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ON THE PLANE POTENTIAL FLOW PAST A

LATTICE OF ARBITRARY AIRFOILS

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ADVANCE RESTRICTED REPORT

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SUMMARY

The two-dimensional, incompressible potential flow past a symmetrical lattice of airfoils of arbitrary shape is investigated theoretically. The problem is treated by usual methods of conformal mapping in several stages, one stage corresponding to the mapping of the framework of the arbitrary line lattice and another significant stage corresponding to the Theodorsen method for the mapping of the arbitrary single wing profile into a circle. A particular feature in the theoretical treatment is the special handling of the regions at an infinite distance in front of and behind the lattice. Expressions are given for evaluation of the velocity and pressure distribution at the airfoil boundary. An illustrative numerical example is included.

INTRODUCTION

This paper treats the problem of determining the flow pattern, or the velocity and pressure fields, associated with the uniform flow past an infinite row of symmetrically placed airfoils of the same shape. This airfoil-lattice problem occurs in the design of turbine blades, wind-tunnel vanes or grids, and elsewhere. There is a purely mathematical interest in the problem that concerns the field of conformal mapping of infinitely connected regions. Analogous two-dimensional "lattice" problems occur in the steady flow of heat and electricity.

Considerable ingenuity has been devoted to the airfoil-lattice problem, especially in the turbomachine studies in the German literature and more recently in the British studies; nevertheless, a survey of the available literature indicates that nearly all the treatments employed and the results obtained are of a special or indirect nature which involve, for example, lattices of thin lines or approximate graphical procedures. Recently, however, A. R. Howell in a British paper of limited circulation has written briefly on the theory of arbitrary airfoils in cascade. Howell applies a special transformation to an airfoil lattice to convert the lattice region to a somewhat random, simply connected region and, with the aid of several stages of conformal mapping, obtains a region about a circle.

The problem of determining the incompressible potential flow past an arbitrary single wing section was studied by Theodorsen (reference 1), who gave a practical procedure for its solution. The case of two wing sections, or the arbitrary biplane, was later treated in reference 2. The determination of the flow past an infinite lattice of airfoils of the same shape is a problem intermediate in difficulty in comparison with the afore-mentioned ones. The treatment for resolving this problem given in the present report is similar to that for the arbitrary single wing section but the calculations are more involved.

The problem will herein be studied by the usual method of conformal mapping. It is convenient to accomplish the result in three or four stages: The airfoil lattice is first replaced by its skeleton, or framework of line segments. The initial mapping function employed transforms the lattice skeleton into a circle. In the plane of this circle there are two singular points, known as branch points. These points have dual significance: They correspond to infinite regions in front of and behind the lattice of lines, and they enter in the problem of reducing the lattice region (multiply connected region) to the region of a single body (simply connected region). If now an arbitrary airfoil shape is generated or given around the framework of lines, then in the plane of the circle a circular-like contour is generated around the original circle. This contour may be transformed into an exact circle by the well-known procedure given in reference 1 or 3. The original two significant points are then traced by a transformation due to H. A. Schwarz. A final elementary transformation will bring the circle into a standard circle for which the two characteristic branch points are symmetrically placed. The region of this circle is considered the standard region for determining the flow pattern.

L-313

For illustrative purposes an outline of a procedure for calculating pressure distributions is included. The method may be followed without reference to the theory by readers interested mainly in numerical results. For convenience, a list of symbols is given in appendix A.

ANALYSIS

Initial transformation for lattice of straight lines.- Consider the transformation (reference 4)

$$\zeta_1 = \frac{g}{2\pi}\left(\log \frac{b + z'}{b - z'} + \log \frac{z' + \frac{a^2}{b}}{z' - \frac{a^2}{b}}\right) \tag{1}$$

where g, b, and a are real numbers and $b > a$. Introduce coordinates ψ and θ by means of the relation

$$z' = ae^{\psi + i\theta} \tag{2}$$

and let

$$b/a = e^{\gamma_o} \tag{3}$$

Equation (1) may then be expressed as

$$\zeta_1 = \frac{g}{2\pi} \log \left[\frac{\cosh \gamma_o + \cosh (\psi + i\theta)}{\cosh \gamma_o - \cosh (\psi + i\theta)}\right] \tag{4}$$

If $\psi = 0$, according to equation (2), z' lies on a circle of radius a (fig. 1(a)). According to equation (4), $\zeta_1 = x_1 + iy_1$ is the logarithm of a real positive function and consequently represents a real function (its principal value) and the infinite sequence of values differing from this function by $\frac{g}{2\pi}\,2k\pi i$, where k is any integer. The transformation illustrated in figure 1(b) is that of an infinite lattice of unstaggered lines of gap g in the ζ_1-plane into the circle of radius a in the z'-plane. The points $z' = b$ and $z' = -b$ correspond to infinity in front of and behind the lattice, respectively. The inverse points $z' = \frac{a^2}{b}$ and $z' = -\frac{a^2}{b}$ are inside the circle of radius a.

In order to introduce stagger, it is convenient to consider the transformation

$$\zeta_2 = -\frac{ih}{2}\left(\log \frac{b + z'}{b - z'} - \log \frac{z' + \frac{a^2}{b}}{z' - \frac{a^2}{b}}\right)$$

where h is real. This transformation can be written with the use of equations (2) and (3) as

$$\zeta_2 = -i \frac{h}{2\pi} \log \left[\frac{\sinh \gamma_o + \sinh (\psi + i\theta)}{\sinh \gamma_o - \sinh (\psi + i\theta)}\right] \qquad (5)$$

If $\psi = 0$, the expression within the brackets is a complex number of unit magnitude; hence, the logarithm is a pure imaginary number plus an infinite sequence of numbers differing by $2\pi i$. Then $\zeta_2 = x_2 + iy_2$ represents a sequence of real numbers differing by h and the lattice is one of horizontal lines displaced from each other by h (fig. 1(c)).

The transformation for the general staggered-line lattice is a combination of equations (4) and (5)

$$\zeta = \zeta_1 + \zeta_2 \qquad (6a)$$

or

$$\zeta = \frac{d}{2\pi}\left(e^{-i\beta} \log \frac{b + z'}{b - z'} + e^{i\beta} \log \frac{z' + \frac{a^2}{b}}{z' - \frac{a^2}{b}}\right) \qquad (6b)$$

L-313

where

gap $g = d \cos \beta$

stagger $h = d \sin \beta$

stagger ratio $h/g = \tan \beta$

the parameter d may be called the slant gap (fig. 1(d)), and β the stagger angle.

The geometry of the lattice may be expressed in terms of the parameters γ_o and β by noting that the chord length may be obtained from the (singular or critical) values of θ which correspond to the end points of the chord and are solutions of the equation $d\zeta/dz' = 0$. This equation gives the result

$$\tan \theta = \tanh \gamma_o \tan \beta \tag{7a}$$

or, for later reference,

$$\left.\begin{aligned} \cos \theta &= \frac{\cosh \gamma_o \cos \beta}{Q} \\ \sin \theta &= \frac{\sinh \gamma_o \sin \beta}{Q} \end{aligned}\right\} \tag{7b}$$

where

$$Q = \left(\cosh^2\gamma_o - \sin^2\beta\right)^{1/2}$$

Relations (7) may be employed in two ways: (1) When the parameters γ_o and β are given, the relation determines the two critical values of θ, θ_l and θ_t, where the subscripts l and t refer to leading edge and trailing edge, respectively, and $\theta_t = \theta_l + \pi$. (2) When θ_l or $\tan \theta_l$ and the stagger angle β are given, the relation determines the parameter γ_o.

The chord c may be obtained by putting $\theta = \theta_l$ and $\theta = \theta_t$ in equation (6a) and taking the difference in abscissas x_l and x_t. From equations (4) to (7),

$$c = x_l - x_t$$

$$= \frac{2d}{\pi}\left(\cos \beta \log \frac{Q + \cos \beta}{\sinh \gamma_0} + \sin \beta \tan^{-1} \frac{\sin \beta}{Q}\right) \qquad (8)$$

By means of equation (8), the parameter γ_0 can be presented directly in terms of given values of the gap-chord ratio for any stagger ratio. A representative chart relating gap-chord ratio, stagger angle, and γ_0 is shown in figure 2; some values are given in table I.

Inversion of equations (4) to (6).- The initial transformations may be thought of as mapping a framework of chords of an arbitrary lattice into a circle. If a shape is generated around the chords in the z'-plane, a contour is generated around the circle of radius a. This contour, which must exclude the points $z' = -b$ and $z' = b$ and must enclose the points $z' = -\frac{a^2}{b}$ and $z' = \frac{a^2}{b}$, may be considered to be completely defined by the function $\psi(\theta)$. If a lattice of airfoils is preassigned, the function $\psi(\theta)$ must be found from the given coordinates of the airfoil shape. In order not to interrupt the sequence of main ideas, the details of this problem are relegated to appendix B, with certain remarks on the practical achievement of a nearly circular contour.

Transformation of contour in z'-plane to circle in z-plane.- It is assumed now that the circular-like contour in the z'-plane which corresponds to the airfoil contour of the lattice is either given or determined; that is, the function $\psi(\theta)$ is known in the boundary expression $z' = ae^{\psi+i\theta}$. By the procedure of reference 1 or 3, the transformation

$$z' = ze^{f(z)} \qquad (9a)$$

where

$$f(z) = \sum_{1}^{\infty} \frac{c_n}{z^n} = \log \frac{z'}{z} \qquad (9b)$$

and c_n are complex coefficients determined by the boundary, is then employed to transform the z'-contour into a circle $z = ae^{\psi_o + i\varphi}$ in the z-plane. The transformation (9a) keeps the regions alike at infinity in the z'- and z-planes; that is, $z = z'$ and $dz'/dz = 1$ at infinity. The correspondence of the boundaries is determined by the functional equation

$$\varphi - \theta = \epsilon(\varphi)$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} \psi(\varphi') \cot\frac{\varphi' - \varphi}{2}\, d\varphi' \qquad (10)$$

for which a convenient numerical solution has been outlined in reference 3. The radius of the circle $R = ae^{\psi_o}$ is determined by the relation

$$\psi_o = \frac{1}{2\pi}\int_0^{2\pi} \psi(\varphi)\, d\varphi \qquad (11)$$

For consistency, the functional symbol $\psi(\varphi)$ is here used to denote the quantity ψ expressed as a function of φ - that is, $\psi[\theta(\varphi)]$. In reference 3 the notations $\psi(\varphi)$ and $\bar{\psi}[\theta(\varphi)]$ are used.

It is necessary also to trace the correspondence of the points $z' = b$ and $z' = -b$. Let $z = \beta_1$ correspond to $z' = b$ and let $z = -\beta_2$ correspond to $z' = -b$. The values β_1 and β_2 may be determined by a relation (due to Schwarz) that expresses the value of a complex function in terms of an integral of the real part of the function along a circle. A simple derivation of the desired relation is shown in appendix C. The expression is

$$\log\frac{z'}{z} = f(z)$$

$$= -\frac{1}{\pi}\int_0^{2\pi} \psi(\varphi) \frac{d\varphi}{1 - \frac{z}{R}e^{-i\varphi}} \qquad (12)$$

The values of β_1 and β_2 may be determined from equation (12) by an iteration process that converges extremely rapidly. The process may be described as follows: In equation (12), let the zeroth approximation to β_1 be $z = z_0 = b$ and let the corresponding value of z' be written $z' = z_0' = be^{f(b)}$, where $f(b)$ is the evaluation of equation (12) for $z = b$. It is actually desired, however, to have $z' = b$ but, because

$$z' = z_0' = b + z_0' - b$$

the initial value of z' differs from the desired value by $z_0' - b$. Furthermore, $z = z_0$ differs from $z = \beta_1$ by approximately the same amount; hence, reducing z_0 by the quantity $z_0' - b$ gives

$$z_1 = z_0 + b - z_0'$$

$$= b\left[2 - e^{f(b)}\right]$$

which may be considered a first approximation to β_1. If it is desired to check this result or to obtain a second approximation, the process may be repeated; thus, from equation (12), find $f(z_1)$ and

$$z_1' = z_1 e^{f(z_1)}$$

Then,

$$z_2 = z_1 + b - z_1'$$

which is a second approximation to β_1 and, in general, the nth approximation is

$$z_n = z_{n-1} + b - z_{n-1}'$$

It is clear that, should z_n correspond to z_{n-1}, z_{n-1}' must correspond to b and the process automatically stops. The numerical process is given in appendix C; relatively elementary steps are involved. In order to determine $-\beta_2$, the process is applied with b replaced by $-b$.

