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TECHNICAL MEMORANDUM 1379

A METHOD OF QUADRATURE FOR CALCULATION OF THE LAMINAR AND TURBULENT BOUNDARY LAYER IN CASE OF PLANE AND ROTATIONALLY SYMMETRICAL FLOW

By E. Truckenbrodt

Translation of "Ein Quadraturverfahren zur Berechnung der laminaren und turbulenten Reibungsschicht bei ebener und rotationssymmetrischer Strömung."
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1. SUMMARY

For calculation of the characteristic parameters of the boundary layer (momentum-loss thickness and form parameter for the velocity profile), two quadrature formulas are given which are valid for the laminar as well as for the turbulent state of flow. These formulas cover both the two-dimensional and the rotationally symmetrical case.

The calculation of the momentum-loss thickness is carried out by a simple integration of the energy theorem. The equation for the form parameter is obtained by coupling of the momentum theorem with the energy theorem. Knowledge of the derivatives of the velocity distribution and of the radius of the body along the length x is not necessary.

2. INTRODUCTION

The calculation of the laminar and turbulent boundary layer in case of pressure drop and pressure rise is the decisive problem in the determination of the flow loss in channels and pipes and of the flow drag of bodies. We shall treat below the plane problem as well as the rotationally symmetrical one. In these considerations, we shall limit ourselves to incompressible flows.

Whereas theoretical treatment of the laminar boundary layer has been fundamentally clarified, one is still dependent on semiempirical connections for the treatment of the turbulent boundary layer. Since the exact methods in the laminar case are rather troublesome, various approximation methods have been developed just like for the turbulent case. All these methods are based on the momentum equation given for the first time by Th. von Kármán (ref. 1).

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*"Ein Quadraturverfahren zur Berechnung der laminaren und turbulenten Reibungsschicht bei ebener und rotationssymmetrischer Strömung." Ingenieur-Archiv, Band XX, Viertes Heft, 1952, pp. 16-228.

The approximation method for the plane laminar case has been perfected by K. Pohlhausen (ref. 2). Later on, it was essentially improved by H. Holstein and T. Bohlen (ref. 3): they introduced as the desired parameter not the boundary-layer thickness as Pohlhausen had done but the momentum-loss thickness. More recently, K. Wieghardt (ref. 4) derived an energy equation in addition to the momentum equation which he uses for developing a two-parameter calculation method. A. Walz (ref. 5) simplified this latter method by reverting to the one-parameter condition as in the methods of Pohlhausen and Holstein-Bohlen. The rotationally symmetrical method which is analogous to the Pohlhausen method was indicated by S. Tomotika (ref. 6). Its simplification in the sense of the Holstein-Bohlen statement was carried out by F. W. Scholkemeyer (ref. 7).

A method for calculation of the plane turbulent boundary layer was indicated for the first time by E. Gruschwitz (ref. 8). Aside from the momentum theorem, Gruschwitz also uses a semiempirical equation obtained from certain energy considerations for determination of a form parameter marking the velocity profile in the boundary layer; this form parameter characterizes the sensitivity to separation of the boundary layer. The empirical parameters appearing in Gruschwitz' were investigated once more by A. Kehl (ref. 9). Another method which was similar to Gruschwitz' method was developed by E. Buri (ref. 10). In the United States, a method by A. E. von Doenhoff and N. Tetervin (ref. 11) proved to be useful. In this method, too, an empirical equation, in addition to the momentum theorem, is used for determination of a form parameter characterizing the velocity profile. Starting from this report, H. C. Garner (ref. 12), England, developed a method which is superior to that of v. Doenhoff-Tetervin in its numerical evaluation. The transfer of the momentum theorem to the rotationally symmetrical case was performed by C. B. Millikan (ref. 13).

Owing to recent investigations by H. Ludwieg and W. Tillmann (ref. 14) and J. Rotta (ref. 15) on the theoretical properties of turbulent flows, particularly of the wall shear stress and the energy loss in the boundary layer, it is possible to find a better basis for and to improve the existing semiempirical methods.

It is the aim of the present report to develop a calculation method which is equally valid for the four cases of laminar and turbulent as well as plane and rotationally symmetrical flow. The most recent results will be taken into consideration.

3. THE MOMENTUM THEOREM AND ENERGY THEOREM

As is well known, Prandtl's boundary-layer equations and Bernoulli's equation represent the fundamental equations for calculation of boundary layers. As stated in the Introduction, we are going to consider incompressible flows. Since, however, the following derivations can be given for variable density ρ without particular difficulties, we shall make the transfer to incompressible flow only in the final result.

We assume that, in the rotationally symmetrical case, the radius of the body or pipe R is large compared to the thickness of the boundary layer δ. The equations mentioned then read

$$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = - \frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \tag{1}$$

$$\frac{\partial(\rho u R)}{\partial x} + \frac{\partial(\rho v R)}{\partial y} = 0 \tag{2}$$

$$\frac{\partial p}{\partial x} = - \rho_a U \frac{dU}{dx} \tag{3}$$

Therein, u and v signify the velocity components within the boundary layer

$$0 \leqq y \leqq \delta$$

in x and y direction, U the velocity outside of the boundary layer $(y \geq \delta)$, figure 1. p is the pressure, assumed constant across the boundary-layer thickness, τ the shear stress, and ρ the density of the flowing medium. For the plane (two-dimensional) case, the radius R is to be omitted in equation (2) as well as in all following formulas.

The pertaining boundary conditions are

$$\left.\begin{aligned} y = 0 : u = 0, \quad & v = v_0, \quad \tau = \tau_0, \quad \rho = \rho_0 \\ y = \delta : u = U, \quad & \qquad\quad\ \tau = 0, \quad \rho = \rho_a \end{aligned}\right\} \tag{4}$$

If $v_0 \neq 0$, one is dealing with the case of suction $(v_0 < 0)$ or of blowing $(v_0 > 0)$.

We now combine the three equations (1) to (3) by adding to equation (1) - which has been multiplied by u^r - the equation (2) which has been multiplied by $u^{r+1}/(r + 1)R$; one may choose arbitrarily $r = 0, 1, 2. . . .$ One then obtains

$$\frac{1}{R}\frac{\partial}{\partial x}(\rho u^{r+2}R) + \frac{\partial}{\partial y}(\rho u^{r+1}v) = (r + 1)u^r\left(\rho_a U \frac{dU}{dx} + \frac{\partial \tau}{\partial y}\right) \tag{5}$$

The continuity equation (2) is integrated over the distance y from the wall, and one obtains

$$\rho v = - \frac{1}{R}\frac{\partial}{\partial x}\int_0^y \rho uR \, dy + \rho_0 v_0 \tag{6}$$

If we now assume the velocity distribution $u(y)$ and the shear-stress distribution $\tau(y)$ in y direction to be known, we may integrate equation (5) over y from $y = 0$ to $y = \delta$. We then obtain, if we, moreover, substitute equation (6)

$$\frac{1}{\rho_a U^{r+2}R}\frac{d}{dx}(\rho_a U^{r+2}Rf_r) + \frac{g_r}{U}\frac{dU}{dx} - \frac{\rho_0}{\rho_a}\frac{v_0}{U} = e_r \tag{7}$$

Therein, the newly introduced abbreviations signify

$$\left.\begin{aligned}
f_r &= \int_0^\delta \frac{\rho}{\rho_a}\frac{u}{U}\left[1 - \left(\frac{u}{U}\right)^{r+1}\right]dy \\
g_r &= - (r + 1) \int_0^\delta \frac{u}{U}\left[\frac{\rho}{\rho_a} - \left(\frac{u}{U}\right)^{r-1}\right]dy \\
e_r &= - (r + 1) \int_0^\delta \left(\frac{u}{U}\right)^r \frac{\partial}{\partial y}\left(\frac{\tau}{\rho_a U^2}\right)dy
\end{aligned}\right\} \tag{8}$$

Equation (7) is valid quite generally for laminar and turbulent, incompressible and compressible, plane and rotationally symmetrical flow with and without suction or blowing. As said before, the radius R is to be omitted in the plane case.

If r is put equal to zero, there results the well-known momentum equation of von Kármán. K. Wieghardt (ref. 4) and J. Rotta (ref. 15) have shown that, for $r = 1$, equation (7) may be given a physical interpretation, namely that of the energy equation. We shall now assume that the flowing medium is incompressible, ρ = Constant, and also that neither suction nor blowing occur $v_0 = 0$. Then r becomes

$r = 0$: momentum theorem (v. Kármán, Pohlhausen)

$$\frac{1}{U^2 R} \frac{d}{dx}\left(U^2 R \vartheta\right) + \frac{\delta^*}{U} \frac{dU}{dx} = \frac{\tau_0}{\rho U^2}, \tag{9}$$

with

$$\vartheta = f_0 = \int_0^\delta \frac{u}{U}\left[1 - \frac{u}{U}\right] dy \quad \text{as momentum-loss thickness} \tag{10}$$

$$\delta^* = g_0 = \int_0^\delta \left[1 - \frac{u}{U}\right] dy \quad \text{as displacement thickness} \tag{11}$$

$$\frac{\tau_0}{\rho U^2} = e_0 \quad \text{as wall shear stress}$$

$r = 1$: energy theorem (Wieghardt, Rotta)

$$\frac{1}{U^3 R} \frac{d}{dx}\left(U^3 R \bar{\delta}\right) = 2 \frac{d + t}{\rho U^3} \tag{12}$$

with

$$\bar{\delta} = f_1 = \int_0^\delta \frac{u}{U}\left[1 - \left(\frac{u}{U}\right)^2\right]dy \qquad \text{as energy-loss thickness} \tag{13}$$

$$\frac{d + t}{\rho U^3} = \frac{e_1}{2} = \int_0^\delta \frac{\tau}{\rho U^2} \frac{\partial}{\partial y}\left(\frac{u}{U}\right)dy \qquad \text{as shear-stress work}^1 \tag{14}$$

In the laminar case, the shear-stress work equals the energy converted into heat (dissipation d). In case of turbulent flows, not all energy is converted into heat; one part still remains as turbulence energy (t) which is, however, usually negligibly small. The fact that, in the case $r = 1$, the second term on the left side of equation (7) vanishes is important since, with the density assumed to be constant, $g_1 = 0$.

