NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1324

STEADY VIBRATIONS OF WING OF CIRCULAR PLAN FORM*

and

THEORY OF WING OF CIRCULAR PLAN FORM**

By N. E. Kochin

Translation

*"Ob ustanovivshikhsya kolebaniyakh kryla krugovoi formy v plane." Prikladnaya Matematika i Mekhanika, Vol. VI, 1942.

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By N. E. Kochin

The nonvortical motion of an ideal incompressible fluid has been solved (reference 1) for the case of uniform rectilinear motion of a wing of circular plan form. The method developed in reference 1 may also be generalized to the case of the nonsteady motion of such wing. The problem of the steady vibrations of a circular wing is solved herein. The results will be frequently referred to herein. The problem of the steady vibrations of a circular wing was solved by another method by Th. Schade (reference 2).

1. Fundamental equations

The wing, the motion of which is under consideration, is assumed, as in reference 1, to be thin and slightly curved; its projection on the xy-plane has the shape of a circle ABCD of radius a with center at the origin of coordinates. The principal motion of the wing is assumed to be a rectilinear translational motion with constant velocity c parallel to the x-axis. The coordinate axes are assumed as displaced with the same velocity. On the principal motion of the wing is superposed its additional harmonic vibration of frequency ω, where the possibility of deformation of the wing is not excluded. The equation of the surface of the wing may then be represented in the form:

$$z(x,y,t) = \zeta_0(x,y) + \zeta_1(x,y) \cos \omega t + \zeta_2(x,y) \sin \omega t \tag{1.1}$$

where the ratios ζ_k/a as well as the derivatives $\partial\zeta_k/\partial x$ and $\partial\zeta_k/\partial y$, where $k = 0,1,2$, are assumed small magnitudes.

The fluid is assumed incompressible and the motion is assumed nonvortical and occurring in the absence of external forces. The velocity

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*"Ob ustanovivshikhsya kolebaniyakh kryla krugovoi formy v plane." Prikladnaya Matematika i Mekhanika, Vol. VI, 1942, pp. 287-316.

potential will be denoted by $\varphi(x,y,z,t)$ and steady vibrations of the fluid will be assumed; that is, the velocity potential is represented in the form:

$$\varphi(x,y,z,t) = \varphi_0(x,y,z) + \varphi_1(x,y,z)\cos\omega t + \varphi_2(x,y,z)\sin\omega t$$

It is evident that the functions φ_0, φ_1, and φ_2 satisfy the equations of Laplace

$$\frac{\partial^2\varphi_k}{\partial x^2} + \frac{\partial^2\varphi_k}{\partial y^2} + \frac{\partial^2\varphi_k}{\partial z^2} = 0 \qquad (k = 0,1,2)$$

The velocity of the particles of the fluid near the leading edge of the wing DAB is assumed to approach infinity as $\delta^{-1/2}$, where δ is the distance of the particle from the leading edge, but the velocity of the fluid particles near the trailing edge of the wing BCD is assumed as finite. From this edge a surface of discontinuity passes off on which the function φ undergoes a discontinuity. As in reference 1, the problem will be linearized. Since the values of the functions φ_k and their derivatives are assumed to be small quantities of the first order, their squares and products are rejected. The functions $\varphi_k(x,y,z)$ are further assumed to have discontinuities on the infinite half-strip Ξ situated in the xy-plane in the direction of the negative x-axis from the rear semicircumference BCD of the circle S to infinity. The boundary conditions on the surface of the wing are replaced by the conditions on the circle S located in the xy-plane. Everywhere outside the half-strip Ξ and the circle S the functions $\varphi_k(x,y,z)$ are thus regular functions.

The boundary conditions which these functions satisfy are now set up. On the surface of discontinuity Ξ, the kinematic condition expressing the continuity of the normal component of the velocity must first of all be satisfied:

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=+0} = \left(\frac{\partial\varphi}{\partial z}\right)_{z=-0}$$

from which is obtained the conditions

$$\left(\frac{\partial\varphi_k}{\partial z}\right)_{z=+0} = \left(\frac{\partial\varphi_k}{\partial z}\right)_{z=-0} \qquad \text{on } \Xi \qquad (1.2)$$

The dynamical conditions expressing the continuity of the pressure in passing through the surface of discontinuity Ξ are now stated.