Transformation to standard circle in w-plane.- In order to obtain the flow pattern, it is desirable to introduce another function which transforms the circle in the z-plane into another circle in the w-plane in such a way that the characteristic points $z = \beta_1$ and $z = -\beta_2$ map into $w = b$ and $w = -b$, respectively. The region of the circle in the w-plane may be considered the standard region. The desired transformation may be written as (see appendix D)

$$\frac{b - w}{b + w} = K\left(\frac{\beta_1 - z}{\beta_2 + z}\right) \tag{13}$$

where

$$K = \frac{b^2 - S^2}{b^2 + S^2} \frac{\beta_2\overline{\beta}_1 + R^2}{\beta_1\overline{\beta}_1 - R^2} \tag{14}$$

and

$R = ae^{\psi_o}$ is the radius of the original circle in the z-plane, $\overline{\beta}_1$ is the complex conjugate to β_1, and S is the radius of the new circle in the w-plane. The radius S is determined by

$$S = be^{-\gamma_1} \tag{15}$$

where γ_1 is obtained from

$$\cosh \gamma_1 = \frac{1}{R}\left|\frac{R^2 + \beta_2\overline{\beta}_1}{\beta_1 + \beta_2}\right| \tag{16}$$

Complex velocity potential in w-plane.- Consider the flow function $\Omega(w) = \Phi + i\Psi$, which is defined as

$$\Omega(w) = -\frac{Vd}{2\pi}\left(e^{i\alpha} \log \frac{b + w}{b - w} + e^{-i\alpha} \log \frac{w + \frac{S^2}{b}}{w - \frac{S^2}{b}}\right) - \frac{i\Gamma}{4\pi} \log \frac{w^2 - \frac{S^4}{b^2}}{b^2 - w^2} \tag{17}$$

The flow pattern may be regarded as due to a combination of singularities, sinks, sources, and vortices, placed at the points $w = \pm b$ and $w = \pm\frac{S^2}{b}$ as indicated in figure 3. It may be readily verified that the circle of radius S - that is, $w = Se^{i\sigma}$ - is part of a streamline and it may further be observed from figure 3 that the circulation around any contour which encloses the points $w = \pm\frac{S^2}{b}$ and for which the points $w = \pm b$ are exterior points, is Γ (positive if counterclockwise). The parameter α will be interpreted later as an angle of attack.

The value of the circulation Γ may be determined by means of the Kutta-Joukowski condition for smooth flow at the trailing edge of the lattice. Let σ_o be the value of σ on the boundary circle $Se^{i\sigma}$ that corresponds to the trailing edge of the lattice. The Kutta-Joukowski condition then requires that the flow separate at $\sigma = \sigma_o$, or that a stagnation point exist there.

With $d\Omega/dw = 0$ and $w = Se^{i\sigma_o}$, the following relation for Γ is found:

$$\Gamma = -\frac{4VSd}{b^2 - \frac{S^4}{b^2}}\left[b \sin(\sigma_o + \alpha) + \frac{S^2}{b}\sin(\sigma_o - \alpha)\right] \qquad (18)$$

If S/b is replaced by $e^{-\gamma_1}$ (equation (15)), equation (18) may be expressed as

$$\Gamma = -2Vd\left(\frac{\cos \sigma_o}{\cosh \gamma_1}\sin \alpha + \frac{\sin \sigma_o}{\sinh \gamma_1}\cos \alpha\right) \qquad (19)$$

Expressions for velocity in lattice field.- In order to obtain the flow pattern in the lattice field (ζ-plane), the component factors of the following expression are required:

$$\frac{d\Omega}{d\zeta} = \frac{d\Omega}{dw}\frac{dw}{dz}\frac{dz}{dz'}\frac{dz'}{d\zeta} \qquad (20)$$

These terms may be obtained from equations (17), (13), (9), and (6).

It is of particular interest to evaluate equation (20) explicitly for the regions at infinity in front of and behind the lattice and also on the lattice boundary itself. It is recalled that $\zeta = \infty$ corresponds to $z' = b$, $z = \beta_1$, $w = b$ and that $\zeta = -\infty$ corresponds to $z' = -b$, $z = -\beta_2$, $w = -b$. By combining terms according to equation (20), the (reflected) inlet-velocity vector is obtained as

$$\left[\frac{d\Omega}{d\zeta}\right]_{\infty} = V_{x_1} - iV_{y_1}$$

$$= - Ve^{i(\alpha+\beta)} - \frac{i\Gamma}{2d} e^{i\beta} \tag{21}$$

and the corresponding expression for the outlet-velocity vector is

$$\left[\frac{d\Omega}{d\zeta}\right]_{-\infty} = V_{x_2} - iV_{y_2}$$

$$= - Ve^{i(\alpha+\beta)} + \frac{i\Gamma}{2d} e^{i\beta} \tag{22}$$

By addition of equations (21) and (22), it becomes clear that the velocity vector of magnitude V and angle of attack $\alpha + \beta$ with respect to the x-axis is one-half the vector sum of the inlet and outlet velocities (fig. 4).

If the angle of attack of the mean velocity vector with respect to the x-axis (chord direction) is denoted by $\alpha_x = \alpha + \beta$, the velocity components in equations (21) and (22) are

$$V_{x_1} = -V \cos \alpha_x + \frac{\Gamma}{2d} \sin \beta$$

$$V_{y_1} = -V \sin \alpha_x + \frac{\Gamma}{2d} \cos \beta$$

and

$$V_{x_2} = -V \cos \alpha_x - \frac{\Gamma}{2d} \sin \beta$$

$$V_{y_2} = V \sin \alpha_x - \frac{\Gamma}{2d} \cos \beta$$

The conventional angle of attack α is measured with respect to the normal to the slant line of the lattice. The components normal to and along the slant line of the lattice, sometimes referred to as "axial" and "whirl" components, respectively, are found by rotating all vectors in the xy-plane by angle β (fig. 4). These components are, for the inlet velocity,

$$V_{N_1} = -V \cos \alpha$$

$$V_{L_1} = V \sin \alpha + \frac{\Gamma}{2d}$$

and, for the outlet velocity,

$$V_{N_2} = -V \cos \alpha = V_{N_1}$$

$$V_{L_2} = V \sin \alpha - \frac{\Gamma}{2d}$$

The squares of the magnitudes of the inlet and outlet velocities are

$$V_1{}^2 = V^2 \left[1 + 2 \frac{\Gamma}{2Vd} \sin \alpha + \left(\frac{\Gamma}{2Vd}\right)^2\right]$$

$$V_2{}^2 = V^2 \left[1 - 2 \frac{\Gamma}{2Vd} \sin \alpha + \left(\frac{\Gamma}{2Vd}\right)^2\right]$$

where $\Gamma/2Vd$ may be obtained from equation (19). Observe that the inlet and outlet speeds are equal, $V_1 = V_2$, when $\alpha = 0^\circ$ for any value of Γ. The inlet and outlet angles of attack with respect to the normal to the lattice line are

$$\alpha_1 = \tan^{-1} \frac{\sin \alpha + \frac{\Gamma}{2Vd}}{\cos \alpha}$$

$$\alpha_2 = \tan^{-1} \frac{\sin \alpha - \frac{\Gamma}{2Vd}}{\cos \alpha}$$

and the angle through which the stream is turned is

$$\alpha_1 - \alpha_2 = \tan^{-1} \frac{2 \frac{\Gamma}{2Vd} \cos \alpha}{1 - \left(\frac{\Gamma}{2Vd}\right)^2} \tag{23}$$

The component factors in equation (20) are now to be evaluated at the lattice boundary and, as the boundary itself is part of a streamline, only the magnitudes of the factors are of interest.

From equations (17) and (19) and with $w = Se^{i\sigma}$,

$$\left|\frac{d\Omega}{dw}\right| = \frac{2Vd}{\pi S} \frac{1}{\cosh 2\gamma_1 - \cos 2\sigma} \Big[\sinh \gamma_1 \sin \alpha(\cos \sigma - \cos \sigma_0)$$

$$+ \cosh \gamma_1 \cos \alpha(\sin \sigma - \sin \sigma_0)\Big] \tag{24}$$

where the parameter γ_1 is defined in equation (15).

In order to obtain dw/dz, it is convenient first to express equation (13) explicitly in w as

$$w = \frac{b(1 + K)z - b\left(K\beta_1 - \beta_2\right)}{(1 - K)z + K\beta_1 + \beta_2} \tag{25a}$$

A standard form for the transformation of a circular region $|z| \geqq R$ into $|w| \geqq S$ is

$$w = RSe^{i\lambda} \frac{z - \delta}{R^2 - \bar{\delta} z} \tag{25b}$$

Comparison of equations (25a) and (25b) makes it clear that the complex parameter δ and the real parameter λ may be obtained from the following relations:

$$\delta = \frac{K\beta_1 - \beta_2}{1 + K} \tag{26a}$$

or, as a check relation,

$$\bar{\delta} = \frac{R^2(K - 1)}{K\beta_1 + \beta_2} \tag{26b}$$

and

$$\frac{S}{R} e^{i\lambda} = \frac{(1 + K)b}{K\beta_1 + \beta_2}$$

or, by equating angles on both sides,

$$\lambda = \arg(1 + K) - \arg\left(K\beta_1 + \beta_2\right) \tag{27}$$

From equation (25b), the explicit correspondence of a point on the circle $w = Se^{i\sigma}$ to a point on the circle $z = Re^{i\varphi}$ can be obtained as follows:

$$e^{i\sigma} = e^{i(\varphi+\lambda)} \frac{1 - \frac{\delta}{R} e^{-i\varphi}}{1 - \frac{\bar{\delta}}{R} e^{i\varphi}} \tag{28}$$

Let the complex number δ be expressed as $|\delta| e^{i\tau}$ and let

$$1 - \frac{\delta}{R} e^{-i\varphi} = me^{i\mu} \tag{29}$$

where

$$m(\varphi) = 1 - 2 \frac{|\delta|}{R} \cos(\varphi - \tau) + \frac{|\delta|^2}{R^2}$$

L-313

and

$$\mu(\varphi) = \tan^{-1} \frac{\frac{|\delta|}{R} \sin (\varphi - \tau)}{1 - \frac{|\delta|}{R} \cos (\varphi - \tau)}$$