If we now introduce the boundary-layer thickness ratios

$$H = \frac{\delta^*}{\vartheta} \qquad \text{and} \qquad \bar{H} = \frac{\bar{\delta}}{\vartheta} \tag{15}$$

we may write for equations (9) and (12)

$$\frac{1}{U^2R} \frac{d}{dx}(U^2R\vartheta) + H \frac{\vartheta}{U} \frac{dU}{dx} = \frac{\tau_0}{\rho U^2}, \tag{16}$$

$$\frac{1}{U^3R} \frac{d}{dx}(U^3R\bar{H}\vartheta) = 2 \frac{d + t}{\rho U^3} \tag{17}$$

We subtract equations (16) from (17) and find[2]

[1]This equation is obtained by partial integration from (8).

[2]For the laminar case, a corresponding formula has already been given by A. Walz (ref. 5). In the turbulent case, too, one can show that the empirical equations indicated by E. Gruschwitz, A. E. von Doenhoff and N. Tetervin, and also H. C. Garner for calculation of the form parameter of the velocity profile (in Gruschwitz' paper $\eta = 1 - (u_\vartheta/U)^2$, in those of the others H) have a structure very similar to that of equation (18).

$$\vartheta \frac{d\bar{H}}{dx} = (H - 1)\bar{H} \frac{\vartheta}{U} \frac{dU}{dx} + 2 \frac{d + t}{\rho U^3} - \bar{H} \frac{\tau_0}{\rho U^2} \tag{18}$$

As will be shown in the following section, the shear stress $\tau_0 / \rho U^2$ and the shear-stress work $(d + t) / \rho U^3$ may be expressed as functions of the dimensionless quantities $U\vartheta/\nu$ (Reynolds number formed with the momentum thickness) and H. Furthermore, a one-parameter condition is valid in good approximation for the velocity distributions in the boundary layer; that is, there exists a fixed relation between the boundary-layer thickness ratios $\bar{H}$ and H. With consideration of these facts, the equations (17) and (18) represent two equations for the two unknowns ϑ and $\bar{H}$.

In contrast to the existing methods, we shall calculate the momentum-loss thickness not from the momentum theorem (16) but from the energy theorem (17). This offers decisive advantages for the performance of the integrations indicated in section 4. We resort to the momentum theorem in connection with the energy theorem for calculating the parameter $\bar{H}$, equation (18), which is characteristic for the velocity profile.

4. SHEAR STRESS, SHEAR-STRESS WORK, AND BOUNDARY-LAYER THICKNESS RATIO

In order to be able to operate with the equations (17) and (18), one must know the dependence of the shear stress and the shear-stress work on the quantities $U\vartheta/\nu$ and H, and likewise the connection $\bar{H}(H)$.

(a) Laminar Flow

Assuming single-parameter velocity profiles we may write

$$\frac{u}{U} = f\left(\frac{y}{\vartheta}, H\right)$$

whence then follows

$$\frac{\tau_0}{\rho U^2} = \frac{\alpha_l(H)}{\frac{U\vartheta}{\nu}} \qquad \text{with} \qquad \alpha_l = \left(\frac{\partial \frac{u}{U}}{\partial \frac{y}{\vartheta}}\right)_0 \tag{19}$$

$$\frac{d}{\rho U^3} = \frac{\beta_l(H)}{\frac{U\vartheta}{\nu}} \qquad \text{with} \qquad \beta_l = \int_0^{\delta/\vartheta} \left(\frac{\partial \frac{u}{U}}{\partial \frac{y}{\vartheta}} \right)^2 d\,\frac{y}{\vartheta} \tag{20}$$

and

$$H = \int_0^{\delta/\vartheta} \left[1 - \frac{u}{U}\right] d\,\frac{y}{\vartheta} \qquad \bar{H} = \int_0^{\delta/\vartheta} \frac{u}{U}\left[1 - \left(\frac{u}{U}\right)^2\right] d\,\frac{y}{\vartheta} \tag{21}$$

For evaluation of these formulas, we select the so-called Hartree profiles; these are the exact profiles for the velocity variation $U \sim x^m$. In figure 2, the quantities α, β, $\bar{H}$ are plotted against H [3]. It is of interest that β is almost independent of H. For the case without pressure gradient, one has $\alpha = 0.220$, $\beta = 0.173$, and $H = 2.60$.

(b) Turbulent Flow

Recently, H. Ludwieg and W. Tillmann (ref. 14), as well as J. Rotta (ref. 15), dealt in detail with the determination of the wall-shear stress in case of turbulent boundary layers with pressure gradient.

Ludwieg-Tillmann indicate for the range of the Reynolds numbers $1 \times 10^3 < U\vartheta/\nu < 4 \times 10^4$ the following interpolation formula

$$\frac{\tau_0}{\rho U^2} = \frac{0.123}{\left(\frac{U\vartheta}{\nu}\right)^{0.268}} 10^{-0.678H} \tag{22}$$

Rotta finds for the wall-shear stress the relation

$$\frac{U}{u^*} = \frac{1}{\kappa} \ln\left(H \frac{U\vartheta}{\nu}\right) + B \tag{23}$$

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[3] We took over the numerical values of A. Walz (ref. 5). The following identities are valid: $\alpha_l \equiv \epsilon^*$, $\beta_l \equiv D^*$ and $\bar{H} \equiv H_{32}$.

Therein $u^* = \sqrt{\tau_0/\rho}$ signifies the shear velocity. Thus

$$\frac{\tau_0}{\rho U^2} = \left(\frac{u^*}{U}\right)^2 \tag{24}$$

is valid. In equation (23) $1/\kappa = 2.5$ and $B = B(I_1)$ is a function of the quantity $I_1 = \frac{H - 1}{H} \frac{U}{u^*}$. The function B was evaluated and graphically represented by Rotta.

We calculated the shear-stress values for various values of H and plotted them against the Reynolds number $U\vartheta/\nu$ in figure 3. The agreement between the values according to Ludwieg-Tillmann and to Rotta is quite satisfactory.

J. Rotta also dealt with the calculation of the shear-stress work (dissipation and turbulence energy). For the dissipation, there applies

$$\frac{d}{\rho U^3} = \left(\frac{u^*}{U}\right)^3 \left[\frac{1}{\kappa} \ln\left(H \frac{U\vartheta}{\nu}\right) + G\right] \tag{25}$$

The function appearing therein, $G = G(I_1)$, has been evaluated by Rotta. With consideration of equation (23), one may write for equation (25) also

$$\frac{d}{\rho U^3} = \left(\frac{u^*}{U}\right)^2 \left[1 + \frac{u^*}{U}(G - B)\right] \tag{25a}$$

Since $\frac{u^*}{U} = f\left(\frac{U\vartheta}{\nu}, H\right)$ and $I_1 = \frac{H - 1}{H} \frac{U}{u^*}$, the dissipation may also be represented as a function of the Reynolds number $U\vartheta/\nu$ and of the form parameter H; this has been done in figure 4. It is found that the differences for different values H are only slight; whereas, for the wall-shear stress, they are very considerable. We see from figure 4 that the dissipation may be approximated by the statement

$$\frac{d}{\rho U^3} = \frac{\beta_t(H)}{\left(\frac{U\vartheta}{\nu}\right)^n} \tag{26}$$

A suitable value for n is $n = 1/6$. The dependence of the value β_t on H is only slight so that we may assume for it the constant value $\beta = 0.56 \times 10^{-2}$.

For calculation of the turbulence energy, J. Rotta indicates an approximation formula. With the aid of this formula, one can show that the turbulence energy is negligibly small compared to the dissipation; this fact has already been pointed out by Rotta. We put, therefore, for our further calculations

$$\frac{t}{\rho U^3} = 0 \tag{27}$$

The boundary-layer thickness ratio $\bar{H}$ (energy-loss thickness/momentum-loss thickness) may also be determined from J. Rotta's results. The equation

$$\bar{H} = 3 - H + \frac{I_2}{I_1^{\,2}} \frac{(H - 1)^2}{H} \tag{28}$$

is valid. Therein, I_2 is a form parameter which is in a fixed relation (indicated by Rotta) with I_1. We calculated according to equation (28) the values $\bar{H}$ in dependence on the quantity H and on the Reynolds number $U\vartheta/\nu$. The influence of the Reynolds number was shown to be vanishingly small. The result of our calculation is represented in figure 5. K. Wieghardt (ref. 16) also has dealt with the connection $\bar{H}(H)$. He finds

$$\bar{H} = \frac{AH}{H - B} \qquad \text{with} \qquad A = 1.269 \qquad \text{and} \qquad B = 0.379 \tag{29}$$

The resulting curve also has been plotted in figure 5. Except for the larger values of H, the agreement with Rotta's curves is satisfactory.

