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CRITICAL SHEAR STRESS OF AN INFINITELY LONG FLAT PLATE WITH EQUAL ELASTIC RESTRAINTS AGAINST ROTATION ALONG THE PARALLEL EDGES

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SUMMARY

A chart for the values of the coefficient in the formula for the critical shear stress at which buckling may be expected to occur in an infinitely long flat plate with parallel edges is presented. The plate is assumed to have supported edges with equal elastic restraints against rotation along their length. The mathematical derivations of the formulas required for the construction of the chart are given in two appendixes.

An approximate method is presented for the evaluation of the critical shear stress when the elastic restraints on the two parallel edges are not equal. Approximate methods are also given for the evaluation of the critical shear stress for a plate of finite length with the same restraint against rotation along all edges. It is pointed out that, if a plate is longer than five times its width, the error that is involved in assuming it to be infinitely long for the purpose of calculating its critical shear strength is less than 10 percent and is on the conservative side.

INTRODUCTION

In the design of stressed-skin structures for aircraft, it is often necessary to evaluate the critical stress at which buckling occurs. One of the loading conditions for which the buckling stress for a flat plate has not been adequately evaluated theoretically is that of shear forces in the plane of the plate. In reference 1, Southwell and Skan evaluated theoretically the critical shear stress for an infinitely long flat sheet with two conditions of the edges, namely, that of simply supported and fixed edges.

In the present paper, the theoretical work of Southwell and Skan has been extended to provide a solution for the critical shear stress for an infinitely long flat plate having parallel supported edges with equal elastic restraints against rotation along their length. The theoretical work for this solution is presented in Appendix A. An energy solution, presented in Appendix B, was used in conjunction with the exact solution of Appendix A in the numerical evaluation of the shear stress; the final results of this mathematical work are presented in a design chart which is discussed in the following sections of this report.

EVALUATION OF THE CRITICAL SHEAR STRESS

Within the elastic range.- Within the elastic range, in which the effective modulus of elasticity is equal to Young's modulus, the critical shear stress τ_{cr} for a thin flat rectangular plate is expressed (reference 2, p. 359) as:

$$\tau_{cr} = \frac{k_s \pi^2 E t^2}{12(1-\mu^2) b^2} \qquad (1)$$

in which

k_s nondimensional coefficient that depends upon conditions of edge restraint and relative dimensions of plate

E Young's modulus

t thickness of plate

μ Poisson's ratio

b width of plate

<u>Beyond the elastic range</u>.- When a thin flat plate is stressed in shear beyond the elastic range, the effective modulus of elasticity for the plate is less than Young's modulus. If a single, over-all effective plate modulus $\eta_s E$ is substituted for Young's modulus E, the critical stress, when the material of the plate is loaded beyond the elastic range, can be obtained from equation (1). The nondimensional coefficient η_s has a value that lies between zero and unity and is determined by the stress.

For stresses within the elastic range, $\eta_s = 1$; for stresses beyond the elastic range, experimental research is needed to establish properly the variation of η_s with stress. For stresses beyond the elastic range, equation (1) cannot be used to solve for τ_{cr} directly because the value of τ_{cr} must be known before the value of $\eta_s E$, which replaces E, can be determined. If the equation is divided by η_s, however, τ_{cr}/η_s is given directly by the geometrical dimensions of the plate, Young's modulus, and Poisson's ratio. Thus

$$\frac{\tau_{cr}}{\eta_s} = \frac{k_s \pi^2 E t^2}{12(1-\mu^2) b^2} \qquad (2)$$

A possible relationship between τ_{cr} and τ_{cr}/η_s for 24S-T aluminum alloy is given in figure 1. This figure was prepared from compression-test data on Z-section columns, presented in reference 3, by use of the affinity relations between tension and shear stress-strain curves developed in reference 4. The curve plotted in figure 1 was obtained from figure 8 of reference 3 in accordance with these affinity relations by use of the equations

$$\tau_{cr} = \frac{1}{2} \sigma_{cr}$$

$$\frac{\tau_{cr}}{\eta_s} = \frac{1}{2} \frac{\sigma_{cr}}{\eta}$$

where σ_{cr} is the critical compressive stress for local buckling of the Z-section column and σ_{cr}/η is computed from equation (1) of reference 5. The coefficient has the same significance regarding compression of plates in reference 5 as η_s has regarding shear of plates in this report.

It is not known whether the use of data from column tests gives a true indication of the probable action of thin plates under shear. The relationship shown in figure 1 is, however, believed to be slightly conservative. In the absence of data on thin plates in shear, therefore, the curve of figure 1 can be used as a guide in estimating critical shear stresses for 24S-T aluminum alloy.

EVALUATION OF k_s FOR PLATE OF INFINITE LENGTH

The value of τ_{cr}/η_s at which buckling occurs is given by equation (2), in which all of the quantities are known except the value of the coefficient k_s. The values of k_s can be obtained from figure 2. In this figure, k_s is plotted against λ/b, the ratio of the half wave length of the buckle to the width of the plate, for different values of a parameter ϵ denoting the restraint coefficient and defined by the following equations:

$$\text{Within the elastic range} \quad \epsilon = \frac{4S_o b}{D} \tag{3}$$

$$\text{Beyond the elastic range} \quad \epsilon = \frac{4S_o b}{\eta_s D} \tag{4}$$

In these equations, D is the flexural stiffness of the plate per unit length $\frac{Et^3}{12(1-\mu^2)}$ and S_o is the stiffness per unit length of the elastic restraining medium at the edge of the plate or the moment required to rotate a unit length of the medium through one-fourth radian.

The solution of the differential equation for the critical shear stress of the infinitely long plate reveals that, when the plate buckles, the moments and the rotations at an edge of the plate vary sinusoidally along the edge of the plate and are in phase with each other. In order to evaluate S_o for a stiffener, plate, or other restraining structure, the conditions of continuity of rotations and equilibrium of moments between plate and restraining structure as well as the conditions of internal equilibrium within the restraining structure must be satisfied. These conditions of continuity and equilibrium require that the differential equation of equilibrium of the restraining structure for a sinusoidally distributed applied moment give a sinusoidally distributed rotation of the restraining structure in phase with the moment. Structures of uniform section, such as stiffeners and plates, satisfy the foregoing requirement. In such cases, the ratio of the moment per unit length at any point to the rotation at that point in quarter radians is S_o.

When the elastic restraining structure consists of a sturdy stiffener, defined as a stiffener of such proportions that it does not suffer cross-sectional distortion when moments are applied to some part of the cross section, the value of S_o is, from reference 6,

$$S_o = \frac{\pi^2}{4\lambda^2}\left(\tau_2 GJ - \sigma I_p + \frac{\pi^2}{\lambda^2}\tau_1 EC_{BT}\right) \qquad (5)$$

where

GJ torsional rigidity of stiffener

σ uniformly distributed compressive stress in stiffener

I_p polar moment of inertia of stiffener sectional area about axis of rotation

C_{BT} torsion-bending constant of stiffener sectional area about axis of rotation at or near edge of plate

τ_1, τ_2 nondimensional coefficients equal to or less than unity that take into account the effect of stress in the stiffener on the bending and shear moduli of the stiffener

It is not likely that any stiffener will be under a compressive stress σ when attached to a plate that is under shear only. In such cases, σ in equation (5) is zero, and both τ_1 and τ_2 are equal to unity.

