NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1364

THE PLANE PROBLEM OF THE FLAPPING WING

By Walter Birnbaum

Translation of "Das ebene Problem des schlagenden Flugels"
Zeitschrift für augewandte Mathematik und Mechanik,
Band 4, 1924.

Washington
January 1954

1U

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1364

THE PLANE PROBLEM OF THE FLAPPING WING*[1]

By Walter Birnbaum

In connection with my report on the lifting vortex sheet[2] which forms the essential basis of the following investigations, I shall show how the methods developed there are also suitable for dealing with the air forces for a wing with a circulation variable with time. I shall, in particular, develop the theory of a propulsive wing flapping up and down periodically in the manner of a bird's wing. I shall show how the lift and its moment result as a function of the flapping motion, what thrust is attainable, and how high is the degree of efficiency of this flapping propulsion unit if the air friction is disregarded. Finally, I shall treat an interesting case of dynamic instability for a spring-suspended wing; this phenomenon was confirmed by experiments at the Göttingen aerodynamic test laboratory. Professor Prandtl gave me his guidance concerning the present report, and I want to express here also my sincere gratitude for the abundant stimulation and energetic assistance he gave to me at all times.

1. General statements.- The calculations refer to the two-dimensional problem, that is, to the wing of infinite length, or bounded by plane side walls, or to a wing with so large an aspect ratio that the boundary effect is negligible. For the rest, the same assumptions as in the first report are valid concerning smallness of the air forces, slight camber of the wing, and so forth. The wing is assumed to extend from $x = -1/2$ to $x = +3/2$ so that the point $x = 0$ becomes the center of pressure of a plane wing with fixed angle of attack. Let v, in the direction of the positive X-axis, be the air velocity at a large distance. Let the positive Y-axis point downward. By $\gamma = \gamma(x,t)$, I denote the density of the lifting vortices (now a function of the time). At every variation with time of the density of circulation, free vortices of the density $\epsilon(x,t)$ will separate from the lifting vortices and will drift away with the air flow. On the basis of the theorem about the conservation of circulation, or by integration of Euler's differential equation once along the pressure side and once along the suction side of the wing (compare the unabbreviated

*"Das ebene Problem des schlagenden Flugels" Zeitschrift für augewandte Mathematik und Mechanik, Band 4, 1924, pp. 277-292.

[1]Abstract from the Göttinger dissertation of the same title. Available in the University library at Göttingen and the state library at Berlin. Referent: Professor Prandtl.

[2]This periodical (Z.f.a.M.M.), vol. 3, 1923, pp. 290-297.

report), one finds easily that the vortices must satisfy at every point and at every time the following continuity equation of the vortex density[3]:

$$\frac{\partial \gamma}{\partial t} + \frac{\partial \epsilon}{\partial t} + v\,\frac{\partial \epsilon}{\partial x} = 0 \tag{1}$$

Generally one will make suitable assumptions regarding $\gamma(x,t)$ so that ϵ may be found from the equation. If $\partial\gamma/\partial t$ is designated by $\gamma(x,t)$, its integral is

$$\epsilon(x,t) = -\frac{1}{v}\int_{\varphi(x-vt)}^{x} \dot{\gamma}\left(\xi, t + \frac{\xi - x}{v}\right) d\xi \tag{2}$$

Where φ is an arbitrary function to be determined by boundary conditions. After ϵ has thus been found, there results the induced vertical velocity in first approximation as principal value of the integral

$$w(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\gamma + \epsilon}{x - \xi}\, d\xi \tag{3}$$

Similarly to the former case, w now has to be put in relation with the moved wing contour. If the motion of the wing is indicated by the value of the ordinate at the point x, thus by $y = y(x,t)$ (with this formulation one could also include a moved and simultaneously deforming wing), the kinematic boundary condition reads

$$\frac{\partial y}{\partial t} + v\,\frac{\partial y}{\partial x} = w(x,t) \tag{4}$$

---

[3]Details on the derivation of this equation may be found in a lecture of Prandtl "On the Formation of Vortices in the Ideal Fluid" at the hydro-aerodynamic conference at Innsbruck 1922. (Lectures concerning the field of hydro and aerodynamics, edited by Th. v. Kármán and T. Levi-Civite, Berlin 1924) and in the original report quoted.

This is the equation from which contour and motion of the wing are obtained if γ, and therewith w, is prescribed. In analogy to the above said

$$y = \frac{1}{v} \int_{\psi(x-vt)}^{x} w\left(\xi, t + \frac{\xi - x}{v}\right) d\xi \tag{5}$$

If one takes into consideration that the air velocity relative to the supporting vortex line has the components v and $w - \frac{\partial y}{\partial t}$, there result as before the air forces for the unit length of the wing

$$K = \rho v \int_{-\frac{1}{2}}^{1\frac{1}{2}} \gamma \, dx \tag{6a}$$

$$M = \rho v \int_{-\frac{1}{2}}^{1\frac{1}{2}} \gamma x \, dx \tag{6b}$$

$$W = \rho \int_{-\frac{1}{2}}^{1\frac{1}{2}} \gamma \left(w - \frac{\partial y}{\partial t}\right) dx - \rho\pi \frac{a^2}{2} = \rho v \int_{-\frac{1}{2}}^{1\frac{1}{2}} \gamma \frac{\partial y}{\partial x} dx - \rho\pi \frac{a^2}{2} \tag{7}$$

The last term in W represents as in my first report the suction force at the leading edge of the wing, now of course with a coefficient a variable with time of the first fundamental function.

2. Periodical wing motion.- From these general expressions, I now pass to treating the special problem of the flapping wing. It is to be expected that γ itself in this case will be periodical, except for a constant mean value. Thus, I determine first from the expression

$$\gamma = \gamma_0(x) e^{i\nu t} = \gamma_0 e^{i\omega v t} = \gamma_0 e^{\omega' v t} \tag{8}$$

the free vortices ϵ; for γ which alone enters into ϵ is independent of the constant mean value $\overline{\gamma}$ so that this expression is sufficiently general. As customary in oscillation theory, one tacitly deals with the imaginary constituent of γ. ν is the circular frequency of the oscillations, $\omega = \frac{\nu}{v}$ a dimensionless quantity which I call "reduced frequency" (v has the dimensions of an angular velocity since it equals the flight velocity, measured with half the wing chord as unit length). If one designates the distance travelled by the wing after a full period as the "wave length" λ, one has

$$\lambda = 2\pi \frac{v}{\nu} = \frac{2\pi}{\omega}$$

The further results are obtained by series developments with respect to the reduced frequency ω which have the form

$$\Sigma c_{mn}\omega^{m}(\log \omega)^{n} \qquad m \geqq n$$

These series converge satisfactorily only for small ω up to about $\omega = 0.1$ corresponding to a wave length of 30 wing chords or more. Thus I presuppose for the present report slow quasi-steady oscillations of the wing; this does not imply, of course, that the actual frequency cannot assume high values, too, if only the wing chord is sufficiently small. If I calculate, finally, with $\omega' = i\omega$, the formulas obtained, at least outwardly, real coefficients which offer many advantages for the calculation.

I now assume that parallel flow free from vortices prevails ahead of the wing. Then I obtain with $\overline{x} = x - 1/2$ for ϵ and w, under the presupposition that the "switching-on process" of the wing motion has already run its course (the motion thus has become steady), the following expressions:

$$\left.\begin{aligned}
&\epsilon = 0; \quad \overline{x} < -1 \\
&\epsilon = \epsilon_{\alpha} = \omega' e^{\omega'(vt-\overline{x})} \int_{\overline{x}}^{-1} e^{\omega'\overline{\xi}}\gamma_{0}(\overline{\xi})d\overline{\xi} \qquad -1 \leqq \overline{x} \leqq 1 \\
&\epsilon = \epsilon_{\beta} = \omega' e^{\omega'(vt-\overline{x})} \int_{1}^{-1} e^{\omega\overline{\xi}}\gamma_{0}(\overline{\xi})d\overline{\xi} \qquad \overline{x} \geqq 1
\end{aligned}\right\} \quad (9)$$

$$2\pi w(\bar{x},t) = \int_{-1}^{+1} \frac{\gamma(\bar{\xi},t)}{\bar{x} - \bar{\xi}}\, d\bar{\xi} + \int_{-1}^{+1} \frac{\epsilon_\alpha(\bar{\xi},t)}{\bar{x} - \bar{\xi}}\, d\bar{\xi} + \int_{1}^{\infty} \frac{\epsilon_\beta(\bar{\xi},t)}{\bar{x} - \bar{\xi}}\, d\bar{\xi} \tag{10}$$

As in my first report, I determine $\gamma_0(x)$ from the three first fundamental functions[4]

$$\gamma_0 = a\gamma_{0a} + b\gamma_{0b} + c\gamma_{0c}$$

Where a, b, c now are complex numbers. One has then also

$$w = w_0 e^{\omega' vt} \qquad w_0 = aw_{0a} + bw_{0b} + cw_{0c}$$

The integrals (9) and (10) cannot be evaluated by elementary functions. The last integral (10) in particular leads to an integral logarithm. The Euler constant C which appears as a consequence always occurs in combination with log 2 so that I shall introduce as transcendent number

$$Z = \log 2 - C = 0.11593\ .\ .\ .$$

especially for the present report. If one takes this fact into consideration one obtains the w in the form

$$w_{0a} = \alpha_0 + \alpha_1\omega' + \alpha_2\omega' \log \omega' + \alpha_3\omega'^2 + \alpha_4\omega'^2 \log \omega' + \alpha_5\omega'^3 + \alpha_6\omega'^3 \log \omega' \tag{11}$$

[4]One obtains

$$\gamma_{0a} = \sqrt{\frac{1 - \bar{x}}{1 + \bar{x}}} \qquad \gamma_{0b} = \sqrt{1 - \bar{x}^2} \qquad \gamma_{0c} = \bar{x}\sqrt{1 - \bar{x}^2}$$

and corresponding expressions with β_i and γ_i are valid for w_{0b} and w_{0c}. The coefficients α_i, β_i, γ_i are polynomials in x the coefficients of which are linear and rational in Z (regarding their values compare the original report). Besides, the appearance of $\log \omega'$ shows that a treatment of the problem with a lifting line instead of the supporting surface must fail since to the transition to the lifting line there corresponds a transition to $\omega' = 0$ (v increases arbitrarily for decreasing wing chord). For this reason, the theory of the flapping wing here presented is the simplest possible one.

Bypassing the integral for y, we are now concerned with determining the wing motion in an appropriate manner and bringing it into accord with the expression for γ. Since the methods of the lifting vortex sheet are linear and since a deformation oscillation will not be taken into consideration, shape and mean angle of attack of the wing may be disregarded: It is then sufficient to calculate the oscillations of a plane wing with the mean angle of attack 0 and to superimpose these oscillations as small fluctuations with time of the profile chord linearly on the arbitrarily prescribed profile. For the amplitudes which are assumed to be small, one may disregard the fluctuation of the abscissa of a wing point as small of higher order, and has then as the most general motion still possible (compare fig. 1)

$$y = p(t) + x\varphi(t) \tag{12}$$

p and φ are to be developed as Fourier series with respect to ωv; I retain only their first term

$$y = (A + Bx)e^{\omega' vt} \tag{13}$$

Higher-harmonic oscillations would again have to be superimposed linearly. Fluctuations in the flight direction (which could be expressed by periodica fluctuations of v) also will be disregarded.