5. THE APPROXIMATION METHOD

We shall summarize the result of the previous section: The representations

$$\frac{\tau_0}{\rho U^2} = f\left(\frac{U\vartheta}{\nu}, H\right), \qquad \frac{d + t}{\rho U^3} = \frac{d}{\rho U^3} = g\left(\frac{U\vartheta}{\nu}, H\right) \qquad \text{and} \qquad \bar{H} = h(H)$$

given in figures 2 to 5, are valid. In the laminar case, the values $\tau_0/\rho U^2$ and $d/\rho U^3$ are inversely proportional to the Reynolds number.

(a) The Calculation of the Momentum-Loss Thickness

As mentioned above, we shall determine the momentum-loss thickness from the energy theorem (17).

We write for the shear-stress work according to equations (20), (26), and (27)

$$\frac{d + t}{\rho U^3} = \frac{\beta(H)}{\left(\frac{U\vartheta}{\nu}\right)^n} \tag{30}$$

with $n = 1$ being valid for the laminar, $n = 1/6$ for the turbulent case.

We now substitute this expression into equation (17) and obtain with

$$\chi = U^{3+2n}R^{1+n}\bar{H}^{1+n}\left(\frac{U\vartheta}{\nu}\right)^n \vartheta \tag{31}$$

$$\frac{d\chi}{dx} = 2(1 + n)\beta(H)\bar{H}^n U^{3+2n}R^{1+n} \tag{32}$$

If we now assume that H is known from x and that $\bar{H}$, too, is known by the unique relation $\bar{H}(H)$, equation (32) may be integrated with respect to x, and there results, if we introduce in addition the quantity

$$\Theta = \left(\frac{U\vartheta}{\nu}\right)^n \vartheta \tag{33}$$

and take equation (31) into consideration

$$\Theta(x) = \frac{\chi_1 + \int_{x_1}^{x} E(H)U^{3+2n}R^{1+n}dx'}{\bar{H}(x)^{1+n}U(x)^{3+2n}R(x)^{1+n}} \tag{34}$$

As a new abbreviation we introduced

$$E(H) = 2(1 + n)\beta(H)\bar{H}^n \tag{35}$$

We plotted this function in figure 6 for the two cases of laminar and turbulent flow.

By way of approximation, we shall now assume constant mean values for $E(H)$ and $\bar{H}$. This assumption proves correct in a particularly satisfactory manner for the turbulent case. We then obtain from equation (34)

$$\Theta(x) = \left(\frac{U\vartheta}{\nu}\right)^n \vartheta = \frac{C_1 + A\int_{x_1}^{x} U^{3+2n}R^{1+n}dx'}{U^{3+2n}R^{1+n}} \tag{36}$$

Herein

$$A = \frac{E}{\bar{H}^{1+n}} = 2(1 + n)\frac{\beta}{\bar{H}} \tag{37}$$

signifies a mean value suitable for the range $x_1 < x' < x$.

The integration constant is determined to be

$$C_1 = U_1^{3+2n}R_1^{1+n}\left(\frac{U_1\vartheta_1}{\nu}\right)^n \vartheta_1 \tag{38}$$

with $n = 1/6$ if, starting from the point x_1, a turbulent boundary layer is present. As the laminar or, respectively, turbulent mean value for A, we shall choose the value which results when we assume $U = U_\infty$ to be constant and consider the flat plate for fully laminar or fully turbulent flow. Equation (36) then becomes, with $x_1 = 0$ and $C_1 = 0$

$$\Theta_p = \left(\frac{U_\infty \vartheta_p}{\nu}\right)^n \vartheta_p = Ax \tag{39}$$

Between the momentum-loss thickness $\vartheta_p(l)$ and the drag coefficient c_f of a plate of the length l wetted on one side and approached by a flow of the velocity U_∞ there exists the connection

$$\frac{\vartheta_p(l)}{l} = \frac{c_f}{2} \tag{40}$$

If we put in equation (39) $x = l$, there follows by comparison of equations (39) and (40)

$$A = \left(\frac{U_\infty l}{\nu}\right)^n \left(\frac{c_f}{2}\right)^{1+n} \tag{37a}$$

According to the existing flow state, c_f is to be taken for laminar or turbulent flow (fig. 7). The notation (37a) offers the additional advantage that the surface roughness also can easily be taken into consideration, merely by substitution of the corresponding c_f values of a rough plate. For smooth surfaces, there result the values of table 1 for the constant A.

TABLE 1.- THE QUANTITIES n AND A

FOR LAMINAR AND TURBULENT FLOW

	Laminar (Blasius)	Turbulent (Falkner)[4]
n	1	$\frac{1}{6}$
A	0.441	0.760×10^{-2}

We now solve equation (36) for the momentum thickness ϑ and obtain - if we introduce, in addition, dimensionless quantities and take equation (37a) into consideration - the following expression

$$\frac{\vartheta(x)}{l} = \frac{\left[C_1^* + \left(\frac{c_f}{2}\right)^{1+n} \int_{x_1/l}^{x/l} \left(\frac{U}{U_\infty}\right)^{3+2n} \left(\frac{R}{l}\right)^{1+n} d\,\frac{x'}{l}\right]^{\frac{1}{1+n}}}{\left(\frac{U}{U_\infty}\right)^3 \frac{R}{l}} \tag{41}$$

[4]V. M. Falkner, Aircraft Engineering, 15, 1943, p. 65. $c_f = \frac{0.0306}{\left(\frac{U_\infty l}{\nu}\right)^{1/7}}$ is valid.

The integration constant is determined as

$$C_1{}^* = \left[\left(\frac{U_1}{U_\infty}\right)^3 \frac{R_1}{l} \frac{\vartheta_1}{l}\right]^{1+n} = \left[\frac{c_{fl}}{2}\left(\int_0^{x_1/l} \left(\frac{U}{U_\infty}\right)^5 \left(\frac{R}{l}\right)^2 d\,\frac{x'}{l}\right)^{1/2}\right]^{1+n} \tag{42}$$

Summarizing, we repeat once more: If the flow, starting from the initial point $x_1 = 0$, is fully laminar or fully turbulent, one has $C_1{}^* = 0$. One has to insert accordingly the laminar or turbulent drag coefficients. If a laminar starting length precedes the turbulent boundary layer, the value for ϑ in equation (42) is to be taken from the laminar calculation (second formula in equation (42)). For the laminar case $n = 1$, for the turbulent case $n = 1/6$ is valid. The two-dimensional case results if one omits in equations (41) and (42) the radius R. As can be seen from equation (41), the quantity n does not play a significant role in the turbulent case so that one may assume for rough calculations also $n = 0$.

(b) The Calculation of the Form Parameter

We shall determine the form parameter $\bar{H}$ which is characteristic for the velocity profile in the boundary layer from equation (18); we write this equation as follows:

$$\Theta(x)\,\frac{d\bar{H}}{dx} = F(\bar{H})\Gamma(x) + G(\bar{H}) \tag{43}$$

Here Θ is given according to equation (33). For the other abbreviations

$$\Gamma(x) = \frac{\Theta}{U}\frac{dU}{dx} \tag{44}$$

and

$$\left.\begin{aligned} F(\bar{H}) &= (H - 1)\bar{H} \\ G(\bar{H}) &= \left(2\,\frac{d + t}{\rho U^3} - \bar{H}\,\frac{\tau_0}{\rho U^2}\right)\left(\frac{U\vartheta}{\nu}\right)^n \end{aligned}\right\} \tag{45}$$

are valid. Furthermore, we transform equation (43) by introducing the substitution

$$L(\bar{H}) = \int \frac{d\bar{H}}{F(\bar{H})} = L(H) \tag{46}$$

and the abbreviation

$$K(\bar{H}) = -\frac{G(\bar{H})}{F(\bar{H})} = K(L) \tag{47}$$

We then obtain a differential equation for the new form parameter L

$$\Theta(x)\frac{dL}{dx} = \Gamma(x) - K(L) \tag{48}$$

We determined the quantity L by graphical integration of the function $1/F(\bar{H})$ over $\bar{H}$. The value $L = 0$ we have placed in the domain of vanishing pressure gradients (constant pressure, flat plate). In figures 8 and 9, we represent the relation $H(L)$, also for the laminar case $\alpha(L)$, and, in figures 10 and 11, the relation $K(L)$ for the cases of laminar and turbulent flow. Whereas no influence of the Reynolds number exists for the function $K(L)$ in the laminar case, that influence is rather considerable for the turbulent case. One can show that it is possible to give to the equations for calculation of the form parameter appearing in the reports of E. Gruschwitz, A. Kehl, and H. C. Garner the form of our equation (48)[5]. The resulting connections for $H(L)$ and $K(L)$ have also been plotted in the figures named above. The differences between the individual methods are therefore based on the deviations of the curves $H(L)$ and $K(L)$.

For solution of equation (48), we set up a linear expression for $K(L)$, (compare the figures 10 and 11):

$$K(L) = a(L - b) \tag{49}$$

The quantities a and b are obtained, for instance, from table 2. In the turbulent case, b is, in addition, dependent on the Reynolds number and therewith on the length x.

[5] A detailed comparison of the individual methods has been carried out in an unpublished report of the author.

If we substitute equation (49) into equation (48), we obtain a linear differential equation of the first order for L which we can solve in closed form. For this purpose, we make the substitution

$$\xi = e^{a\int\frac{dx}{\Theta(x)}} = \left(C_1 + A\int_{x_1}^{x} U^{3+2n}R^{1+n}dx'\right)^{\frac{a}{A}} \qquad (50)$$

The last relation follows from the fact that in formula (36) for Θ the denominator is exactly equal to the derivative of the numerator multiplied by $1/A$. The numerical values a/A for smooth surfaces are also plotted in table 2.