If a stationary system of coordinates $x_1y_1z_1$ is employed, connected with the coordinates xyz of the moving system of coordinates by the relations

$$x = x_1 - ct_1 \qquad y = y_1 \qquad z = z_1 \qquad t = t_1$$

then the pressure may be determined by the following formula:

$$p = -\rho \frac{\partial \varphi}{\partial t_1} - \frac{\rho}{2}\left[\left(\frac{\partial \varphi}{\partial x_1}\right)^2 + \left(\frac{\partial \varphi}{\partial y_1}\right)^2 + \left(\frac{\partial \varphi}{\partial z_1}\right)^2\right] + F(t_1) \tag{1.3}$$

Since

$$\frac{\partial \varphi}{\partial t_1} = \frac{\partial \varphi}{\partial t} - c\frac{\partial \varphi}{\partial x} \quad \frac{\partial \varphi}{\partial x_1} = \frac{\partial \varphi}{\partial x} \quad \frac{\partial \varphi}{\partial y_1} = \frac{\partial \varphi}{\partial y} \quad \frac{\partial \varphi}{\partial z_1} = \frac{\partial \varphi}{\partial z} \tag{1.4}$$

the following equation will apply in the movable xyz system:

$$p = -\rho \frac{\partial \varphi}{\partial t} + \rho c\frac{\partial \varphi}{\partial x} - \frac{\rho}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + F(t) \tag{1.5}$$

When small quantities of the second order are rejected and the magnitude $F(t)$ is not dependent on the coordinates,

$$p = -\rho \frac{\partial \varphi}{\partial t} + \rho c\frac{\partial \varphi}{\partial x}$$

or, on account of equation (1.2),

$$p(x,y,z,t) = \rho c\frac{\partial \varphi_0}{\partial x} + \left(\rho c\frac{\partial \varphi_1}{\partial x} - \rho\omega\varphi_2\right)\cos \omega t + \left(\rho c\frac{\partial \varphi_2}{\partial x} + \rho\omega\varphi_1\right)\sin \omega t \tag{1.6}$$

For briefness, the following notation is introduced:

$$\omega/c = k \tag{1.7}$$

The condition of continuity of the pressure on Ξ then leads to the three equations:

$$\begin{aligned}
&\left(\frac{\partial\varphi_0}{\partial x}\right)_{z=+0} \quad \left(\frac{\partial\varphi_0}{\partial x}\right)_{z=-0} \\
&\left(\frac{\partial\varphi_1}{\partial x} - k\varphi_2\right)_{z=+0} = \left(\frac{\partial\varphi_1}{\partial x} - k\varphi_2\right)_{z=-0} \qquad \text{on } \Xi \\
&\left(\frac{\partial\varphi_2}{\partial x} + k\varphi_1\right)_{z=+0} = \left(\frac{\partial\varphi_2}{\partial x} + k\varphi_1\right)_{z=-0}
\end{aligned} \tag{1.8}$$

The condition on the circle S is now written. Equation (1.1) in the stationary system of coordinates has the form:

$$z_1 = \zeta_0(x_1-ct_1,y_1) + \zeta_1(x_1-ct_1,y_1) \cos \omega t_1 + \zeta_2(x_1-ct_1,y_1) \sin \omega t_1$$

Hence, for the normal component of the velocity of the fluid particles adjacent to the surface of the wing,

$$\frac{dz_1}{dt_1} = - c \frac{\partial\zeta_0}{\partial x} - c \frac{\partial\zeta_1}{\partial x} \cos \omega t - \omega\zeta_1 \sin \omega t - c \frac{\partial\zeta_2}{\partial x} \sin \omega t + \omega\zeta_2 \cos \omega t$$

The notations

$$- c \frac{\partial\zeta_0}{\partial x} = Z_0(x,y) \qquad - c\left(\frac{\partial\zeta_1}{\partial x} - k\zeta_2\right) = Z_1(x,y) \qquad - c\left(\frac{\partial\zeta_2}{\partial x} + k\zeta_1\right) = Z_2(x,y)$$

yield the boundary condition

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = Z_0(x,y) + Z_1(x,y) \cos \omega t + Z_2(x,y) \sin \omega t$$

which must be satisfied on both the upper and the lower sides of the circle S and which breaks down into the three conditions:

$$\left(\frac{\partial\varphi_k}{\partial z}\right)_{z=0} = Z_k(x,y) \quad \text{on } S \quad (k = 0,1,2) \tag{1.9}$$

The presence of conditions (1.2) and (1.9) permits consideration of the functions $\varphi_k(x,y,z)$ as odd functions of Z:

$$\varphi_k(x,y,-z) = - \varphi_k(x,y,z) \tag{1.10}$$

If it is assumed, in particular, that $z = 0$,

$$\varphi_k(x,y,0) = 0 \tag{1.11}$$

in the entire xy-plane with the exception of the circle S and the half strip Ξ on which φ_k undergoes a discontinuity.

