, c	gas-turbine-engine characteristic constants
C	constant
E	function of λ
F, H	functions of y and $\dot{y}$
$F(t_1)$	indicates $F[y(t_1), \dot{y}(t_1)]$
$F_y(y, \dot{y})$	indicates partial differentiation with respect to y
f, f_o	arbitrary function
$f_1, f_2, \ldots$	weighting functions used for gas-turbine control criterions
G, G_1, G_2	functions of y
g	function used for gas-turbine-engine surge criterion

[1] Supersedes NACA TN 2378, "Automatic Control Systems Satisfying Certain General Criterions on Transient Behavior" by Aaron S. Boksenbom and Richard Hood, 1951.

h	function used for gas-turbine-engine blow-out criterion
K	controller gain
L, L_1, L_2	temperature limits for gas-turbine engine
N, P, T, w_f	deviations of gas-turbine-engine speed, compressor-discharge pressure, characteristic temperature, and fuel flow, respectively, from values at some common equilibrium condition
N_e	actual engine acceleration
n	exponent
t	time
t_1	time at end of transient
δt_1	variation in time at end of transient
$w, y, z,$ $\delta w, \delta y, \delta z$	independent variables, functions of time
ϵ	small number
$\lambda, \lambda_1, \lambda_2, \lambda_3$	arbitrary constants
σ	gas-turbine-engine time constants for response to temperature
τ	transient time constant of controlled system
Subscripts:	
0	initial condition of variable
f	final value of variable
s	setting or desired value of variable
Superscripts:	
*	indicates case different from optimum

The dot indicates differentiation with respect to time.

The prime indicates differentiation with respect to the argument shown.

ANALYSIS

SURVEY OF PROBLEM

Control problem.—An important aspect of the control-synthesis problem is a clear definition of the criterions of desired controlled behavior. If a variable y is to be controlled, a reasonable criterion is that the time integral of some function of y is to be a minimum or a constant; that is,

$$\int_0^{t_1} f(y)\,dt = \text{constant or minimum} \tag{1}$$

or

$$\int_0^{t_1} (y-y_s)^2 dt = \text{constant or minimum} \tag{2}$$

Equation (2), for instance, weights the error in y as the square and according to the time duration of that error. Another type of criterion may be that which requires a certain time duration to be a minimum or a constant; that is,

$$\int_0^{t_1} dt = \text{constant or minimum} \tag{3}$$

The use of a single criterion, such as equation (1), will usually yield $f(y)$=constant. This result is reasonable because $f(y)$ can usually be made identically a constant if no additional criterions are imposed on other variables in the system. Certain limiting conditions usually exist on other variables in the system and these conditions must be included in the original criterions.

Thus, a possible criterion could be written

$$\left.\begin{aligned} &\int_0^{t_1} (y-y_s)^2\,dt = \text{minimum} \\ \text{for} \quad &\int_0^{t_1} f(z)\,dt = \text{constant} \end{aligned}\right\} \tag{4}$$

If, for instance, y=engine speed and z=characteristic temperature of a gas-turbine engine, the criterion of equation (4) states that it is desired to design a control system such that, for a particular value of a temperature integral, the integral of the speed-error squared is a minimum. This criterion may be used if, for instance, it is known that an over-temperature condition can be tolerated for a certain period of time and it is desired to keep the average speed error at a minimum during a transient.

The general theory will show that as many criterions as desired of the type shown in equations (1) to (4) can be included together and a control system can be derived that automatically satisfies all these criterions simultaneously.

Another aspect of the control criterions is the end conditions of the integrals of equations (1) to (4). The time interval for which these integrals are to be a minimum or a constant must be chosen. A reasonable time interval is any duration during which essential external disturbances are constant and during which the system to be controlled moves from one essential level of operation to another. The essential external disturbances are those that cannot be immediately corrected by the control system. If an essential external disturbance were allowed within the time interval of the criterions, no physically realizable system could be expected to anticipate this disturbance so as to behave properly before this disturbance occurs. An essential level of operation is any specific condition of only those variables that must be continuous. It will be shown that the essential level of operation appears as a natural boundary condition for the type of criterion used. In the case of a turbojet engine the transient behavior of which can be described by a first-order differential equation, the engine speed determines the level of operation. If a lag exists in the fuel system or between temperature and engine speed, then both engine speed and acceleration are required to describe the essential operating level of the engine.

Analytic problem.—The control system resulting from any design method must be physically realizable. There are two aspects to this problem. First, it is possible to set down criterions that are not realizable with any system or are incompatible with each other. If such criterions are used, the unrealizability will appear either as a requirement on the control to look ahead into the future or as an inability to satisfy the boundary conditions of some differential equation. In most cases, a clear understanding of the criterions used and of the system to be controlled will indicate incompatibilities of this sort.

The second aspect of physical realizability is purely mathematical. It is desired to derive a description (a differential equation) of the control or the controlled system that satisfies the criterions of control and all the necessary boundary conditions that arise in the derivation of this equation. Although the mathematical solution of the problem may be any derivative or integral of this differential equation, the physical solution of the problem requires the differential equation that itself satisfies the boundary conditions and for which no undetermined constants of integration exist. Thus, such forms as

$$\left.\begin{aligned} \dot{y}&=C\dot{x} \\ \text{and} \qquad y&=Cx \end{aligned}\right\} \tag{5}$$

are not necessarily interchangeable as descriptions of some part of a controlled system because the forms differ by an undetermined constant of integration. For stable linear systems, the effect of this constant becomes vanishingly small; for the general nonlinear systems presented herein, however, this constant must be considered.

Stability problem.—The requirement of stability is a special criterion that does not enter into the main body of the methods of this report. It may enter in the final steps of the method where the final differential equation describing the controlled system may be the integral of a higher-order differential equation that satisfies the necessary boundary condition for stability. In addition, it is always necessary to add to the controlled system a stability device that does not affect the behavior of the system as far as satisfying the other criterions is concerned. This device can be described as follows:

$$\dot{y}=0 \text{ when } y=y_s \tag{6}$$

or, for a second-order system,

$$\ddot{y}=0 \text{ when } y=y_s \text{ and } \dot{y}=0 \tag{7}$$

GENERAL THEORY AND RESULTS

It has been shown that the criterions for control can be developed in the following forms:

$$\left.\begin{aligned} &\int_0^{t_1} f(y)\,dt \\ &\int_0^{t_1} (y-y_s)^2\,dt \\ &\int_0^{t_1} f_o(z)\,dt \\ &\int_0^{t_1} dt \end{aligned}\right\} \tag{8}$$

and so forth. If, for such a list of criterions, one of the integrals is to be a minimum under the condition that the other integrals are to be constant, it is sufficient (reference 5) to make

$$\int_0^{t_1} f(y)\,dt+\lambda_1\int_0^{t_1}(y-y_s)^2\,dt+ \lambda_2\int_0^{t_1} f_o(z)\,dt+\lambda_3\int_0^{t_1} dt=\text{minimum} \tag{9}$$

or

$$\int_0^{t_1}[f(y)+\lambda_1(y-y_s)^2+\lambda_2 f_o(z)+\lambda_3]\,dt=\text{minimum} \tag{10}$$

The λ's are arbitrary constants that enter into the control system as the adjustable parameters and are precisely determined by the choice of values that the constant integrals are to have.

The technique of the λ multipliers is widely used for problems of this type where one condition is to be a minimum under other restrictive conditions. Indeed, the conditions need not be in integral form and any functional or differential relation among variables can be handled in a similar manner (reference 6).

Equation (10) can be made very general when all possible restrictive conditions are included. In the final equations, which are derived later, if any one criterion is not to be used, then the corresponding $\lambda\rightarrow 0$. If any of the criterions is to be zero (as for the case of a variable having an absolute limit), then the corresponding $\lambda\rightarrow\infty$.

If, for this development, the system to be controlled is of first order and of one degree of freedom (has one independent variable), then the variables y and z are such that $z=z(y,\dot{y})$. Equation (10) can then be written, in general, as

$$\int_0^{t_1} F(y,\dot{y})\,dt=\text{minimum} \tag{11}$$

In equation (11), F is a continuous function of y and $\dot{y}$, and y is a continuous function of time.