Observe that the denominator in equation (28) is the conjugate of equation (29) and is therefore equal to $me^{-i\mu}$. There results for the correspondence of σ and φ

$$\sigma = \varphi + \lambda + 2\mu \tag{30}$$

In particular, if the (trailing-edge) value of φ that corresponds to θ_t as determined by equations (7) is written as $\varphi_o = \theta_t + \epsilon_t$, where ϵ_t is the value of $\epsilon(\varphi)$ at $\theta = \theta_t$ from equation (10), then

$$\sigma_o = \varphi_o + \lambda + 2\mu_o$$

By differentiation of equation (25b),

$$\frac{dw}{dz} = \frac{RS\left(R^2 - \delta\bar{\delta}\right)e^{i\lambda}}{\left(R^2 - \bar{\delta}z\right)^2} \tag{31}$$

On the boundary, put $z = Re^{i\varphi}$; then, the magnitude of equation (31) is

$$\left|\frac{dw}{dz}\right| = \frac{S}{R}\left(1 - \frac{|\delta|^2}{R^2}\right)\frac{1}{m^2} \tag{32}$$

The expression for $\left|\frac{dz'}{dz}\right|$ on the boundary is obtained from equation (9) in terms of the functions $\epsilon(\varphi)$ and $\psi(\varphi)$ of equation (10) as follows (see reference 3):

$$\frac{dz'}{dz} = \frac{z'}{z}\left(1 + z\frac{df}{dz}\right) \tag{33a}$$

and, because $f(z)$ on the boundary is

$$f(z) = \psi(\varphi) - \psi_o + i(\theta - \varphi)$$

where

$$\theta - \varphi = \epsilon(\varphi)$$

then

$$\left|\frac{dz'}{dz}\right| = \left|\frac{z'}{z}\right|\left[\left(1 - \frac{d\epsilon}{d\varphi}\right)^2 + \left(\frac{d\psi}{d\varphi}\right)^2\right]^{1/2} \tag{33b}$$

The last factor of equation (20) is expressed from equation (6) on the boundary $z' = ae^{\psi+i\theta}$ as

$$\frac{d\zeta}{dz'} = \frac{2d}{\pi}\,\frac{E}{D}\,\frac{1}{z'} \tag{34}$$

where

$$E = \left[\cos^2\beta\,\cosh^2\gamma_o\left(\cosh^2\psi - \cos^2\theta\right) + \sin^2\beta\,\sinh^2\gamma_o\left(\cosh^2\psi - \sin^2\theta\right) - \frac{1}{4}\sin 2\beta\,\sinh 2\gamma_o\,\sin 2\theta\right]^{1/2}$$

$$D = \left|\cosh 2\gamma_o - \cosh 2(\psi + i\theta)\right|$$

$$= \left[\left(\cosh 2\gamma_o - \cosh 2\psi\,\cos 2\theta\right)^2 + (\sinh 2\psi\,\sin 2\theta)^2\right]^{1/2}$$

Finally, combining in equation (21) the factors given in equations (24), (32), (33b), and (34) yields

$$\left|\frac{d\Omega}{d\zeta}\right| = v$$

$$= ABCD\,\frac{1}{E}\,V \tag{35}$$

where

$$A = \frac{1}{\cosh 2\gamma_1 - \cos 2\sigma}\Big[\sinh\gamma_1\,\sin\alpha(\cos\sigma - \cos\sigma_o) + \cosh\gamma_1\,\cos\alpha(\sin\sigma - \sin\sigma_o)\Big]$$

L-313

$$B = \left(1 - \frac{|\delta|^2}{R^2}\right)\frac{1}{m^2}$$

$$C = \left[\left(1 - \frac{d\epsilon}{d\varphi}\right)^2 + \left(\frac{d\psi}{d\varphi}\right)^2\right]^{-1/2}$$

$$D = \left[\left(\cosh 2\gamma_o - \cosh 2\psi \quad \cos 2\theta\right)^2 + (\sinh 2\psi \sin 2\theta)^2\right]^{1/2}$$

$$E = \left[\cos^2\beta \cosh^2\gamma_o\left(\cosh^2\psi - \cos^2\theta\right) + \sin^2\beta \sinh^2\gamma_o\left(\cosh^2\psi - \sin^2\theta\right)\right.$$
$$\left. - \frac{1}{4} \sin 2\beta \sinh 2\gamma_o \sin 2\theta\right]^{1/2}$$

An application of equation (35) for the purpose of illustrating the various steps involved in a calculation of the surface velocity and pressure of the airfoil lattice is given in appendix E and illustrated in figures 5 and 6. For the sake of comparison, the single-airfoil case is given in figure 7.

Some special results from equation (35) for a lattice of lines.- In the case of a lattice of straight lines, the z'-, z-, and w-planes merge; hence $\theta = \varphi = \sigma$ and $R = S = a$.

From equations (19) and (7) and with $\alpha + \beta = \alpha_x$, which is the angle of attack with respect to the chord,

$$\frac{\Gamma}{2Vd} = \frac{\sin \alpha_x}{\left(\cosh^2\gamma_o - \sin^2\beta\right)^{1/2}} \tag{36}$$

The lift per unit span on a single member of the lattice is given by

$$L = \rho V\Gamma$$

where ρ is the air density. The lift vector is perpendicular to the mean velocity vector (fig. 4). This result is general and not limited to a straight-line lattice. The lift coefficient is

$$C_L = \frac{\rho V\Gamma}{c\left(\frac{1}{2}\rho V^2\right)} = \frac{2\Gamma}{cV} = 4\,\frac{1}{c/d}\,\frac{\Gamma}{2Vd} \tag{37}$$

where $\Gamma/2Vd$ is given in equation (36) and c/d can be found by equation (8).

The local velocity on the surface (equation (35)) becomes

$$v = V\left(\cos\,\alpha_x + \frac{N}{M}\sin\,\alpha_x\right) \tag{38}$$

where

$$N = \frac{1}{\left(\cosh^2\gamma_o - \sin^2\beta\right)^{1/2}} + \frac{\cos\,\beta\,\cos\,\theta}{\cosh\,\gamma_o} + \frac{\sin\,\beta\,\sin\,\theta}{\sinh\,\gamma_o}$$

$$M = \frac{\cos\,\beta\,\sin\,\theta}{\sinh\,\gamma_o} - \frac{\sin\,\beta\,\cos\,\theta}{\cosh\,\gamma_o}$$

In the special cases in which $\beta = 0^o$ and $\beta = 90^o$, the relations (36) to (38) are simpler.

For stagger angle $\beta = 0^o$ and with $d = g$,

$$\frac{\Gamma}{2Vg} = \frac{\sin\,\alpha_x}{\cosh\,\gamma_o}$$

From equation (8),

$$\cosh\,\gamma_o = \coth\,\frac{\pi c}{2g}$$

and

$$L = 2\rho V^2 g\,\tanh\,\frac{\pi c}{2g}\,\sin\,\alpha_x$$

$$= \pi\rho c V^2\,\frac{\tanh\,\frac{\pi c}{2g}}{\frac{\pi c}{2g}}\,\sin\,\alpha_x$$

L-313

The lift coefficient, according to equation (37), is

$$C_L = 2\pi \frac{\tanh \frac{\pi c}{2g}}{\frac{\pi c}{2g}} \sin \alpha_x$$

For $\beta = 0^\circ$, therefore, the slope of the lift curve is always less than 2π. Note that, for large gap, $c/g \rightarrow 0$ and the lift coefficient is

$$C_L = 2\pi \sin \alpha_x$$

When the gap g is small compared with the chord c,

$$C_L \rightarrow 4 \frac{g}{c} \sin \alpha_x$$

The local velocity at the surface, by equation (38), is

$$v = V\left(\cos \alpha_x + \tanh \gamma_o \cot \frac{\theta}{2} \sin \alpha_x\right)$$

This result may be compared with that for the single-line airfoil $(\gamma_o = \infty)$

$$v = V\left(\cos \alpha_x + \cot \frac{\theta}{2} \sin \alpha_x\right)$$

For stagger angle $\beta = 90^\circ$ and with $d = h$,

$$\frac{\Gamma}{2Vh} = \frac{\sin \alpha_x}{\sinh \gamma_o}$$

From equation (8),

$$\sinh \gamma_o = \cot \frac{\pi c}{2h}$$

and

$$L = 2\rho V^2 h \tan \frac{\pi c}{2h} \sin \alpha_x$$

$$= \pi\rho c V^2 \frac{\tan \frac{\pi c}{2h}}{\frac{\pi c}{2h}} \sin \alpha_x$$

The lift coefficient, according to equation (37), is

$$C_L = 2\pi \frac{\tan \frac{\pi c}{2h}}{\frac{\pi c}{2h}} \sin \alpha_x$$

For $\beta = 90^\circ$, therefore, the slope of the lift curve is always greater than 2π. The local velocity at the surface is

$$v = V\left[\cos \alpha_x + \coth \gamma_o \cot \frac{1}{2}\left(\theta - \frac{\pi}{2}\right) \sin \alpha_x\right]$$

It may be noted in passing that, for $c = \frac{1}{2}h$,

$$C_L = 8 \sin \alpha_x$$

as compared with

$$C_L = 2\pi \sin \alpha_x$$

for the single airfoil.

For the limiting case in which b and d approach ∞, the transformation (6) becomes

$$\zeta = \frac{d}{2\pi b}\left(z'e^{-i\beta} + \frac{a^2}{z'e^{-i\beta}}\right)$$

and, with limit $\frac{d}{2\pi b} \rightarrow 1$ and a new variable $z'' = z'e^{-i\beta}$

$$\zeta = z'' + \frac{a^2}{z''}$$

which is the familiar Joukowski transformation. If the variables ψ and θ are introduced, the corresponding result is expressed as

$$\zeta = 2a \cosh\left[\psi + i(\theta - \beta)\right]$$

where the limit, as $\gamma_o \longrightarrow \infty$, of $\frac{d}{2\pi a \cosh \gamma_o}$ has been put equal to 1.

L-513

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APPENDIX A

MAIN SYMBOLS

ζ complex plane of airfoil lattice $(x + iy)$

ζ_1, ζ_2 complex planes of airfoil lattice for stagger angles $\beta = 0^\circ$ and $\beta = 90^\circ$, respectively $(x_1 + iy_1 ; x_2 + iy_2)$

z' complex plane of circular-like contour $\left(ae^{\psi+i\theta}\right)$

z circle of radius $R = ae^{\psi_o}$ in z-plane $\left(ae^{\psi_o+i\varphi}\right)$

w circle of radius $S = be^{-\gamma_1}$ in w-plane $\left(be^{-\gamma_1+i\sigma}\right)$

$\zeta = \infty$, $z' = b$, $z = \beta_1$, $w = b$ corresponding points

$\zeta = -\infty$, $z' = -b$, $z = -\beta_2$, $w = -b$ corresponding points

a, b reference lengths

γ_o gap-chord parameter $\left(b = ae^{\gamma_o}\right)$

β stagger angle

d lattice spacing, or "slant" gap for any value of β

g lattice spacing, or gap for $\beta = 0^\circ$

h lattice spacing, or stagger for $\beta = 90^\circ$

V magnitude of mean of inlet- and outlet-velocity vectors (fig. 4)

α_x angle of attack with respect to x-axis of mean velocity vector

α angle of attack with respect to normal to slant line of lattice of mean velocity vector

α_1, α_2 inlet and outlet angles of attack with respect to normal to slant line of lattice, respectively

V_1, V_2 magnitudes of inlet and outlet velocities, respectively

L-313

APPENDIX B

INVERSION OF EQUATIONS (4) TO (6)

AND CHOICE OF COORDINATES

It is desired to find from a given airfoil lattice in the ζ-plane the contour defined by $\psi(\theta)$ in the z'-plane. This problem corresponds to an inversion of equations (4) to (6) and can be exactly treated for the cases in which $\beta = 0^\circ$ and $\beta = 90^\circ$ (equations (4) and (5), respectively) but an iteration or successive-approximation method is required for equation (6). Furthermore, although the parameters g and h are fixed by the geometry of the lattice, a choice exists in the definition of the chords and the origin of coordinates. This choice is discussed following equation (B17).