I am designating the quantities A and B as complex stroke amplitude and amplitude of rotary oscillation, respectively, since equation (13) is formed by combination of the special translatory and rotary oscillations

$$A = 1, \quad B = 0 \quad (\alpha) \qquad \text{and} \qquad A = 0, \quad B = 1 \qquad (\beta)$$

Thus γ_0 and all quantities linearly connected with it will be linear and homogeneous in A and B. For simplification of the calculation, the cases (α) and (β) may therefore be calculated separately and afterwards be combined linearly, for instance

$$a = Aa_\alpha + Ba_\beta \quad \text{etc.}$$

The complex circulation coefficients are found from equation (4) in the following manner: a, b, c are expressed in the form of the series

$$\left.\begin{aligned}
a &= a_0 + a_1\omega' + a_2\omega' \log \omega' + a_3\omega'^2 + a_4\omega'^2\log \omega' + \\
&\quad a_5\omega'^2\log^2\omega' + a_6\omega'^3 + a_7\omega'^3\log \omega' + \\
&\quad a_8\omega'^3\log^2\omega' + a_9\omega'^3\log^3\omega' + \ldots \\
b &= b_0 + \ldots \\
c &= c_0 + \ldots
\end{aligned}\right\} \tag{14}$$

Equation (4) has in our case the form

$$\begin{aligned}
w(x,t) &= (A\omega' + B\omega'x + B)ve^{\omega' vt} \\
&= \left(aw_{0a} + bw_{0b} + cw_{0c}\right)e^{\omega' vt}
\end{aligned}$$

This is to be valid identically in x and t. If both sides are interpreted as series in $\omega'^m(\log \omega')^n$, their coefficients must be identical. a, b, c therein are unknowns. Each coefficient yields a relation in x which at first cannot be expected to be identically fulfillable by only three free values. It was intended to fulfill the condition at least at three suitably selected locations x_1, x_2, and x_3. Surprisingly, these conditions are now shown to be precisely of the second and not of a higher degree in x. At least this is the case up to the terms of third order, and there is no doubt that this will remain so for the higher terms as well. The coefficients of second degree in x determine, therefore, with their three subcoefficients each one triplet of values, each of the a, b, c. In every new comparative coefficient there appears a new value triplet of this kind so that the thirty unknowns for which a formulation has been set up may be found uniquely from linear relations by successive evaluation. The calculation results in the following values:

$$a_0 = 2Bv$$

$$a_1 = 2v\left[A + \frac{1}{2}B(1 - 2Z)\right]$$

$$a_2 = 2Bv$$

$$a_3 = 2v\left[-AZ - B(Z - Z^2)\right]$$

$$a_4 = 2v\left[A + B(1 - 2Z)\right]$$

$$a_5 = 2Bv$$

$$a_6 = 2v\left[AZ^2 - B\left(\frac{1}{4} + \frac{1}{2}Z - \frac{1}{2}Z^2 + Z^3\right]\right.$$ (See footnote 5.)

$$a_7 = 2v\left[-2AZ + B\left(\frac{1}{2} - Z + 3Z^2\right)\right]$$

$$a_8 = 2v\left[A + B\left(\frac{1}{2} - 3Z\right)\right]$$

$$a_9 = 2Bv$$

$$b_0 = 0$$

$$b_1 = 4Bv$$

$$b_2 = 0$$

$$b_3 = 2v\left(A + \frac{1}{2}B\right)$$

$$c_0 = 0$$

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = Bv$$

$$b_i = c_i = 0, \quad i \geqq 4$$

---

[5]The dimensions of all quantities are affected by the selection of the wing chord as unit length. A treatise of the author in the Zeitschrift für Motorluftschiffahrt on the same subject has been arranged so that no objections are possible from the viewpoint of similarity mechanics.

I want to point out here briefly that approximated values for the circulation coefficients are obtained also, if the effect of the free vortices is disregarded and the elementary calculation made in such a manner as if the momentary apparent angle of attack of each wing element were decisive for its circulation, namely the angle formed by the air velocity v relative to the moved wing element and the direction of the latter. Since in case of rotary oscillations ($B \neq 0$) every wing element has another vertical velocity, the apparent angles of attack of the elements are all different, that is, the wing assumed to be plane behaves as if it had an apparent (dynamic) curvature which is periodical. The simple calculation (compare the original report, third part, beginning of section II) yields accordingly circulation contributions of the two first fundamental functions, that is, only the following terms:

$$a_0 = 2Bv \qquad b_0 = 0$$

$$a_1 = 2v\left(A + \frac{1}{2} B\right) \qquad b_1 = 2Bv$$

Thus except for higher terms and with consideration of the order of magnitude of B (see below) a good approximation is obtained.

All the rest follows readily from the circulation coefficients. I had introduced the quantity ω' in order to enable an easier calculation and consideration also of complex ω', that is, damped or excited oscillations. It is true that for damped oscillations the integrals (10) lose their meaning, since the oscillation had been assumed to have been going on for an infinitely long time so that here infinitely large amplitudes would have had to precede. However, the formulas obtained may be retained as approximations, provided a correction is made for the starting process. For the following consideration of the steady oscillations, I continue the calculation for reasons of physical clarity with the real frequency ω.

Lift and moment are purely periodical quantities, that is, their temporal mean value is 0. Nevertheless, their amplitude and their phase angle are of interest. According to formula (6), there is

$$\left.\begin{aligned} K &= \rho v^2\pi\left(Ak_\alpha + Bk_\beta\right)e^{i\omega vt} = \rho v^2\pi\left(Ak_\alpha + Ck_\gamma\right)e^{i\omega vt} \\ &= \rho v^2\pi\omega\left(A\bar{k}_\alpha + C\bar{k}_\gamma\right)e^{i\omega vt} \\ M &= \rho v^2\pi\left(Am_\alpha + Bm_\beta\right)e^{i\omega vt} = \rho v^2\pi\left(Am_\alpha + Cm_\gamma\right)e^{i\omega vt} \\ &= \rho v^2\pi\omega\left(A\bar{m}_\alpha + C\bar{m}_\gamma\right)e^{i\omega vt} \end{aligned}\right\} \quad (15)$$

Therein

$$B = C\omega; \quad k_\gamma = \omega k_\beta; \quad m_\gamma = \omega m_\beta; \quad k_\alpha = \omega \bar{k}_\alpha;$$

$$m_\alpha = \omega \bar{m}_\alpha; \quad \bar{k}_\beta = \bar{k}_\gamma; \quad m_\beta = \bar{m}_\gamma$$

I have introduced here the quantity C as new amplitude of rotary oscillation. This was done because in case of ordinary flapping motions B is of the order of magnitude ωA as a simple consideration shows. The value of ω is assumed to be small; thus A and C will be of the same order of magnitude so that calculation with C instead of B will be more convenient in practice. The third form of the air forces permits, for cases of equal stroke velocity $Ai\omega e^{i\omega vt}$, comparison of these forces in a simple manner; the roughest approximation theory would yield constant coefficients $\bar{k}_\alpha = 2i$, $\bar{k}_\gamma = 2$. The coefficients k and m are complex numbers the constituents of which are given by the following series developments. It is noteworthy that the series for m are finite.

$$k_\sigma = k_\sigma' + ik_\sigma''; \quad m_\sigma = m_\sigma' + m_\sigma''; \quad \sigma = \alpha, \beta, \gamma$$

$$\left.\begin{aligned}
\omega\bar{k}_\alpha' &= k_\alpha' = -(1 - 2Z)\omega^2 - 2\omega^2 \log \omega - 2\pi Z\omega^3 + 2\pi\omega^3 \log \omega + \ldots \\
\omega\bar{k}_\alpha'' &= k_\alpha'' = 2\omega - \pi\omega^2 + \left(\frac{\pi^2}{2} - 2Z^2\right)\omega^3 + 4Z\omega^3 \log \omega - \\
&\quad 2\omega^3 \log^2 \omega + \ldots
\end{aligned}\right\} \quad (16)$$

$$
\left.
\begin{aligned}
k_\beta' &= \bar{k}_\gamma' = \frac{1}{\omega} k_\gamma' = 2 - \pi\omega + \left(\frac{\pi^2}{2} + 2Z - 2Z^2 - \frac{1}{2}\right)\omega^2 - \\
&\quad 2(1 - 2Z)\omega^2\log\omega - 2\omega^2\log^2\omega - \left(\frac{\pi^3}{4} - \frac{\pi}{2} + \pi Z - 3\pi Z^2\right)\omega^3 + \\
&\quad \pi(1 - 6Z)\omega^3\log\omega + 3\pi\omega^3\log^2\omega + \ldots \\
k_\beta'' &= \bar{k}_\gamma'' = \frac{1}{\omega} k_\gamma'' = (3 - 2Z)\omega + 2\omega\log\omega - \pi(1 - 2Z)\omega^2 - \\
&\quad 2\pi\omega^2\log\omega + \left(\frac{\pi^2}{4} + \frac{1}{2} + Z - \frac{3}{2}\pi^2 Z - Z^2 + 2Z^3\right)\omega^3 + \\
&\quad \left(\frac{3}{2}\pi^2 - 1 + 2Z - 6Z^2\right)\omega^3\log\omega - (1 - 6Z)\omega^3\log^2\omega - \\
&\quad 2\omega^3\log^3\omega + \ldots
\end{aligned}
\right\} \tag{17}
$$

$$
m_\alpha' = \omega\bar{m}_\alpha' = -\frac{1}{2}\omega^2 \qquad m_\alpha'' = \omega\bar{m}_\alpha'' = 0 \tag{18}
$$

$$
m_\beta' = \bar{m}_\gamma' = \frac{1}{\varpi} m_\gamma' = -\frac{3}{8}\omega^2 \qquad m_\beta'' = \bar{m}_\gamma'' = \frac{1}{\varpi} m_\gamma'' = \omega \tag{19}
$$

It suggests itself to represent k_σ and m_σ by the initial position of "time vectors," visualized as rotating, in the complex number plane (as customary in alternating-current techniques) and to combine from them - with the parameters A and C (of which A may be assumed real without impairing the generality) - linearily, in the known manner, the amplitude coefficients $k = Ak_\alpha + Ck_\gamma$ (and correspondingly m) according to magnitude and phase. The diagram (figs. 2 and 3) shows the curves of the end points of the vectors k and m as functions of the parameter ω. In the representation of the curves for $\overline{k}$ and $\overline{m}$ which would yield the most accurate values for the graphical evaluation, the curves of the various approximations have been plotted side by side for comparison of the convergence of the series. Of particular interest is the case where the wing - without being affected by significant air forces - glides over an undulating streamline course, clinging to it as much as possible. To this corresponds the parameter $C = -iA$, which in fact yields small air forces of second order in ω, namely

$$\frac{1}{A} k' = 2\omega^2 + \ldots \qquad \frac{1}{A} k'' = \left(\frac{1}{2} - 2Z\right)\omega^3 + \ldots$$

$$\frac{1}{A} m' = \frac{1}{2} \omega^2 \qquad \frac{1}{A} m'' = \frac{3}{8} \omega^3$$