The calculus of variations (reference 5) is used to determine y as a function of time such that the integral of equation (11) is a minimum; that is, if the solid curve of figure 1 makes the integral a minimum, any other curve (such as the dashed one) will make the integral equal to or greater than the first

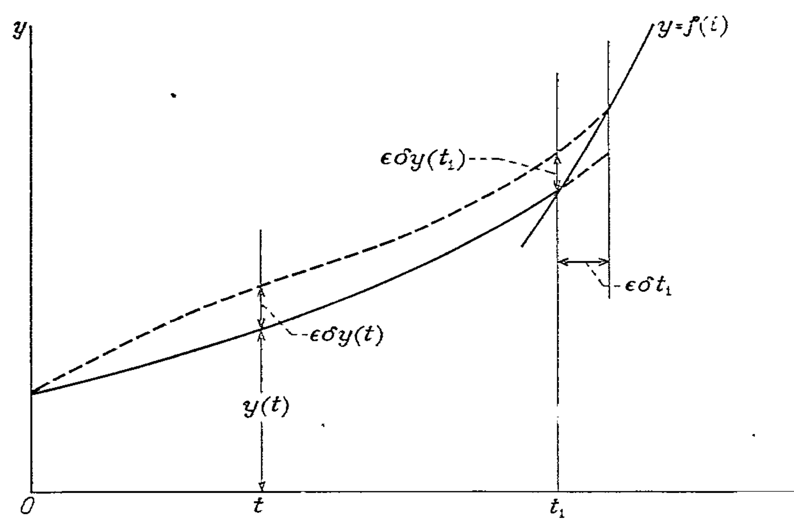


FIGURE 1.—Illustration of curves for minimization of $\int_0^{t_1} F(y,\dot{y})\,dt$.

integral. If the two curves are very close, the condition that the two integrals are equal will make the integral of equation (11) stationary (maximum, minimum, or inflection point).

The problem is similar to that of finding the maximum or minimum point on a curve by setting the first derivative equal to zero. Whether a maximum or minimum point exists is decided by the second derivative at that point. In the variational problem, proving a true minimum involves taking the "second variation." As in the problem of finding a minimum on a curve, the second variation proves, at most, a local minimum and not an absolute minimum. For the specific examples discussed herein, an absolute minimum is proved by another method. In many cases, the physical meaning of the equations will indicate the existence of a unique minimum that obviates going further than the "first variation."

On figure 1, the curve for minimizing the integral of equation (11) is the solid curve having the value $y(t)$ at any time. Any other curve differing from it by a small amount is shown as the dashed curve having the value $y(t)+\epsilon\ \delta y(t)$ at any time, where ϵ is a small number and $\delta y(t)$ is an arbitrary continuous function of time. The condition for the integral of equation (11) to be stationary is

$$\frac{d}{d\epsilon}\int_0^{t_1+\epsilon\delta t_1} F(y+\epsilon\delta y,\ \dot{y}+\epsilon\delta\dot{y})\,dt=0 \tag{12}$$

at

$$\epsilon=0$$

where $\frac{d}{dt}[\delta y(t)]=\delta\dot{y}(t)$ (reference 5). The time duration for the integral of equation (12) is such as to start at some definite time ($t=0$) but not to end at a definite time but rather along some curve $y=f(t)$ (fig. 1), in order to allow the proper boundary conditions of moving from one essential level of operation to another, as previously discussed.

Performing the operation indicated by equation (12) leads to

$$\int_0^{t_1} F_y\,\delta y\,dt+\int_0^{t_1}F_{\dot{y}}\,\delta\dot{y}\,dt+F(t_1)\,\delta t_1=0 \tag{13}$$

Integrating the second term of equation (13) by parts gives

$$\int_0^{t_1}\left[F_y-\frac{d}{dt}(F_{\dot{y}})\right]\delta y\,dt+F_{\dot{y}}\,\delta y\Big]_0^{t_1}+F(t_1)\,\delta t_1=0 \tag{14}$$

Because δy is an arbitrary function, the integral and the boundary-condition terms must vanish separately. The geometry at the end condition ($t=t_1$, fig. 1) gives

$$\delta y(t_1)=[f'(t_1)-\dot{y}(t_1)]\ \delta t_1 \tag{15}$$

as $\epsilon\rightarrow 0$. The two conditions that follow from equation (14) then become

$$\int_0^{t_1}\left[F_y-\frac{d}{dt}(F_{\dot{y}})\right]\delta y\,dt=0 \tag{16}$$

and

$$\delta t_1\,\{F(t_1)+F_{\dot{y}}(t_1)\,[f'(t_1)-\dot{y}(t_1)]\}-F_{\dot{y}}(0)\,\delta y(0)=0 \tag{17}$$

The time interval during which the criterion of equation (11) is to hold is considered as that during which the system moves from one essential operating level to another; in this case, from one definite value of y to another definite value of y. Thus,

$$\left.\begin{aligned}\delta y(0)&=0\\ \text{and}\qquad f'(t_1)&=0\end{aligned}\right\} \tag{18}$$

Equation (16) is satisfied only if the integrand is zero, and, because $\delta t_1\neq 0$, the two conditions of equations (16) and (17) become

$$F_y=\frac{d}{dt}(F_{\dot{y}}) \tag{19a}$$

and

$$F(t_1)=F_{\dot{y}}(t_1)\,\dot{y}(t_1) \tag{19b}$$

Equation (19a) need not hold at $t=0$ because $\delta y(0)=0$. The only condition that need hold at $t=0$ is that $F_{\dot{y}}(0)$ is finite, which will be true if y is continuous at $t=0$. At the start of a new transient, $\dot{y}$, F, F_y, and $F_{\dot{y}}$ may be discontinuous, whereas at other points ($0<t\leq t_1$) $F_{\dot{y}}$ will be continuous because of equation (19a).

Equation (19a) is the differential equation for the $y(t)$ that satisfies the original criterion of equation (11). The physical answer to the problem is the first integral of equation (19a), which satisfies the boundary-condition equation (19b). This solution is

$$F(y,\dot{y})=\dot{y}F_{\dot{y}}(y,\dot{y}) \tag{20}$$

and whenever y, $F_{\dot{y}}$, and so forth are continuous,

$$\dot{y}\left[F_y-\frac{d}{dt}(F_{\dot{y}})\right]=0 \tag{21}$$

by differentiating equation (20) with respect to time. Thus, either $\dot{y}=0$ (which is true only during stability) or equation (19a) is satisfied.

Thus, equation (20) is the description of that physically realizable system the behavior of which will automatically and simultaneously satisfy those criterions included in the function F during that time interval for which the external disturbances are constant and during which the system goes from one operating level to any other operating level. A stability device must be added to the system; the description of such an ideal device is

$$\dot{y}=0 \text{ when } y=y_s \tag{22}$$

An additional condition must be met if $\dot{y}$ is discontinuous in the interval $0<t<t_1$. In this case, $F_{\dot{y}}$ must be continuous during the discontinuity in $\dot{y}$; F_y, however, will usually be discontinuous when $\dot{y}$ is discontinuous. This discontinuity in F_y usually means that some essential external disturbance has entered the system and the point of discontinuity must be the start of a new time interval.

APPLICATION TO TURBOJET-ENGINE CONTROLS

In the usual case of designing turbojet-engine controls, the engine speed that sets the essential operating level of the engine and, in the main, other pertinent characteristics such as thrust, is to be set or controlled. Limiting conditions of the engine are those of overspeed, overtemperature, compressor surge, and rich burner blow-out. The following criterions on the behavior of this engine are typical:

$$\left.\begin{array}{ll}
\int_0^{t_1} f_1(N-N_s)\,dt & \text{for speed control} \\
\int_0^{t_1} f_2(N)\,dt & \text{for speed overshoot} \\
\int_0^{t_1} f_3(T)\,dt & \text{for temperature overshoot and undershoot} \\
\int_0^{t_1} f_4[P-g(N)]\,dt & \text{for compressor surge} \\
\int_0^{t_1} f_5[P-h(N)]\,dt & \text{for blow-out} \\
\int_0^{t_1} dt & \text{for rise time}
\end{array}\right\} \tag{23}$$

Figures 2(a) to 2(e) show the nature of the functions appearing in expressions (23). The variable involved is essentially weighted by the function shown and integrated with respect to time. The quantity $P-g(N)$ is the amount the compressor-discharge pressure exceeds the safe pressure for surge and $g(N)$ (shown in fig. 2(f)) is the compressor-discharge pressure for each engine speed at a safe value below surge. Rich burner blow-out can be handled in a similar manner ($h(N)$ is shown in fig. 2(g)). The rise time is the total time for the system to move from one essential operating level to the other.