Without dealing in detail with the intermediate calculation - for which we perform, in addition, a partial integration, in such a manner that the derivative $dU/d\xi$ does no longer appear - we find finally

$$L = \frac{\xi_1}{\xi} L_1 + \ln\frac{U(\xi)}{U_1} + \frac{1}{\xi}\int_{\xi_1}^{\xi}\left[b(\xi') - \ln\frac{U(\xi')}{U_1}\right]d\xi' \qquad (51)$$

At the initial point $x_1 = 0$, there is also $\xi_1 = 0$. The first term in equation (51) then disappears. Especially for the laminar case there applies with $b = 0$ the simple expression

$$L = -\int_0^1 \ln\frac{U(\xi')}{U(\xi)}\, d\,\frac{\xi'}{\xi} \qquad (52)$$

We shall report later on regarding the initial values for L at the stagnation point $x_1 = 0$ and at the transition point.

As can be seen from equation (51), one may provide the new variable ξ with an arbitrary factor without causing thereby a change in L. We may therefore write

$$\xi = \left[C_1{}^* + \left(\frac{c_f}{2}\right)^{1+n}\int_{x_1/l}^{x/l}\left(\frac{U}{U_\infty}\right)^{3+2n}\left(\frac{R}{l}\right)^{1+n} d\,\frac{x'}{l}\right]^{\frac{a}{A}} \qquad (50a)$$

TABLE 2.- THE QUANTITIES a, b, AND a/A

FOR LAMINAR AND TURBULENT FLOW

	Laminar		Turbulent
	$L > 0$ [1]	$L < 0$ [2]	
a	2.87	3.53	3.04×10^{-2}
b	0	0	$0.071\lg \frac{U\vartheta}{\nu} - 0.23$
$\frac{a}{A}$	6.5	8.0	4.0

[1]Pressure drop.
[2]Pressure rise.

The calculation of ξ thus will be very simple, since the expression in brackets already occurs in the calculation of the momentum-loss thickness (eq. (41)).

A complete calculation is carried out as follows.

Prescribed:

General quantities: $U(x)$, $R(x)$, $\frac{U_\infty l}{\nu}$.

State of flow n, c_f, $\frac{a}{A}$ (laminar or turbulent); initial values U_1, R_1, ϑ_1, L_1.

Desired:

Momentum-loss thickness $\vartheta(x)$; form parameters L, $H(x)$.
First, one calculates the momentum-loss thickness according to equation (41) by performing a simple quadrature. Hence one forms, if one has to calculate turbulent flows, the Reynolds number $U\vartheta/\nu$ with which one then calculates the quantity b according to table 2. Next, one determines according to equation (50a) the new variable ξ over which one integrates the function $b - \ln U/U_1$. According to equation (51), one then obtains the form parameter L and, by means of figure 8 or figure 9, the boundary-layer thickness ratio H. Due to the large exponent a/A in equation (50a), the quantity ξ increases very rapidly with growing

length x. This signifies, however, that the term $(\xi_1/\xi)L_1$ in equation (51) loses its significance more and more with increasing distance; therewith it is shown that the initial value L_1 has, in case of larger distances x (for instance in the neighborhood of the separation point), only slight influence. The advantage of our method lies in the fact that only simple quadratures have to be performed in the individual case. Beyond that, no derivatives of the initial values U and R with respect to x occur.

(c) The Initial Values

The initial values for the momentum-loss thickness ϑ as well as for the form parameter L may be different according to the case to be dealt with.

1. Flow toward a body with stagnation point.- The boundary-layer calculation is started at the stagnation point $x_1 = 0$. There $U = 0$ and $R = 0$. From equation (33) then follows immediately

$$\Theta_0 = 0 \tag{53}$$

In the two-dimensional as well as in the rotationally symmetrical case, the potential velocity varies linearly with the length x, that is

$$U = cx \qquad \text{with} \qquad c = \left(\frac{dU}{dx}\right)_0 \tag{54}$$

The constant c is different for the two-dimensional and for the rotationally symmetrical case.

For the variation of the radius, there results also a linearity in x, namely

$$R = x \tag{55}$$

The integration constant in equations (34), (36), and (41) disappears

$$X_1 = C_1 = C_1{}^* = 0 \tag{56}$$

If we substitute equations (54) and (55) into equation (41), there follows, if we put all quantities which additionally enter the formulas in the rotationally symmetrical case into braces $\{\ \}$, after a brief intermediate calculation

$$x \to 0: \frac{\vartheta}{l} = \frac{1}{2\left[2(2 + n) + \left\{1 + n\right\}\right]^{\frac{1}{1+n}}} \frac{c_f}{c^{*\frac{n}{1+n}}}\left(\frac{x}{l}\right)^{\frac{1-n}{1+n}} \tag{57}$$

Therein $c^* = \frac{l}{U_\infty}\left(\frac{dU}{dx}\right)_0$ signifies the dimensionless expression for c. In the laminar case ($n = 1$), there results a value different from zero for the momentum-loss thickness in the neighborhood of the stagnation point

$$\left(\frac{\vartheta}{l}\right)_0 = \frac{1}{2\sqrt{2(3 + \left\{1\right\})}} \frac{c_f}{\sqrt{c^*}} \tag{58}$$

If one takes into consideration that $c^* = \frac{l}{U_\infty} c$ and $c_f = \frac{1.328}{\sqrt{\frac{U_\infty l}{\nu}}}$, one may write for equation (58) also

$$\sqrt{\frac{c}{\nu}}\, \vartheta_0 = 1.328 \frac{\sqrt{c^*}}{c_f}\left(\frac{\vartheta}{l}\right)_0 = \frac{0.470}{\sqrt{3 + \left\{1\right\}}} \tag{59}$$

In the turbulent case ($n = 1/6$), the momentum-loss thickness at the stagnation point itself ($x = 0$) has the value zero.

For the momentum-loss thickness ratio at the stagnation point for equal velocity increase $(dU/dx)_0$ in the rotationally symmetrical and in the two-dimensional case (c = Constant), one then has

$$\left(\frac{\vartheta_0 \text{rot}}{\vartheta_0 \text{eb}}\right)_{c=\text{const}} = \left(\frac{2(2 + n)}{5 + 3n}\right)^{\frac{1}{1+n}} \tag{60}$$

If the numerical values for n are substituted, there results in the laminar case the value 0.867 and in the turbulent case 0.816. The exact value for the laminar flow which one can calculate from the Hartree profile is 0.845.

The initial value of the form parameter L is determined according to equation (48) by putting therein according to equation (53) $\Theta_0 = 0$,

$$\Gamma_0 = K(L_0) \tag{61}$$

The value Γ_0 is obtained from the following boundary-layer determination

$$\Gamma_0 = \lim_{x \to 0} \frac{\Theta}{U} \frac{dU}{dx} = \lim_{x \to 0} \frac{\Theta}{x} = \frac{A}{2(2 + n) + \left\{1 + n\right\}} \tag{62}$$

In the laminar case, there results with equation (59)

$$\Gamma_0 = \frac{c}{\nu} \vartheta_0^2 = \frac{0.220}{3 + \left\{1\right\}} \tag{62a}$$

and from equation (49)

$$L_0 = \frac{K(L_0)}{a} = \frac{\Gamma_0}{a} = \frac{0.077}{3 + \left\{1\right\}}^{6} \tag{63}$$

(See table 3.) Although, according to equation (62), in the turbulent case a finite value different from zero does result for Γ_0, it is of no

[6] It can be shown that this value follows also from equation (52).

help in calculating the value L_0 according to equation (61) since the values $K(L)$ for the Reynolds numbers $\frac{U\vartheta}{\nu} \longrightarrow 0$ are not known. Since the flow at the stagnation point always will be initially laminar, we refer to section 5, c3, where we report on the initial values at the transition point.

For the variation of the form parameter L with the length x at the stagnation point, there applies also (as can be derived from equation (48))

$$\left(\frac{dL}{dx}\right)_0 = 0 \tag{64}$$

2. Flow into a channel.- The boundary layer begins at a point $x_1 = x_k = 0$ where the velocity, and likewise the radius in the rotationally symmetrical case, are of finite magnitude. Since again $X_1 = C_1 = C_1{}^* = 0$, there follows immediately from equations (34), (36), and (41)

$$\Theta_k = 0 \qquad \text{and} \qquad \vartheta_k = 0 \tag{65}$$

and from equation (44), when $dU/dx \neq \infty$

$$\Gamma_k = 0 \tag{66}$$

The form parameter then follows from equation (48) as

$$K(L_k) = 0 \tag{67}$$

In the laminar case, this form parameter becomes

$$L_k = 0 \tag{68}$$

that is, the value of the flat plate in longitudinal approach flow. There is no difference between the two-dimensional and the rotationally symmetrical case. (See table 3.) For the turbulent flow, there applies what was said above in the discussion of the stagnation-point flow.

The variation of the form parameter L with x is obtained from equation (48)

$$\left(\frac{dL}{dx}\right)_k = \frac{\left(\frac{1}{U}\frac{dU}{dx}\right)_k}{1 + \frac{1}{A}\left(\frac{dK}{dL}\right)_k} = \frac{1}{1 + \frac{a}{A}}\left(\frac{1}{U}\frac{dU}{dx}\right)_k \tag{69}$$

The numerical value a/A is to be taken from table 2.