The conditions (1.8), because of equation (1.10), assume the form:

$$\frac{\partial\varphi_0}{\partial x} = 0 \qquad \frac{\partial\varphi_1}{\partial x} - k\varphi_2 = 0 \qquad \frac{\partial\varphi_2}{\partial x} + k\varphi_1 = 0 \quad \text{on } \Xi \tag{1.12}$$

Finally, the absence of a disturbance of the fluid far ahead of the wing leads to the evident conditions at infinity:

$$\lim_{x\to+\infty} \frac{\partial\varphi_k}{\partial x} = \lim_{x\to+\infty} \frac{\partial\varphi_k}{\partial y} = \lim_{x\to+\infty} \frac{\partial\varphi_k}{\partial z} = 0 \tag{1.13}$$

The problem of determining the function $\varphi_0(x,y,z)$ satisfying all obtained conditions for this function was considered in reference 1.

The following equality is set up:

$$\Phi(x,y,z) = \varphi_1(x,y,z) + i\varphi_2(x,y,z) \tag{1.14}$$

so that

$$\varphi(x,y,z,t) = \varphi_0(x,y,z) + \mathrm{Re}\left\{\Phi(x,y,z)e^{-i\omega t}\right\} \tag{1.15}$$

Also,

$$\zeta_1(x,y) + i\zeta_2(x,y) = \zeta(x,y)$$

$$Z_1(x,y) + iZ_2(x,y) = Z(x,y) = -c\left(\frac{\partial\zeta}{\partial x} + ik\zeta\right) \tag{1.16}$$

The shape of the wing will be determined by the equation

$$z(x,y,t) = \zeta_0(x,y) + \mathrm{Re}\left\{\zeta(x,y)e^{-i\omega t}\right\} \tag{1.17}$$

The functions $\Phi(x,y,z)$ will then be a harmonic function, regular in the entire half-space $z > 0$ and satisfying the conditions:

$$\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} = Z(x,y) \quad \text{on} \quad S \tag{1.18}$$

$$\frac{\partial \Phi}{\partial x} + ik\Phi = 0 \quad \text{on} \quad \Xi \tag{1.19}$$

following from equations (1.9) and (1.12). In the entire remaining part of the plane xy the following condition must be satisfied:

$$\Phi(x,y,0) = 0 \tag{1.20}$$

Moreover, the following conditions must be satisfied at infinity:

$$\lim_{x \to +\infty} \frac{\partial \Phi}{\partial x} = \lim_{x \to +\infty} \frac{\partial \Phi}{\partial y} = \lim_{x \to +\infty} \frac{\partial \Phi}{\partial z} = 0 \tag{1.21}$$

which are the boundary conditions of the first derivatives of the function $\Phi(x,y,z)$ near the rear semicircumference BCD of the circle S and the condition that near the forward semicircumference DAB these derivatives may become infinite to the order of $\delta^{-1/2}$.

2. Fundamental formulas

In reference 1 an expression was constructed, which depended on an arbitrary function $f_0(x,y)$, which determined a harmonic function $\varphi_0(x,y,z)$ satisfying all the conditions imposed in the preceding section

$$\varphi_0(x,y,z) = \frac{1}{2\pi} \iint_S f_0(\xi,\eta) \left\{K(x,y,z,\xi,\eta) + \frac{1}{\pi^2\sqrt{2}} \int_{+\infty}^{x} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{G(x,y,z,\gamma)\sqrt{a^2 - \xi^2 - \eta^2} \cos \gamma \, d\gamma \, dx}{(\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2a\eta \sin \gamma)}\right\} d\xi \, d\eta \tag{2.1}$$

The functions $K(x,y,z,\xi,\eta)$ and $G(x,y,z,\gamma)$ for $z > 0$ are given by

$$\begin{aligned} K(x,y,z,\xi,\eta) &= \frac{2}{\pi r} \arc\tan \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\, ar} \\ G(x,y,z,\gamma) &= \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{x^2 + y^2 + z^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma} \end{aligned} \tag{2.2}$$

which are harmonic functions of x,y,z where

$$
\begin{aligned}
r &= \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2} \\
R &= \sqrt{(a^2 - x^2 - y^2 - z^2)^2 + 4a^2z^2}
\end{aligned}
\tag{2.3}
$$

In order to satisfy boundary condition (1.9)

$$
\frac{\partial \varphi_0}{\partial z} = Z_0(x,y) \qquad \text{on} \quad S \tag{2.4}
$$

it is necessary to take

$$
f_0(x,y) = - Z_0(x,y) + g_0(y) \tag{2.5}
$$

where $g_0(y)$ is determined from a Fredholm integral equation of the second kind.

The solution of the more general problem of steady vibrations may be presented in a similar form.