Stagger angle $\beta = 0^\circ$.- From equation (3), there is obtained

$$\cosh (\psi + i\theta) = \cosh \gamma_o \tanh \frac{\pi}{g} \zeta_1 \tag{B1}$$

Putting $\zeta_1 = x_1 + iy_1$ and denoting the real and imaginary parts of equation (B1) by ξ_1 and η_1, respectively, leads to

$$\left.\begin{aligned} \cosh \psi \cos \theta = \xi_1 &= \frac{\cosh \gamma_o \sinh \frac{2\pi}{g} x_1}{\cosh \frac{2\pi}{g} x_1 + \cos \frac{2\pi}{g} y_1} \\ \sinh \psi \sin \theta = \eta_1 &= \frac{\cosh \gamma_o \sin \frac{2\pi}{g} y_1}{\cosh \frac{2\pi}{g} x_1 + \cos \frac{2\pi}{g} y_1} \end{aligned}\right\} \tag{B2}$$

The expressions containing x_1 and y_1 in equation (B2) are considered given since these quantities are known from the coordinates of the airfoil lattice. If ψ and θ are eliminated successively,

$$\left.\begin{aligned} \left(\frac{\xi_1}{\cos\theta}\right)^2 - \left(\frac{\eta_1}{\sin\theta}\right)^2 &= 1 \\ \text{and} \qquad & \\ \left(\frac{\xi_1}{\cosh\psi}\right)^2 + \left(\frac{\eta_1}{\sinh\psi}\right)^2 &= 1 \end{aligned}\right\} \qquad (B3)$$

From equation (B3), there result the following expressions, which serve to define the function $\psi(\theta)$ in terms of the airfoil coordinates:

$$\left.\begin{aligned} \sin^2\theta &= p + \sqrt{p^2 + {\eta_1}^2} \\ \sinh^2\psi &= -p + \sqrt{p^2 + {\eta_1}^2} \end{aligned}\right\} \qquad (B4)$$

where

$$p = \frac{1}{2}\left(1 - {\xi_1}^2 - {\eta_1}^2\right)$$

For small values of θ, the relation $\sinh\psi = \dfrac{\eta_1}{\sin\theta}$ may be used.

It is useful for computational purposes to record the real and imaginary parts of equation (3)

$$\left.\begin{aligned} x_1 &= \frac{g}{2\pi}\left(\frac{1}{2}\log\frac{{\rho_1}^2}{{\rho_2}^2}\right) \\ y_1 &= \frac{g}{2\pi}\left(\varphi_1 - \varphi_2\right) \end{aligned}\right\} \qquad (B5)$$

where

$${\rho_1}^2 = (\cosh\gamma_o + \cosh\psi\cos\theta)^2 + (\sinh\psi\sin\theta)^2$$

$${\rho_2}^2 = (\cosh\gamma_o - \cosh\psi\cos\theta)^2 + (\sinh\psi\sin\theta)^2$$

$$\sin \varphi_1 = \frac{1}{\rho_1} \sinh \psi \sin \theta$$

$$\sin \varphi_2 = -\frac{1}{\rho_2} \sinh \psi \sin \theta$$

The angles are to be chosen between $-\pi$ and π, and the quadrants may be determined by noting also the relations

$$\cos \varphi_1 = \frac{1}{\rho_1} (\cosh \gamma_o + \cosh \psi \cos \theta)$$

$$\cos \varphi_2 = \frac{1}{\rho_2} (\cosh \gamma_o - \cosh \psi \cos \theta)$$

<u>Stagger angle $\beta = 90^\circ$</u>.- From equation (5), there is obtained

$$\sinh (\psi + i\theta) = \sinh \gamma_o \tan \frac{\pi}{h} \zeta_2 \tag{B6}$$

With $\zeta_2 = x_2 + iy_2$ and the real and imaginary parts of equation (B6) denoted by ξ_2 and η_2, respectively,

$$\left.\begin{aligned}
\sinh \psi \cos \theta = \xi_2 &= \frac{\sinh \gamma_o \sin \frac{2\pi}{h} x_2}{\cosh \frac{2\pi}{h} y_2 + \cos \frac{2\pi}{h} x_2} \\
\cosh \psi \sin \theta = \eta_2 &= \frac{\sinh \gamma_o \sinh \frac{2\pi}{h} y_2}{\cosh \frac{2\pi}{h} y_2 + \cos \frac{2\pi}{h} x_2}
\end{aligned}\right\} \tag{B7}$$

If ψ and θ are eliminated successively,

$$\left.\begin{aligned}\left(\frac{\xi_2}{\cos\theta}\right)^2 - \left(\frac{\eta_2}{\sin\theta}\right)^2 &= -1\\ \left(\frac{\xi_2}{\sinh\psi}\right)^2 + \left(\frac{\eta_2}{\cosh\psi}\right)^2 &= 1\end{aligned}\right\}\qquad\text{(B8)}$$

From equations (B8) there result finally the following expressions, which serve to define the function $\psi(\theta)$ in terms of the airfoil coordinates:

$$\left.\begin{aligned}\cos^2\theta &= q + \sqrt{q^2 + \xi_2{}^2}\\ \sinh^2\psi &= -q + \sqrt{q^2 + \xi_2{}^2}\end{aligned}\right\}\qquad\text{(B9)}$$

where

$$q = \frac{1}{2}\left(1 - \xi_2{}^2 - \eta_2{}^2\right)$$

For values of θ near $\pm 90^\circ$, the relation $\sinh\psi = \dfrac{\xi_2}{\cos\theta}$ may be used.

It is useful for computational purposes to write the real and imaginary parts of equation (5)

$$\left.\begin{aligned}x_2 &= \frac{h}{2\pi}\left(\varphi_3 - \varphi_4\right)\\ y_2 &= -\frac{h}{2\pi}\left(\frac{1}{2}\log\frac{\rho_3{}^2}{\rho_4{}^2}\right)\end{aligned}\right\}\qquad\text{(B10)}$$

where

$$\rho_3{}^2 = (\sinh\gamma_o + \sinh\psi\cos\theta)^2 + (\cosh\psi\sin\theta)^2$$

$$\rho_4{}^2 = (\sinh\gamma_o - \sinh\psi\cos\theta)^2 + (\cosh\psi\sin\theta)^2$$

$$\sin \varphi_3 = \frac{1}{\rho_3} \cosh \psi \sin \theta$$

$$\sin \varphi_4 = -\frac{1}{\rho_4} \cosh \psi \sin \theta$$

The angles are to be chosen between $-\pi$ and π, and the quadrants may be determined by noting also the relations

$$\cos \varphi_3 = \frac{1}{\rho_3} (\sinh \gamma_o + \sinh \psi \cos \theta)$$

$$\cos \varphi_4 = \frac{1}{\rho_4} (\sinh \gamma_o - \sinh \psi \cos \theta)$$

<u>Arbitrary stagger angle β and choice of coordinates</u>.- Because of the transcendental nature of equation (6), a direct inversion expression seems unobtainable; however, the values (ψ, θ) that correspond to coordinates (x, y) may be obtained without difficulty by an iterative process. For this purpose and for the purpose of choosing the coordinate axes, expansions of x_1, x_2, y_1, and y_2 in powers of ψ are useful. The following expansions may be readily verified:

$$x_1 \approx \frac{d}{2\pi} \cos \beta \left[\log \frac{\cosh \gamma_o + \cos \theta}{\cosh \gamma_o - \cos \theta} + \psi^2 \cosh \gamma_o \cos \theta \, \frac{\sinh^2 \gamma_o - \sin^2 \theta}{\left(\cosh^2 \gamma_o - \cos^2 \theta\right)^2} + \ldots \right] \tag{B11a}$$

$$x_2 \approx \frac{d}{2\pi} \sin \beta \left[2 \tan^{-1} \frac{\sin \theta}{\sinh \gamma_o} + \psi^2 \sinh \gamma_o \sin \theta \, \frac{\cosh^2 \gamma_o + \cos^2 \theta}{\left(\cosh^2 \gamma_o - \cos^2 \theta\right)^2} + \ldots \right] \tag{B11b}$$

$$y_1 \approx \frac{d}{2\pi} \cos \beta \, \frac{2 \cosh \gamma_o \sin \theta}{\cosh^2 \gamma_o - \cos^2 \theta} \psi \tag{B11c}$$

$$y_2 \approx -\frac{d}{2\pi} \sin \beta \, \frac{2 \sinh \gamma_o \cos \theta}{\cosh^2 \gamma_o - \cos^2 \theta} \psi \tag{B11d}$$

Then

$$\begin{aligned} y &= y_1 + y_2 \\ &\approx \frac{d}{\pi} \psi F(\theta) \end{aligned} \tag{B12}$$

where

$$F(\theta) = \frac{\cosh \gamma_o \cos \beta \sin \theta - \sinh \gamma_o \sin \beta \cos \theta}{\cosh^2 \gamma_o - \cos^2 \theta}$$

If the x-coordinate of the straight-line lattice, which is considered the skeleton of the airfoil lattice, is denoted by x_0, then x_0 is given by the value of $x = x_1 + x_2$ for $\psi = 0$, or

$$x_0 = \frac{d}{2\pi} \left(\cos \beta \log \frac{\cosh \gamma_o + \cos \theta}{\cosh \gamma_o - \cos \theta} + 2 \sin \beta \tan^{-1} \frac{\sin \theta}{\sinh \gamma_o} \right) \tag{B13}$$

and

$$x \approx x_0 + \frac{d}{2\pi} \psi^2 G(\theta) \tag{B14}$$

where

$$\begin{aligned} G(\theta) &= \frac{\cosh \gamma_o \cos \beta \cos \theta + \sinh \gamma_o \sin \beta \sin \theta}{\cosh^2 \gamma_o - \cos^2 \theta} \\ &\quad - \frac{(\cosh \gamma_o \cos \beta \sin \theta - \sin \gamma_o \sin \beta \cos \theta) \, 2 \sin \theta \cos \theta}{\left(\cosh^2 \gamma_o - \cos^2 \theta\right)^2} \\ &= F'(\theta) \end{aligned}$$

In particular, the leading- and trailing-edge points $x = x_l$ and $x = x_t$, are determined by the values of $\theta = \theta_l$ and $\theta = \theta_t$ that may be obtained from equations (7b). Then,

$$x_l \approx x_{0_l} + \frac{d}{2\pi} \psi^2 G_0 \tag{B15}$$

where

$$G_0 = Q\left(\frac{\cos^2\beta}{\sinh^2\gamma_0} + \frac{\sin^2\beta}{\cosh^2\gamma_0}\right)$$

and x_{0_l} denotes the leading edge of the line given by $\psi = 0$. A similar expression holds for x_t.