TABLE 3.- LAMINAR INITIAL VALUES. THE EXACT VALUES - INSOFAR THEY DO NOT AGREE WITH THE APPROXIMATION VALUES - ARE PUT IN PARENTHESES

	Stagnation-point flow			Channel flow
	Plane	Rotationally symmetrical		Plane, rotationally symmetrical
$\sqrt{\frac{c}{\nu}}\,\vartheta_0$	0.271 (.292)	0.235 (.247)	ϑ_k	0
L_0	.0260 (.0292)	.0195 (.0208)	L_k	0
H_0	2.25 (2.22)	2.32 (2.30)	H_k	2.60
α_0	.345 (.360)	.320 (.324)	α_k	.220

3. Transition laminar-turbulent.- Behind a certain transition region, the laminar boundary layer is transformed into the turbulent layer at the point $x_l = x_u$. From the theory of the origin of turbulence, compare H. Schlichting (ref. 17), one can find, in agreement with measurements, that the transition occurs at places which lie somewhat downstream with respect to the velocity maximum. Since the phenomena in the transition region have not yet been explored, determination of the initial values for ϑ and H or L which are required for the turbulent calculation is only approximately possible. What is fundamental in our considerations

has been shown in figure 12 on the example of the flat plate. Up to the point $x_{ü}$ the boundary layer is laminar and obeys the regularities according to Blasius, that is, $\vartheta \sim \sqrt{x}$. The corresponding H-value is constant and amounts to $H = 2.60$. Starting from the point x_u the boundary layer is fully turbulent. Here applies approximately $\vartheta \sim \vartheta_u + c(x - x_u)$. The corresponding H-value depends, in addition, to some extent on the Reynolds number $U_\infty \vartheta/\nu$ prevailing at the transition point. According to measurements, for instance, by F. Schultz-Grunow (ref. 18), also from the similar solutions in case of constant pressure by J. Rotta (ref. 15), one finds about $1.2 < H < 1.4$, with the H-value being the smaller, the larger the Reynolds number. Thus the H-value decreases compared to the laminar value. In the transition region, it must therefore vary continuously from the laminar to the turbulent value. In figure 13, we plotted the difference ΔH against $U\vartheta/\nu$. For the sake of simplicity, we let x_u coincide with $x_{ü}$; then the initial value for the momentum-loss thickness of the turbulent calculation is equal to the momentum-loss thickness which would result at the point x_u if the flow were fully laminar up to this point. We shall therefore put

$$\vartheta_t(x_u) = \vartheta_u = \vartheta_l(x_u) \tag{70}$$

For the H-value, we write

$$H_t(x_u) = H_u = H_l(x_u) - \Delta H \tag{71}$$

We shall now assume that ΔH may be taken from figure 13 also for the cases with pressure gradient. This assumption we deem justified since the transition point lies in the proximity of the point of vanishing pressure gradient and, thus, the values of the flat plate may be used in suitable approximation. Having thus determined H_u, we ascertain according to figure 9 the value $L_u = L(x_u)$ which then enters equation (51) as L_1. We want here to point out once more that in places lying further downstream from transition point the initial value for L is only of slight significance; therefore, a somewhat rougher estimate seems justified when it is a matter of determining the separation point.

(d) The Separation Point

Knowledge of the position of the separation point also is important. Separation results when the wall-shear stress assumes the value zero. Whereas this point in the laminar case corresponding to the prescribed velocity profile is fixed by $\alpha_l = 0$ according to equation (19), it is not yet possible to make a perfectly clear statement regarding the separation point in the turbulent case. According to the wall-shear-stress statements of Ludwieg-Tillmann and Rotta (compare fig. 3), the shear stress decreases with increasing value H, however, without attaining the value zero. As approximation rule one may, for instance, assume according to v. Doenhoff-Tetervin (ref. 11) that the separation starts at the earliest when $H \approx 1.8$ [7] and has certainly taken place when H has attained the value 2.4. Table 4 presents a compilation of the occurring values.

TABLE 4.- VALUES OF THE FORM PARAMETER

AT THE SEPARATION POINT

	Laminar	Turbulent
H	4.038	1.8 to 2.4
L	-0.018	-0.13 to -0.18

(e) Examples

The usefulness of the method described above is shown on one example each of laminar and of turbulent flow.

1. Howarth - flow (laminar).- As example for the laminar flow, we choose the well-known Howarth flow (ref. 19). In this case, the velocity distribution is

$$\frac{U}{U_\infty} = 1 - \frac{x}{l} \tag{72}$$

If equation (72) is substituted into equation (41) where the radius distribution R is omitted, the quadrature may be carried out in closed form. There results for the momentum-loss thickness

$$\frac{\vartheta}{l} = \frac{c_f}{2\sqrt{6}} \frac{\sqrt{1 - \left(1 - \frac{x}{l}\right)^6}}{\left(1 - \frac{x}{l}\right)^3} = \frac{0.2711}{\sqrt{\frac{U_\infty l}{\nu}}} \frac{\sqrt{1 - \left(1 - \frac{x}{l}\right)^6}}{\left(1 - \frac{x}{l}\right)^3} \tag{73}$$

[7]This statement corresponds approximately to Gruschwitz' assumption $\eta = 1 - (u_\vartheta/U)^2 = 0.8$.

In figure 14, we show the comparison of this approximation with the exact values of Howarth. Within drawing accuracy, the agreement is perfect.

For determination of the form parameter, we first calculate the new variable ξ according to equation (50a)

$$\xi = Z^{\frac{a}{A}} \qquad \text{with} \qquad Z = 1 - \left(1 - \frac{x}{l}\right)^6 = 1 - \left(\frac{U}{U_\infty}\right)^6 \tag{74}$$

The form parameter itself also may be calculated analytically according to equation (52). Since we find ourselves, corresponding to the prescribed velocity distributions, in the region of pressure increase, we choose according to table 2 the value $a/A = 8$. We obtain after a brief intermediate calculation

$$L = \frac{1}{\xi}\int_0^{\xi} \frac{1}{U}\frac{dU}{d\xi'}\,\xi'\,d\xi' = -\frac{1}{6Z^8}\int_0^Z \frac{Z'^8}{1 - Z'}\,dZ' = -\frac{1}{6}\sum_{m=1}^{\infty}\frac{Z^m}{m + 8} \tag{75}$$

Taking equation (74) into consideration, we represented $L(x)$ also in figure 14. From these values we determined, with the aid of figure 8, the values α for the wall-shear stress. They, too, are represented in figure 14 and are there compared with the exact values of Howarth. The agreement is quite satisfactory. We also showed the curve which results according to the Pohlhausen method.[8] One achieves, as already stated by A. Walz, a considerably better determination of the separation point, $\alpha = 0$, if one uses in addition to the momentum theorem also the energy theorem, as in our method.

2. Profile NACA 65(216)-222 (turbulent).- As an example for the turbulent calculation, we choose the profile NACA 65(216)-222 (approximately) for which measurements (ref. 11) as well as theoretical calculations have been carried out. The conditions refer to the upper side of the profile placed at an angle of attack $\alpha = 10.1^\circ$. The Reynolds number is $U_\infty l/\nu = 2.64 \times 10^6$. The graphical representation of the velocity distribution, the momentum-loss thickness, and the form parameter according to measurements, likewise the result of our calculation, are given in

[8]This curve we took from A. Walz (ref. 5).

figure 15. We started our calculation at the first point measured, that is, $x/l = 0.075$. The agreement between our approximation and the measurement may be called satisfactory.[9] For comparison, we also plotted the results according to the calculation of v. Doenhoff-Tetervin and according to the methods of Garner, Gruschwitz, and Gruschwitz-Kehl. Regarding the form parameter H, the last two methods deviate greatly from the other methods and from the measurement.

[9]As A. E. v. Doenhoff and N. Tetervin pointed out, the larger differences between measurement and calculation for the momentum-loss thickness in the neighborhood of the separation point $x/l \approx 0.55$ might be caused by systematic measuring inaccuracies.

APPENDIX

DISCUSSION OF THE FORMULA FOR CALCULATION OF THE MOMENTUM-LOSS THICKNESS

As mentioned before, the momentum-loss thickness was calculated, so far, from the momentum theorem (16). With the use of certain simplifications, it is possible to find for the calculation of the momentum-loss thickness from the momentum theorem a formula similar to our equation (36). Two possibilities exist:

1. The wall-shear stress is determined according to the laws of the flat plate in longitudinal approach flow where the length x is expressed by the momentum-loss thickness ϑ, and the approach-flow velocity U_∞ by the local velocity. For the laminar as well as for the turbulent case, one may write

$$\frac{\tau_0}{\rho U^2} = \frac{\alpha}{\left(\frac{U\vartheta}{\nu}\right)^n}$$

If one substitutes, moreover, in the momentum theorem a constant value for the boundary-layer thickness ratio H, one can integrate the momentum theorem in closed form, as was shown by E. Truckenbrodt in an unpublished report (compare H. Schlichting, Boundary-layer theory, page 430; one obtains the result

$$\Theta = \left(\frac{U\vartheta}{\nu}\right)^n \vartheta = \frac{C_1 + (1 + n)\alpha \int_{x_1}^{x} U^a R^{1+n}\, dx'}{U^a R^{1+n}}$$

where $a = (1 + n)(H + 2) - n$.

The corresponding numerical values may be taken from table 5.