3. The induced drag.- The induced drag is no longer linear in the circulation so that the complex method could not be retained without new stipulations. It offers no longer any simplifications, and it is advisable to continue from here on the calculation with the imaginary constituent of all quantities in real form. If the calculation is carried out according to equation (7), W assumes the form

$$W = W_0 + W_1 \sin(\omega vt + \varphi_1) + W_2 \sin(2\omega vt + \varphi_2)$$

Here W_1 is different from zero only when the oscillation is superimposed on a constant angle of attack different from zero. W is in ω small, of second order. The purely periodical terms are, therefore, hardly significant; however, the temporal mean value W_0, which is different from zero is important. I calculate only this value and write for it, for reasons of simplicity, again W. I equate

$$A = A' + iA'', \quad B = B' + iB'', \quad C = C' + iC''$$

and may assume, without restricting the generality, $A'' = 0$. Then W becomes a quadratic form in $A' = A$, B', B'' (or A, C', C''): a simple deliberation shows that the coefficients of B'^2 and B''^2 (or C'^2 and C''^2) are equal and that the coefficient of $B'B''$ (or $C'C''$) is zero. Thus W becomes

$$\left.\begin{aligned} W &= \rho v^2\pi\left\{A^2 w_{\alpha\alpha} + 2AB' w_{\alpha\beta}' + 2AB'' w_{\alpha\beta}'' + (B'^2 + B''^2) w_{\beta\beta}\right\} \\ &= \rho v^2\pi\left\{A^2 w_{\alpha\alpha} + 2AC' w_{\alpha\gamma}' + 2AC'' w_{\alpha\gamma}'' + (C'^2 + C''^2) w_{\gamma\gamma}\right\} \end{aligned}\right\} \qquad (20)$$

The w_{ik} are again series in $\omega^n(\omega \log \omega)^m$ and have up to higher terms the values

$$\left.\begin{aligned}
w_{\alpha\alpha} &= -\omega^2 + \pi\omega^3 - \left(\frac{3}{4}\pi^2 - Z^2\right)\omega^4 - 2Z\omega^4\log\omega + \omega^4\log^2\omega + \ldots \\
w_{\alpha\beta}{}' &= \frac{1}{\omega} w_{\alpha\gamma}{}' = -\frac{1}{4}(3 - 2Z)\omega^2 - \frac{1}{2}\omega^2\log\omega + \\
&\quad \frac{\pi}{4}(3 - 2Z)\omega^3 + \frac{\pi}{2}\omega^3\log\omega + \ldots \\
w_{\alpha\beta}{}'' &= \frac{1}{\omega} w_{\alpha\gamma}{}'' = -\frac{1}{2}\omega + \frac{3}{4}\pi\omega^2 - \left(\frac{5}{8}\pi^2 - \frac{1}{2}Z^2 + \frac{1}{2}Z\right)\omega^3 + \\
&\quad \frac{1}{2}(1 - 2Z)\omega^3\log\omega = \frac{1}{2}\omega^3\log^2\omega + \ldots \\
w_{\beta\beta} &= \frac{1}{\omega^2} w_{\gamma\gamma} = \frac{\pi}{2}\omega - \frac{1}{2}(\pi^2 + 1)\omega^2 + \frac{\pi}{8}(3\pi^2 + 2 - 4Z^2 - 4Z)\omega^3 + \\
&\quad \frac{\pi}{2}(1 + 2Z)\omega^3\log\omega - \frac{\pi}{2}\omega^3\log^2\omega + \ldots
\end{aligned}\right\rbrace \quad (21)$$

W may be positive or negative, according to the selection of parameters. Depending on the type of motion, one has, therefore, to expect drag or thrust. I postpone detailed discussion until after calculation of the power requirement and the efficiency. The case of gliding over an undulating flow mentioned above, corresponding to $C' = 0$, $C'' = -A$, gives

$$W = A^2\rho v^2\pi\left[-\omega^4\log\omega - \left(\frac{1}{2} - Z\right)\omega^4\right] > 0$$

Thus the selection of parameters made does not yet correspond exactly to the case $W = 0$; a small correction would have to be provided for this case.

4. Work done at the wing.- In the free vortices behind the wing, energy is contained which must be produced by mechanical work on the airplane. This may be done in two ways. The flapping motion may result, as mentioned before, in positive or negative thrust. In the case of negative thrust a propeller which overcomes this and all other resistances to flight is required for maintenance of equilibrium of motion. In the case of positive thrust a propeller is needed only until the thrust due to flapping exceeds the resistances to flight, whether the flight be uniform or accelerated. As to the work performed at the wing itself, the wing motion consumes, of couse, energy if thrust exists; the ratio of thrust power $-L_w = -Wv$ and the total mechanical power L_f to be applied to the wing may then be denoted as aerodynamic efficiency of the flapping wing. In case of drag, two more possibilities exist. First, the flapping motion may require additional work. The efficiency defined above then becomes negative and arbitrarily large when the wing power L_f decreases more and more. Second, the case may occur that L_f becomes negative, that is, the wing then is supplied with energy from the air (indirectly by the propeller) and may use that energy for surmounting the resistances in the oscillation mechanism, or may store it in the oscillation itself, that is, increase its amplitudes. Aside from this "incremental power" the propeller must, of course, in this case yield additionally the energy of the free vortices so that one may define the quotient of $-L_f$ and the total propeller power as efficiency referred to the power absorption of the wing. This is then exactly the reciprocal value of the efficiency defined above which in this case, as the quotient of two negative numbers becomes positive but larger than one, thus loses its physical meaning. L_f is divided into two parts. One has

$$\frac{\partial y}{\partial t} = \dot{p} + \dot{\varphi}x = i\omega v(A + Bx)e^{i\omega vt}$$

Therewith $L_t = K\dot{p}$ becomes the "flapping power" and $L_r = M\dot{\varphi}$ the power opposed to the rotary oscillation; thus the wing power is $L_f = L_t + L_r$. The power opposed to the drag is denoted by $L_w = Wv$. In the air there then remains in all $L = L_f + L_w = L_t + L_r + L_w$. I indicate of all L_i again only the temporal mean values for which I obtain quadratic forms of the same type as for W

$$\left.\begin{aligned} L_i &= \rho v^3\pi\left\{A^2 l_{i\alpha\alpha} + 2AB' l_{i\alpha\beta'} + 2AB'' l_{i\alpha\beta''} + (B'^2 + B''^2) l_{i\beta\beta}\right\} \\ &= \rho v^3\pi\left\{A^2 l_{i\alpha\alpha} + 2AC' l_{i\alpha\gamma'} + 2AC'' l_{i\alpha\gamma'} + (C'^2 + C''^2) l_{i\gamma\gamma}\right\} \end{aligned}\right\} \quad (22)$$

L_r has finite series and is, in general, small compared to L_t. The coefficients are individually

$$l_{wik} = w_{ik} \tag{23}$$

$$\left.\begin{aligned}
l_{t\alpha\alpha} &= \omega^2 - \frac{\pi}{2}\omega^3 + \left(\frac{\pi^2}{4} - Z^2\right)\omega^4 + 2Z\omega^4\log\omega - \omega^4\log^2\omega + \ldots \\
l_{t\alpha\beta'} &= \frac{1}{\omega}\, l_{t\alpha\gamma'} = \frac{1}{4}(3 - 2Z)\omega^2 + \frac{1}{2}\omega^2\log\omega - \\
&\quad \frac{\pi}{4}(1 - 2Z)\omega^3 - \frac{\pi}{2}\omega^3\log\omega + \ldots \\
l_{t\alpha\beta''} &= \frac{1}{\omega}\, l_{t\alpha\gamma''} = \frac{1}{2}\omega - \frac{\pi}{4}\omega^2 + \left(\frac{\pi^2}{8} + \frac{1}{2}Z - \frac{1}{2}Z^2 - \frac{1}{8}\right)\omega^3 - \\
&\quad \frac{1}{2}(1 - 2Z)\omega^3\log\omega - \frac{1}{2}\omega^3\log^2\omega + \ldots \\
l_{t\beta\beta} &= \frac{1}{\omega^2}\, l_{t\gamma\gamma} = 0
\end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned}
l_{r\alpha\alpha} &= 0 \\
l_{r\alpha\beta'} &= \frac{1}{\omega}\, l_{r\alpha\gamma'} = 0 \\
l_{r\alpha\beta''} &= \frac{1}{\omega}\, l_{r\alpha\gamma''} = \frac{1}{8}\omega^3 \\
l_{r\beta\beta} &= \frac{1}{\omega^2}\, l_{r\gamma\gamma} = \frac{1}{2}\omega^2
\end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned}
l_{f\alpha\alpha} &= \omega^2 - \frac{\pi}{2}\,\omega^3 + \left(\frac{\pi^2}{4} - Z^2\right)\omega^4 + 2Z\omega^4\log\,\omega - \omega^4\log^2\omega + \ldots \\
l_{f\alpha\beta'} &= \frac{1}{\omega}\,l_{f\alpha\gamma'} = \frac{1}{4}\,(3 - 2Z)\omega^2 + \frac{1}{2}\,\omega^2\log\,\omega - \\
&\quad \frac{\pi}{4}\,(1 - 2Z)\omega^3 - \frac{\pi}{2}\,\omega^3\log\,\omega + \ldots \\
l_{f\alpha\beta''} &= \frac{1}{\omega}\,l_{f\alpha\gamma''} = \frac{1}{2}\,\omega - \frac{\pi}{4}\,\omega^2 + \left(\frac{\pi^2}{8} + \frac{1}{2}\,Z - \frac{1}{2}\,Z^2\right)\omega^3 - \\
&\quad \frac{1}{2}\,(1 - 2Z)\omega^3\log\,\omega - \frac{1}{2}\,\omega^3\log^2\omega + \ldots \\
l_{f\beta\beta} &= \frac{1}{\omega^2}\,l_{f\gamma\gamma} = \frac{1}{2}\,\omega^2
\end{aligned}\right\} \qquad (26)$$

$$\left.\begin{aligned}
l_{\alpha\alpha} &= \frac{\pi}{2}\,\omega^3 - \frac{\pi^2}{2}\,\omega^4 + \ldots \\
l_{\alpha\beta'} &= \frac{1}{\omega}\,l_{\alpha\gamma'} = \frac{\pi}{2}\,\omega^3 + \ldots \\
l_{\alpha\beta''} &= \frac{1}{\omega}\,l_{\alpha\gamma''} = \frac{\pi}{2}\,\omega^2 - \frac{\pi^2}{2}\,\omega^3 + \ldots \\
l_{\beta\beta} &= \frac{1}{\omega^2}\,l_{\gamma\gamma} = \frac{\pi}{2}\,\omega - \frac{\pi^2}{2}\,\omega^2 + \left(\frac{3}{8}\,\pi^3 + \frac{\pi}{4} - \frac{\pi}{2}\,Z^2 - \frac{\pi}{2}\,Z\right)\omega^3 + \\
&\quad \frac{\pi}{2}\,(1 + 2Z)\omega^3\log\,\omega - \frac{\pi}{2}\,\omega^3\log^2\omega + \ldots
\end{aligned}\right\} \qquad (27)$$