From equation (B12), for constant ψ,

$$\frac{\partial y}{\partial \theta} \approx \frac{d}{\pi} \psi F'(\theta)$$

$$= \frac{d}{\pi} \psi G(\theta)$$

In the neighborhood of the leading edge, therefore,

$$y \approx \frac{d}{\pi} \psi G_0(\theta - \theta_l) \tag{B16}$$

For x_0 near x_{0_l}, there is obtained from equation (B13),

$$x_0 = x_{0_l} + (\theta - \theta_l)\, x_{0_l}' + \frac{(\theta - \theta_l)^2}{2}\, x_{0_l}'' + \ldots$$

where the following relations are found to hold:

$$x_{0_l}' = \frac{d}{\pi}\Big[F(\theta)\Big]_{\theta=\theta_l} = 0$$

$$x_{0_l}'' = \frac{d}{\pi}\Big[-G(\theta)\Big]_{\theta=\theta_l} = -\frac{d}{\pi} G_0$$

Hence,

$$x_0 \approx x_{0_l} - \frac{d}{2\pi} G_0(\theta - \theta_l)^2$$

Then, from equation (B14),

$$x - x_{0_l} \approx x_0 - x_{0_l} + \frac{d}{2\pi} \psi^2 G(\theta)$$

$$\approx \frac{d}{2\pi} G_0 \left[-(\theta - \theta_l)^2 + \psi^2\right]$$

It follows from equation (B16) that, for $x = x_{0_l}$,

$$\theta - \theta_l \approx \psi$$

and

$$y = y_{0_l} \approx \frac{d}{\pi} \psi^2 G_0$$

With this value of y_{0_l} and equation (B15),

$$\frac{y_{0_l}}{x_l - x_{0_l}} \approx 2$$

If the total ordinate for both upper and lower sides at $x = x_{0_l}$ is denoted by y_t,

$$\frac{y_t}{x_l - x_{0_l}} \approx 4 \qquad \text{(B17)}$$

This result leads to a simple and convenient way of choosing axes of coordinates in order that $\psi(\theta)$ will behave smoothly at the edges; that is, that the value of ψ at the leading edge is approximately the mean of the values of ψ at nearby ordinates on the upper and lower surfaces. For a parabola the latus rectum, or ordinate through the focus, is four times the distance from the vertex to the focus. Equation (B17) states that the end point of the skeleton chord should be approximately the focus of a parabola at the nose.

L-313

The scheme for choice of axes is as follows: Locate a point F near the leading edge where the ordinate through F is four times the distance of F from the leading edge. Similarly locate a point F' near the trailing edge. The origin of coordinates then bisects the line FF', which is on the x-axis and represents the chord of the skeleton line airfoil $\psi = 0$. (To the order of approximation employed, the afore-mentioned choice of axes coincides with that given for the single wing section in reference 1 or 3.)

<u>Procedure for finding (ψ, θ) from (x, y) for arbitrary stagger angle β</u>.- An iterative procedure is given herein for finding $\psi(\theta)$ from (x, y) for arbitrary β, in which the knowledge of the case for $\beta = 0^{\circ}$ is employed to help in formulating the initial approximation. In brief, values of θ are obtained for stagger angle $\beta = 0^{\circ}$ for both the airfoil and its line skeleton. Values of θ are then found for the skeleton, in the case of stagger angle β. These functions permit approximate values of θ to be found for the airfoil, for stagger angle β. Equation (B12) then enables approximate values of ψ to be obtained. These values of (ψ, θ) are then readily checked and improved, if necessary. The steps are as follows:

(1) Choose the axes as outlined and express the airfoil coordinates in percent chord, where the chord for this purpose is the part of the x-axis intercepted by the airfoil. Denote the coordinates thus obtained by (x_p, y_p). Find $k = FF'$ in percent chord. Find $x_l - x_{0_l}$, the distance from the leading edge to F in percent chord, and denote this value by e. Obtain the ratio c/d, where c means here FF' and d is the spacing between corresponding points on adjacent airfoils of the lattice. Find conversion factor m by

$$m = 2\pi \frac{c}{d} \frac{1}{k}$$

(2) Convert coordinates of the airfoil from (x_p, y_p) to $\left(2\pi \frac{x}{d}, 2\pi \frac{y}{d}\right)$ as follows:

$$2\pi \frac{x}{d} = m\left(e + \frac{k}{2} - x_p\right)$$

$$2\pi \frac{y}{d} = my_p$$

(3) Find the parameter γ_o that corresponds to the determined value of c/d for the given value of β from graph or by calculation (equation (8)). Also find for later use the value of c/g corresponding to this value of γ_o for $\beta = 0^o$.

(4) Consider, for this value of γ_o, the two straight-line cases $(\psi = 0, \beta = 0^o)$ and $(\psi = 0, \beta = \beta)$; associate values of $\theta = \theta_o$ for $\beta = 0^o$ with values $\theta = \theta_\beta$ for the stagger angle β by referring associated values of θ to geometrically similar points of the lines (equation (B13)).

(5) Multiply coordinates in step (2) by the ratio $\frac{(c/g)_o}{(c/d)_\beta}$ where the chord-gap values are from step (3) for $\beta = 0^o$ and for $\beta = \beta$. Using equation (B4), find values of θ for $\beta = 0^o$.

(6) With the aid of step (4), obtain approximate values of θ_β associated with the values of θ obtained in step (5). Then, with $\theta = \theta_\beta$, use equation (B12) to obtain an approximate value of ψ, where

$$\psi = \frac{2\pi y}{d} \frac{F(\theta)}{2}$$

and the leading- and trailing-edge values of ψ are obtained from equation (B15).

(7) Calculate, from equations (B5) and (B10), exact values of $\left(2\pi \frac{x}{d}, 2\pi \frac{y}{d}\right)$, associated with the initial values of (ψ, θ) in step (6) where $x = x_1 + x_2$ and $y = y_1 + y_2$.

(8) If, on comparison of the coordinates in step (7) with the coordinates in step (2), it is deemed necessary to approximate (ψ, θ) more closely for several of the points (x, y), one procedure is the following: An expression for $\dfrac{d\zeta}{d(\psi + i\theta)}$ can be found from equations (4) to (6) as

$$\frac{d\zeta}{d(\psi + i\theta)} = \frac{d}{2\pi} \cos \beta \left[\frac{\sinh (\psi + i\theta)}{\cosh \gamma_0 + \cosh (\psi + i\theta)} + \frac{\sinh (\psi + i\theta)}{\cosh \gamma_0 - \cosh (\psi + i\theta)} \right]$$

$$- i \frac{d}{2\pi} \sin \beta \left[\frac{\cosh (\psi + i\theta)}{\sinh \gamma_0 + \sinh (\psi + i\theta)} + \frac{\cosh (\psi + i\theta)}{\sinh \gamma_0 - \sinh (\psi + i\theta)} \right]$$

With the notation of equations (B5) and (B10), this expression may be written

$$\frac{d\left(\zeta \frac{2\pi}{d}\right)}{d(\psi + i\theta)} = p + iq$$

$$= \cos \beta \sinh (\psi + i\theta) \left(\frac{1}{\rho_1} e^{-i\varphi_1} + \frac{1}{\rho_2} e^{-i\varphi_2} \right)$$

$$- i \sin \beta \cosh (\psi + i\theta) \left(\frac{1}{\rho_3} e^{-i\varphi_3} + \frac{1}{\rho_4} e^{-i\varphi_4} \right)$$

where

$$p = \cos \beta \left[\sinh \psi \cos \theta \left(\frac{\cos \varphi_1}{\rho_1} + \frac{\cos \varphi_2}{\rho_2} \right) \right.$$

$$\left. + \cosh \psi \sin \theta \left(\frac{\sin \varphi_1}{\rho_1} + \frac{\sin \varphi_2}{\rho_2} \right) \right]$$

$$+ \sin \beta \left[\sinh \psi \sin \theta \left(\frac{\cos \varphi_3}{\rho_3} + \frac{\cos \varphi_4}{\rho_4} \right) \right.$$

$$\left. - \cosh \psi \cos \theta \left(\frac{\sin \varphi_3}{\rho_3} + \frac{\sin \varphi_4}{\rho_4} \right) \right]$$

and

$$q = \cos\beta\left[\cosh\psi\sin\theta\left(\frac{\cos\varphi_1}{\rho_1} + \frac{\cos\varphi_2}{\rho_2}\right)\right.$$

$$\left. - \sinh\psi\cos\theta\left(\frac{\sin\varphi_1}{\rho_1} + \frac{\sin\varphi_2}{\rho_2}\right)\right]$$

$$- \sin\beta\left[\cosh\psi\cos\theta\left(\frac{\cos\varphi_3}{\rho_3} + \frac{\cos\varphi_4}{\rho_4}\right)\right.$$

$$\left. + \sinh\psi\sin\theta\left(\frac{\sin\varphi_3}{\rho_3} + \frac{\sin\varphi_4}{\rho_4}\right)\right]$$

The following relation may then be noted

$$\Delta\psi + i\,\Delta\theta \approx \frac{\Delta\left(\frac{2\pi x}{d}\right) + i\,\Delta\left(\frac{2\pi y}{d}\right)}{p + iq} \qquad \text{(B18)}$$

Let

$$\Delta\left(\frac{2\pi x}{d}\right) = \left(2\pi\,\frac{x}{d}\right)_0 - \left(2\pi\,\frac{x}{d}\right)_1$$

$$\Delta\left(\frac{2\pi y}{d}\right) = \left(2\pi\,\frac{y}{d}\right)_0 - \left(2\pi\,\frac{y}{d}\right)_1$$

where the subscripts 0 and 1 refer to the coordinates given in steps (2) and (7), respectively. If the values (ψ, θ) obtained in step (6) are used, evaluation of equation (B18) gives values $(\Delta\psi, \Delta\theta)$, and $(\psi + \Delta\psi,\ \theta + \Delta\theta)$ represents the next approximation to the desired coordinates. The process in steps (7) and (8) can be repeated if deemed necessary.

L-313

APPENDIX C

DERIVATION OF EQUATION (12)

The transformation (equation (9)) from the z'- to the z-plane may be rewritten

$$\log \frac{z'}{z} = f(z)$$

$$= \sum_{1}^{\infty} \frac{c_n}{z^n} \tag{C1}$$

where the complex constants c_n may be defined as

$$c_n = a_n + ib_n$$

On the boundaries, $z' = ae^{\psi + i\theta}$ and $z = ae^{\psi_o + i\varphi}$; hence,

$$\log \frac{z'}{z} = \psi - \psi_o + i(\theta - \varphi)$$

and

$$\psi - \psi_o = \sum_{1}^{\infty} \left(\frac{a_n}{R^n} \cos n\varphi + \frac{b_n}{R^n} \sin n\varphi \right) \tag{C2}$$

where

$$R = ae^{\psi_o}$$

With ψ considered as a function of φ denoted by $\psi(\varphi)$, the coefficients in equation (C2) are obtained as

$$\left.\begin{aligned}
\frac{a_n}{R^n} &= \frac{1}{\pi} \int_o^{2\pi} \psi(\varphi) \cos n\varphi \, d\varphi \\
\frac{b_n}{R^n} &= \frac{1}{\pi} \int_o^{2\pi} \psi(\varphi) \sin n\varphi \, d\varphi \\
\frac{c_n}{R^n} &= \frac{1}{\pi} \int_o^{2\pi} \psi(\varphi) \, e^{in\varphi} \, d\varphi
\end{aligned}\right\} \tag{C3}$$

Substituting equation (C3) in equation (C1) yields

$$f(z) = \frac{1}{\pi} \int_0^{2\pi} \psi(\varphi) \sum_1^{\infty} \frac{R^n e^{in\varphi}}{z^n} \, d\varphi \tag{C4}$$

For $\left|\frac{R}{z}\right| < 1$, the geometric series in equation (C4) can be summed and

$$f(z) = \frac{1}{\pi} \int_0^{2\pi} \psi(\varphi) \frac{Re^{i\varphi}}{z - Re^{i\varphi}} \, d\varphi \tag{C5}$$

which can immediately be expressed as in equation (12).