TABLE 5

		Laminar	Turbulent	
	n	1	$\frac{1}{4}$(Blasius)	$\frac{1}{6}$(Falkner)
Factor ahead of the integral	A $(1 + n)\alpha$ c	0.441 .441 .470	------ 0.0160 .0160	0.0076 .0076 ------
Exponent of the velocity distribution	3 + 2n a b	5.0 8.2 5.0	--- 4.0 4.0	3.33 3.67 ----

2. According to K. Pohlhausen for the laminar case and to E. Buri for the turbulent case, the wall-shear stress and the boundary-layer ratio are functions of the quantity

$$\Gamma = \frac{\Theta}{U} \frac{dU}{dx}$$

that is,

$$\frac{\tau_0}{\rho U^2} = \frac{f_1(\Gamma)}{\left(\frac{U\vartheta}{\nu}\right)^n}$$

and

$$H = f_2(\Gamma)$$

If one puts

$$\Lambda = \Theta R^{1+n} \qquad \text{and} \qquad x^* = \int_0^x R^{1+n}\, dx$$

one can find after some intermediate calculations the following equation:

$$\frac{d\Lambda}{dx^*} = \Theta(\Gamma)$$

with

$$\Gamma = \frac{\Lambda}{U}\frac{dU}{dx^*}$$

Therein

$$\Theta(\Gamma) = (1 + n)f_1(\Gamma) - \left[2 + n + (1 + n)f_2(\Gamma)\right]\Gamma$$

represents a universal function for the laminar or, respectively, turbulent state which, in the first case, may be calculated analytically (for instance, for the Pohlhausen polynomials) and, in the second case, may be determined from measurements. In both cases, one may find by way of approximation a linear connection between Φ and Γ (compare A. Walz, Lilienthal-Bericht 141 (1941) and E. Buri).

$$\Phi(\Gamma) = -b\Gamma + c$$

If one substitutes this expression in the above equation, the integration may be carried out in closed form and yields after a simple intermediate calculation

$$\Theta = \left(\frac{U\vartheta}{\nu}\right)^n \vartheta = \frac{C_1 + c\int_{x_1}^{x} U^b R^{1+n}\, dx'}{U^b R^{1+n}}$$

The numerical values c and b are also indicated in table 5. (Compare H. Schlichting, Boundary-layer theory, pp. 199 and 424.) The constants according to equation (36) also have been indicated for comparison.

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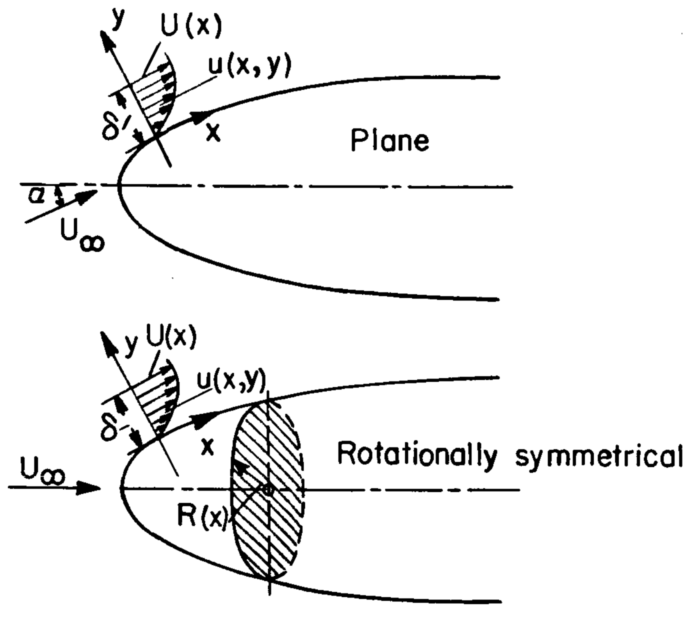

Figure 1.- Survey sketch.

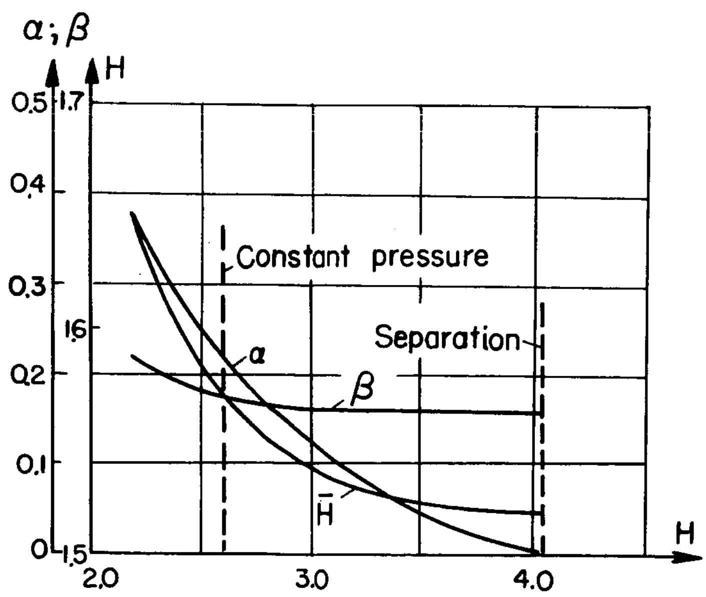

Figure 2.- Laminar friction-layer parameters (Hartree profiles).

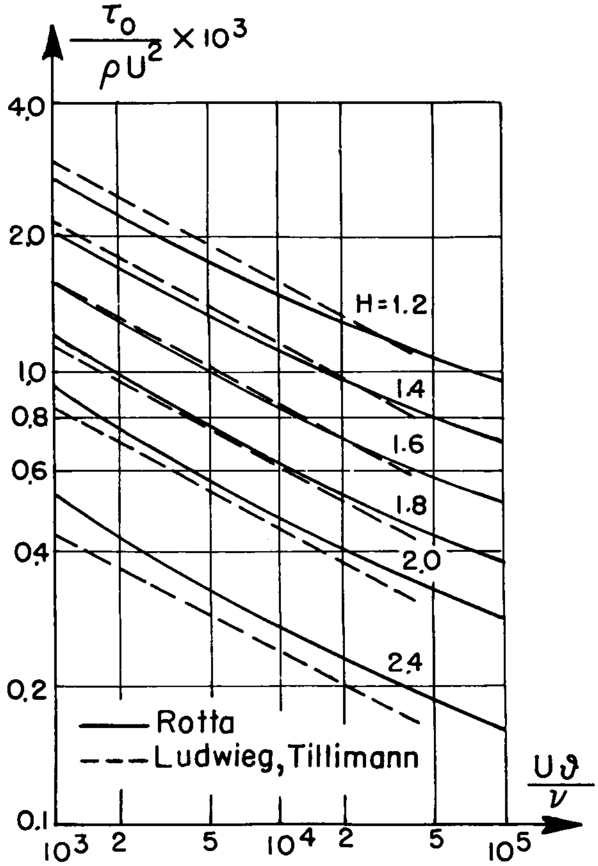

Figure 3.- Turbulent wall-shear stress (according to H. Ludwieg, W. Tillmann, and J. Rotta).

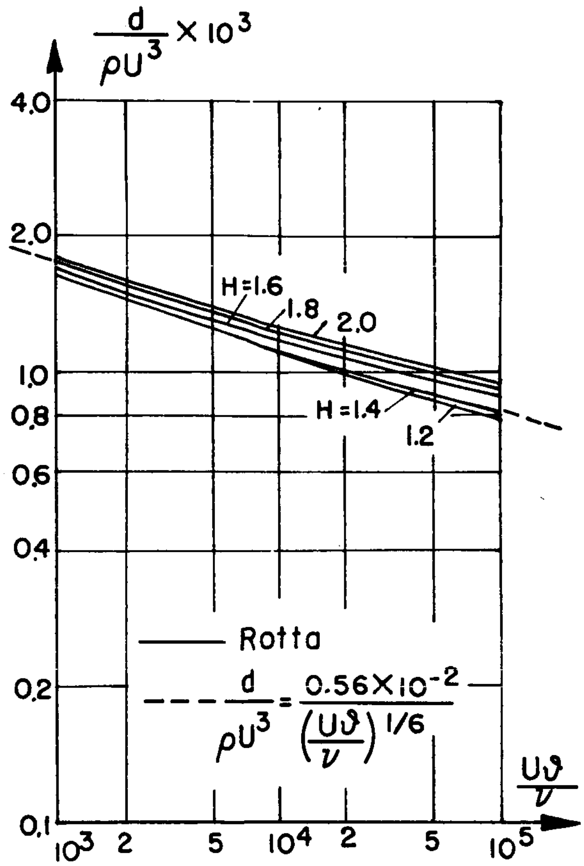

Figure 4.- Turbulent dissipation (according to J. Rotta).