L, as the energy of the vortex trail, can of course never become negative and must therefore be a positive quadratic form - definitive in the amplitudes. The fact that L in the form here noted is capable also of small negative values is caused by the neglect of higher terms in the series development. What is obtained in this case, is therefore only the error, accidentally negative, of the almost vanishing vortex power. Since, when C is used, all coefficients contain the factor ω^2, the latter has been cancelled in the graphical plotting which is in agreement with the presentation of the power for constant flapping velocity (fig. 4). The curves show that the essential terms always stem from the stroke amplitude A, possibly in combination of the latter with the amplitude of rotary oscillation, and that the corresponding coefficients increase somewhat more slowly than ω^2. By rotary oscillation I meant above an oscillation about the z-axis. More generally, every oscillation where the ratio of A and C is real is a rotary oscillation about the fixed axis with the abscissa a where A then is $A = -aC\omega$. It is shown that for not too large values of $a\omega$, that is, for axes which do not lie at too great a distance, W and L are always positive; that is, it is not possible to obtain thrust or power absorption by rotary oscillations about fixed axes. Production of thrust always requires a stroke amplitude different from zero, power absorption requires additionally a rotary oscillation lagging by about 90°. Pure stroke oscillation without rotation also produces thrust which in roughest approximation results as $-W = A^2 \rho v^2 \pi \omega^2$, similar to the so-called Knoller-Betz effect (if one calculates with the y-axis as the "polar" in the plane problem).

One obtains good insight into the variation of drag and power if one varies, for fixed absolute value of the amplitude ratio $C/A = c$, only the phase angle φ between the two oscillation components where one then has to put

$$C' = Ac' = A|c| \cos \varphi, \qquad C'' = Ac'' = A|c| \sin \varphi$$

W is shown to become a minimum, the thrust thus a maximum, when the rotation leads by somewhat more than 90°, namely by $\varphi = \text{arc tan} \frac{w_{\alpha\gamma}''}{w_{\alpha\gamma}'}$.

It is plausible physically, too, that the thrust will assume large values precisely then when the phase is shifted by about 180° compared to the phase which is present for gliding free from air forces over the wave course. For every $|c|$ there exists an ω and vice versa for which the thrust maximum is absolute. One then has

$$c'(\omega) = - \frac{w_{\alpha\gamma}'}{w_{\gamma\gamma}}$$

$$c''(\omega) = - \frac{w_{\alpha\gamma}''}{w_{\gamma\gamma}}$$

$$W_{min} = A^2 \rho v^2 \pi w_{min}$$

$$w_{min} = w_{\alpha\alpha} - \frac{w_{\alpha\gamma}'^2 + w_{\alpha\gamma}''^2}{w_{\gamma\gamma}}$$

$$= - \frac{1}{2\pi} \omega - \frac{1}{2\pi^2} \omega^2 - \frac{1}{2\pi^3}\left(1 + \frac{3}{4} \pi^2\right)\omega^3 - \cdot \cdot \cdot$$

L becomes a minimum and disappears for suitable ω except for terms of the fifth order when the rotation is lagging by somewhat more than 90°. Finally, the wing power L_f becomes a minimum - thus, the absorbed power a maximum - when the rotation lags by somewhat less than 90°, namely for $\varphi = \text{arc tan} \frac{l_{f\alpha\gamma}''}{l_{f\alpha\gamma}'}$. This maximum too becomes absolute when between c and ω the following condition is satisfied

$$c'(\omega) = - \frac{l_{f\alpha\gamma}'}{l_{f\gamma\gamma}}$$

$$c''(\omega) = - \frac{l_{f\alpha\gamma}''}{l_{f\gamma\gamma}}$$

$$L_{fmin} = A^2 \rho v^3 \pi l_{fmin}$$

$$l_{fmin} = l_{f\alpha\alpha} - \frac{l_{f\alpha\gamma}'^2 + l_{f\alpha\gamma}''^2}{l_{f\gamma\gamma}} = -1 + \frac{\pi}{2} \omega + \cdot \cdot \cdot$$

Altogether, L becomes negative only when $|c| > 1$.

The efficiency

$$\eta_1 = \frac{1}{\eta_2} = -\frac{L_w}{L_f} = 1 - \frac{L}{L_f} \quad \text{where} \quad \left.\begin{array}{l} \eta_1 \text{ is valid for } L_f > 0 \text{ and} \\ \eta_2 \text{ for } L_f < 0. \end{array}\right\} \qquad (28)$$

is the quotient of two quadratic forms and capable of a great many values. Generally, it becomes negative for small or completely vanishing stroke amplitudes, and arbitrarily large with $\omega \longrightarrow 0$. The same is valid for rotations about a fixed not too remote axis. If, however, $A \neq 0$ and also $\lim_{\omega=0} A \neq 0$, η approaches for $\omega \longrightarrow 0$ to the limit 1.

By a rotary component $(c'' = 0)$ of equal or opposite phase η is always deteriorated compared to $c = 0$; the same is true for $c' = 0$, $c'' > 0$. In contrast, the efficiency is improved for $c' = 0$, $-1 \leqq c'' \leqq 0$, as can be seen from the diagram (fig. 5). For $c' = 0$, $c'' = -1$, η becomes identically 1, and for $c' = 0$, $c'' < -1$ there results power absorption. The representation for fixed $|c|$ in dependence on φ as a discontinuous single-wave-harmonic function is very graphical for the efficiency as well. Figure 6 shows clearly at what phase angles the transition from power absorption to power production takes place. The most important ones among the coefficients found from the series developments have been compiled in the numerical table (table I).

5. Application to the flutter of elastically supported wings.- The derived laws could be practically applied in the investigation of a phenomenon our pilots observed in the last war. In the so-called sesquiplanes, the lower wing was fastened to one single spar only, thus was only slightly elastic against small deflections and rotations. In case of increased flight velocity, for instance in steep dives, there occurred sometimes vigorous flutter of the lower wing tips which underwent obviously unstable oscillations in the increased air flow. Of course, such unstable oscillations are possible only if energy is supplied to the oscillating system, and this occurs, according to my investigations, only when the vector of the amplitude of rotation lags by about 90° and when the amplitude of rotation itself is sufficiently large $\left(|C| \text{ must be } > |A|\right)$. Let us visualize again the gliding - almost free from air forces - of the wing over an undulating flow course where the airspeed (relative to the wing elements) has no vertical component. If the amplitude of rotation is smaller than corresponds to this case, the motion is damped by the counteracting air force. If the amplitude of rotation is, on the contrary, larger than in the case above and the wing therefore scoops more deeply into the air, the air force always acts in the direction of the motion, and the motion is excited.

I consider a wing supported on spars, elastic with respect to translation in the y-direction and with respect to twist. In order to be able to go on from my formulas used so far, I introduce the directional forces per unit length in z-direction; I calculate therefore as if these forces were distributed continuously over the length of the wing. This assumption does not lead to any contradictions if the wing in itself, aside from its support, is sufficiently stiff. Since I disregard deflections in the flight direction, I can show that a wing supported on spars always has only three essential elasticity parameters, corresponding to the three constants of the work of deformation quadratic in p and φ. (Compare fig. 1 and the original report.) With respect to its elastic properties, this wing may therefore always be replaced by a wing which is supported only on one spar (the "elastic axis") with the abscissa a, elastic with respect to translation by means of the directional force c, and with respect to rotation by means of the directional moment γ, as schematically indicated in figure 1. The resultant of the elastic forces and its moment at the origin then are

$$K = -c(p + a\varphi)$$

$$M = -cap - (ca^2 + \gamma)\varphi$$

If s is the abscissa of the center-of-gravity axis (point S, fig. 1), the momentum and its moment at the origin have the values (with the mass m per unit length and the corresponding moment of inertia $\vartheta = mr^2$ for the center-of-gravity axis)

$$G = m(\dot{p} + s\dot{\varphi})$$

$$U = ms\dot{p} + (ms^2 + \vartheta)\dot{\varphi} = ms\dot{p} + m(s^2 + r^2)\dot{\varphi}$$

The equations of motion of the wing then read

$$\dot{G} = K \qquad \dot{U} = M$$

For K and M the air forces have to be added. If I put as before

$$p = Ae^{\omega' vt} \qquad \varphi = Be^{\omega' vt}$$

and take into consideration that the air forces have on the wing the opposite effect from the one the forces of the wing calculated above have on the air, K and M become

$$K = -e^{\omega' vt}\left\{cA + caB + \rho v^2\pi(Ak_\alpha + Bk_\beta)\right\}$$

$$M = -e^{\omega' vt}\left\{caA + (ca^2 + \gamma)B + \rho v^2\pi(A_{m\alpha} + B_{m\beta})\right\}$$

For abbreviation, I introduce the following designations

$$\frac{\rho\pi}{m} = \rho_0 \qquad \frac{c}{mv^2} = \omega_0^2 \qquad \frac{\gamma}{c} = q^2$$

Therewith there result finally as the equations of oscillation

$$A(\omega'^2 + \omega_0^2 + \rho_0 k_\alpha) + B(s\omega'^2 + a\omega_0^2 + \rho_0 k_\beta) = 0$$

$$A(s\omega'^2 + a\omega_0^2 + \rho_0 m_\alpha) + B\left((r^2 + s^2)\omega'^2 + (a^2 + q^2)\omega_0^2 + \rho_0 m_\beta\right) = 0 \qquad (29)$$

Equating the determinant to zero yields the equation for the frequencies $\omega = -i\omega'$. If I first disregard the forces (for this purpose I put $\rho_0 = 0$), there result two main oscillations as a consequence of the coupling of the oscillation of the center of gravity with the frequency ω_0 and the oscillation about the center of gravity with the frequency $\frac{q}{r}\omega_0$. With $\delta = (s - a)^2 + r^2 + q^2$ there is

$$\left.\begin{aligned} \omega_{1.2}^2 &= -\omega_{1.2}'^2 = \frac{\omega_0^2}{2r^2}\left(\delta \pm \sqrt{\delta^2 + 4q^2r^2}\right) \\ \frac{A}{B} &= -\frac{s\left(\delta \pm \sqrt{\delta^2 - 4q^2r^2}\right) - 2ar^2}{\delta \pm \sqrt{\delta^2 - 4q^2r^2} - 2r^2} = -a_{1.2} \\ y_k &= B_k(x - a_k)e^{i\omega_k vt}; \quad k = 1.2 \end{aligned}\right\} \qquad (30)$$

The main oscillations therefore are rotary oscillations about fixed axes at the distances a_1, a_2. The quantity $s - a$ forms a measure for the coupling. In general, there develop beats from both main oscillations. If I write those in the form

$$y = \left\{ -a_1 B_1 e^{i\epsilon vt} - a_2 B_2 e^{-i\epsilon vt} + x\left[B_1 e^{i\epsilon vt} + B_2 e^{-i\epsilon vt}\right] \right\} e^{i\omega vt}$$

$$\omega_1 + \omega_2 = 2\omega$$

$$\omega_1 - \omega_2 = 2\epsilon \ll \omega$$

The motion may be interpreted as an ordinary oscillation with an amplitude ratio slowly variable as to magnitude and phase. Therein there appears of course, periodically recurrent, the phase angle which corresponds in the air flow to the power absorption. If the air forces are to counteract the change of this phase angle, corresponding to continuous supply of energy, it can be shown that the case of slight coupling ($s - a$ small) for balanced or almost balanced frequencies of the uncoupled system is the best presupposition for this phenomenon. I am anticipating from the results of the following calculation with consideration of the air forces that without coupling ($s = a$) no unstable oscillation at all would be possible.