The evaluation of S_0 for other types of elastic restraining structural elements is a subject for further theoretical study.

In general, S_0 will vary with the half-wave length λ. If, however, the elastic restraining medium is such that rotation at one point does not affect rotation at another point, the value of S_0 is independent of λ. In this case, the value of λ/b that the plate assumes upon buckling will be the value for which τ_{cr} is a minimum. This minimum value of τ_{cr} will be obtained if the minimum value of k_s for the particular value of ϵ established by S_0 is used in equations (1) or (2).

As stated previously, S_0 is not usually independent of λ/b for the structural elements that provide elastic restraint against rotation along the edges of the plate. (See equation (6) for the sturdy stiffener.) In this more usual case, a series of values of λ/b must be assumed, from which the corresponding value of S_0, ϵ, and τ_{cr} may be found, until a value of λ/b is found that gives a minimum value for τ_{cr}.

Beyond the elastic range, where ϵ is defined by equation (4), the value of τ_{cr} must be known before ϵ can be evaluated, because η_s is a function of τ_{cr}. As the problem is to determine τ_{cr}, a trial-and-error method of solution must obviously be employed. In order to facilitate such a solution a curve showing the variation of η_s with τ_{cr} has been included in figure 1.

If S_0 is zero, ϵ is also zero and the condition of simple support or of zero restraint is obtained. If S_0 is infinite, ϵ is also infinite, and the condition of a fixed edge or of infinite restraint is obtained.

APPROXIMATE EVALUATION OF CRITICAL SHEAR

STRESS FOR SPECIAL CASES

Infinitely long plate with unequal restraints against rotation along the two parallel edges.- The chart of figure 2 was drawn on the assumption that equal restraints exist along the edges. When the two edge restraints are not equal, a value of k_s is found for each of the values of restraint by the method used for equal restraints. The average of the two values of k_s thus obtained should be a reasonably good approximation of the true value of k_s. This average may be either the arithmetic mean $(k_{s_1} + k_{s_2})/2$, or the geometric mean $\sqrt{k_{s_1} k_{s_2}}$. As the geometric mean will always give a slightly lower value for k_s, it will be more conservative to use the geometric mean that the arithmetic mean.

The foregoing method for establishing an approximate value of k_s for an infinitely long plate with unequal restraints along the edges is suggested by the fact that application of this method to plates under edge compression with unequal elastic restraints along two edges gives critical stresses which are less than 3 percent in error. (See reference 7.) It therefore appears reasonable to assume that a somewhat similar procedure should also give good results for plates under shear.

Beyond the elastic range it is more conservative to average the critical shear stresses than to average the corresponding values of k_s and, with this average, to compute the critical shear stress. This average of the critical shear stress should be the geometric average.

<u>Rectangular plate of finite length with equal restraints along all four edges</u>.- For a rectangular plate of finite length supported, or supported and restrained against rotation at both the side and the end edges, the critical shear stress is greater than for a plate of the same width but of infinite length. An approximate value of k_s for a flat plate of finite length with equal restraint on the four edges can be obtained from the product of the critical shear stress for an infinitely long plate and the factor K read from the curves of figure 3. The two curves in this figure for $\epsilon = 0$ and $\epsilon = \infty$ were obtained from numerical values given in references 2 and 8, respectively. These curves are not exact, but they are good engineering approximations. The fact that the two curves for $\epsilon = 0$ and $\epsilon = \infty$ so nearly coincide indicates that, for engineering use, the lower of the two curves can be used for any intermediate value of ϵ without excessive conservatism.

From figure 3, it may be concluded that if a plate is longer than five times its width ($b/a < 0.2$), the error that is involved in assuming it to be infinitely long for the purpose of calculating its critical shear strength is less than 10 percent and is on the conservative side.

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APPENDIX A

SOLUTION BY DIFFERENTIAL EQUATION

The exact solution for the critical stress at which buckling occurs in a flat rectangular plate subjected to a shear force in its own plane may be obtained by solving the differential equation which expresses the equilibrium of the buckled plate. The plate is assumed to be infinitely long, and equal elastic restraints against rotation are assumed to be present along the two edges of the plate.

Figure 4 shows the coordinate system used. The differential equation for equilibrium of a plate element is (reference 2, p. 305):

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + 2\tau t\frac{\partial^2 w}{\partial x \partial y} = 0 \qquad \text{(A-1)}$$

where

w deflection of plate at (x,y) from unstressed position

τ uniformly distributed shear stress

D flexural stiffness of plate per unit length $\left[\frac{Et^3}{12(1-\mu^2)}\right]$

t thickness of plate

E modulus of elasticity

μ Poisson's ratio

It is known that the formula for the critical shear stress τ is of the form (reference 2, p. 359)

$$\tau = \frac{k_s \pi^2 E t^2}{12(1-\mu^2) b^2} \qquad \text{(A-2)}$$

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where k_s is a constant to be determined. Substitution of equation (A-2) for τ in equation (A-1) and elimination of the constant factor D gives the equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{2\pi^2 k_s}{b^2}\frac{\partial^2 w}{\partial x \partial y} = 0 \qquad \text{(A-3)}$$

If the plate is infinitely long in the x-direction, the disturbing effects due to the local conditions at the ends will not effect the deflection at any particular point, and the deflection may be taken to be periodic in x. It is therefore assumed that

$$w = Ye^{i\frac{\pi x}{\lambda}} \qquad \text{(A-4)}$$

where λ is the half wave length of the buckles, and Y is a function of y only. If expression (A-4) is substituted into equation (A-3), and the variable $z = y/b$ is introduced, there results an equation in nondimensional form which defines the function Y:

$$\frac{d^4 Y}{dz^4} - 2\left(\frac{\pi b}{\lambda}\right)^2 \frac{d^2 Y}{dz^2} + \left(\frac{\pi b}{\lambda}\right)^4 Y + 2i\left(\frac{\pi b}{\lambda}\right)\pi^2 k_s \frac{dY}{dz} = 0 \qquad \text{(A-5)}$$

Equation (A-5) is satisfied by

$$Y = e^{imz}$$

provided

$$\left[m^2 + \left(\frac{\pi b}{\lambda}\right)^2\right]^2 - 2\left(\frac{\pi b}{\lambda}\right)\pi^2 k_s m = 0 \qquad \text{(A-6)}$$

Solution of equation (A-6) will result in four values of m, which may be temporarily designated as m_1, m_2, m_3, m_4. Thus

$$Y = Pe^{im_1 z} + Qe^{im_2 z} + Re^{im_3 z} + Se^{im_4 z} \qquad \text{(A-7)}$$

where the coefficients P, Q, R, and S are to be determined from the boundary conditions. The boundary conditions along the two loaded edges are:

$$(Y)_{z=\frac{1}{2}} = 0 \qquad \text{(A-8)}$$

$$(Y)_{z=-\frac{1}{2}} = 0 \qquad \text{(A-9)}$$

$$\left(D\frac{d^2Y}{dz^2}\right)_{z=\frac{1}{2}} = -4S_o b\left(\frac{dY}{dz}\right)_{z=\frac{1}{2}} \qquad \text{(A-10)}$$

$$\left(D\frac{d^2Y}{dz^2}\right)_{z=-\frac{1}{2}} = 4S_o b\left(\frac{dY}{dz}\right)_{z=-\frac{1}{2}} \qquad \text{(A-11)}$$

where S_o is the stiffness per unit length of the elastic restraining medium along the edges, or the moment required to rotate a unit length of this medium through 1/4 radian.