For computational purposes, equation (12) may be separated into real and imaginary parts. Let $f(z) = p + iq$ and $z = x + iy$ (where, for example, in the zeroth approximation $x = b$, $y = 0$). Then,

$$p = \frac{1}{\pi} \int_0^{2\pi} \psi(\varphi) \frac{N_1}{D} \, d\varphi$$

$$q = \frac{1}{\pi} \int_0^{2\pi} \psi(\varphi) \frac{N_2}{D} \, d\varphi$$

where

$$N_1 = \frac{x}{R} \cos \varphi + \frac{y}{R} \sin \varphi - 1$$

$$N_2 = \frac{x}{R} \sin \varphi - \frac{y}{R} \cos \varphi$$

$$D = 1 - 2\left(\frac{x}{R} \cos \varphi + \frac{y}{R} \sin \varphi\right) + \frac{x^2 + y^2}{R^2}$$

and the integrations can be conveniently performed by Simpson's rule.

APPENDIX D

TRANSFORMATION FROM z-PLANE TO w-PLANE

The linear fractional transformation

$$w = \frac{az + b}{cz + d}$$

on which the derivation of equation (13) is based, has the following well-known properties:

(1) When z traverses a circle C_z, w traverses a circle C_w.

(2) Two points w_1 and w_2 inverse with respect to a circle C_w correspond to two points z_1 and z_2 inverse with respect to the circle C_z.

(3) The anharmonic ratio of four points is preserved; that is, if z_1, z_2, z_3, and z_4 correspond to w_1, w_2, w_3, and w_4,

$$\frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)} = \frac{(w_1 - w_4)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w_4)}$$

For the desired correspondence it is known that four points $w_1 = b$, $w_2 = -b$, and their inverse points $w_3 = \frac{S^2}{b}$, $w_4 = \frac{-S^2}{b}$ are to correspond to $z_1 = \beta_1$, $z_2 = -\beta_2$ and their inverse points $z_3 = \frac{R^2}{\bar{\beta}_1}$, $z_4 = \frac{-R^2}{\bar{\beta}_2}$. Property (3) yields a relation that may be used to solve for the radius S and that can be expressed by equations (15) and (16). When the radius of the circle in the w-plane has been determined, property (3) can again be used by replacing - say, w_4 by w and z_4 by z. This procedure will yield a result that is equivalent to equation (13).

APPENDIX E

OUTLINE OF CALCULATION PROCEDURE

(1) List airfoil-section coordinates in percent chord.

(2) Choose axes (appendix B, paragraph following equation (B17)).

(3) List stagger angle β and find γ_o and value of c/d for the skeleton line lattice (table I, fig. 2, and equation (8)).

(4) Find (ψ, θ) (appendix B).

(5) Find $\epsilon(\varphi)$ (equation (10)) by method given in appendix of reference 3.

(6) Plot ψ against φ where $\varphi = \theta + \epsilon$. Find constant ψ_o (equation (11)) and $R = ae^{\psi_o}$.

(7) Find complex constants β_1 and β_2 (equation (12) and appendix C).

(8) Find constants $\cosh \gamma_1$, γ_1, S, and $K = k_1 + ik_2$ (equations (16), (15), and (14)).

(9) Find complex constant $\delta = |\delta| e^{i\tau}$ (equation (26)) and real constant λ (equation (28)). Also obtain functions $m(\varphi)$ and $\mu(\varphi)$ from equation (29).

(10) Find σ and, in particular, σ_o (equation (30)).

(11) Evaluate factors B, C, D, and E (equation (35)).

(12) Evaluate factor A in equation (35), first choosing the angle of attack α as indicated in the following paragraphs:

The lift coefficient is as in equation (37)

$$C_L = 4 \frac{1}{c/d} \frac{\Gamma}{2Vd}$$

Here c/d refers to the value of x/d at 0-percent chord minus x/d at 100-percent chord. By using equation (19) for $\Gamma/2Vd$, C_L may be expressed as

$$C_L = H \sin(\alpha + \eta) \tag{E1}$$

where

$$H = 4\frac{d}{c}\left[\left(\frac{\cos\sigma_o}{\cosh\gamma_1}\right)^2 + \left(\frac{\sin\sigma_o}{\sinh\gamma_1}\right)^2\right]^{1/2}$$

and

$$\eta = \tan^{-1}\left(\frac{\sin\sigma_o}{\cos\sigma_o}\,\frac{\cosh\gamma_1}{\sinh\gamma_1}\right)$$

This relation may be used to find α for any desired value of C_L and it is further noted that $\alpha = -\eta$ is the angle of zero lift.

The "ideal" angle of attack, introduced by Theodorsen, is defined for a thin section as the angle of attack for which a stagnation point exists not only at the sharp trailing edge but also at the sharp leading edge. For thick airfoils, the ideal angle of attack is defined in the same manner (the pressure difference at the leading edge vanishes) although the point that is considered the leading-edge point is not precisely defined. If this point is taken to be the intersection of the x-axis with the airfoil leading edge, the ideal lift and ideal angle of attack are found as follows: Let σ_1 be the value of σ corresponding to the leading-edge point. The quantity $d\Omega/dw$ in equation (24) (or the factor A in equation (35)) vanishes for $\sigma = \sigma_1$. The relation that gives the value of the ideal angle of attack $\alpha = \alpha_I$ is then

$$\frac{\sin\alpha}{\cos\alpha} = -\frac{\cosh\gamma_1}{\sinh\gamma_1}\,\frac{\sin\sigma_1 - \sin\sigma_o}{\cos\sigma_1 - \cos\sigma_o}$$

and the ideal lift coefficient, from equation (E1), is

$$C_{L_I} = -4\frac{d}{c}\frac{1}{J}\cos\frac{1}{2}\left(\sigma_1 - \sigma_o\right)$$

where

$$J^2 = \left[\cosh \gamma_1 \cos \frac{1}{2}(\sigma_1 + \sigma_o)\right]^2 + \left[\sinh \gamma_1 \sin \frac{1}{2}(\sigma_1 + \sigma_o)\right]^2$$

(13) The surface velocity ratio v/V is now found from equation (35) and the (mean) superstream pressure is found from Bernoulli's equation as

$$\frac{p}{q} = 1 - \left(\frac{v}{V}\right)^2$$

The angle through which the stream is turned may be found from equation (23).

A remark may be inserted here regarding an inverse calculation procedure. Instead of starting with a given lattice, it may be convenient to start with given function $\psi(\varphi)$, (quantity ψ as a function of φ) and given parameters γ_o and β. Then both the lattice arrangement and the flow properties follow uniquely, and in this way, systematic families of lattices can be studied.

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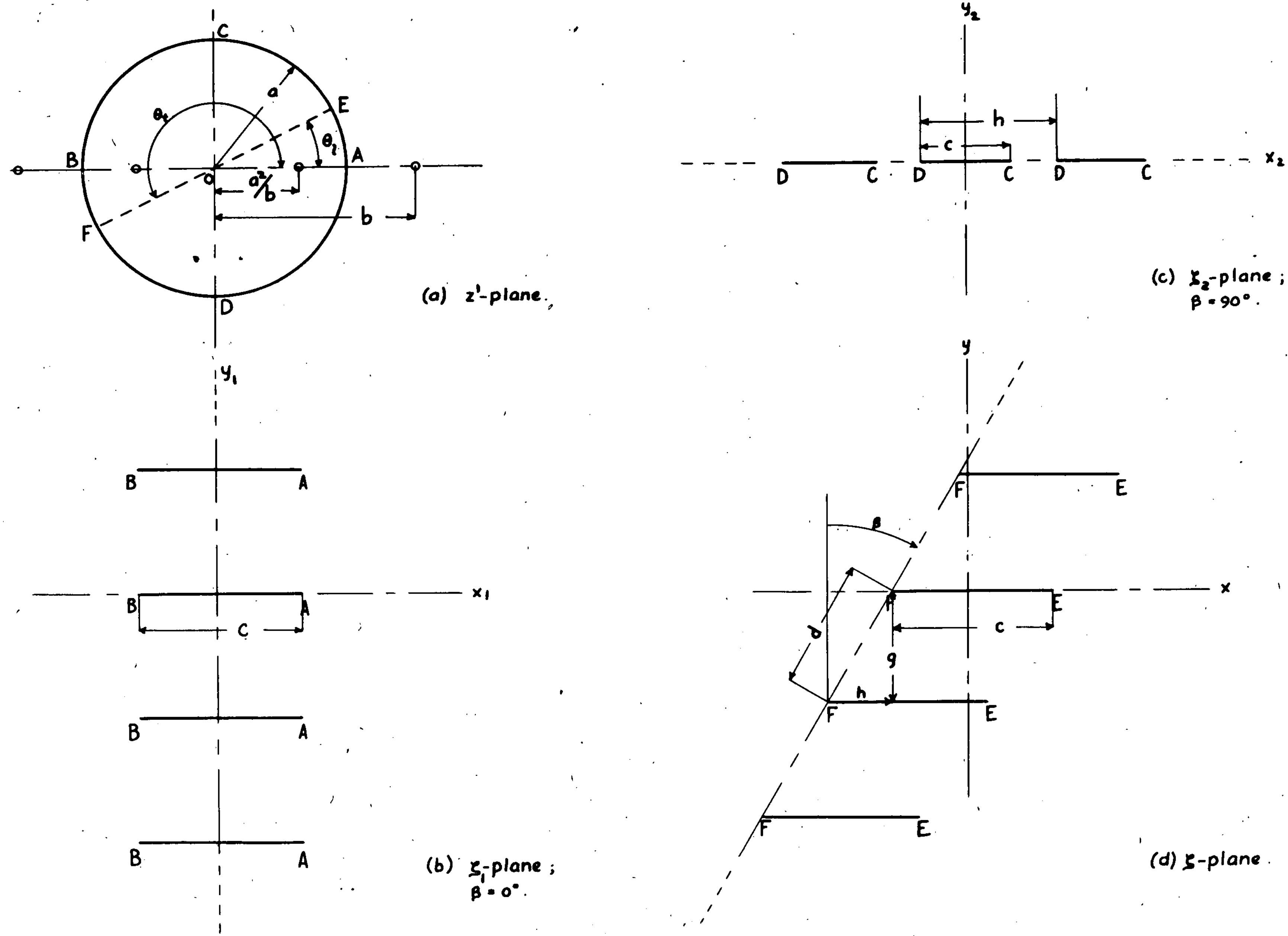
TABLE I

GAP-CHORD RATIO, PARAMETER γ_o, AND CORRESPONDING VALUES

OF θ_l FOR VARIOUS STAGGER ANGLES

				$\beta=0^\circ$; $\theta_l=0^\circ$	$\beta = 30^\circ$		$\beta = 60^\circ$		$\beta=90^\circ$; $\theta_l=90^\circ$
b/a	γ_o	sinh γ_o	cosh γ_o	d/c	d/c	θ_l (deg)	d/c	θ_l (deg)	d/c
1.005	0.004988	0.004988	1.000012	0.26207	0.29481	0.13	0.44157	0.50	1.00319
1.01	.009950	.009950	1.000050	.29619	.33208	.33	.48904	.99	1.00637
1.02	.019803	.019804	1.000196	.34036	.37997	.66	.54774	1.97	1.01277
1.05	.048790	.048809	1.001191	.42299	.46845	1.61	.65010	4.83	1.03204
1.10	.095310	.095455	1.004545	.51594	.56641	3.14	.75547	9.35	1.06450
1.15	.139762	.140217	1.009783	.58995	.64331	4.58	.83357	13.52	1.09732
1.20	.182322	.183333	1.016667	.65507	.71028	5.94	.89906	17.35	1.13051
1.25	.223144	.225000	1.025000	.71490	.77129	7.22	.95724	20.82	1.16400
1.30	.262364	.265385	1.034615	.77118	.82829	8.42	1.01064	23.96	1.19781
1.35	.300105	.304630	1.045370	.82489	.88241	9.55	1.06073	26.78	1.23191
1.40	.336472	.342857	1.057143	.87668	.93433	10.61	1.10841	29.33	1.26627
1.45	.371564	.380172	1.069828	.92694	.98456	11.59	1.15433	31.61	1.30086
1.50	.405465	.416667	1.083333	.97599	1.03340	12.52	1.19890	33.67	1.33570
1.60	.470004	.487500	1.112500	1.07124	1.12792	14.20	1.28505	37.20	1.40601
1.70	.530628	.555882	1.144118	1.16362	1.21927	15.67	1.36849	40.08	1.47708
1.80	.587787	.622222	1.177778	1.25387	1.30830	16.96	1.45020	42.46	1.55123
1.90	.641854	.686842	1.213158	1.34250	1.39561	18.10	1.53067	44.44	1.62114
2.0	.693147	.750000	1.250000	1.42980	1.48157	19.11	1.61036	46.10	1.69397
2.5	.916291	1.050000	1.450000	1.85391	1.89894	22.69	2.00292	51.44	2.06407
3.0	1.098613	1.333333	1.666667	2.26619	2.30553	24.79	2.39240	54.18	2.44105
4.0	1.386294	1.875000	2.125000	3.07484	3.10588	27.00	3.17118	56.80	3.20605
5.0	1.609438	2.400000	2.600000	3.87402	3.89894	28.06	3.95163	57.98	3.97883
10.0	2.302585	4.950000	5.050000	7.82840	7.84191	29.51	7.86040	59.50	7.88022



(a) z'-plane.