The air-force coefficients occurring in the oscillation determinant are themselves functions of the frequency ω' which cannot be indicated by simple analytical expressions so that the roots of the oscillation determinant cannot be obtained in a simple manner. However, since their existence is secured by the physical meaning of the problem, it is permissible to introduce into the equation instead of the coefficients k and m, the first terms of their series developments, all the more so since the occurring factor ρ_0 generally is a small number; for it is sensible to break off the series for $\cos x$ in order to find from the polynomial obtained for instance approximately the first zero of this function whereas the same method is meaningless for e^x, since no zeros of this function exist in the finite domain.

The oscillation determinant obtains the following form

$$r^2\omega'^4 + \delta\omega_0^2\omega'^2 + q^2\omega_0^4 + \rho_0\omega'^2\left[(r^2 + s^2)k_\alpha - sk_\beta - sm_\alpha + m_\beta\right] +$$

$$\rho_0\omega_0^2\left[(a^2 + q^2)k_\alpha - ak_\beta - am_\alpha + m_\beta\right] + \rho_0^2(k_\alpha m_\beta - k_\beta m_\alpha) = 0 \qquad (31)$$

The roughest approximation is the result of the calculation which bases the determination of the lift and moment coefficients - in the manner of the theory of the Knoller-Betz effect - on the momentary apparent angle of attack and the apparent (dynamic) curvature of the wing. Even with this procedure, there result in certain cases complex roots ω' with positive real parts which result in an "increment" of the oscillations and thus correspond to dynamic instability. I shall not here discuss this approximation more closely. Also, I shall only briefly mention the case of the greatest instability since this case is trivial. It occurs if the elastic axis lies so far to the rear that at the slightest displacement of the wing from equilibrium position the air flow simply causes the apparatus to tip over aperiodically toward the rear. This always occurs as soon as a is positive and the air velocity sufficiently large, namely for $a > \frac{q^2\omega_0^2}{2\rho_0}$ or $v^2 > \frac{\gamma}{2\rho\pi a}$ for $a > 0$.

If I retain of k and m all terms up the second order in ω', the period-equation of the wing becomes

$$f(\omega') = a_0\omega'^4 + a_0'\omega'^4 \log \omega' + a_0''\omega'^4 \log^2\omega' + b_0\omega'^3 + b_0'\omega'^3 \log \omega' +$$

$$c_0\omega'^2 + c_0'\omega'^2 \log \omega' + c_0''\omega'^2 \log^2\omega' + d_0\omega' +$$

$$d_0'\omega' \log \omega' + e_0 = 0 \tag{32}$$

With the coefficients

$$a_0 = r^2 + \rho_0\epsilon + \rho_0^2\left(\frac{1}{8} + \frac{1}{4} Z - Z^2\right) > 0$$

$$b_0 = \rho_0\mu + \rho_0^2\left(\frac{1}{4} - Z\right) > 0$$

$$c_0 = \omega_0^2\left(\delta + \rho_0\nu\right) - 2\rho_0 s + \rho_0^2$$

$$d_0 = \omega_0^2\rho_0\lambda > 0$$

$$e_0 = \omega_0^2\left(q^2\omega_0^2 - 2\rho_0 a\right)$$

$$a_0' = \rho_0\left[2(r^2 + s^2) - 2s(1 - 2Z) - \rho_0\left(\frac{1}{4} - 2Z\right)\right]$$

$$a_0'' = -\rho_0(2s + \rho_0)$$

$$b_0' = -\rho_0\left(2s - \rho_0\right)$$

$$c_0' = 2\rho_0\omega_0^2\left[a^2 + q^2 - a(1 - 2Z)\right]$$

$$c_0'' = d_0' = -2\rho_0 a\omega_0^2$$

$$\delta = (s - a)^2 + r^2 + q^2 > 0$$

$$\mu = 1 + 2(r^2 + s^2) - s(3 - 2Z) > 0$$

$$\lambda = 1 + 2(a^2 + q^2) - a(3 - 2Z) > 0$$

$$\epsilon = (r^2 + s^2)(1 - 2Z) - s(1 - 2Z + 2Z^2) + \frac{3}{8} > 0$$

$$\nu = (q^2 + a^2)(1 - 2Z) - a(1 - 2Z + 2Z^2) + \frac{3}{8} > 0$$

$$Z = 0.11593 \ . \ . \ .$$

6. Numerical evaluation.- The equation can be solved only approximately, of course. For this purpose, I first omitted the logarithmic terms and determined the roots $\bar{\omega}'$ of the algebraic equation

$$g(\bar{\omega}') = a_0\bar{\omega}'^4 + b_0\bar{\omega}'^3 + c_0\bar{\omega}'^2 + d_0\bar{\omega}' + e_0 = 0 \tag{33}$$

I regard this equation as an approximation, equate $\omega' = \bar{\omega}' + \chi$, and can now develop the logarithmic terms retaining the linear terms in χ. Thus I obtain

$$\chi = -\bar{\omega}' \log \bar{\omega}' \frac{R}{S} \tag{34}$$

$$R = a_0'\bar{\omega}'^3 + a_0''\bar{\omega}'^3\log \bar{\omega}' + b_0'\bar{\omega}'^2 + c_0'\bar{\omega}' + d_0'(1 + \bar{\omega}' \log \bar{\omega}')$$

$$S = 4a_0\bar{\omega}'^3 + a_0'\bar{\omega}'^3(1 + 4 \log \bar{\omega}') + 2a_0''\bar{\omega}'^3\log \bar{\omega}'(1 + 2 \log \bar{\omega}') +$$

$$3b_0\bar{\omega}'^2 + b_0'\bar{\omega}'^2(1 + 3 \log \bar{\omega}) + 2c_0\bar{\omega}' + c_0'\bar{\omega}'(1 + 2 \log \bar{\omega}') +$$

$$d_0 + d_0'(1 + 2\bar{\omega}' \log \bar{\omega}')(1 + \log \bar{\omega}')$$

The procedure may be continued and yields the further approximation

$$\chi' = -\frac{R(\omega')\omega' \log \omega' + g(\omega')}{S(\omega')}$$

Finally, there follows the complex amplitude ratio $B:A = b$ from one of the equations (29).

Finally one now has to learn the conditions for which the equation for ω' has complex roots with positive real part, corresponding to excited oscillations or critical support of the wing. For the limiting case of dynamic indifference, I formulate the roots of the equation (32) as purely imaginary and obtain, by setting the real and the imaginary constituent of the equation equal to zero, the conditions

$$\left.\begin{aligned}&\left(a_0 - \frac{\pi^2}{4} a_0''\right)\omega^4 + a_0'\omega^4\log\omega + a_0''\omega^4\log^2\omega + b_0'\frac{\pi}{2}\omega^3 - \\ &\left(c_0 - \frac{\pi^2}{4} c_0''\right)\omega^2 - c_0'\omega^2\log\omega - d_0'\omega\left(\frac{\pi}{2} + \omega\log^2\omega\right) + e_0 = 0 \\ &a_0'\frac{\pi}{2}\omega^3 + a_0''\pi\omega^3\log\omega - b_0\omega^2 - b_0'\omega^2\log\omega - \\ &c_0'\frac{\pi}{2}\omega + d_0'\log\omega(1 - \pi\omega) + d_0 = 0\end{aligned}\right\} \tag{35}$$

From them ω would have to be eliminated in every special case whereby a conditional equation for any of the parameters s, a, r^2, q^2, ω_0, ρ_0 or for the flight velocity v results. As an approximation, it is sufficient to investigate the equation (33). Since the approximation (34) shows in the case of instability, a small additional damping, the criteria for instability derived from the following equations are necessary but not always sufficient; they are therefore valid only with the reservation of checking with the approximation (34). When they are satisfied, the wing may at any rate be regarded as critically supported. Before I set down the most general criterion for instability, I shall mention one which is very simple but suffices only for cases of great instability. This is the condition that the coefficient c_0 becomes negative

$$v^2 > \frac{c(\delta + \rho_0\nu)}{\rho\pi(2s - \rho_0)} \tag{36}$$

Written as a function of s, the same condition reads

$$s^2\omega_0^2 - 2s\left(\rho_0 + a\omega_0^2\right) + \rho_0^2 + \omega_0^2\left(a^2 + r^2 + q^2 + \rho_0\nu\right) < 0$$

Further limits are yielded by the Routh condition

$$c_0 < a_0 \frac{d_0}{b_0} + e_0 \frac{b_0}{d_0}$$

or

$$\omega_0^2\left(\delta + \rho_0\nu\right) + \rho_0^2 - 2\rho_0 s < \left[r^2 + \rho_0\epsilon + \rho_0^2\left(\frac{1}{8} + \frac{1}{4} Z - Z^2\right)\right]\frac{\lambda\omega_0^2}{\mu + \rho_0\left(\frac{1}{4} - Z\right)} + \left(q^2\omega_0^2 - 2\rho_0 a\right)\frac{\mu + \rho_0\left(\frac{1}{4} - Z\right)}{\lambda} \qquad (37)$$

In order to satisfy this equation, it will above all be required that v be sufficiently large, that is, ω_0 sufficiently small. Furthermore, s must be positive; therewith s - a is positive, too, because, as mentioned before, $a > 0$ would for sufficiently large v always result in great instability. One can readily understand that this must be so. The centroidal axis of the wing as axis of inertia generally lags behind the elastic axis as line of application of the directional force. If the centroidal axis therefore lies, in flight direction, behind the elastic axis $(s - a > 0)$, the wing has in its upward motion, on the average, a positive angle of attack; the opposite is true for the downward motion. Thus the air forces always take effect in the direction of the motion and amplify the oscillation; whereas in the case $s - a < 0$ the opposite, that is, damping occurs.