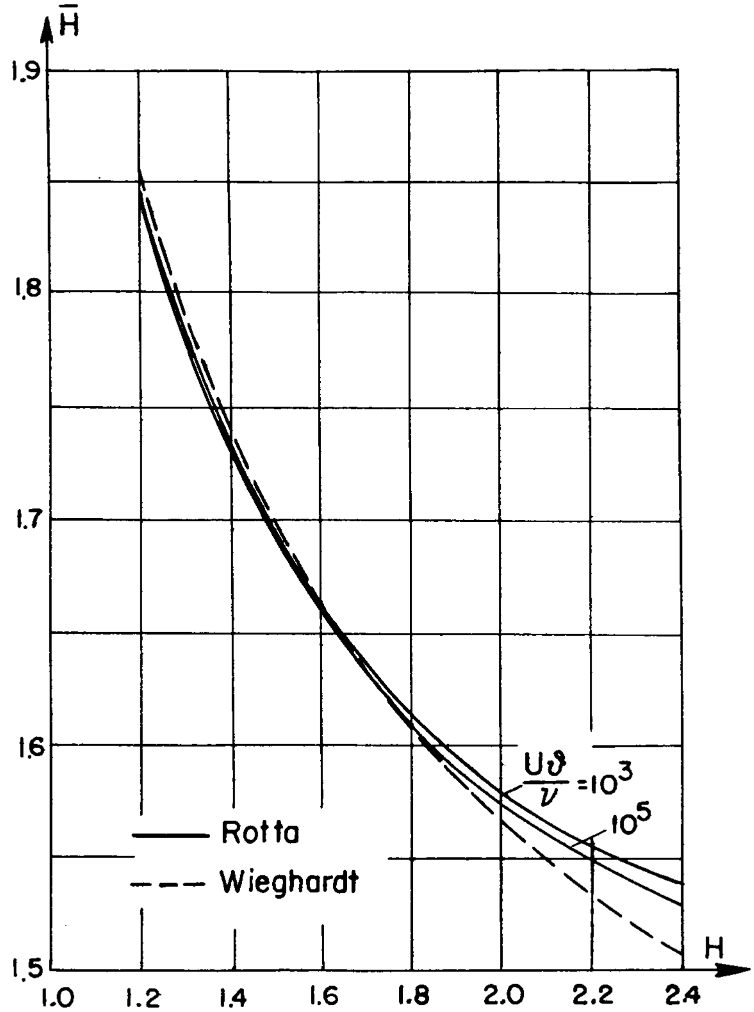

Figure 5.- Turbulent boundary-layer thickness ratio (according to J. Rotta and K. Wieghardt).

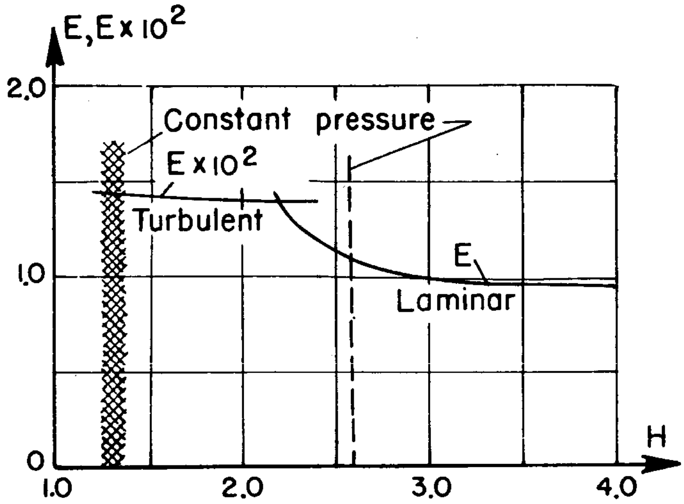

Figure 6.- The function E(H) for laminar and turbulent flow.

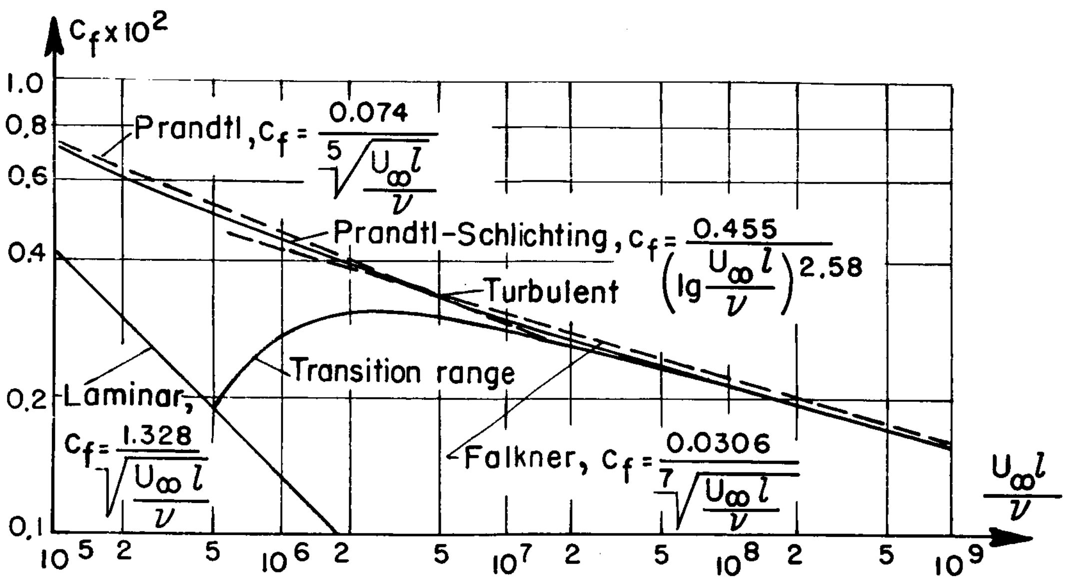

Figure 7.- Drag law of the smooth flat plate in longitudinal approach flow.

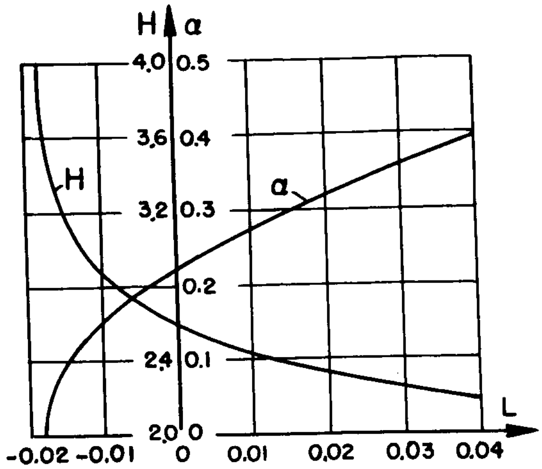

Figure 8.- Relation between the boundary-layer thickness ratio H and the wall shear-stress coefficient α, on one hand, and the form parameter L on the other for laminar flow.

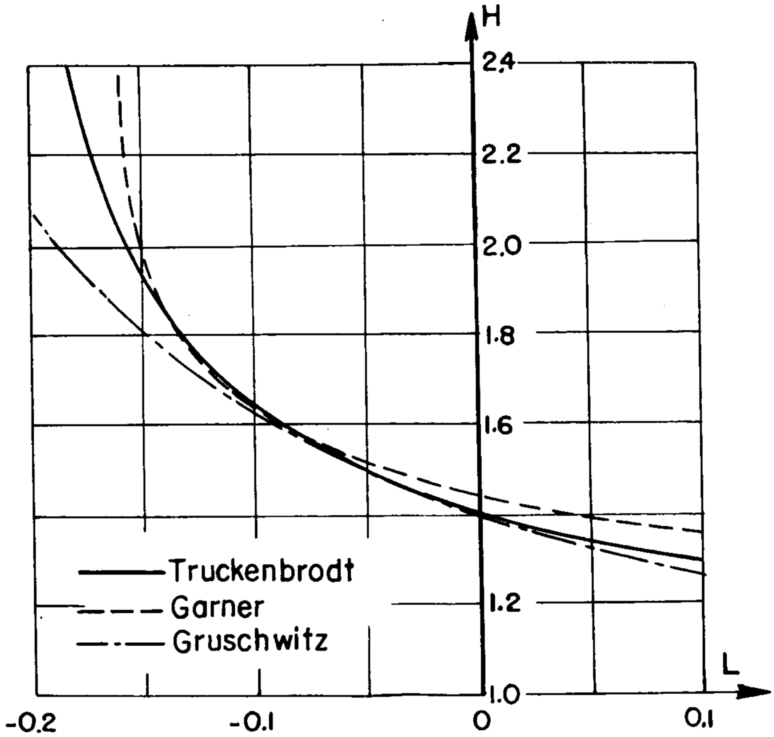

Figure 9.- Relation between the boundary-layer thickness ratio H and the form parameter L for turbulent flow.

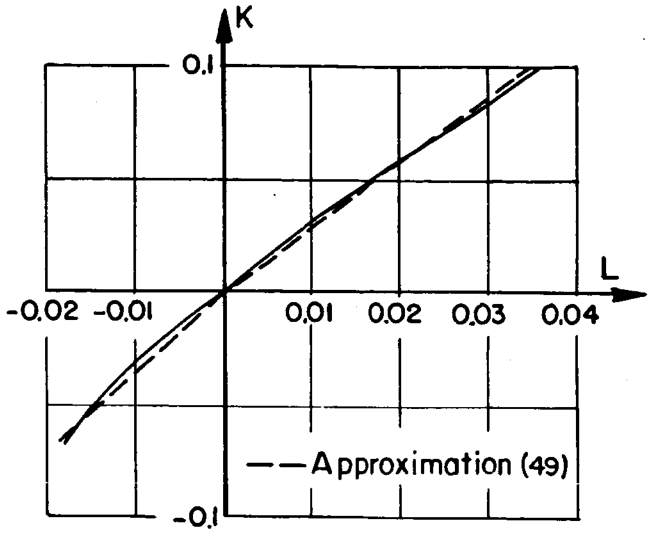

Figure 10.- The function K(L) for laminar flow.