(b) ζ_1-plane; $\beta = 0°$.

(c) ζ_2-plane; $\beta = 90°$.

(d) ζ-plane

Figure 1. - The properties of initial transformation.

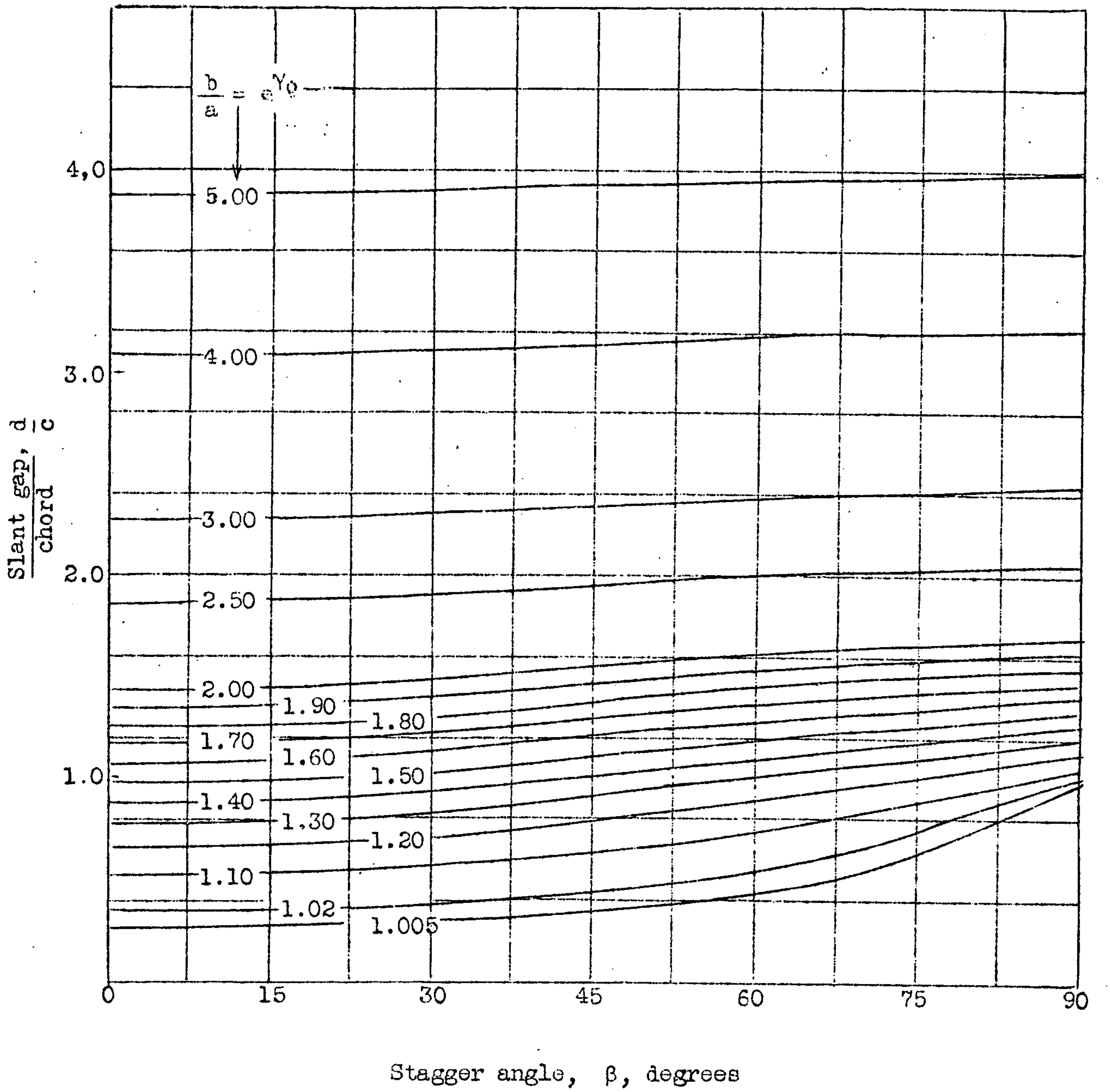


Figure 2.- Gap-chord ratio against stagger angle for various values of $\frac{b}{a} = e^{\gamma_0}$

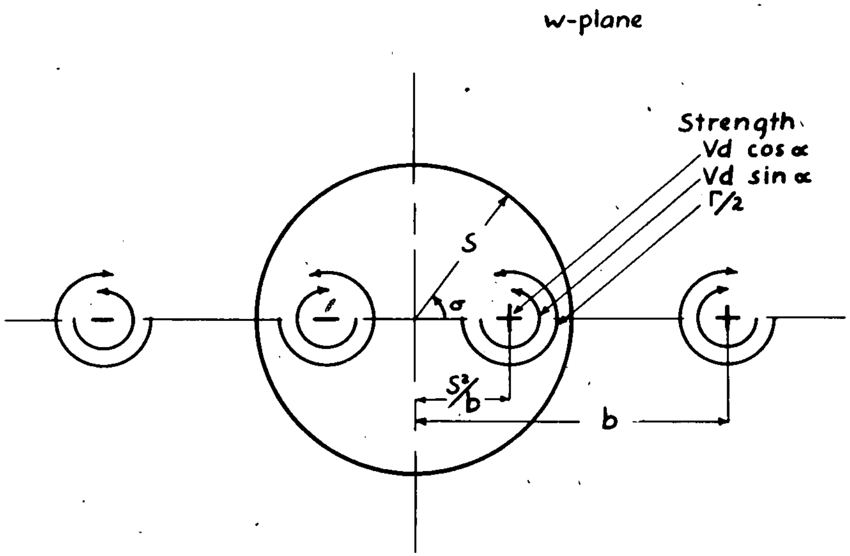


Figure 3. - Flow singularities in standard w-plane.

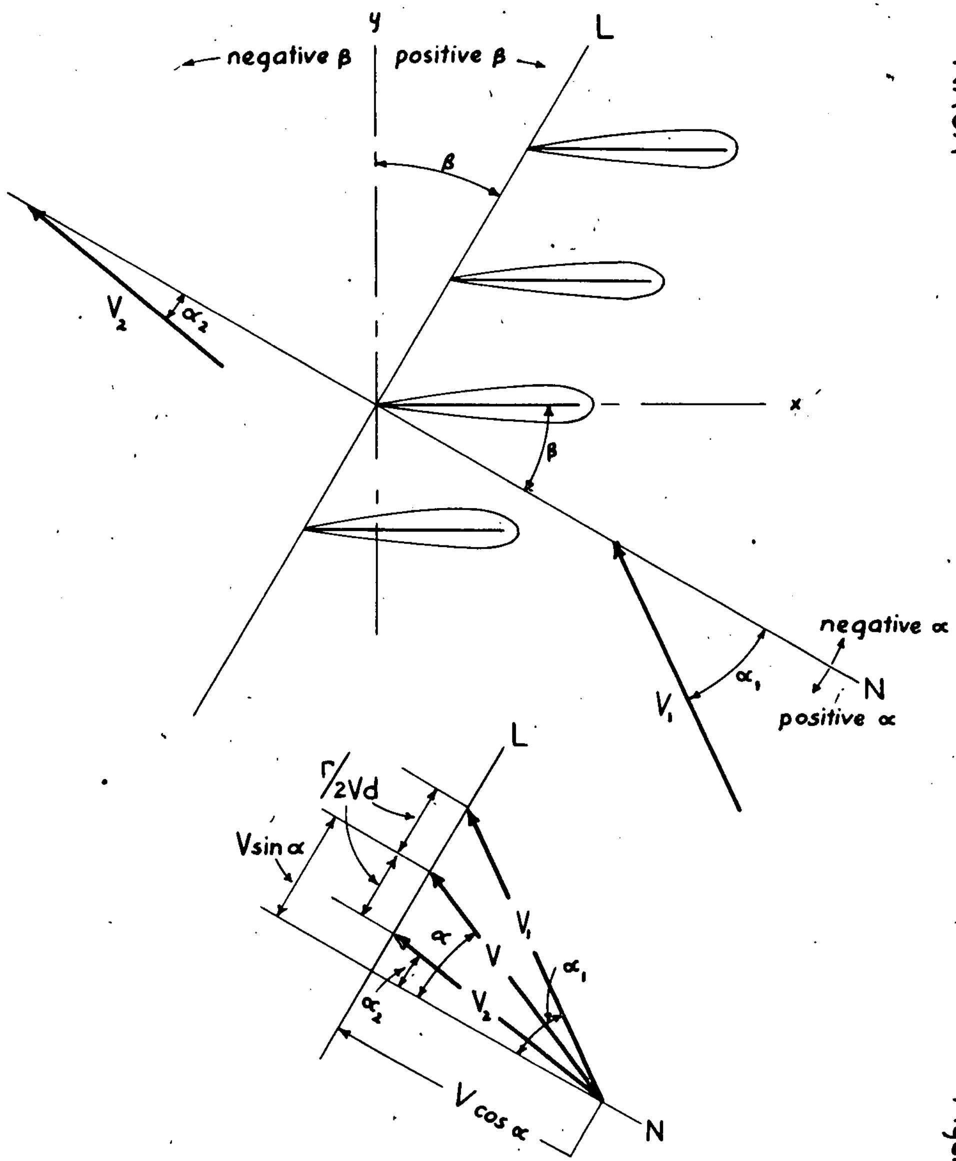


Figure 4. - Inlet, outlet, and mean velocity vectors and angles of attack.