It can easily be confirmed that both degrees of freedom must act together for achievement of the oscillation. The calculation always yields roots with negative real parts if one of the degrees of freedom is suppressed (corresponding to $\omega_0 = \infty$ or $q^2 = \infty$). This fact is confirmed by the failure of tests undertaken formerly in Göttinger with a wing with only one degree of freedom.

The power L produced or, respectively, absorbed by the wing is according to previous formulas

$$\left.\begin{aligned} L_f &= A^2 \rho\pi v^3 l_f \\ l_f &= l_{f\alpha\alpha} + 2b' l_{f\alpha\beta'} + \ldots \end{aligned}\right\} \qquad (38a)$$

If I put temporarily $\omega' = \beta + i\omega$, the mean value of the entire wing energy for constant amplitude A is

$$\overline{E} = \frac{1}{4} mv^2 A^2 Q$$

$$Q = \omega_0^2 + \omega^2 + 2b'\left(a\omega_0^2 + s\omega^2\right) + \left(b'^2 + b''^2\right)\left[\left(a^2 + q^2\right)\omega_0^2 + \left(r^2 + s^2\right)\omega^2\right]$$

$A = A_0 e^{v\beta t}$, and hence, in a different form

$$\left.\begin{aligned} -L_f &= \frac{d\overline{E}}{dt} = \frac{1}{2} mv^2 A\dot{A} Q = \frac{\beta}{2} mv^3 A^2 Q \\ l_f &= -\frac{\beta}{2\rho_0} Q \end{aligned}\right\} \qquad (38b)$$

If the oscillation calculation has been carried out, the agreement of the values l_f found by different methods offers therefore a control for the calculation.

7. Comparison with a test result.- For confirmation of the theory, tests with a light wooden wing of 60 cm length and 10 cm chord were performed in the small wind tunnel of the Göttinger aerodynamic test institute.

This wing was suspended on an axis between plane side walls by means of springs. The springs acted on cross-shaped metal sheets which carried moreover sliding weights for variation of the distance from the center of gravity and of the moment of inertia. The suspension springs or, respectively, their initial compression transferred the directional force or, respectively, the directional moment corresponding to my formulation to the suspension axis. I shall not discuss here further details of the apparatus; I refer instead to the drawing figure 7. The purpose of the tests was the determination of the increments or decrements of the oscillations; for this it was sufficient to plot the oscillation of a point, for instance the suspension axis. This was done with the aid of a registering drum kindly put at our disposal by the physiological Institute of the University Göttingen. A few of the diagrams of damped and increasing oscillations thus obtained are reproduced in figure 8. From these diagrams the ratios $\sigma = e^{\pi\frac{\beta}{\omega}}$ of amplitudes succeeding one another were determined and were compared with the values found by calculation with consideration of the frictional damping produced by σ_{ot} and σ_{or} of the translatory or rotary oscillations. Within the considerable limits of error which could have been cut down only by a major expenditure of time and means, the agreement may be called satisfactory.

The variation of the numbers was in excellent agreement with the theory. An increase or damping of the small oscillations resulted according to whether the wing was overweight toward the rear or the front; the ratio of amplitudes succeeding one another increased when the overweight was increased or when the air was blowing more strongly against the wing. A precession of the leading edge of the wing was clearly evident when the wing was supported so as to be unstable. This corresponds in the usual notation to a lag of the rotary oscillation as must be the case for power absorption.

Finally, it is noteworthy that for small air velocities of about 5 m per second the influence of the viscosity was shown by the fact that the lift was not yet fully developed at small angles of attack. For the smaller characteristic values of about 500 m/sec. mm a symmetrical wing has a small range "dead angle of attack" where the lift remains near zero (as shown also by other tests in Göttingen). The small oscillations were therefore still damped when oscillations with a larger initial amplitude were already increasing. The limiting amplitude lying between these two cases decreased of course more and more with increasing v.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

TABLE I

NUMERICAL TABLE OF CALCULATED COEFFICIENTS

ω =	0.02	0.04	0.06	0.08	0.10	0.12
k_α' =	0.00262	0.00773	0.01351	0.01892	0.02317	0.02572
k_α'' =	0.03852	0.07387	0.10605	0.13532	0.16182	0.18601
k_γ' =	0.03860	0.07449	0.10806	0.13982	0.17029	0.19989
k_γ'' =	-0.00184	-0.00468	-0.00694	-0.00788	-0.00710	-0.00452
m_α' =	-0.0002	-0.0008	-0.0018	-0.0032	-0.0050	-0.0072
m_α'' =	0	0	0	0	0	0
m_γ' =	-0.000003	-0.000024	-0.000081	-0.000192	-0.000375	-0.000648
m_γ'' =	0.0004	0.0016	0.0036	0.0064	0.0100	0.0144
$\omega^{-2}w_{\alpha\alpha}$ =	-0.9336	-0.8684	-0.8073	-0.7514	-0.7014	-0.6576
$\omega^{-2}w_{\alpha\gamma}'$ =	0.02369	0.03209	0.03480	0.03419	0.03150	0.02752
$\omega^{-2}w_{\alpha\gamma}''$ =	-0.4529	-0.4094	-0.3707	-0.3371	-0.3089	-0.2862
$\omega^{-2}w_{\gamma\gamma}$ =	0.02924	0.05414	0.07468	0.09088	0.10273	0.11024
$\omega^{-2}l_{\alpha\alpha}$ =	0.02944	0.05494	0.07648	0.09408	0.10773	0.11744
$\omega^{-2}l_{\alpha\gamma}'$ =	0.00063	0.00251	0.00566	0.01005	0.01571	0.02262
$\omega^{-2}l_{\alpha\gamma}''$ = $\omega^{-2}l_{\gamma\gamma}$ =	0.02944	0.05494	0.07648	0.09408	0.10773	0.11744
$\omega^{-2}l_{f\alpha\alpha}$ =	0.9631	0.9233	0.8837	0.8455	0.8091	0.7750
$\omega^{-2}l_{f\alpha\gamma}'$ =	-0.02306	-0.02958	-0.02914	-0.02414	-0.01579	-0.00490
$\omega^{-2}l_{f\alpha\gamma}''$ =	0.48235	0.4643	0.4471	0.4312	0.4166	0.4036
$\omega^{-2}l_{f\gamma\gamma}$ =	0.0002	0.0008	0.0018	0.0032	0.0050	0.0072

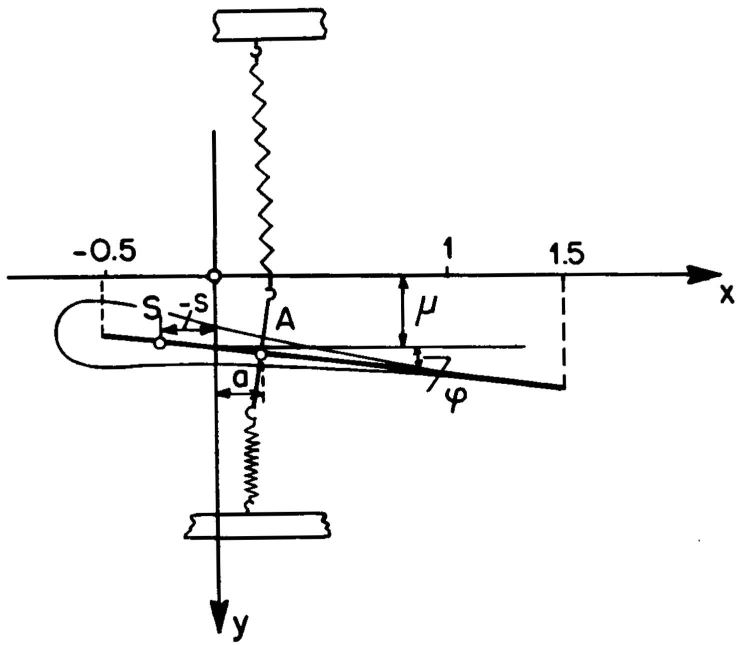

Figure 1.- Elastic suspension of a Joukowsky profile with plane "spine."

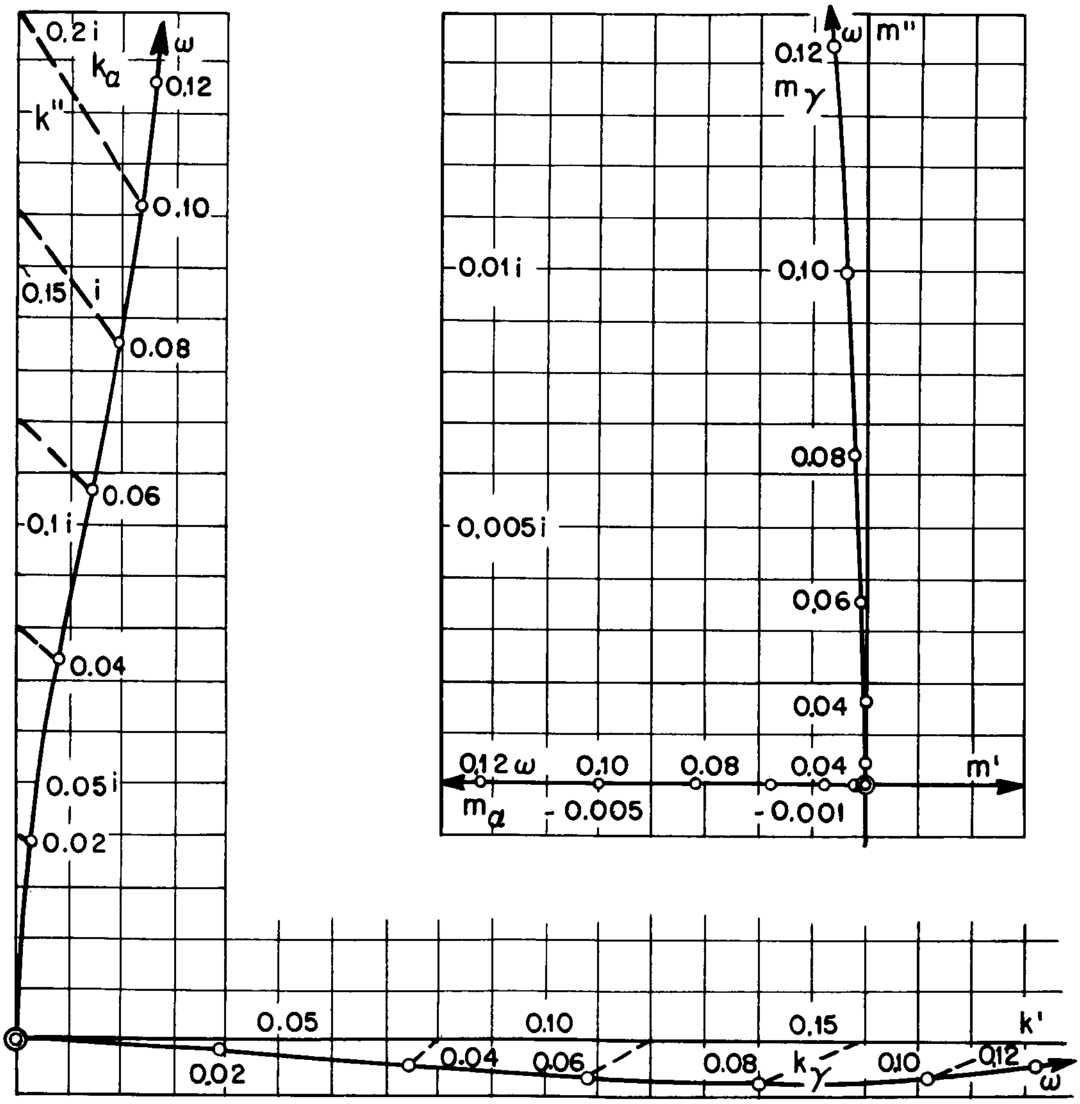

Figure 2.- Vectorial lift and moment coefficients for equal beat amplitudes.