If the conditions given in equations (A-8) to (A-11) are imposed upon equation (A-7), four equations in P, Q, R, and S result as follows:

$$Pe^{i\frac{m_1}{2}} + Qe^{i\frac{m_2}{2}} + Re^{i\frac{m_3}{2}} + Se^{i\frac{m_4}{2}} = 0 \qquad \text{(A-12)}$$

$$Pe^{-i\frac{m_1}{2}} + Qe^{-i\frac{m_2}{2}} + Re^{-i\frac{m_3}{2}} + Se^{-i\frac{m_4}{2}} = 0 \qquad \text{(A-13)}$$

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$$m_1{}^2Pe^{i\frac{m_1}{2}} + m_2{}^2Qe^{i\frac{m_2}{2}} + m_3{}^2Re^{i\frac{m_3}{2}} + m_4{}^2Se^{i\frac{m_4}{2}}$$

$$-i\epsilon\left(m_1Pe^{i\frac{m_1}{2}} + m_2Qe^{i\frac{m_2}{2}} + m_3Re^{i\frac{m_3}{2}} + m_4Se^{i\frac{m_4}{2}}\right) = 0 \qquad \text{(A-14)}$$

$$m_1{}^2Pe^{-i\frac{m_1}{2}} + m_2{}^2Qe^{-i\frac{m_2}{2}} + m_3{}^2Re^{-i\frac{m_3}{2}} + m_4{}^2Se^{-i\frac{m_4}{2}}$$

$$+ i\epsilon\left(m_1Pe^{-i\frac{m_1}{2}} + m_2Qe^{-i\frac{m_2}{2}} + m_3Re^{-i\frac{m_3}{2}} + m_4Se^{-i\frac{m_4}{2}}\right) = 0 \qquad \text{(A-15)}$$

where

$$\epsilon = \frac{4S_0b}{D}$$

The quantity ϵ specifies the degree of fixity along the edges and is termed the "restraint coefficient."

The buckled form of equilibrium of the plate becomes possible when the determinant formed by the coefficients of P, Q, R, and S in equations (A-12), (A-13), (A-14), and (A-15) equals zero, that is,

$$\begin{vmatrix} e^{i\frac{m_1}{2}} & e^{-i\frac{m_1}{2}} & (m_1{}^2 - i\epsilon m_1)e^{i\frac{m_1}{2}} & (m_1{}^2 + i\epsilon m_1)e^{-i\frac{m_1}{2}} \\ e^{i\frac{m_2}{2}} & e^{-i\frac{m_2}{2}} & (m_2{}^2 - i\epsilon m_2)e^{i\frac{m_2}{2}} & (m_2{}^2 + i\epsilon m_2)e^{-i\frac{m_2}{2}} \\ e^{i\frac{m_3}{2}} & e^{-i\frac{m_3}{2}} & (m_3{}^2 - i\epsilon m_3)e^{i\frac{m_3}{2}} & (m_3{}^2 + i\epsilon m_3)e^{-i\frac{m_3}{2}} \\ e^{i\frac{m_4}{2}} & e^{-i\frac{m_4}{2}} & (m_4{}^2 - i\epsilon m_4)e^{i\frac{m_4}{2}} & (m_4{}^2 + i\epsilon m_4)e^{-i\frac{m_4}{2}} \end{vmatrix} = 0$$

or

$$\left[(m_1^2-m_2^2)(m_3^2-m_4^2)+\epsilon^2(m_1-m_2)(m_3-m_4)\right]\sin\frac{m_1-m_3}{2}\sin\frac{m_2-m_4}{2}$$

$$-\left[(m_1^2-m_3^2)(m_2^2-m_4^2)+\epsilon^2(m_1-m_3)(m_2-m_4)\right]\sin\frac{m_1-m_2}{2}\sin\frac{m_3-m_4}{2}$$

$$-\epsilon\left\{\left[m_1^2(m_2-m_3)-m_2^2(m_1-m_3)+m_3^2(m_1-m_2)\right]\sin\frac{m_1-m_4}{2}\cos\frac{m_2-m_3}{2}\right.$$

$$-\left[m_1^2(m_2-m_4)-m_2^2(m_1-m_4)+m_4^2(m_1-m_2)\right]\sin\frac{m_1-m_3}{2}\cos\frac{m_2-m_4}{2}$$

$$\left.+\left[m_1^2(m_3-m_4)-m_3^2(m_1-m_4)+m_4^2(m_1-m_3)\right]\sin\frac{m_1-m_2}{2}\cos\frac{m_3-m_4}{2}\right\} = 0 \quad \text{(A-16)}$$

The coefficients m_1, m_2, m_3, and m_4 are solutions of equation (A-6), which is of the form

$$f(u) = u^n + a_1 u^{n-1} + a_2 u^{n-2} + \ldots a_{n-1} u + a_n = 0$$

The sum of the products of the roots of this equation, taken r at a time, is $(-1)^r a_r$. Thus, from equation (A-6),

$$\left.\begin{aligned}
&m_1 + m_2 + m_3 + m_4 = 0 \\
&m_1m_2 + m_1m_3 + m_1m_4 + m_2m_3 + m_2m_4 + m_3m_4 = 2\left(\frac{\pi b}{\lambda}\right)^2 \\
&m_1m_2m_3 + m_1m_2m_4 + m_1m_3m_4 + m_2m_3m_4 = 2\pi^2\left(\frac{\pi b}{\lambda}\right)k_s \\
&m_1m_2m_3m_4 = \left(\frac{\pi b}{\lambda}\right)^4
\end{aligned}\right\} \quad \text{(A-17)}$$

From the first of equations (A-17) it appears that the sum of the roots is zero. It is therefore possible to substitute in place of m_1, m_2, m_3, and m_4 the expressions

$$\left.\begin{aligned}
\frac{m_1}{2} &= \gamma + \beta \\
\frac{m_2}{2} &= \gamma - \beta \\
\frac{m_3}{2} &= -\gamma + i\alpha \\
\frac{m_4}{2} &= -\gamma - i\alpha
\end{aligned}\right\} \qquad \text{(A-18)}$$

Upon substitution of the values of (A-18) in equations (A-16) and (A-17) there is obtained for the stability criterion

$$\begin{aligned}
&2\alpha\beta\left(4\gamma^2 - \frac{\epsilon^2}{4}\right)(\cosh 2\alpha \cos 2\beta - \cos 4\gamma) \\
&\quad - \left[4\gamma^2(\beta^2-\alpha^2) - (\beta^2+\alpha^2)^2 - (4\gamma^2-\beta^2+\alpha^2)\frac{\epsilon^2}{4}\right] \sinh 2\alpha \sin 2\beta \\
&\quad + \epsilon\Big[\alpha(4\gamma^2+\alpha^2+\beta^2) \cosh 2\alpha \sin 2\beta \\
&\quad + \beta(4\gamma^2-\alpha^2-\beta^2) \sinh 2\alpha \cos 2\beta - 4\alpha\beta\gamma \sin 4\gamma\Big] = 0 \qquad \text{(A-19)}
\end{aligned}$$

and for the relations between α, β, γ, and the parameters,

$$\alpha^2 - \beta^2 - 2\gamma^2 = \frac{1}{2}\left(\pi\frac{b}{\lambda}\right)^2 \qquad \text{(A-20a)}$$

$$\gamma(\alpha^2+\beta^2) = \frac{\pi^2}{8}\left(\pi\frac{b}{\lambda}\right) k_s \qquad \text{(A-20b)}$$

$$(\alpha^2+\gamma^2)(\beta^2-\gamma^2) = -\frac{1}{16}\left(\pi\frac{b}{\lambda}\right)^4 \qquad \text{(A-20c)}$$

Inspection of equation (A-19) discloses that the value of the left-hand side is independent of the sign of γ. In equations (A-20a), (A-20b), and (A-20c) which define α, β, and γ, only in equation (A-20b) does γ occur to the first power. As the sign of γ makes no difference in equation (A-19), it is evident that no difference will result in the stability criterion if the sign of k_s is reversed. The physical significance of this result is that the critical shear stress for the plate is independent of the direction of the applied shear force.