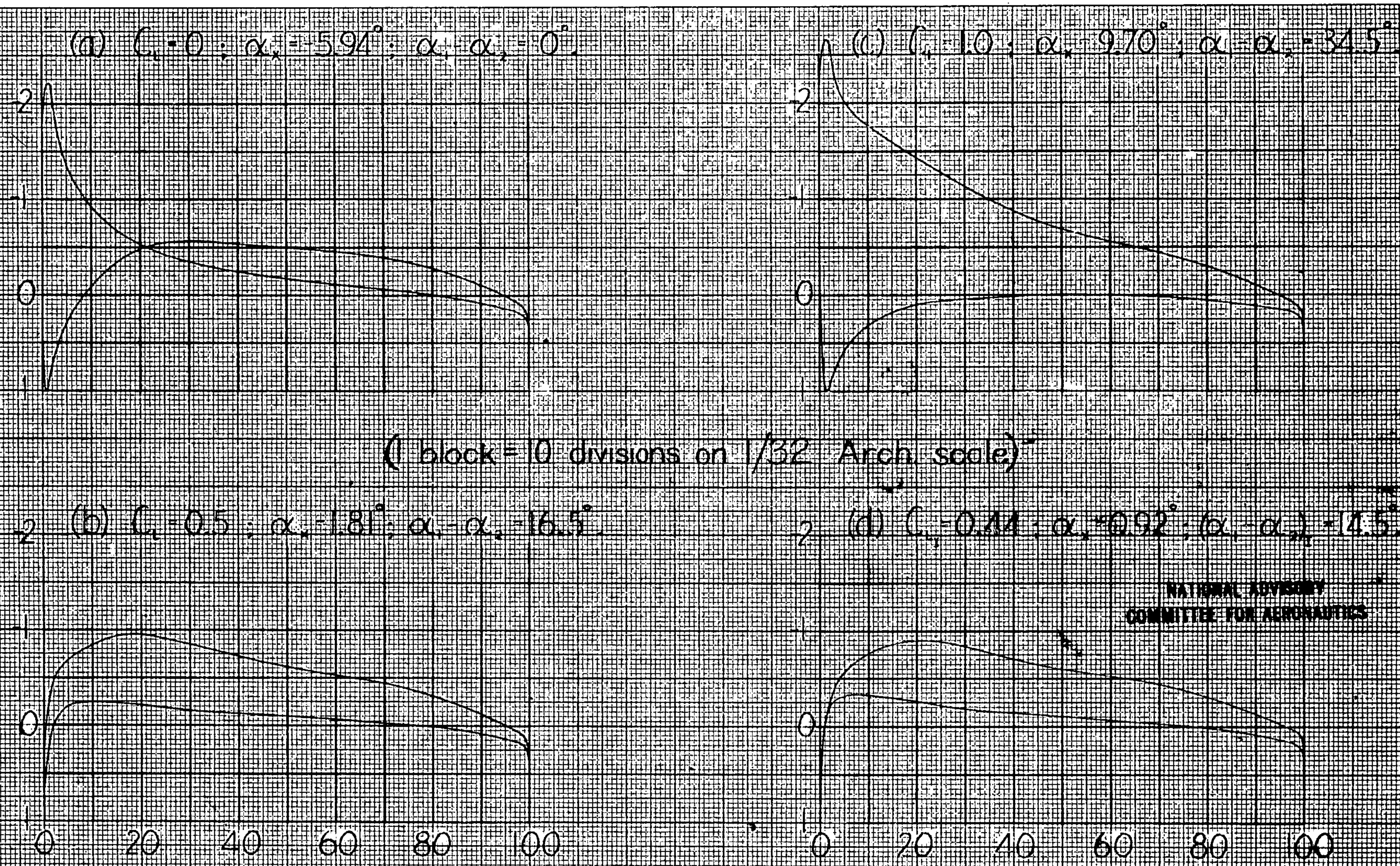


Figure 5.- Pressure distribution for NACA 4412 airfoil in a lattice arrangement.
Stagger angle $\beta = 0°$; gap-chord ratio, 0.968 ; slope of lift curve $dC_L/d\alpha = 3.71$

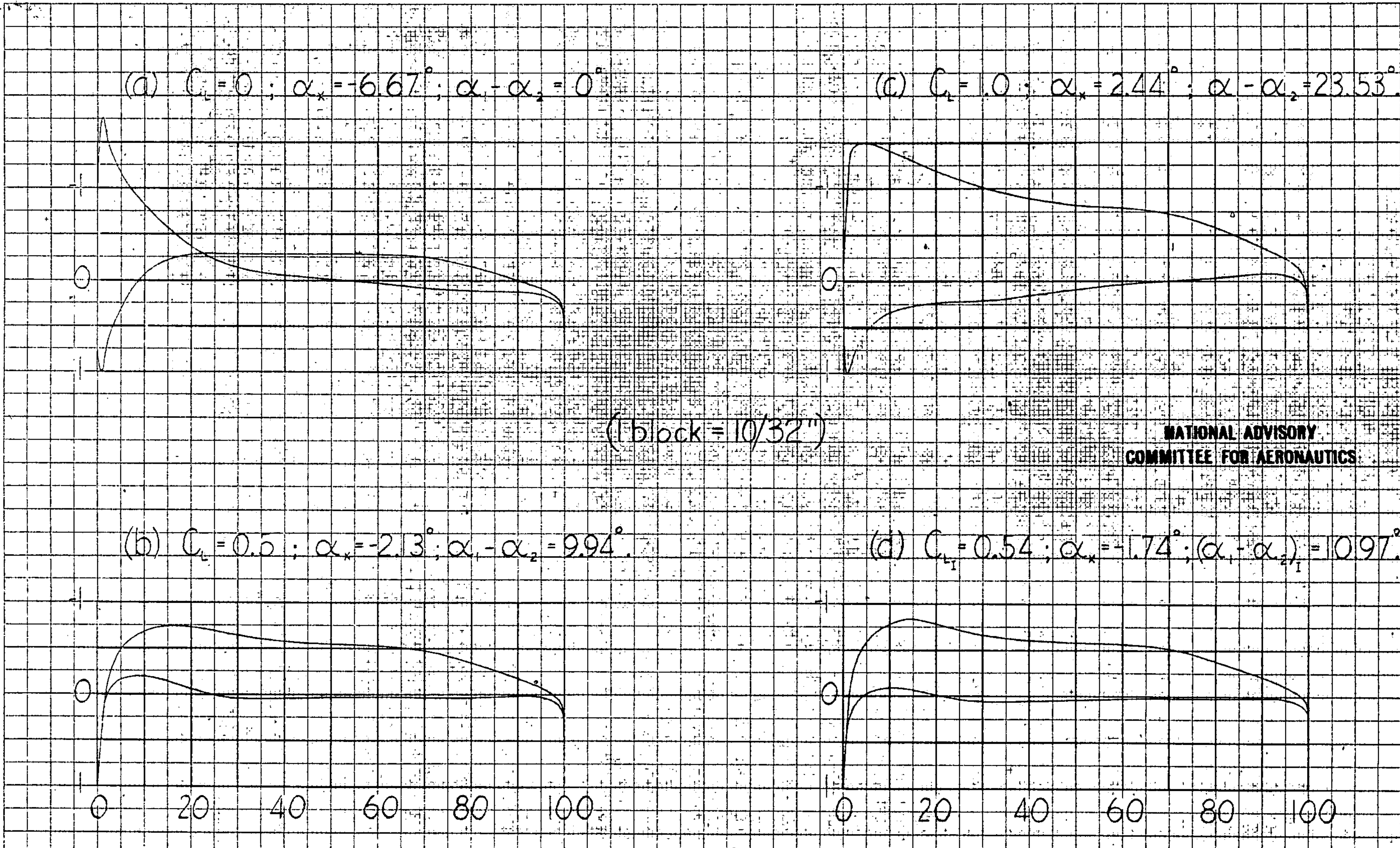


Figure 6.- Pressure distribution for NACA 4412 airfoil in a lattice arrangement.
Stagger angle $\beta = 45°$; gap-chord ratio, 1.096; slope of lift curve $dC_L/d\alpha = 6.31$.

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Fig. 7

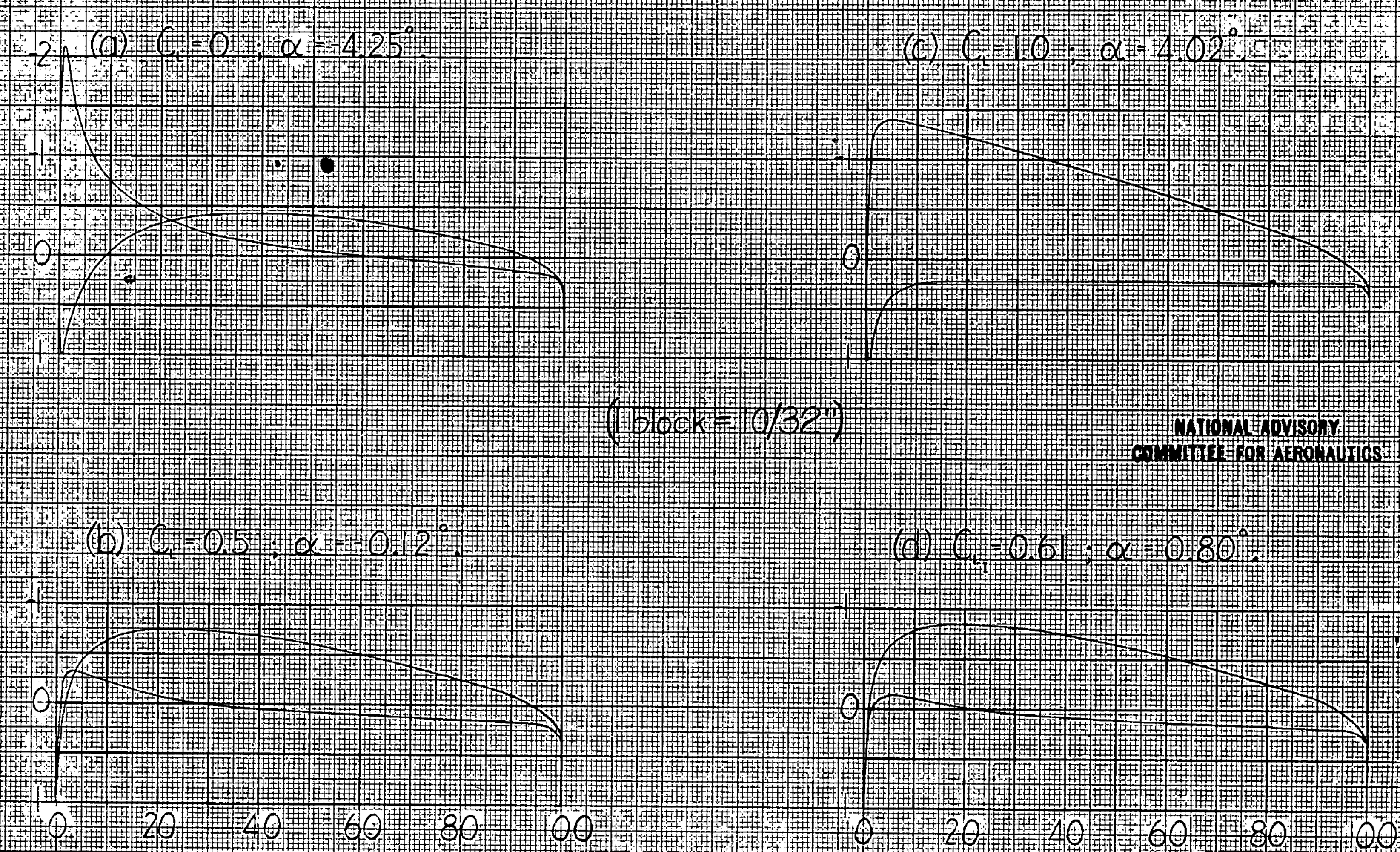

Figure 7.- Pressure distribution for NACA 4412 airfoil. Single-airfoil case; gap-chord ratio, infinite; slope of lift curve $dC_L/d\alpha = 6.95$.