5U

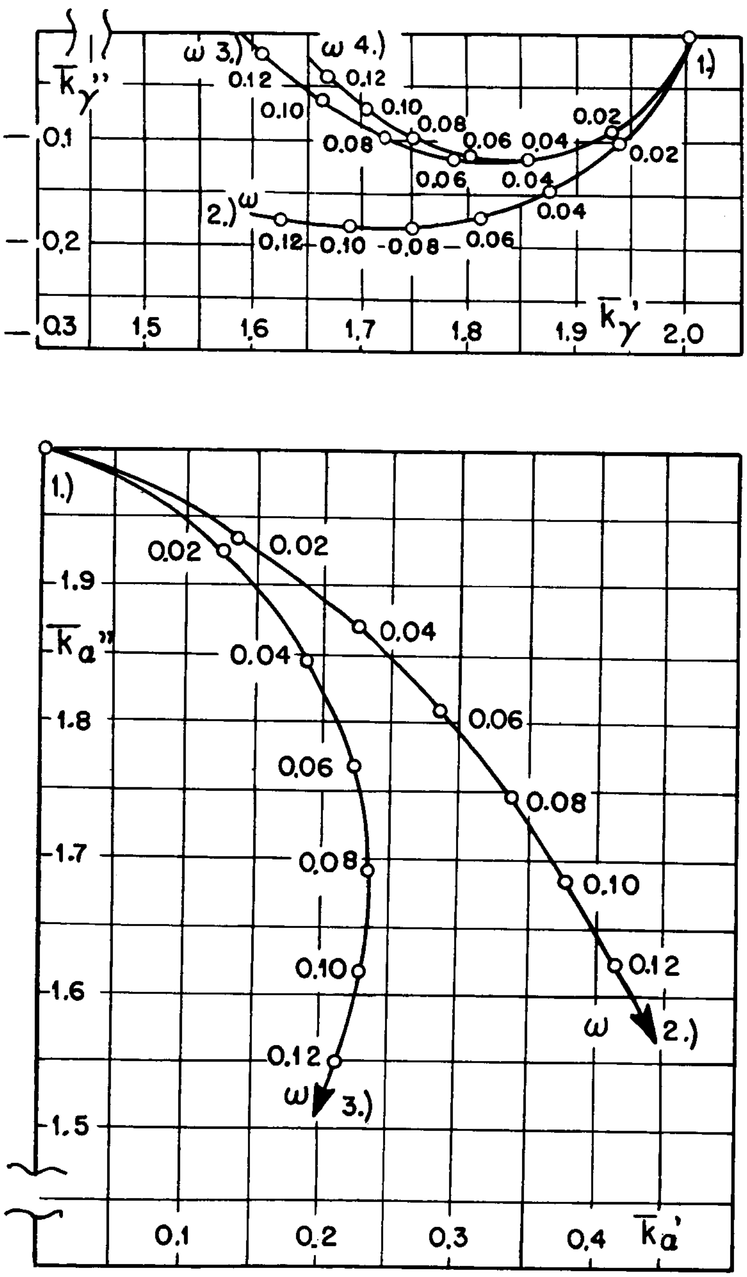

Figure 3.- Vectorial lift and moment coefficients for equal beat velocity.

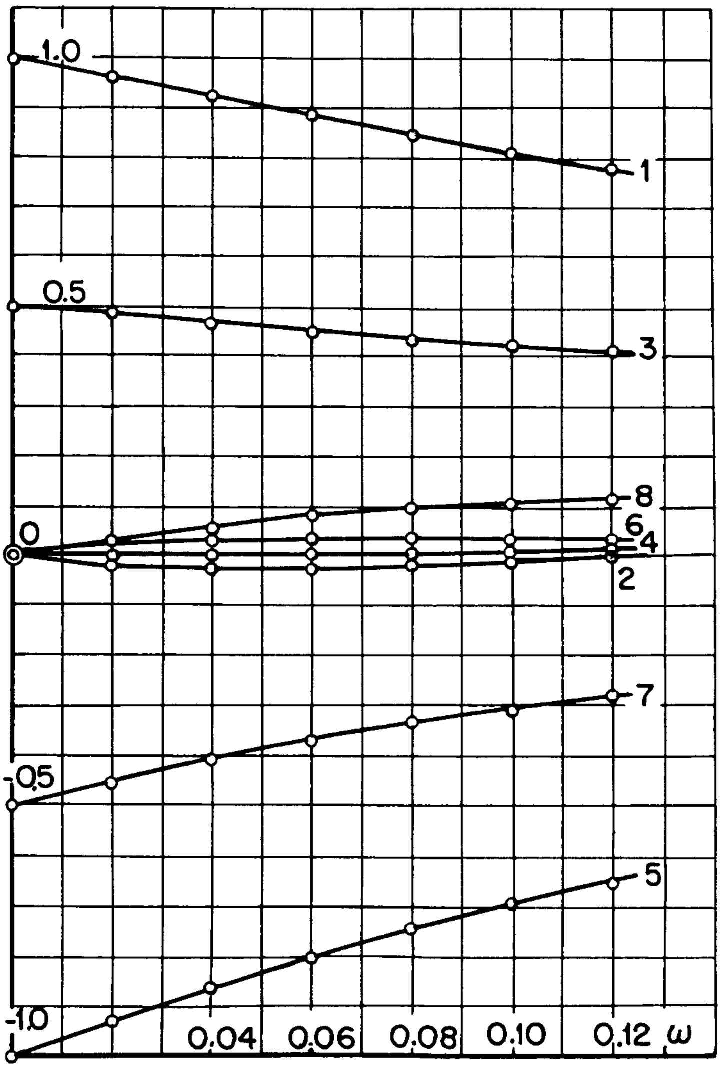

Figure 4.- Power and drag coefficients for equal beat velocity.
(1) $\omega^{-2} l f_{\alpha\alpha}$; (2) $\omega^{-2} l f_{\alpha\gamma}'$; (3) $\omega^{-2} l f_{\alpha\gamma}''$; (4) $\omega^{-2} l f_{\gamma\gamma}$;
(5) $\omega^{-2} w_{\alpha\alpha}$; (6) $\omega^{-2} w_{\alpha\gamma}'$; (7) $\omega^{-2} w_{\alpha\gamma}''$; (8) $\omega^{-2} w_{\gamma\gamma}$.

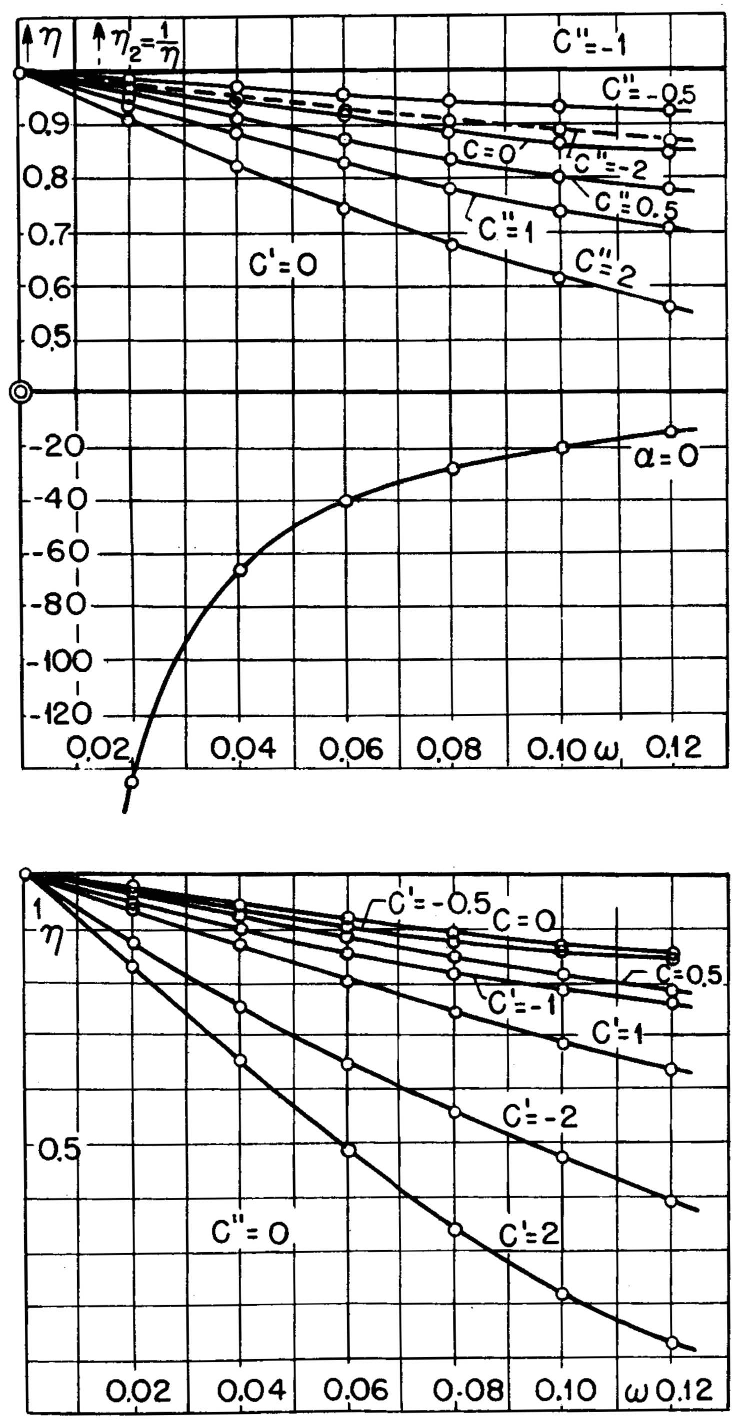

Figure 5.- Degrees of efficiency as a function of ω for different amplitude ratios c.