The defining equations (A-20a) and (A-20c) may be solved for α and β in terms of γ and b/λ for purposes of computation; thus,

$$\alpha = \sqrt{\gamma^2 + \frac{1}{4}\left(\frac{\pi b}{\lambda}\right)^2 + 2\sqrt{\gamma^2\left[\gamma^2 + \frac{1}{4}\left(\frac{\pi b}{\lambda}\right)^2\right]}} \qquad \text{(A-21)}$$

$$\beta = \sqrt{-\gamma^2 - \frac{1}{4}\left(\frac{\pi b}{\lambda}\right)^2 + 2\sqrt{\gamma^2\left[\gamma^2 + \frac{1}{4}\left(\frac{\pi b}{\lambda}\right)^2\right]}} \qquad \text{(A-22)}$$

The following procedure was employed to compute the exact values of k_s for selected values of b/λ and ϵ: By use of an assumed value of γ, the values of α and β were calculated from equations (A-21) and (A-22). Values of α, β, and γ were then substituted in the stability criterion (A-19) together with the selected value of ϵ. If the criterion was not satisfied, the process was repeated until a value of γ was found (generally by plotting the results of previous computations) that would satisfy equation (A-19). When the criterion was finally satisfied, a consistent set of values of α, β, and γ was then available which, when substituted into equation (A-20), determined k_s:

$$k_s = \frac{8\gamma(\alpha^2+\beta^2)}{\pi^2 \frac{\pi b}{\lambda}} \qquad \text{(A-23)}$$

Values of k_s determined from this procedure are marked (c) in columns (b) of table I.

For the extreme cases of simply-supported edges ($\epsilon = 0$) and of clamped edges ($\epsilon = \infty$), equation (A-16) reduces to the following forms:

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For $\epsilon = 0$,

$$(m_1{}^2-m_2{}^2)(m_3{}^2-m_4{}^2)\sin\frac{m_1-m_3}{2}\sin\frac{m_2-m_4}{2}$$

$$-(m_1{}^2-m_3{}^2)(m_2{}^2-m_4{}^2)\sin\frac{m_1-m_2}{2}\sin\frac{m_3-m_4}{2} = 0$$

For $\epsilon = \infty$,

$$(m_1-m_2)(m_3-m_4)\sin\frac{m_1-m_3}{2}\sin\frac{m_2-m_4}{2}$$

$$-(m_1-m_3)(m_2-m_4)\sin\frac{m_1-m_2}{2}\sin\frac{m_3-m_4}{2} = 0$$

These relations were given by Southwell and Skan (reference 1), whose classic paper furnished the basis for the work described in this appendix.

APPENDIX B

SOLUTION BY ENERGY METHOD

Because the exact solution of the differential equation given in appendix A yields critical values of k_s only with considerable labor, an energy solution was made to aid in the construction of the chart of figure 2. The energy method gives approximate values for k_s, the accuracy of which depends upon how closely the assumed deflection surface describes the true deflection surface.

The energy method as applied to the calculation of critical stresses is given in reference 2 (p. 327). The plate is stable when $V_1 + V_2 > T$ and unstable when $V_1 + V_2 < T$, where T is the work done by the shearing force on the plate, V_1 is the strain energy in the plate, and V_2 is the strain energy in the two elastic restraining mediums along the edges of the plate. The critical stress is obtained from the condition of neutral stability,

$$T = V_1 + V_2 \tag{B-1}$$

When a shear force is applied to a plate in its own plane, the nodal lines are inclined at an angle to the sides of the plate. Advantage is taken of this fact by the use of oblique coordinates as shown in figure 5. The oblique coordinates x, y, are related to the more usual Cartesian coordinates x', y' through the transformation equations,

$$x' = x - y \sin \phi$$

$$y' = y \cos \phi$$

where ϕ is the angle of inclination of the nodal lines. As in appendix A, the plate is assumed to be infinitely long, so that the conditions of restraint at the ends do not matter.

In the oblique coordinate system, the expressions for T, V_1, and V_2 are (see reference 2, pp. 307 and 357, and reference 7, equation (B-4) for equivalent Cartesian expressions)

$$T = \tau t \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \left[\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \left(\frac{\partial w}{\partial x}\right)^2 \sin \phi\right] dxdy$$

$$V_1 = \frac{D}{2\cos\phi} \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} \int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} \left\{\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] \frac{1}{\cos^2\phi} + \left[2(1-\mu) + 4\tan^2\phi\right]\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right.$$
$$\left. + 2(\mu + \tan^2\phi)\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 4 \frac{\partial^2 w}{\partial x \partial y} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \frac{\sin \phi}{\cos^2\phi}\right\} dxdy$$

$$V_2 = \frac{4S_0}{2\cos^2\Phi}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\left[\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\sin\Phi\right]^2_{y=\frac{b_1}{2}} dx$$

$$+ \frac{4S_0}{2\cos^2\Phi}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\left[\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\sin\Phi\right]^2_{y=-\frac{b_1}{2}} dx$$

In order to evaluate T, V_1, and V_2 it is necessary to assume a deflected surface w consistent with the boundary conditions. These conditions specify that along the two edges of the infinitely long plate there be no deflection and equal restraint against rotation.

In a solution of the critical compressive stress for a flat rectangular plate (reference 7) by the energy method, precisely these same boundary conditions were considered. The same deflection equation used in that reference will therefore serve the purposes of this appendix when it is expressed in oblique coordinates. The deflection surface as given in equation (B-9) of reference 7, but now expressed in oblique coordinates, is

$$w = w_0\left[\frac{\pi\epsilon}{2}\left(\frac{y^2}{b_1^2} - \frac{1}{4}\right) + \left(1 + \frac{\epsilon}{2}\right)\cos\frac{\pi y}{b_1}\right]\cos\frac{\pi x}{\lambda} \qquad \text{(B-2)}$$

