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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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EXTENSION TO THE CASES OF TWO DIMENSIONAL AND SPHERICALLY SYMMETRIC FLOWS OF TWO PARTICULAR SOLUTIONS TO THE EQUATIONS OF MOTION GOVERNING UNSTEADY FLOW IN A GAS

By Lorenzo Poggi

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SUMMARY

The author previously discovered two interesting particular solutions to the equations of motion describing unsteady flow in a gas confined solely to a one-dimensional duct. These solutions are now extended to cover the more noteworthy cases of central symmetry in two- and three-dimensions.

SOLUTIONS

As in a previous study (ref. 1), the equations of motion describing unsteady one-dimensional flow for a gas obeying the adiabatic law of expansion may be cast into the form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{2}{\gamma - 1} a \frac{\partial a}{\partial x} \qquad \text{(1-one)}$$

$$\frac{\partial a}{\partial t} + u\frac{\partial a}{\partial x} = -\frac{\gamma - 1}{2} a \frac{\partial u}{\partial x} \qquad \text{(2-one)}$$

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where u is the flow-velocity, a is the local velocity of sound corresponding to the state of the fluid, x is the distance along the abscissa or duct axis, t is the time, and γ is the ratio of the specific heat at constant pressure to that at constant volume. The first of these equations constitutes the "equation of motion" (arising from the equilibrium condition on the dynamic forces) while the second arises from the "condition of continuity" applied to the fluid masses.

It was shown in reference 1 that equations such as the ones set down here possess, among others, the following two particular solutions:

$$\left.\begin{aligned} u &= \frac{x}{t + t_o} \\ a &= a_o\left(1 + \frac{t}{t_o}\right)^{-\frac{\gamma-1}{2}} \end{aligned}\right\} \qquad \text{(C-one)}$$

and

$$\left.\begin{aligned} u &= \frac{2}{\gamma + 1}\,\frac{x}{t + t_o} \\ a &= a_o\left[\left(1 + \frac{t}{t_o}\right)^{\frac{-2(\gamma-1)}{\gamma+1}} + \left(\frac{\gamma - 1}{\gamma + 1}\right)^2 \frac{x^2}{a_o^2(t + t_o)^2}\right]^{\frac{1}{2}} \end{aligned}\right\} \qquad \text{(D-one)}$$

where a_o and t_o are constants having the dimensions of velocity and time, respectively. These two solutions are here distinguished from each other by naming them type-C and -D, as was done in the cited reference, and they are further differentiated from those to follow by appending the suffix "one" in order to indicate that they refer to the one-dimensional case.

The object of the rest of this paper is to point out what the analogous solutions are for the cases of two- and three-dimensional centrally symmetric flow (which will be denoted by C-two, C-three, and D-two, D-three, respectively).

Retaining the same system of notation as used previously, it is easy to verify that in these cases the equation of motion remains unaltered, provided simply that x now represent the radial distance from the axis or center of symmetry; the continuity equation, however, undergoes a slight change. Indeed, one finds in the two-dimensional case that[2]:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{2}{\gamma - 1} a \frac{\partial a}{\partial x} \tag{1-two}$$

$$\frac{\partial a}{\partial t} + u \frac{\partial a}{\partial x} = - \frac{\gamma - 1}{2} a \frac{1}{x} \frac{\partial (ux)}{\partial x} \tag{2-two}$$

For the case of three-dimensional flow one then obtains in an analogous way that:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - \frac{2}{\gamma - 1} a \frac{\partial a}{\partial x} \tag{1-three}$$

$$\frac{\partial a}{\partial t} + u \frac{\partial a}{\partial x} = - \frac{\gamma - 1}{2} \frac{a}{x^2} \frac{\partial (ux^2)}{\partial x} \tag{2-three}$$

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[2]For the derivation of equations (1-two) and (1-three), which are identical to equation (1-one), it suffices simply to refer back to what was stated in reference 1. In the cases of two- and three-dimensional flow, the equation of continuity can be derived from the forms in which they are more commonly encountered; that is, from

$$x \frac{\partial \rho}{\partial t} = - \frac{\partial (xu\rho)}{\partial x} \qquad \text{(two-dimensional)}$$

and

$$x^2 \frac{\partial \rho}{\partial t} = - \frac{\partial (x^2 u\rho)}{\partial x} \qquad \text{(three-dimensional)}$$

wherein substitution is made for the density ρ and its derivative, when expressed as functions of a, in accordance with the usual practice. These equations then become equations (2-two) and (2-three), respectively.