The linearized engine characteristics can be expressed, by assuming first-order behavior, as follows:

$$T=aN+a\sigma\dot{N} \tag{24a}$$

$$P=bN+cT \tag{24b}$$

Thus, the integrals of expressions (23) are of the form

$$\int_0^{t_1} f(N,\dot{N})\,dt \tag{25}$$

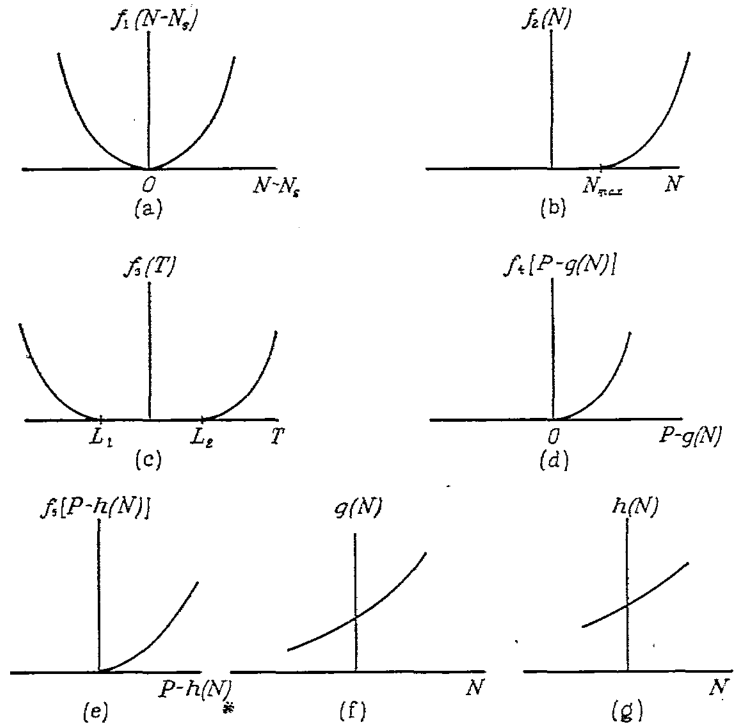
where f is a continuous function of N and $\dot{N}$, and N (barring impulsive temperatures and the like) is a continuous function of time.

Speed control; case A.—If only the error in speed control is considered important, the criterion becomes

$$\int_0^{t_1} f_1(N-N_s)\,dt=\text{minimum} \tag{26}$$

where $F=f_1(N-N_s)$. Equation (20) becomes

$$\left.\begin{array}{r}
f_1(N-N_s)=0 \\
\text{and from the nature of } f_1 \text{ (fig. 2(a))}, \\
N=N_s
\end{array}\right\} \tag{27}$$



(a) Speed control.
(b) Speed overshoot.
(c) Temperature overshoot and undershoot.
(d) Compressor surge.
(e) Blow-out.
(f) Compressor line below surge.
(g) Compressor line below blow-out.

FIGURE 2.—Arbitrary weighting functions for various control criterions and pertinent engine characteristics involved.

This result means that, in the absence of other criterions on the engine behavior, this speed control should keep speed error identically zero, which is physically realizable only in the sense of allowing infinite temperatures and the like to keep the speed error identically zero. This result, however, is inconsistent with the previous development of equation (20) in that N is now a discontinuous function of time and the time interval of the integral of equation (26) is zero. This instance is actually a trivial case of the general problem. The result (equations (27)) does indicate that a criterion like that of equation (26) must be accompanied by an additional criterion (temperature overshoot, for instance) to give a physically realizable system.

Speed control with temperature-limiting criterions; case B.—If the error in speed control and the overshoots and undershoots in temperature are to be considered the primary criterions of control, then from equations (10) and (11),

$$\int_0^{t_1} [f_1(N-N_s)+\lambda f_3(T)]\,dt=\text{minimum} \tag{28}$$

where $F=f_1(N-N_s)+\lambda f_3(T)$. Equation (20) then becomes

$$f_1(N-N_s)+\lambda f_3(T)=\lambda a\sigma\dot{N}f_3'(T) \tag{29}$$

and the ideal stability device is such that

$$\dot{N}=0 \text{ when } N=N_s \tag{30}$$

Equations (29) and (30) describe the complete control system.

In figure 2(c), it is convenient to let $f_3(T)=(T-L_2)^n$ for $T>L_2$ and $f_3(T)=(T-L_1)^n$ for $T<L_1$. In general, the power n should be >1, because, when $n\leq 1$, T may be infinite and of such nature as to make $\dot{N}$ discontinuous and physically unreal even though $\int_0^{t_1} f_3(T)dt$ is finite. In the examples of this report, $n=2$ and $f_1(N-N_s)=(N-N_s)^2$.

A variety of methods of setting up the control system to realize equation (29) exists. When the preceding expressions for f_1 and f_3 are used, equation (29) can be put in the convenient form

$$\left.\begin{array}{l}\dfrac{(N-N_s)^2}{\lambda}+(L-aN)^2=a^2\sigma^2\dot{N}^2 \\ \text{where, for acceleration, when } N<N_s, \\ \qquad \dot{N}>0 \text{ and } L=L_2 \\ \text{and, for deceleration, when } N>N_s, \\ \qquad \dot{N}<0 \text{ and } L=L_1\end{array}\right\} \tag{31}$$

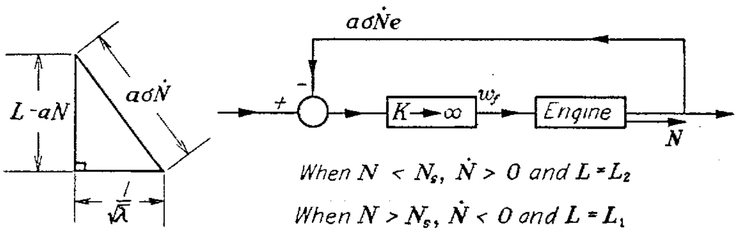


FIGURE 3.—Schematic diagram of control for turbojet engine for case of speed control with temperature-limiting criterions (case B). Stability device must be added to this system.

A schematic diagram of the system is shown in figure 3, where equation (31) is considered to give a desired $\dot{N}$ for any value of N, N_s, and L. Consistent with equation (31), a right-triangle construction is used to give a desired $\dot{N}$. The actual $\dot{N}_e$ can be made very close to $\dot{N}$ by using a high-gain proportional controller, as shown. Provision must be made to change the sign of $\dot{N}$ and the value of L when $N-N_s$ changes sign. In addition, the stability condition requires a provision for making $\dot{N}=0$ whenever speed error is very small or zero.

The use of a high-gain proportional controller, which follows from the requirement that $\dot{N}$ may be discontinuous, means that the fuel-flow rate required may be infinite if lags exist in the fuel system or in the feedbacks. But, as no criterion has been set on fuel-flow rate, this requirement does not violate the original criterions. If necessary, however, a criterion on fuel-flow rate may then be added to equation (28). Even though a criterion on temperature is being satisfied, no direct measurement of temperature is used in figure 3. Actually, the equation for temperature (equation (24a)) is used as an indirect indication of temperature.

The control system of figure 3 has one adjustable parameter λ. For any value of λ, this system will, for the value of integral temperature-overshoot squared obtained, give the minimum value of integral speed-error squared. The value of λ determines the actual value of the integral temperature-overshoot squared.

The integral temperature-overshoot squared as a function of λ is shown in figure 4 for the special case where $aN_s=L$; that is, acceleration or deceleration to the speed that corresponds to limiting temperature. In this case, the system of figure 3 becomes a simple first-order lag system and equation (31) becomes

$$E(L-aN)=a\sigma\dot{N} \tag{32}$$

where

$$E=\left(1+\frac{1}{a^2\lambda}\right)^{1/2}$$

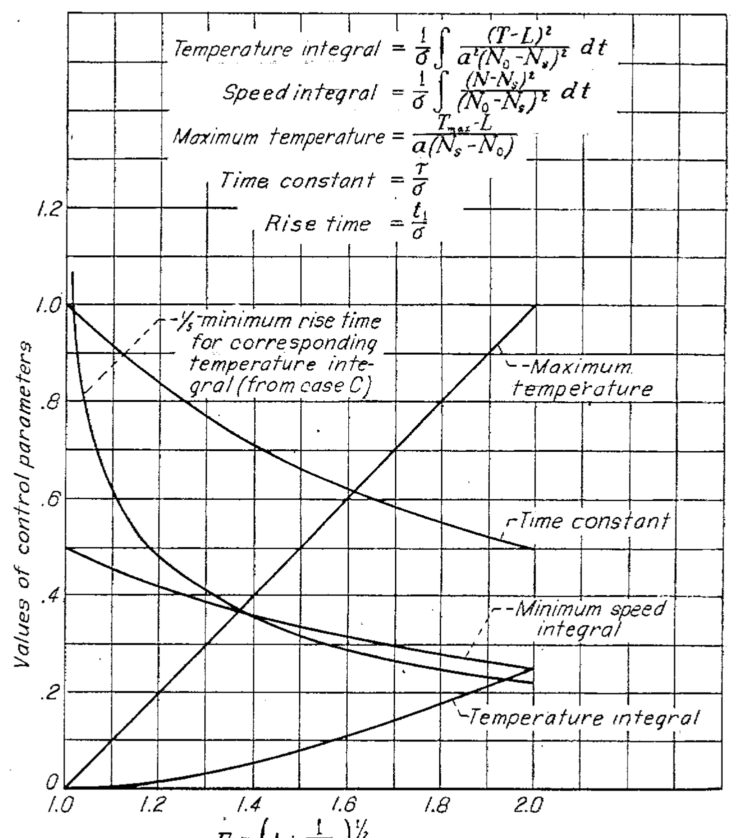


FIGURE 4.—Various control parameters as functions of E for speed control with temperature-limiting criterions (case B) when accelerating to limiting temperature.