Equation (B-2) satisfies the boundary condition of no deflection along the edges of the plate and, on the average over the length of the buckle, satisfies the boundary condition of equilibrium of moments along the edges, provided

$$\epsilon = \frac{4S_0 b}{D}$$

By use of this form of the deflection surface to compute T, V_1, and V_2, it is found that

$$T = w_0^2\,\frac{\pi^2 b_1 t \sin\Phi}{2\lambda}\left[\left(\frac{\pi^2}{120} + \frac{1}{8} - \frac{2}{\pi^2}\right)\epsilon^2 + \left(\frac{1}{2} - \frac{4}{\pi^2}\right)\epsilon + \frac{1}{2}\right]\tau \qquad \text{(B-3)}$$

$$V_1 = w_0^2 \frac{\pi^4 D}{4b_1\lambda\cos^3\Phi}\left\{\left(\frac{b_1}{\lambda}\right)^2\left[\left(\frac{\pi^2}{120}+\frac{1}{8}-\frac{2}{\pi^2}\right)\epsilon^2+\left(\frac{1}{2}-\frac{4}{\pi^2}\right)\epsilon+\frac{1}{2}\right]\right.$$

$$+\frac{1}{\left(\frac{b_1}{\lambda}\right)^2}\left[\left(\frac{1}{8}-\frac{1}{\pi^2}\right)\epsilon^2+\left(\frac{1}{2}-\frac{4}{\pi^2}\right)\epsilon+\frac{1}{2}\right]$$

$$\left.+2(1+2\sin^2\Phi)\left[\left(\frac{5}{24}-\frac{2}{\pi^2}\right)\epsilon^2+\left(\frac{1}{2}-\frac{4}{\pi^2}\right)\epsilon+\frac{1}{2}\right]\right\} \quad \text{(B-4)}$$

$$V_2 = w_0^2\frac{\pi^2 D\lambda\epsilon}{2b_1^3\cos^3\Phi} \quad \text{(B-5)}$$

It is permissible to substitute the values obtained from equations (B-3), (B-4), and (B-5) in equation (B-1) only when the shear stress τ has its critical value τ_{cr}. From this substitution,

$$\tau_{cr} = \frac{k_s\pi^2 E t^2}{12(1-\mu^2)b^2}$$

where

$$k_s = \frac{1}{\sin 2\Phi}\left[\frac{1}{\left(\frac{\lambda}{b}\right)^2\cos^2\Phi}+C_1\left(\frac{\lambda}{b}\right)^2\cos^2\Phi+C_2(1+2\sin^2\Phi)\right] \quad \text{(B-6)}$$

$$C_1 = \frac{\left(\frac{1}{8}-\frac{1}{\pi^2}\right)\epsilon^2+\left(\frac{1}{2}-\frac{2}{\pi^2}\right)\epsilon+\frac{1}{2}}{\left(\frac{\pi^2}{120}+\frac{1}{8}-\frac{2}{\pi^2}\right)\epsilon^2+\left(\frac{1}{2}-\frac{4}{\pi^2}\right)\epsilon+\frac{1}{2}}$$

$$C_2 = 2\,\frac{\left(\frac{5}{24}-\frac{2}{\pi^2}\right)\epsilon^2+\left(\frac{1}{2}-\frac{4}{\pi^2}\right)\epsilon+\frac{1}{2}}{\left(\frac{\pi^2}{120}+\frac{1}{8}-\frac{2}{\pi^2}\right)\epsilon^2+\left(\frac{1}{2}-\frac{4}{\pi^2}\right)\epsilon+\frac{1}{2}}$$

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The coefficient k_s thus is a function of λ/b, of ϵ, and of the unknown angle Φ. For given values of λ/b and ϵ, the angle of inclination Φ will adjust itself to the value that will make k_s a minimum. If the derivative $\frac{\partial k_s}{\partial \Phi}$ is set equal to zero, it is found that

$$\cos \Phi = \sqrt{C_3 + \sqrt{C_3^{\,2} + C_4}}$$

$$C_3 = \frac{\frac{3}{2}C_2 - \frac{2}{\left(\frac{\lambda}{b}\right)^2}}{4C_2 + C_1\left(\frac{\lambda}{b}\right)^2}$$

$$C_4 = \frac{\frac{3}{\left(\frac{\lambda}{b}\right)^2}}{4C_2 + C_1\left(\frac{\lambda}{b}\right)^2}$$

The angle Φ is thus determined as soon as values of λ/b and of ϵ have been selected. This value of Φ is to be used in equation (B-6) for the determination of k_s.

Equation (B-6) was used to calculate the values of k_s listed in the columns designated (a) of table I. With these values of k_s as a guide, a number of correct values of k_s were obtained by satisfying equation (A-19) of appendix A. In this manner, the errors in k_s as given by equation (B-6) were established at regular intervals. From this knowledge of the errors, correction were made to all the values of k_s given in columns (a) of table I. These corrected values of k_s, which are recommended, are listed in the columns designated (b) of table I and were used in the construction of figure 2.

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TABLE I

VALUES OF k_s IN THE EQUATION FOR CRITICAL SHEAR STRESS FOR AN INFINITELY LONG FLAT PLATE WITH EQUAL RESTRAINTS ALONG THE PARALLEL EDGES

ϵ \ $\frac{\lambda}{b}$	0.5		0.6		0.7		0.8		0.9		1.0	
	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
0	9.839	9.634	8.048	7.803[c]	7.021	6.759	6.404	6.126[c]	6.032	5.748	5.813	5.523[c]
.5	-----	-----	8.108	7.876	7.100	6.853	6.505	6.243	6.155	5.888	5.962	5.687
1	9.922	9.724	8.165	7.943	7.175	6.940	6.599	6.351	6.270	6.016	6.097	5.835
1.5	-----	-----	8.220	8.006	7.246	7.021	6.687	6.451	6.375	6.132	6.221	5.969
2	10.003	9.808	8.273	8.066[c]	7.312	7.095	6.769	6.543[c]	6.474	6.239	6.336	6.093[c]
3	------	-----	8.372	8.175	7.435	7.230	6.919	6.707	6.651	6.432	6.542	6.314
4	10.147	9.952	8.460	8.271	7.546	7.350	7.051	6.849	6.807	6.599	6.722	6.507
5	------	------	8.538	8.354	7.644	7.455	7.169	6.975	6.944	6.745	6.879	6.675
6	10.271	10.072	8.614	8.433	7.733	7.548	7.275	7.087	7.067	6.875	7.019	6.824
8	10.377	10.172[c]	8.745	8.565[c]	7.889	7.709	7.456	7.275[c]	7.276	7.095	7.256	7.076[c]
10	10.466	10.256	8.852	8.674	8.017	7.842	7.607	7.431	7.460	7.287	7.450	7.281
12	------	------	8.945	8.766	8.126	7.953	7.733	7.563	7.592	7.425	7.611	7.448
16	------	------	9.094	8.912	8.301	8.132	7.943	7.783	7.809	7.650	7.864	7.708
20	10.768	10.523	9.208	9.020	8.433	8.266	8.084	7.933	7.989	7.836	8.053	7.901
30	------	------	9.404	9.194[c]	8.657	8.486	8.338	8.206[c]	8.273	8.134	8.368	8.222[c]
40	11.028	10.722	9.526	9.302	8.797	8.619	8.496	8.375	8.449	8.320	8.563	8.422
60	11.169	10.853	9.672	9.435	8.962	8.775	8.681	8.570	8.654	8.535	8.788	8.652
80	11.243	10.926	9.755	9.514	9.056	8.864	8.786	8.676	8.771	8.654	8.916	8.782
100	11.290	10.977	9.809	9.567	9.116	8.922	8.853	8.738	8.845	8.725	8.998	8.862
200	11.394	11.126	9.926	9.702[c]	9.248	9.053	9.000	8.825[c]	9.008	8.844	9.176	9.021[c]
∞	11.511	11.247[c]	10.059	9.845	9.397	9.200	9.165	8.989[c]	9.189	9.026	9.374	9.220[c]