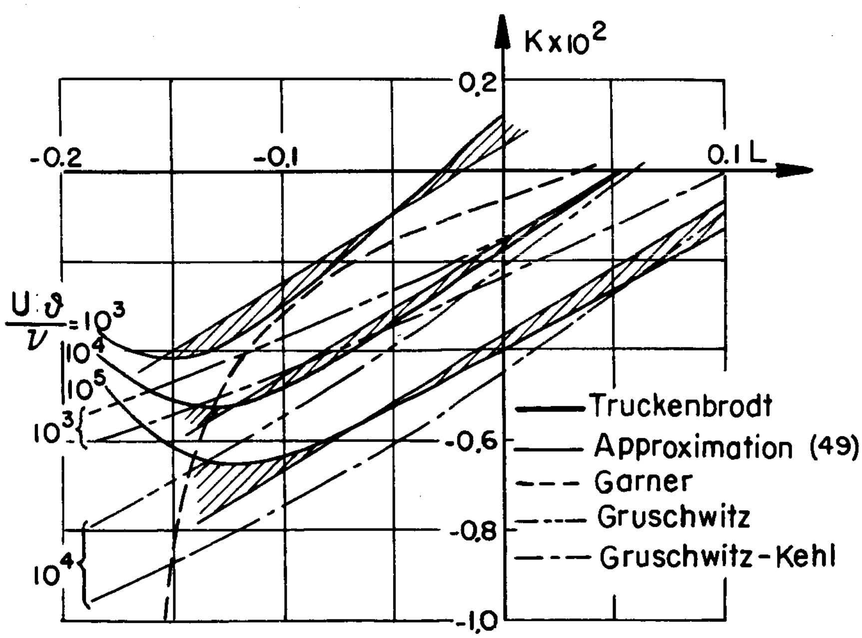

Figure 11.- The function K(L) for turbulent flow.

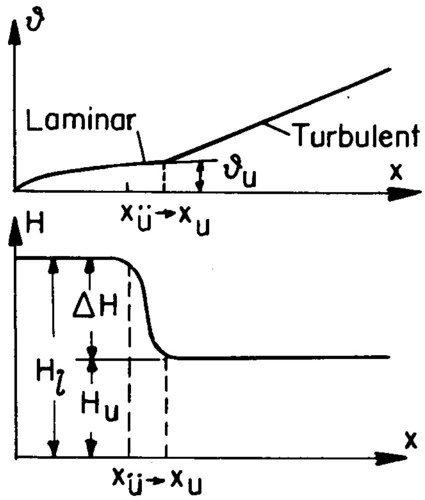

Figure 12.- Survey sketch from the transition point of the flat plate.

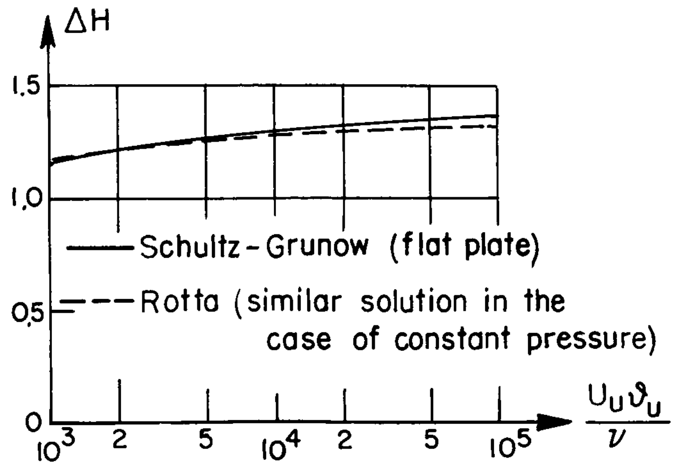

Figure 13.- Variation of the form parameter H in the range of transition from laminar to turbulent flow.

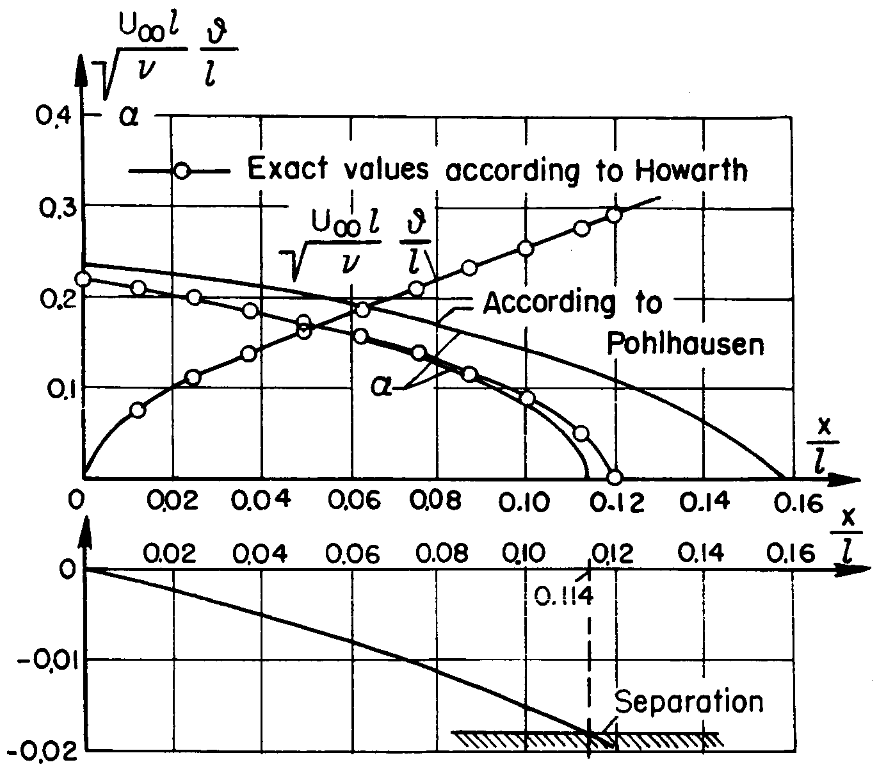

Figure 14.- The laminar boundary-layer parameters of the Howarth flow.

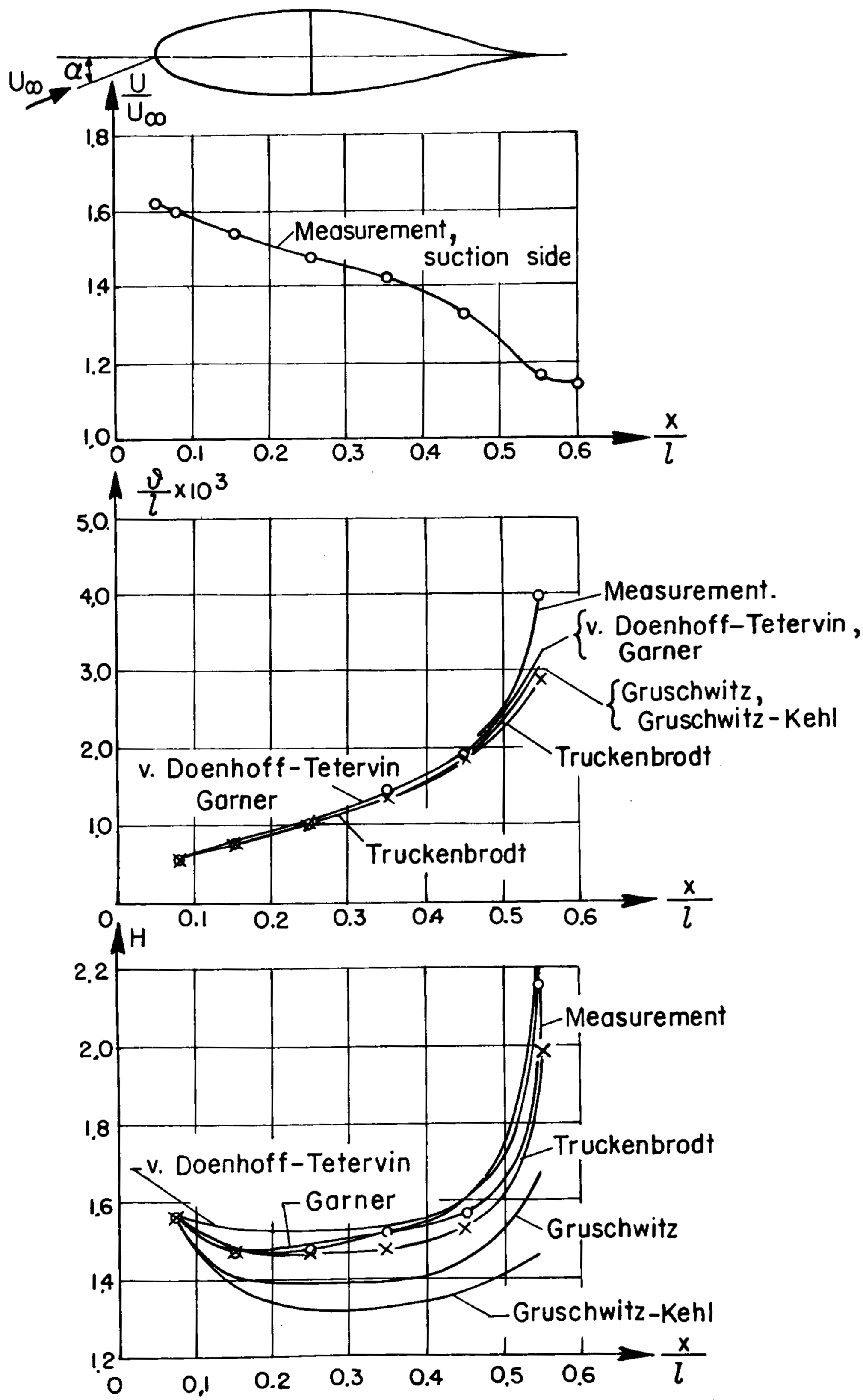

Figure 15.- Turbulent boundary layer on the profile NACA 65(216)-222 $\alpha = 10.1^\circ$, $\frac{U_\infty l}{\nu} = 2.64 \times 10^6$. Measurement according to NACA Rep. 772.

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