In figure 4, the integral speed-error squared, the maximum temperature, and the time constant for this transient are also shown as functions of λ. The curve showing the minimum rise time for the corresponding temperature integral is included for a comparison that will be discussed later. The equations for the curves of figure 4 are

$$\left.\begin{aligned}\frac{1}{\sigma}\int\left(\frac{T-L}{L-aN_0}\right)^2 dt&=\frac{(E-1)^2}{2E}\\ \frac{a^2}{\sigma}\int\left(\frac{N-N_s}{L-aN_0}\right)^2 dt&=\frac{1}{2E}\\ \frac{T_{max}-L}{L-aN_0}&=E-1\\ \text{(time constant) }\frac{\tau}{\sigma}&=\frac{1}{E}\end{aligned}\right\}\tag{33}$$

The left sides of these equations have been put in dimensionless form. The maximum temperature T_{max} occurs at the beginning of the transient. The time constant τ is that for the controlled system and is shown compared with the engine-time constant σ.

From figure 4, the value of λ is chosen as a compromise between the various quantities of equations (33). For $E=1$ ($\lambda=\infty$), the temperature does not overshoot, the speed integral is 0.5, and $\tau=\sigma$. As E increases (λ decreasing), the temperature integral and the maximum temperature increase, whereas the speed integral and the time constant decrease. A value of $E=\sqrt{2}$ (or $a^2\lambda=1$) appears to be a good compromise and is used for the subsequent discussion.

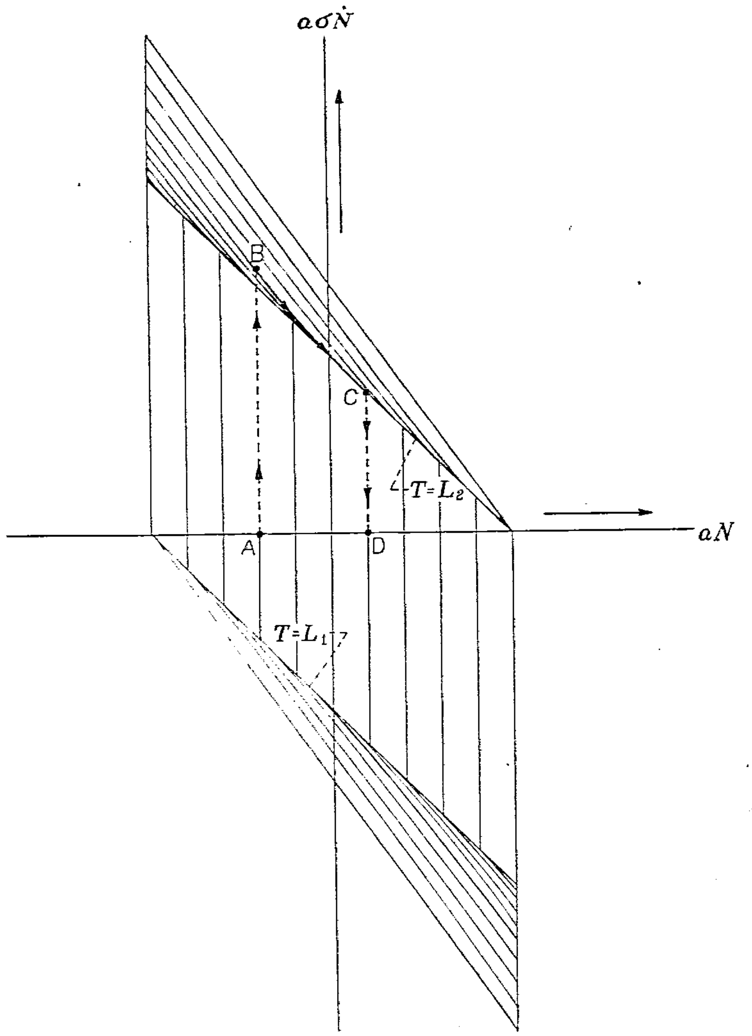


FIGURE 5.—Phase plane showing dynamics of controlled system for speed control with temperature-limiting criterions (case B) where $a^2\lambda=1$.

The behavior of the system of figure 3 can be best seen by drawing the trajectories in the phase plane shown in figure 5 where $a\sigma\dot{N}$ is plotted against aN for lines of constant aN_s according to equation (31). Lines of constant temperature are 45° parallel lines in this plot and the lines of $T=L$ are shown. Each trajectory intersects, and is tangent to, the line $T=L$ at $N=N_s$. Figure 5 completely describes the transient behavior of the system. For any starting point anywhere on the plane (for instance, point A), the system will automatically move the operating point to that trajectory corresponding to the N_s that exists (point B) and then along this trajectory to the point $T=L$ ($N=N_s$, point C), and finally the stability condition will enter to move the operating point along the solid vertical line to $\dot{N}=0$ (point D).

The time sense for these transients is obtained by solving the differential equation (equation (31)) for the speed and the temperature time responses. The equations for these solutions are as follows:

$$\left.\begin{aligned}E^2\left(\frac{aN-aN_s}{L-aN_s}\right)&=1+\sqrt{E^2-1}\,\sinh E\left(\frac{t}{\sigma}+C\right)\\ E^2\left(\frac{T-L}{L-aN_s}\right)&=\sqrt{E^2-1}\,\sinh E\left(\frac{t}{\sigma}+C\right)+\\ &\qquad E\sqrt{E^2-1}\,\cosh E\left(\frac{t}{\sigma}+C\right)+(1-E^2)\end{aligned}\right\}\tag{34}$$

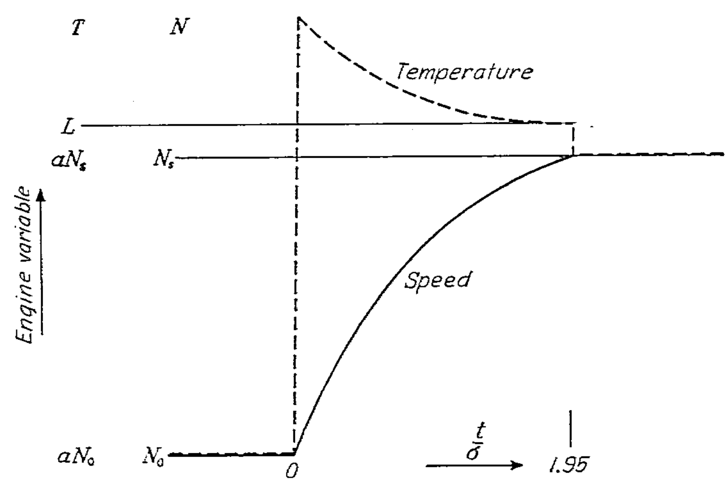


FIGURE 6.—Typical transient of controlled system for speed control with temperature-limiting criterions (case B) where $E=\sqrt{2}$.

These transients are shown in the general case in figure 6 for a step increase in N_s. Maximum temperature occurs at $t=0$ and the temperature overshoot decreases as N increases, and when $N=N_s$, $T=L$ and $\dot{T}=0$. The stability condition then causes T to drop to its equilibrium value at this point. The time scale shown in figure 6 corresponds to the specific relative values of the ordinates shown.

Minimum rise time and temperature-limiting criterions; case C.—In order to obtain a minimum rise time for a constant temperature integral, the requirement is that

$$\int_0^{t_1} [1+\lambda f_3(T)]\, dt = \text{minimum} \tag{35}$$

where $F=1+\lambda f_3(T)$. Equation (20) becomes

$$1+\lambda f_3(T) = \lambda a\sigma \dot{N} f_3'(T) \tag{36}$$

and the ideal stability device is such that

$$\dot{N}=0 \text{ when } N=N_s \tag{37}$$

where N_s is the desired stable N. Equations (36) and (37) describe the complete control system. Equation (36), and therefore the control system proper, does not include N_s and, in this case, the stability device is independent of the control system proper.