ϵ \ $\frac{\lambda}{b}$	1.1		1.2		1.4		1.6		1.8		2.0	
	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
0	-----	-----	5.658	5.343[c]	5.721	5.378	5.907	5.537[c]	6.168	5.768	6.483	6.053[c]
.5	5.872	5.589	5.856	5.562	5.970	5.645	6.207	5.850	6.522	6.137	6.888	6.476
1	-----	-----	6.035	5.758	6.192	5.881	6.475	6.130	6.834	6.463	7.244	6.847
1.5	6.172	5.915	6.197	5.934	6.394	6.096	6.715	6.382	7.113	6.755	7.561	7.178
2	-----	-----	6.346	6.054[c]	6.577	6.290	6.932	6.610[c]	7.364	7.017	7.846	7.475[c]
3	6.539	6.308	6.610	6.375	6.899	6.631	7.312	7.010	7.800	7.475	8.339	7.991
4	6.748	6.529	6.837	6.614	7.178	6.925	7.633	7.350	8.169	7.863	8.753	8.426
5	6.920	6.711	7.035	6.822	7.398	7.159	7.911	7.646	8.485	8.198	9.109	8.801
6	7.077	6.877	7.209	7.004	7.620	7.393	8.154	7.906	8.761	8.491	9.417	9.126
8	7.342	7.155	7.502	7.309[c]	7.968	7.762	8.556	8.338[c]	9.210	8.970	9.929	9.668[c]
10	-----	-----	7.739	7.554	8.248	8.056	8.879	8.682	9.584	9.366	10.337	10.099
12	-----	-----	7.935	7.755	8.479	8.295	9.153	8.967	9.884	9.680	10.672	10.449
16	-----	-----	8.241	8.069	8.837	8.663	9.556	9.381	10.347	10.158	11.187	10.984
20	-----	-----	8.469	8.303	9.104	8.935	9.860	9.688	10.690	10.509	11.567	11.377
30	-----	-----	8.846	8.695[c]	9.543	9.382	10.359	10.188[c]	11.251	11.085	12.190	12.029[c]
40	8.783	8.643	9.076	8.935	9.810	9.656	10.655	10.486	11.591	11.435	12.567	12.421
60	9.029	8.898	9.344	9.213	10.119	9.972	11.015	10.848	11.985	11.838	13.002	12.871
80	9.168	9.040	9.495	9.368	10.284	10.141	11.213	11.048	12.205	12.062	13.247	13.121
100	-----	-----	9.591	9.464	10.404	10.262	11.338	11.175	12.347	12.204	13.402	13.274
200	-----	-----	9.800	9.655[c]	10.645	10.472	11.611	11.461[c]	12.651	12.491	13.739	13.567[c]
∞	-----	-----	10.034	9.885	10.913	10.772	11.914	11.768	12.989	12.838	14.112	13.950[c]

[a]Values obtained from the energy method.

[b]Recommended values.

[c]Values obtained from the exact solution of the differential equation

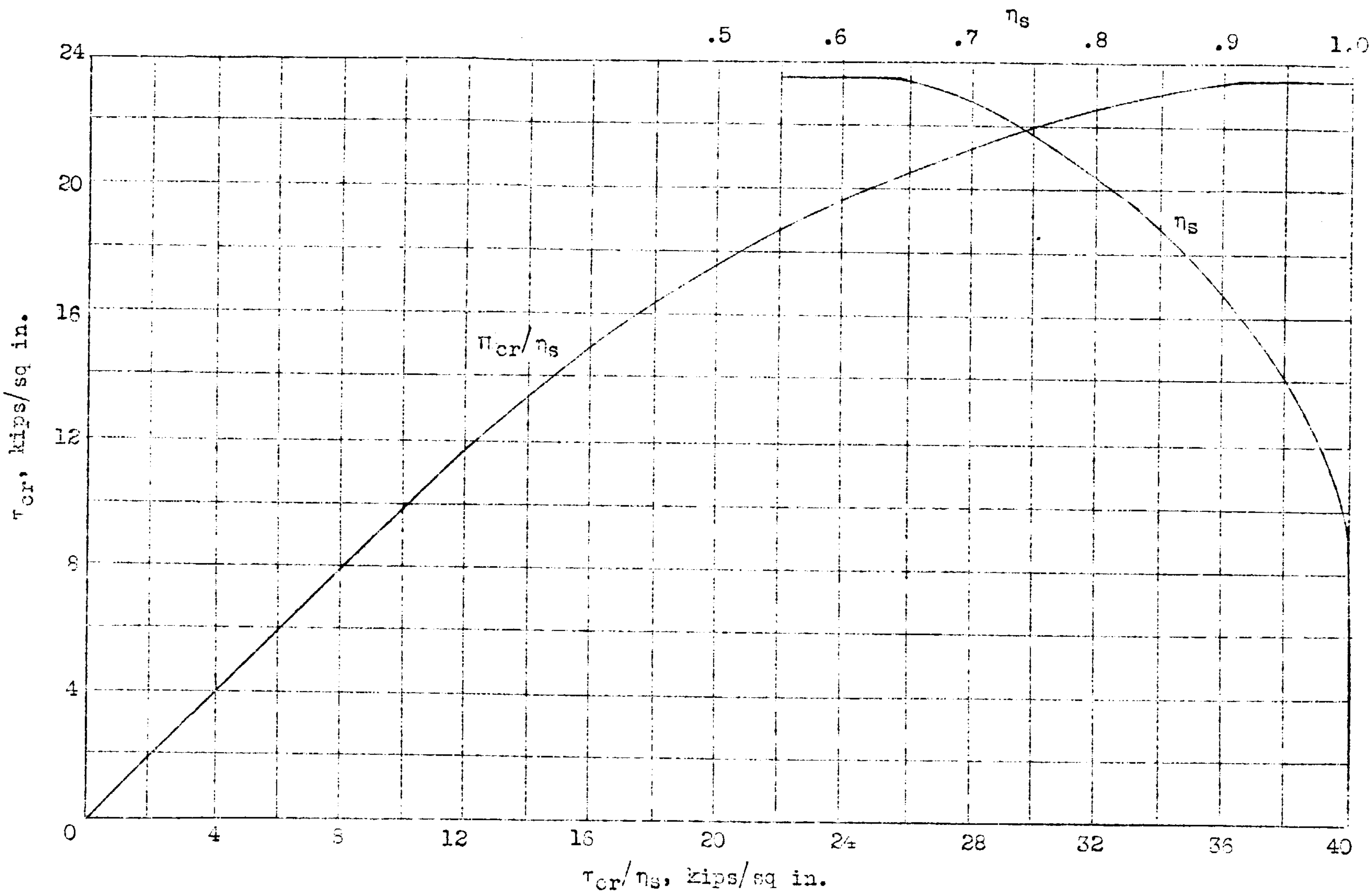

Figure 1.- Tentative curve of τ_{cr} against τ_{cr}/η_s. Data from reference 3.