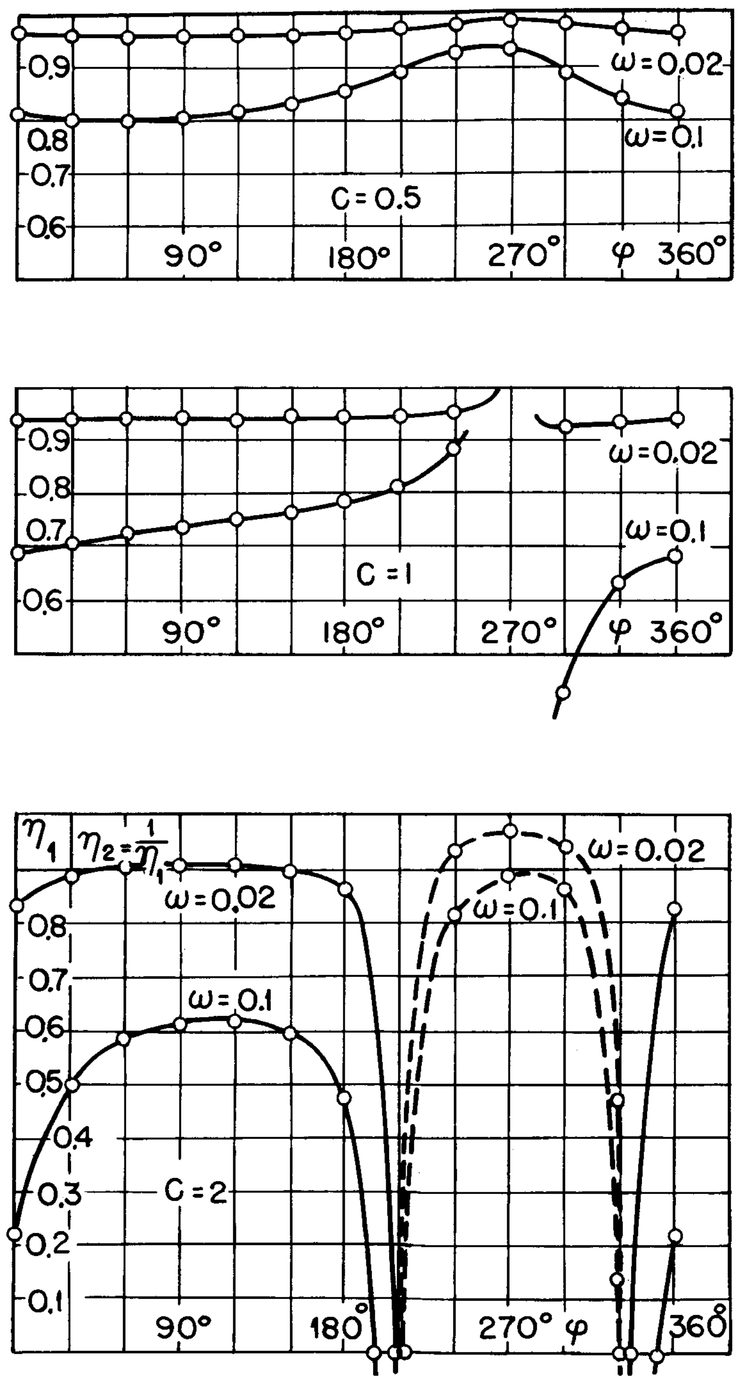

Figure 6.- Degrees of efficiency as a function of the phase φ.

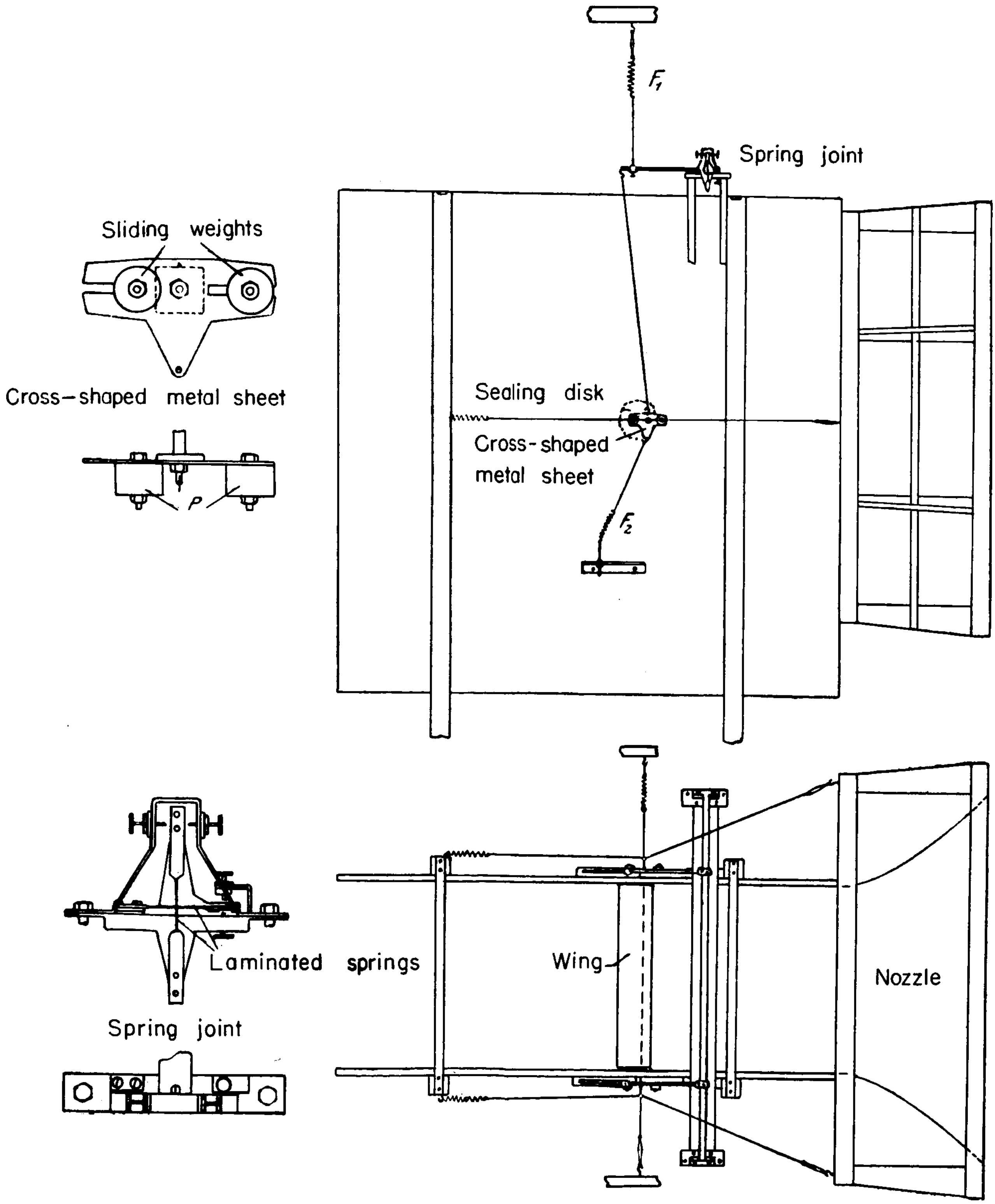

Figure 7.- Test arrangement.

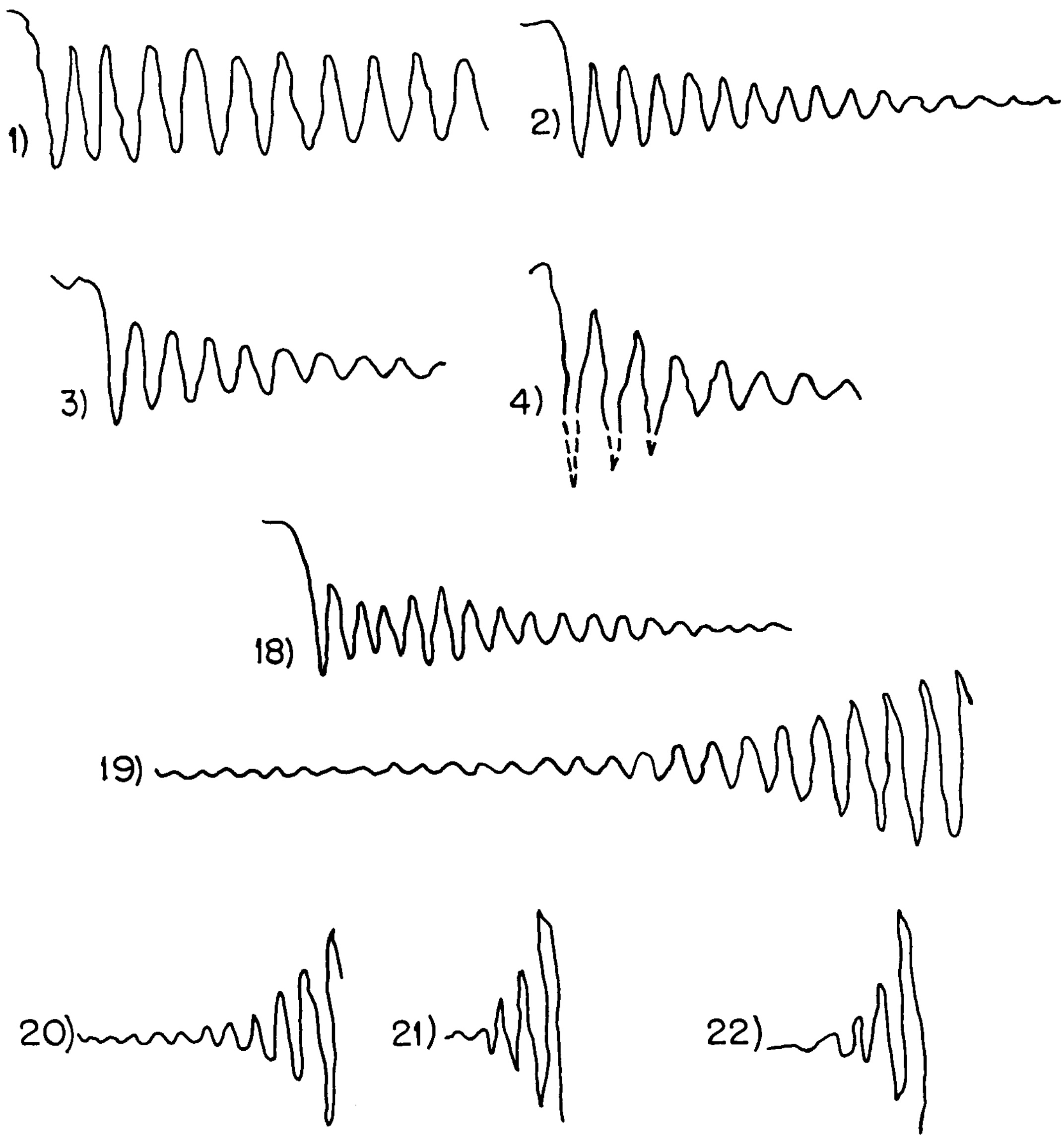

Figure 8.- Oscillation diagrams.

NACA-Langley - 1-14-54 - 1000