When the same $f_3(T)$ is used as was used in the previous case, equation (36) can be put in the convenient form

$$\left.\begin{array}{l} \dfrac{1}{\lambda}+(L-aN)^2=a^2\sigma^2\dot{N}^2 \\ \text{where, for acceleration, when } N<N_s, \\ \qquad \dot{N}>0 \text{ and } L=L_2 \\ \text{and, for deceleration, when } N>N_s, \\ \qquad \dot{N}<0 \text{ and } L=L_1 \end{array}\right\} \tag{38}$$

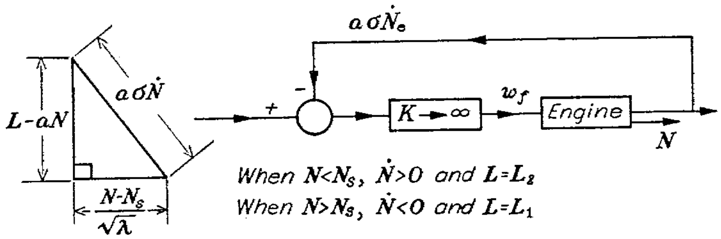


FIGURE 7.—Schematic diagram of control for turbojet engine for minimum rise time with temperature-limiting criterions (case C). Stability device must be added to this sytem.

A schematic diagram of this system satisfying equation (38) is shown in figure 7. This control system is the same as the previous system (fig. 3) except that in figure 7, $\frac{1}{\sqrt{\lambda}}$ replaces $\frac{N-N_s}{\sqrt{\lambda}}$ on one leg of the right-triangle construction and the stability condition must be imposed outside the control system, as N_s does not enter into the criterion of control (equation (35)).

The system of figure 7 has one adjustable parameter λ, which, as before, sets the precise values of the integrals entering into the criterion (equation (35)), as well as all other behavior characteristics.

The temperature integral is shown in figure 8 as a function of λ for the special case where $aN_s=L$; that is, for acceleration or deceleration to the speed that corresponds to limiting temperature. In addition, the rise time, the maximum temperature, and the integral speed-error squared are shown for this transient. A curve showing the minimum speed integral for the corresponding temperature integral (from case B) is also shown and will be discussed later. The equations for the curves of figure 8 are as follows:

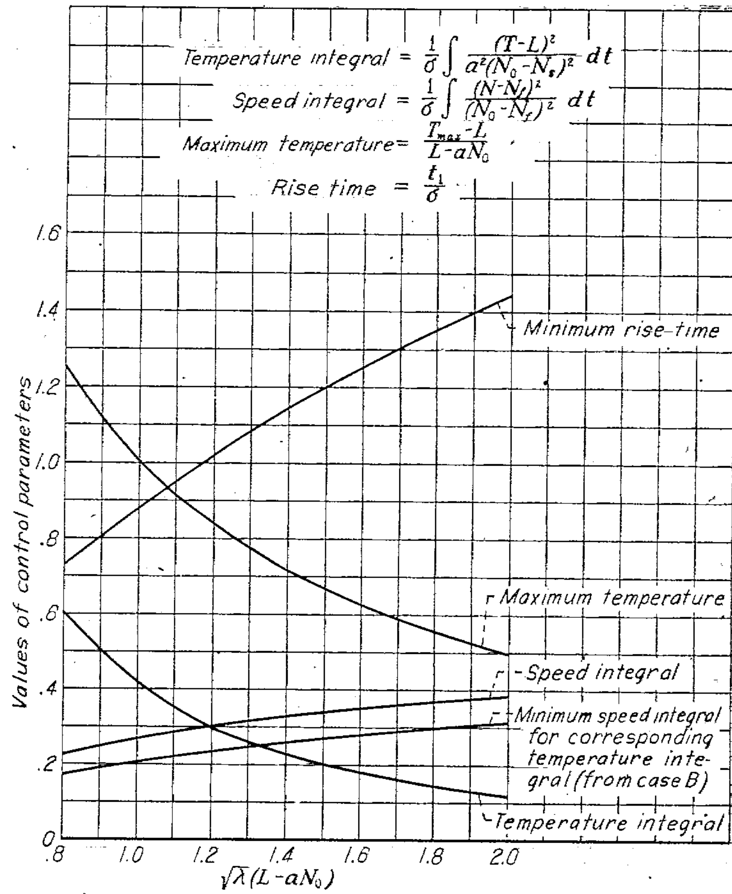


FIGURE 8.—Various control parameters as function of λ for minimum rise time with temperature-limiting criterions (case C) when accelerating to limiting temperature.

$$\left.\begin{array}{l} \dfrac{1}{\sigma}\displaystyle\int_0^{t_1}\left(\frac{T-L}{L-aN_0}\right)^2 dt=\frac{\sqrt{1+\lambda(L-aN_0)^2}}{(L-aN_0)\sqrt{\lambda}}-1 \\ \dfrac{1}{\sigma}\displaystyle\int_0^{t_1} dt=\sinh^{-1}[(L-aN_0)\sqrt{\lambda}] \\ \dfrac{T_{max}-L}{L-aN_0}=\dfrac{1}{\sqrt{\lambda}\,(L-aN_0)} \\ \dfrac{a^2}{\sigma}\displaystyle\int_0^{t_1}\left(\frac{N-N_f}{L-aN_0}\right)^2 dt=\frac{\sqrt{1+\lambda(L-aN_0)^2}}{2(L-aN_0)\sqrt{\lambda}}- \\ \qquad \dfrac{1}{2\lambda(L-aN_0)^2}\sinh^{-1}\sqrt{\lambda}\,(L-aN_0) \end{array}\right\} \tag{39}$$

Again λ is to be chosen as a compromise between these various quantities. In this case, $\frac{1}{\sqrt{\lambda}}$ has the units of temperature and $\sqrt{\lambda}(L-aN_0)$ is the pertinent dimensionless parameter. For a fixed λ, the initial speed N_0 will therefore change the time integrals and an increased $L-aN_0$ will decrease the temperature integral and the maximum temperature and increase the rise time and the speed integral.

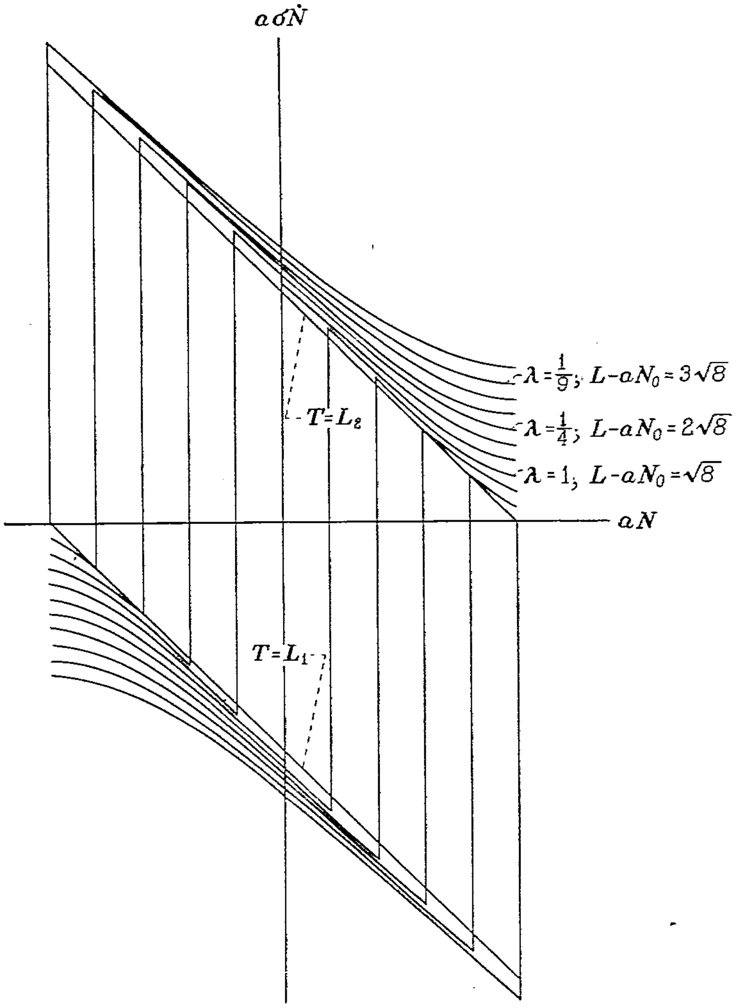


FIGURE 9.—Phase plane showing dynamics of controlled system for minimum rise time with temperature-limiting criterions (case C) where $\sqrt{\lambda}\,(L-aN_0)=\sqrt{8}$.