Fig. 1

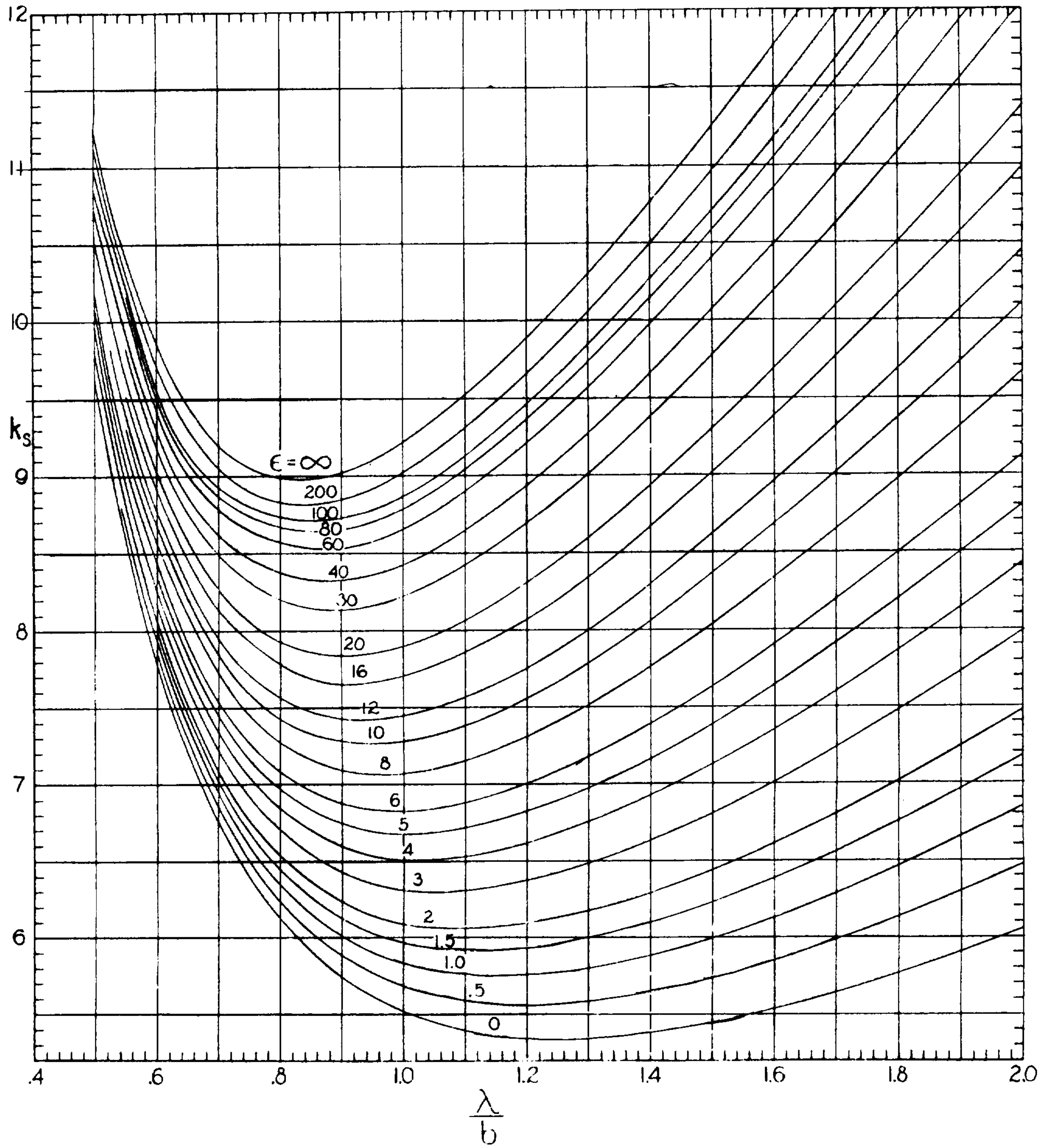

Figure 2.-Chart giving values of k_s in equation for critical shear stress for an infinitely long flat plate with equal restraints along the parallel edges

$$\frac{\tau_{cr}}{\eta_s} = \frac{k_s \pi^2 E t^2}{12(1-\mu^2)b^2}$$

(1 block = 10 divisions on 1/50" Engr. scale)

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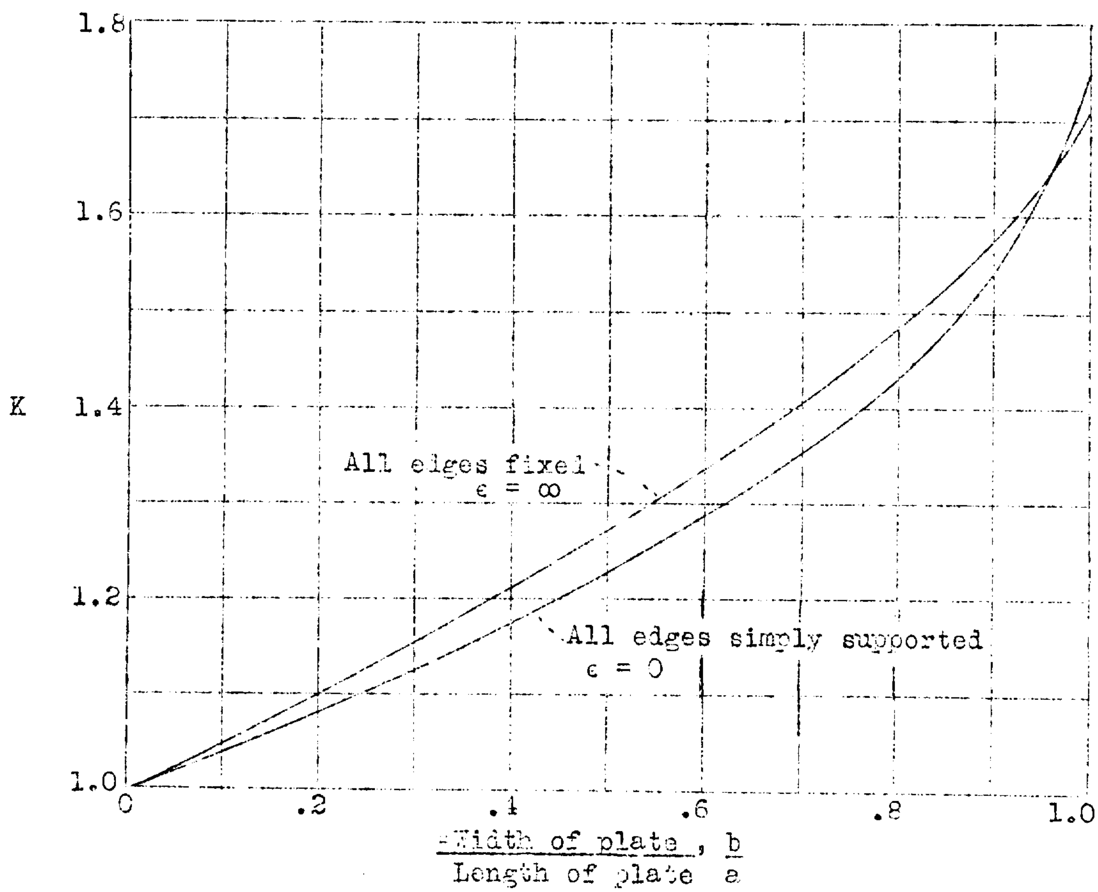

Figure 3.- Variation of K with ratio b/a.