The system of equations (1-two) and (2-two) possess as particular solutions the following two expressions for u and a:

$$\left.\begin{aligned} u &= \frac{x}{t + t_o} \\ a &= a_o\left(1 + \frac{t}{t_o}\right) \end{aligned}\right\} \qquad \text{(C-two)}$$

and

$$\left.\begin{aligned} u &= \frac{x}{\gamma(t + t_o)} \\ a &= a_o\left[\left(1 + \frac{t}{t_o}\right)^{\frac{-2(\gamma-1)}{\gamma}} + \frac{1}{2}\left(\frac{\gamma - 1}{\gamma}\right)^2 \frac{x^2}{a_o^2(t + t_o)^2}\right]^{\frac{1}{2}} \end{aligned}\right\} \qquad \text{(D-two)}$$

This result may be quickly checked by insertion in the differential equations.

The system of equations (1-three) and (2-three) also possess as particular solutions the following two analogous expressions for u and a:

$$\left.\begin{aligned} u &= \frac{x}{t + t_o} \\ a &= a_o\left(1 + \frac{t}{t_o}\right)^{-\frac{3(\gamma-1)}{2}} \end{aligned}\right\} \qquad \text{(C-three)}$$

and

$$\left.\begin{aligned} u &= \frac{2}{3\gamma - 1}\,\frac{x}{t + t_o} \\ a &= a_o\left[\left(1 + \frac{t}{t_o}\right)^{-\frac{6(\gamma-1)}{3\gamma-1}} + 3\left(\frac{\gamma - 1}{3\gamma - 1}\right)^2 \frac{x^2}{a_o^2(t + t_o)^2}\right]^{\frac{1}{2}} \end{aligned}\right\} \qquad \text{(D-three)}$$

The two sets of solutions exhibited here have a rather interesting physical significance, which may not have been brought out with sufficient clarity when the one-dimensional case was examined in reference 1.

To be more exact, when state parameters of the fluid are initially uniform through the flow field, while the velocity is proportional to the abscissa coordinate (in the case of one-dimensional flow this latter relationship may be directly interpreted as the fact that the velocity varies linearly with respect to the location coordinate along the duct axis), the various C-type solutions show the following features: In successive intervals of time this type of wave setup is maintained unaltered, except that a uniformly applied alteration of the fluid properties is seen to occur, and likewise the coefficient of proportionality between the velocity and the abscissa coordinate is changed. If one now examines the motion of the individual particles of the fluid, rather than considering the velocity to be a function of the time and the x-coordinate, it is readily found that the velocity of these particles turns out to be time-independent.

This result follows immediately from the fact that the left-hand member of equation (1) represents the total derivative of the velocity in each one of the cases. When the various C-type solutions are inserted in this same left hand side the result should be zero, because so is the right hand side, inasmuch as "a" is solely a function of time.

As regards the several D-type solutions, we may observe that, from the initial moment on through the subsequent flow process, they are characterized by a linear variation of a^2 (which is proportional to the internal energy of the gas) with respect to u^2 (which is proportional to its kinetic energy).

In fact, through use of quite obvious manipulations, it appears that the various D-type solutions lead to relationships of the general form:

$$a^2 = a_o^2\left(1 + \frac{t}{t_o}\right)^{-Z} + Cu^2$$

where Z and C are constants with different values in the three cases of one-, two-, and three-dimensional flow. Their specific values are derived with little trouble directly from the three D-type solutions already presented explicitly.

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