The dynamic behavior of the system of figure 7 is shown in the phase plane in figure 9. This figure is a plot of $a\sigma\dot{N}$ against aN, according to equation (38), for various values of λ. Lines of constant temperatures are 45° parallel lines and the lines $T=L$ are shown. For any fixed λ, only two trajectories are followed, one for accelerating and one for decelerating. From any starting point on the plane, the system will automatically move to that trajectory corresponding to acceleration or deceleration and will move along this trajectory until the stability condition enters (at $N=N_s$) to make $\dot{N}=0$.

The dependence of the time integrals on N_0 may require λ to vary with N_0. In figure 9, N_0's corresponding to each λ are shown such as to keep $\lambda(L-aN_0)^2=8$. The value of this parameter was so taken as to have the temperature integral in this case equal to that of the pre ious case for purposes of comparison.

The time sense for these transients is obtained by solving the differential equation (38) for the $N-t$ and $T-t$ transients. The equations for these solutions are

$$\left.\begin{aligned}\sqrt{\lambda}\,(aN-L)&=\sinh\left(\frac{t}{\sigma}+C\right)\\ \sqrt{\lambda}\,(T-L)&=e^{\left(\frac{t}{\sigma}+C\right)}\end{aligned}\right\}\tag{40}$$

These transients are shown for the general case in figure 10 for a step increase in N_s. The temperature will jump to some value above L, but in this case, unlike the previous one, the temperature continues to increase as the speed increases. Maximum temperature occurs at the end of the transient. Whenever $N=N_s$ (two such conditions are shown), the stability condition takes over and T is reduced to its equilibrium value. The time scale shown on figure 10 corresponds to the specific relative values of the ordinates as shown.

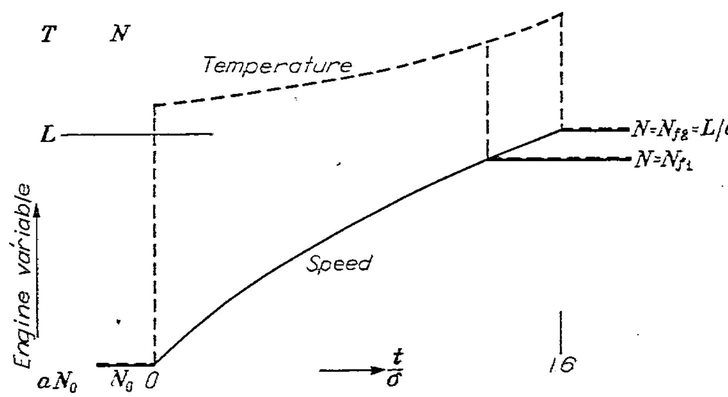


FIGURE 10.—Typical transients for controlled system for minimum rise time with temperature-limiting criterions (case C).

Comparison of cases B and C.—In case B, a system was devised that, for a constant value of the integral temperature-overshoot squared, gives a minimum integral speed-error squared. In case C, a system was devised that, for a constant temperature integral, gives a minimum rise time. For case B, the rise time is not a minimum but can be compared with the minimum rise time of case C. For the special transient of accelerating to $T=L$, the system of case B reduced to a first-order system, the time constant of which is shown in figure 4. Because five time constants are considered as the rise time of an exponential, the time constant of case B should be compared with one-fifth the minimum rise time of case C. In figure 4, for corresponding values of the temperature integral, one-fifth the minimum rise time from case C is plotted. Figure 4 now shows that the rise time for case B is about twice the minimum possible rise time.

At corresponding values of the temperature integral, $\int_0^{t_1}(N-N_f)^2\,dt$ for case C can be compared with the minimum possible value of this quantity from case B. The minimum value of $\int_0^{t_1}(N-N_f)^2\,dt$ is plotted in figure 8. It is seen that, for the same temperature integral, $\int_0^{t_1} (N-N_f)^2\,dt$ when minimum rise time is obtained is about 120 percent of the minimum possible value of $\int_0^{t_1} (N-N_f)^2\,dt$.

Proof of absolute minimum.—As previously noted, the minimization of equation (11) involves not only the condition that the first variation be stationary, which leads to equation (20), but also the condition that the second variation be positive. This second condition, however, would prove at most a local minimum for the condition of equation (11). A special proof of an absolute minimum is shown as follows:

If equation (11) is written in the form

$$\int_{y_0}^{y_f} F(y,\dot{y})\frac{dy}{\dot{y}}=\text{minimum} \tag{41}$$

and $\dot{y}$ is considered as a function of y for the minimum condition and $\dot{y}^*$ is considered as a function of y for any other possible case, then the condition for an absolute minimum is that

$$\int_{y_0}^{y_f} F(y,\dot{y}^*)\frac{dy}{\dot{y}^*}-\int_{y_0}^{y_f} F(y,\dot{y})\frac{dy}{\dot{y}}\geq 0 \tag{42}$$

or

$$\int_{y_0}^{y_f}\left[F(y,\dot{y}^*)-\frac{\dot{y}^*}{\dot{y}}F(y,\dot{y})\right]\frac{dy}{\dot{y}^*}\geq 0 \tag{43}$$

where $dy/\dot{y}^*$ is a positive differential. If the function F is considered in a general form, quadratic in $\dot{y}$, as

$$F(y,\dot{y})\equiv G(y)+\dot{y}G_1(y)+\dot{y}^2G_2(y) \tag{44}$$

then equation (43) becomes

$$\int_{y_0}^{y_f}(\dot{y}^*-\dot{y})\left[\dot{y}^*G_2(y)-\frac{G(y)}{\dot{y}}\right]\frac{dy}{\dot{y}^*}\geq 0 \tag{45}$$

The equation of the control system (equation (20)) becomes, for the form of F assumed in equation (44),

$$\dot{y}^2G_2(y)=G(y) \tag{46}$$

Using this expression for $G(y)$ changes equation (45) to the form

$$\int_{y_0}^{y_f} G_2(y)(\dot{y}^*-\dot{y})^2\frac{dy}{\dot{y}^*}\geq 0 \tag{47}$$

For cases B and C, $G_2=\lambda a^2\sigma^2$ where $\lambda\geq 0$. Thus an absolute minimum is proved for these two cases.

Degree of minimum; case B.—The left side of equation (47) can be used as a measure of the deviation from optimum conditions when the methods of this report are not used. If the two cases ($\dot{N}$ for the optimum case and $\dot{N}^*$ for any other case) are compared at the same value of the temperature integral, the left side of equation (47) is the difference between the integral speed-error squared for any case and the minimum possible value of this quantity. The ratio of this deviation to the minimum value becomes

$$\begin{matrix}\text{Fractional increase}\\ \text{in speed integral}\end{matrix}=\frac{2E}{E^2-1}\int_{y_0}^{y_f}\left(\frac{T^*-T}{T_f-T_0}\right)^2\frac{dy}{\sigma\dot{y}^*} \tag{48}$$

for the transient of acceleration or deceleration to limiting temperature. The coefficient of the integral of equation (48) for the value of λ previously chosen ($a^2\lambda=1$) is $2\sqrt{2}$.

Case B can also be considered as giving a minimum value of the temperature integral for any definite value of the speed integral. If two cases ($\dot{N}$ for the optimum and $\dot{N}^*$ for any other case) are compared at the same value of the speed integral, the left side of equation (45) is proportional to the difference between the integral temperature-overshoot squared for any case and the minimum possible value of this quantity. This deviation can be written

$$\begin{matrix}\text{Fractional increase in}\\ \text{temperature integral}\end{matrix}=\frac{2E}{(E-1)^2}\int_{y_0}^{y_f}\left(\frac{T^*-T}{T_f-T_0}\right)^2\frac{dy}{\sigma\dot{y}^*} \tag{49}$$

for the transient of acceleration or deceleration to limiting temperature. The coefficient of the integral of equation (49) for the value of λ previously chosen ($a^2\lambda=1$) is 16.4.

Degree of minimum; case C.—If the two cases $\dot{N}$ and $\dot{N}^*$ are compared at the same value of the temperature integral, the left side of equation (47) is the difference between the rise time for any case and its minimum possible value. This deviation can be written

$$\begin{matrix}\text{Fractional increase}\\ \text{in rise time}\end{matrix}=\frac{\lambda(1-aN_0)^2}{\sinh^{-1}\sqrt{\lambda}\,(L-aN_0)}\int_{y_0}^{y_f}\left(\frac{T^*-T}{T_f-T_0}\right)^2\frac{dy}{\sigma\dot{y}^*} \tag{50}$$

for the transient of acceleration or deceleration to limiting temperature. The coefficient of the integral of equation (50) for the value of $\sqrt{\lambda}(L-aN_0)=2\sqrt{2}$ previously chosen is 4.54.