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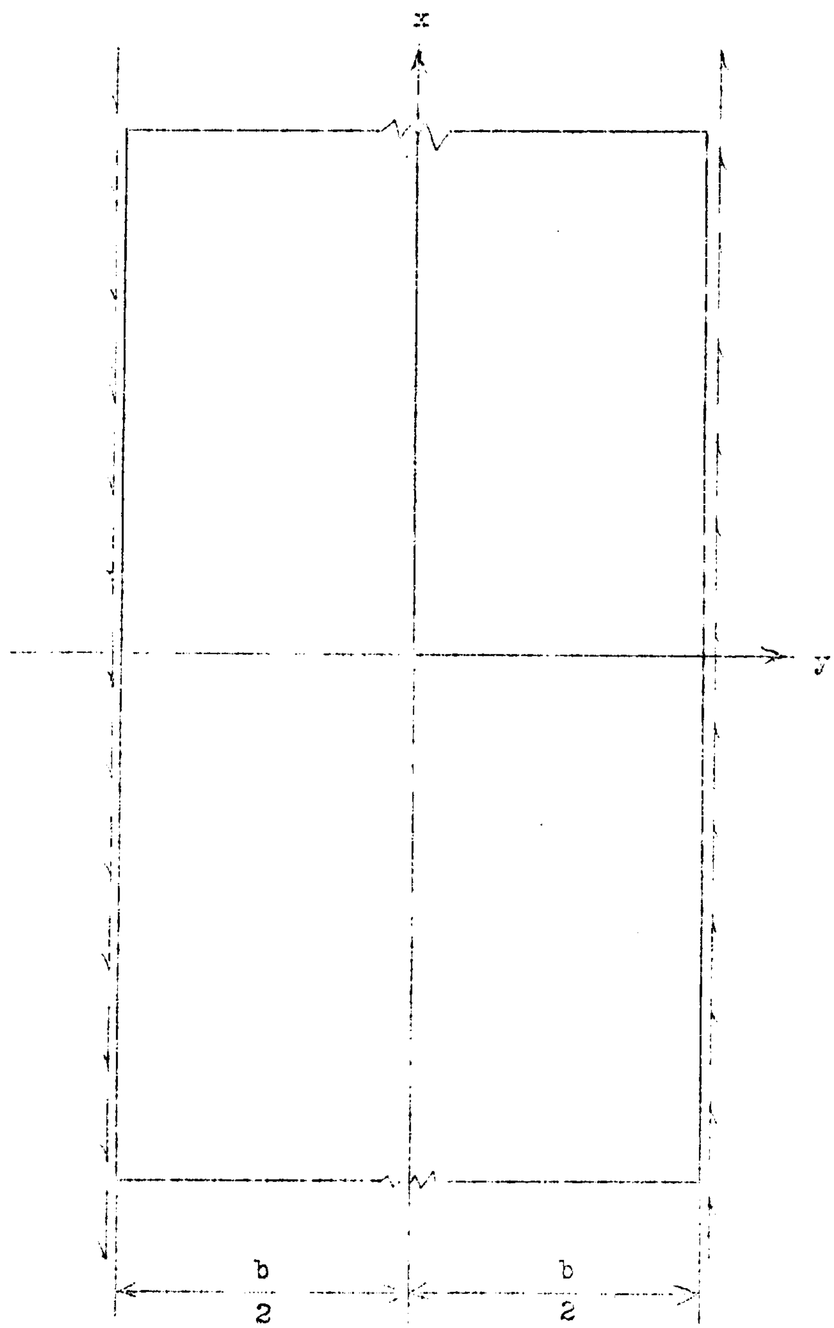

Figure 4.- Infinitely long rectangular plate under shear; coordinate system used in appendix A.

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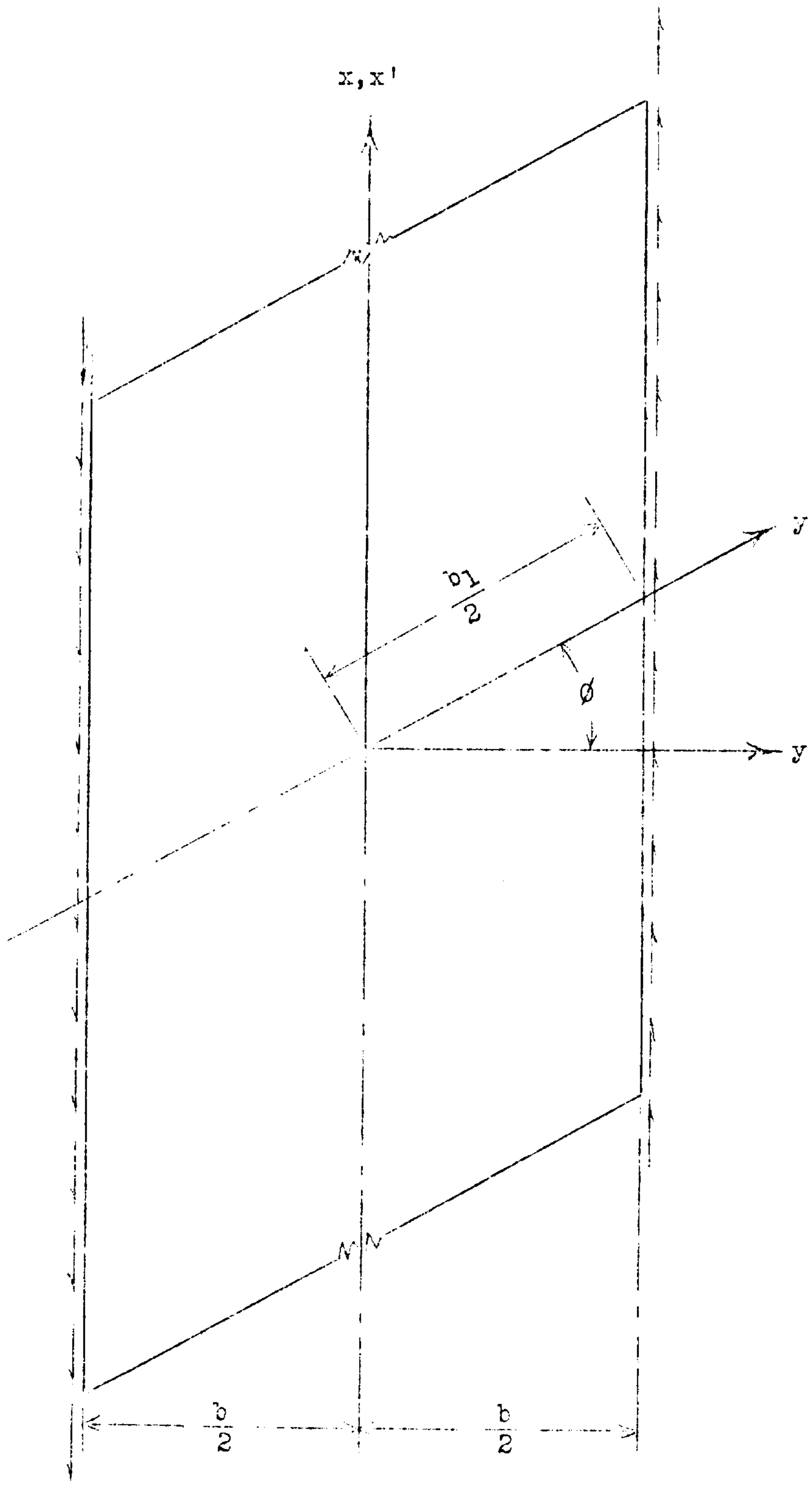

Figure 5.- Oblique coordinate system used in appendix B.