Case C can also be considered as giving a minimum value of the temperature integral for any definite value of rise time. If the two cases $\dot{N}$ and $\dot{N}^*$ are now compared at the same value of rise time, the left side of equation (47) is proportional to the difference between the integral temperature-overshoot squared for any case and the minimum possible value of this quantity. This deviation can be written

$$\begin{matrix}\text{Fractional increase in}\\ \text{temperature integral}\end{matrix}=\frac{\sqrt{\lambda}(L-aN_0)\int_{y_0}^{y_f}\left(\frac{T^*-T}{T_0-T_f}\right)^2\frac{dy}{\sigma\dot{y}^*}}{\sqrt{\lambda(L-aN_0)^2+1}-\sqrt{\lambda}(L-aN_0)} \tag{51}$$

for the transient of acceleration or deceleration to limiting temperature. The coefficient of the integral of equation (51) is 16.4 for the value of $\sqrt{\lambda}(L-aN_0)$ previously chosen.

GENERAL SUMMARY

When the criterions on the behavior of a controlled system can be expressed in certain general forms, as

$$\left.\begin{array}{l}\int_0^{t_1} f(y)\,dt \\ \int_0^{t_1} (y-y_s)^2 dt \\ \int_0^{t_1} f_o(z)\,dt \\ \int_0^{t_1} dt\end{array}\right\} \tag{8}$$

where the time interval is taken as any duration during which essential external disturbances are constant and during which the system moves from one essential operating point to another, the optimum control can be considered as one that minimizes one of the integrals of equations (8) while maintaining the other integrals at prescribed values. The analytical problem, according to the calculus of variations, reduces to the following equation:

$$\int_0^{t_1} f(y)\,dt+\lambda_1\int_0^{t_1}(y-y_s)^2 dt+ \lambda_2\int_0^{t_1} f_o(z)\,dt+\lambda_3\int_0^{t_1} dt=\text{minimum} \tag{9}$$

For general first-order systems, where $z=z(y,\dot{y})$, equation (9) reduces to

$$\int_0^{t_1} F(y,\dot{y})\,dt=\text{minimum} \tag{11}$$

where $F=f(y)+\lambda_1(y-y_s)^2+\lambda_2 f_o(z)+\lambda_3$. The equation necessary for the control system to satisfy equation (11) and all the boundary conditions becomes

$$F(y,\dot{y})=\dot{y}F_{\dot{y}}(y,\dot{y}) \tag{20}$$

This equation should be followed by the control system proper. In addition, a stability device must be added to the system, the idealized characteristics of which would make $\dot{y}=0$ when $y=y_s$.

The arbitrary multipliers λ are then found by evaluating the integral criterions involved in F. The transient behavior of the system derived is found by solving the differential equation (20). The degree of the minimum or the amount suffered when any other control system is used was evaluated for the special cases considered. A summary of these developments follows for a special form of the F function, quadratic in $\dot{y}$, where

$$F(y,\dot{y})\equiv G(y)+\dot{y}G_1(y)+\dot{y}^2G_2(y) \tag{44}$$

Control-system equation.—For F in the form of equation (44), the control-system equation (20) becomes

$$\dot{y}^2=\frac{G(y)}{G_2(y)} \tag{52}$$

The function $G_1(y)$ does not affect the control system and the control-system equation gives an explicit expression for $\dot{y}$ as a function of y.

Evaluation of integrals.—If an integral of a function $H(y,\dot{y})$ is any one of the criterions to be considered, it can be evaluated as follows:

$$\int_0^{t_1} H(y,\dot{y})dt=\int_{y_0}^{y_f} H\left[y,\sqrt{\frac{G(y)}{G_2(y)}}\right]\sqrt{\frac{G_2(y)}{G(y)}}\,dy \tag{53}$$

Thus the integrals can be evaluated without solving the differential equation (52).

Transient behavior.—The differential equation (52) can be easily solved as follows:

$$\int_{x=y_0}^{x=y}\sqrt{\frac{G_2(x)}{G(x)}}\,dx=t-t_0 \tag{54}$$

Degree of minimum.—Equation (47) was derived to prove the absolute minimum for the special cases considered and can be used to evaluate the degree of the minimum found. Thus,

$$\int_0^{t_1^*} H(y,\dot{y}^*)dt^*-\int_0^{t_1} H(y,\dot{y})dt=\int_{y_0}^{y_f} G_2(y)[\dot{y}^*(y)-\dot{y}(y)]^2\frac{dy}{\dot{y}^*(y)} \tag{55}$$

where $\int_0^{t_1} H(y,\dot{y})dt$ is to be a minimum or maximum.

If $G_2(y)>0$ for $y_0<y<y_f$, then an absolute minimum is obtained; if $G_2(y)<0$ for $y_0<y<y_f$, an absolute maximum is obtained. Equation (55) also indicates that the degree of minimum (or maximum) varies with the magnitude of $G_2(y)$. The $\dot{y}^*$ in the denominator indicates that for any deviation $|\dot{y}^*-\dot{y}|$ in $\dot{y}$, it is better to err on the side of a larger $|\dot{y}^*|$.

SUMMARY OF RESULTS

A rational analytic method for the design of automatic control systems was derived. Criterions on the behavior of the controlled system were developed in the form of certain time integrals. When any of these arbitrary but physically realizable criterions were used as a starting point, those equations that the control system must follow were derived. The criterions developed required the minimization of certain time integrals using the calculus of variations. The method gave not only a description of the behavior of the controlled system but also those physically realizable equations that the control system can follow in order to realize this behavior. General results were shown for systems of second order and of any number of degrees of freedom.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, *October 11, 1950*

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APPENDIX

SECOND-ORDER SYSTEMS OF SEVERAL DEGREES OF FREEDOM

It is beyond the scope of this report to detail the cases of higher-order systems and those having several independent variables. The general equations for these cases will be developed and it is expected that subsequent reports will cover their applications in detail.

For the case of a second-order system with two degrees of freedom, equation (11) becomes

$$\int_0^{t_1} F(y,\dot y,\ddot y,z,\dot z,\ddot z)\,dt=\text{minimum} \tag{A1}$$

where y and z are independent functions of time. The condition to satisfy equation (A1) is

$$\frac{d}{d\epsilon}\int_0^{t_1+\epsilon\delta t_1} F(y+\epsilon\delta y,\dot y+\epsilon\delta\dot y,\ddot y+\epsilon\delta\ddot y,z+\epsilon\delta z,\dot z+\epsilon\delta\dot z,\ddot z+\epsilon\delta\ddot z)dt=0 \tag{A2}$$

at

$$\epsilon=0$$

The time duration of the integral of equation (A1) is that beginning at a definite time ($t=0$) but not ending at any definite time but rather along some curves: $y=f_1(t)$, $\dot y=f_2(t)$, $z=g_1(t)$, and $\dot z=g_2(t)$ (fig. 11). The functions δy and δz are arbitrary and independent functions of time.

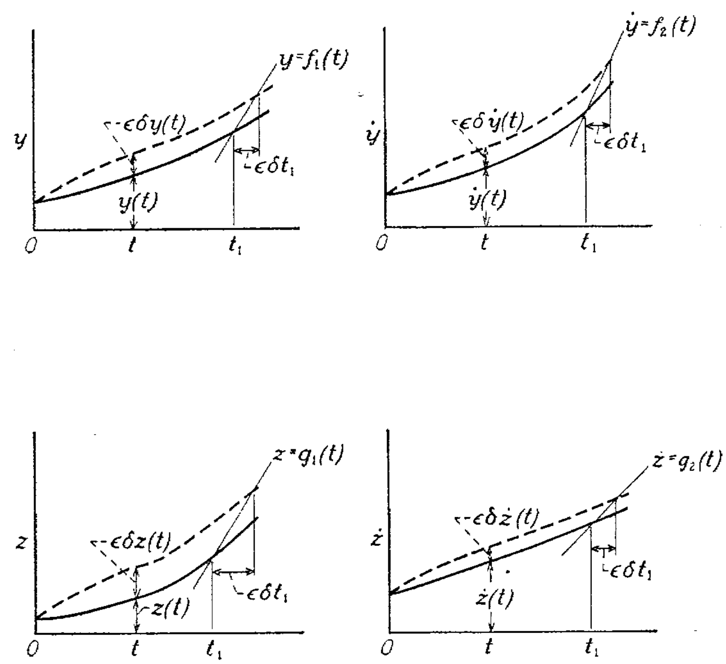

FIGURE 11.—Illustration of curves for minimization of $\int_0^{t_1} F(y,\dot y,\ddot y,z,\dot z,\ddot z)\,dt$.

Performing the operation indicated by equation (A2) gives

$$\int_0^{t_1}(F_y\delta y+F_{\dot y}\delta\dot y+F_{\ddot y}\delta\ddot y+F_z\delta z+F_{\dot z}\delta\dot z+F_{\ddot z}\delta\ddot z)dt+F(t_1)\delta t_1=0 \tag{A3}$$

After integration by parts,

$$\int_0^{t_1}\left[F_y-\frac{d}{dt}(F_{\dot y})+\frac{d^2}{dt^2}(F_{\ddot y})\right]\delta y\,dt+\int_0^{t_1}\left[F_z-\frac{d}{dt}(F_{\dot z})+\frac{d^2}{dt^2}(F_{\ddot z})\right]\delta z\,dt+\left(F_{\dot y}\delta y+F_{\ddot y}\delta\dot y-\frac{d}{dt}F_{\ddot y}\delta y\right)\Bigg]_0^{t_1}+F(t_1)\delta t_1+\left(F_{\dot z}\delta z+F_{\ddot z}\delta\dot z-\frac{d}{dt}(F_{\ddot z})\delta z\right)\Bigg]_0^{t_1}=0 \tag{A4}$$

As before, the integrands of the integrals and the boundary-condition terms must vanish separately. From the geometry at the end condition ($t=t_1$, fig. 11),

$$\left.\begin{aligned}\delta y(t_1)&=[f_1'(t_1)-\dot y(t_1)]\;\delta t_1\\ \delta\dot y(t_1)&=[f_2'(t_1)-\ddot y(t_1)]\;\delta t_1\\ \delta z(t_1)&=[g_1'(t_1)-\dot z(t_1)]\;\delta t_1\\ \delta\dot z(t_1)&=[g_2'(t_1)-\ddot z(t_1)]\;\delta t_1\end{aligned}\right\} \tag{A5}$$

The three conditions from equation (A4) then become

$$\left.\begin{aligned}&F_y-\frac{d}{dt}(F_{\dot y})+\frac{d^2}{dt^2}(F_{\ddot y})=0\\ &F_z-\frac{d}{dt}(F_{\dot z})+\frac{d^2}{dt^2}(F_{\ddot z})=0\\ &\left[F-\dot yF_{\dot y}+\dot y\frac{d}{dt}(F_{\ddot y})-\ddot yF_{\ddot y}-\dot zF_{\dot z}+\dot z\frac{d}{dt}(F_{\ddot z})-\ddot zF_{\ddot z}\right]_{t=t_1}+\\ &f_1'(t_1)\left[F_{\dot y}-\frac{d}{dt}(F_{\ddot y})\right]_{t=t_1}+f_2'(t_1)\,F_{\ddot y}(t_1)+\\ &g_1'(t_1)\left[F_{\dot z}-\frac{d}{dt}(F_{\ddot z})\right]_{t=t_1}+g_2'(t_1)\,F_{\ddot z}(t_1)+\\ &\delta y(0)\left[F_{\dot y}-\frac{d}{dt}(F_{\ddot y})\right]_{t=0}+\delta\dot y(0)\,F_{\ddot y}(0)+\\ &\delta z(0)\left[F_{\dot z}-\frac{d}{dt}(F_{\ddot z})\right]_{t=0}+\delta\dot z(0)\,F_{\ddot z}(0)=0\end{aligned}\right\} \tag{A6}$$

The first two of equations (A6) are the differential equations that satisfy the original criterions of equation (A1). The physical solution to the problem is the pair of solutions of these equations that satisfies the boundary-condition equation of equations (A6). If the first of equations (A6) is multiplied by $\dot y$ and the second, by $\dot z$ and the equations are added, an exact derivative is formed, the integral of which is

$$F-\dot yF_{\dot y}+\dot y\frac{d}{dt}(F_{\ddot y})-\ddot yF_{\ddot y}-\dot zF_{\dot z}+\dot z\frac{d}{dt}(F_{\ddot z})-\ddot zF_{\ddot z}=C \tag{A7}$$

A study of the boundary-condition equation of equations (A6) then shows that the physically reasonable boundary conditions should be as follows:

$$\left.\begin{array}{l}\text{If } F_{\dot{y}}\neq 0 \text{ or } F_{\dot{y}}\neq K\text{, then } \delta y(0)=0 \text{ and } f_1'(t_1)=0\\ \text{If } F_{\ddot{y}}\neq 0\text{, then } \delta\dot{y}(0)=0 \text{ and } f_2'(t_1)=0\\ \text{If } F_{\dot{z}}\neq 0 \text{ and } F_{\dot{z}}\neq K\text{, then } \delta z(0)=0 \text{ and } g_1'(t_1)=0\\ \text{If } F_{\ddot{z}}\neq 0\text{, then } \delta\dot{z}(0)=0 \text{ and } g_2'(t_1)=0\end{array}\right\}\quad\text{(A8)}$$

Then in equation (A7), $C=0$ and the final solution to the problem of equation (A1) for the boundary conditions of equations (A8) is

$$\left.\begin{array}{l}F-\dot{y}F_{\dot{y}}+\dot{y}\dfrac{d}{dt}(F_{\ddot{y}})-\ddot{y}F_{\ddot{y}}-\dot{z}F_{\dot{z}}+\dot{z}\dfrac{d}{dt}(F_{\ddot{z}})-\ddot{z}F_{\ddot{z}}=0\\ \text{and either}\\ \qquad F_y-\dfrac{d}{dt}(F_{\dot{y}})+\dfrac{d^2}{dt^2}(F_{\ddot{y}})=0\\ \text{or}\\ \qquad F_z-\dfrac{d}{dt}(F_{\dot{z}})+\dfrac{d^2}{dt^2}(F_{\ddot{z}})=0\end{array}\right\}\quad\text{(A9)}$$

The meaning of the boundary-condition equations (A8) is to define the original criterions for that duration during which the system moves from one essential operating level to another. Thus, if all conditions of equations (A8) must hold, the system goes from one definite y, $\dot{y}$, z, and $\dot{z}$ to any other definite y, $\dot{y}$, z, and $\dot{z}$. The first differential equation of equations (A9) would be of third order and the second or third equations of (A9) would be of fourth order. If equations (A9) are integrated to obtain a pair of second-order differential equations having three constants of integration, the choice of these three constants can then determine a desired end point, that is, the values of $\dot{y}$, z, and $\dot{z}$ at some final y.

The physical solution to the problem, then, is the pair of second-order differential equations that are solutions of equations (A9) with the constants of integration evaluated so that the system goes through some desired end point. This end point must be such as to allow the possibility of stability. Such an end point may be written as follows:

$$\dot{y}=0,\ z=z_s,\ \text{and}\ \dot{z}=0\ \text{when}\ y=y_s \qquad\text{(A10)}$$

which gives three conditions for the evaluation of the three constants of integration. A stability device must still be added to the system so that, at the point when equations (A10) hold,

$$\text{and}\qquad\left.\begin{array}{l}\ddot{y}=0\\ \ddot{z}=0\end{array}\right\}\qquad\text{(A11)}$$

Equations (A9) are symmetric in the variables y and z, which indicates the nature of the extension for more independent variables. Thus, if a third independent function w exists, the original criterion would be written

$$\int F(y,\dot{y},\ddot{y},z,\dot{z},\ddot{z},w,\dot{w},\ddot{w})\,dt=\text{minimum}\qquad\text{(A12)}$$

This condition is satisfied under boundary conditions similar to equations (A8), where two additional conditions are added on the variable w, and the following equations describe the controlled system:

$$\left.\begin{array}{l}F-\dot{y}F_{\dot{y}}+\dot{y}\dfrac{d}{dt}(F_{\ddot{y}})-\ddot{y}F_{\ddot{y}}-\dot{z}F_{\dot{z}}+\dot{z}\dfrac{d}{dt}(F_{\ddot{z}})-\ddot{z}F_{\ddot{z}}-\\ \qquad\dot{w}F_{\dot{w}}+\dot{w}\dfrac{d}{dt}(F_{\ddot{w}})-\ddot{w}F_{\ddot{w}}=0\\ \text{and any two of the following three equations:}\\ \qquad F_y-\dfrac{d}{dt}(F_{\dot{y}})+\dfrac{d^2}{dt^2}(F_{\ddot{y}})=0\\ \qquad F_z-\dfrac{d}{dt}(F_{\dot{z}})+\dfrac{d^2}{dt^2}(F_{\ddot{z}})=0\\ \qquad F_w-\dfrac{d}{dt}(F_{\dot{w}})+\dfrac{d^2}{dt^2}(F_{\ddot{w}})=0\end{array}\right\}\quad\text{(A13)}$$

The three equations of equations (A13) can then be integrated to give three second-order differential equations where the five constants of integration are evaluated so that the system goes through the desired value of $\dot{y}$, z, $\dot{z}$, w, $\dot{w}$ for some final value of y.

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