Thus, $f_1(x,y)$ and $f_2(x,y)$ denote two arbitrary real functions, continuous, together with their partial derivatives of the first and second order, in the entire circle S;

$$
f_1(x,y) + if_2(x,y) = f(x,y) \tag{2.6}
$$

It will now be shown that the function

$$
\Phi(x,y,z) = \frac{1}{2\pi} \iint_S f(\xi,\eta) \left\{ K(x,y,z,\xi,\eta) + \frac{1}{\pi^2\sqrt{2}} e^{-ikx} \int_{+\infty}^{x} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{G(x,y,z,\gamma)\, e^{ikx} \sqrt{a^2 - \xi^2 - \eta^2} \cos \gamma \, d\gamma \, dx}{(\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2ay \sin \gamma)} \right\} d\xi \, d\eta \tag{2.7}
$$

satisfies all the conditions of the preceding section except condition (1.18).

The function $G(x,y,z,\gamma)$, as shown by equation (2.2), is harmonic; hence the function

$$L(x,y,z) = e^{-ikx}\int_{+\infty}^{x} e^{ikx}\, G(x,y,z)\, dx$$

will be a harmonic function. In fact,

$$\Delta L = \frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} + \frac{\partial^2 L}{\partial z^2} = \frac{\partial G}{\partial x} - ikG + e^{-ikx}\int_{+\infty}^{x} e^{ikx}\left[\frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2} - k^2 G\right] dx$$

$$= \frac{\partial G}{\partial x} - ikG - e^{-ikx}\int_{+\infty}^{x} e^{ikx}\left(\frac{\partial^2 G}{\partial x^2} + k^2 G\right) dx$$

When this expression is integrated by parts, it is easily shown that $\Delta L = 0$, since both G and $\partial G/\partial x$ approach zero for $x \to \infty$.

It then follows that the function $\Phi(x,y,z)$ likewise satisfies the Laplace equation

$$\Delta\Phi = 0 \tag{2.8}$$

where from the form of equation (2.7) it is seen that $\Phi(x,y,z)$ is regular everywhere outside the circle S and half strip Ξ. In exactly the same way it is shown that the conditions at infinity (1.21) and condition (1.20) are satisfied.

Furthermore,

$$\frac{\partial \Phi}{\partial x} + ik\Phi = \frac{1}{2\pi}\iint_S f(\xi,\eta)\left\{\frac{\partial K}{\partial x} + ikK + \right.$$

$$\left.\frac{1}{\pi^2\sqrt{2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{G(x,y,z,\gamma)\sqrt{a^2 - \xi^2 - \eta^2}\cos\gamma\, d\gamma}{(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}\right\} d\xi\, d\eta \tag{2.9}$$

It is clear that if $x^2 + y^2 > a^2$ then

$$\frac{\partial \Phi}{\partial x} + ik\Phi = 0 \qquad \text{for} \quad z = 0 \tag{2.10}$$

so that condition (1.19) is likewise satisfied.

It thus remains to check the finiteness of the derivatives of the function $\Phi(x,y,z)$ at the points of the semicircumference BCD of the circle S and to establish the behavior of these derivatives near the forward semicircumference DAB. But near the forward semicircumference, the inside integral in formula (2.7) evidently remains bounded, as do its partial derivatives; since the first derivatives of the integral

$$\iint_S f(\xi,\eta)\, K(x,y,z,\xi,\eta)\, d\xi\, d\eta$$

as established in reference 1, and as will again be proven, have near the contour of the circle S the order $\delta^{-1/2}$ (where δ is the distance of a point to the contour ABCD of the circle S), it is clear that the first derivatives of the function $\Phi(x,y,z)$ also have the order $\delta^{-1/2}$ near the forward semicircumference DAB of the circle S.

For determining the behavior of the function $\Phi(x,y,z)$ near the rear semicircumference BCD, the right side of equation (2.9) is transformed. Denoting it by $M(x,y,z)$ and making use of formula (2.11) of reference 1 and the formula of integration by parts (2.14) of reference 1,

$$M(x,y,z) = \frac{1}{2\pi} \iint_S f(\xi,\eta) \left\{ ikK - \frac{1}{\pi^2\sqrt{2}} \int_{-\frac{1}{2}\pi}^{+\frac{3}{2}\pi} \frac{G(x,y,z,\gamma)\sqrt{a^2-\xi^2-\eta^2}\cos\gamma\, d\gamma}{(\xi^2+\eta^2+a^2-2a\xi\cos\gamma-2a\eta\sin\gamma)} \right\} d\xi\, d\eta \quad \frac{1}{2\pi} \iint_S \frac{\partial f}{\partial \xi} K\, d\xi\, d\eta \tag{2.11}$$

It is evident that this function remains finite near the rear semicircumference BCD. But when the following equation is integrated,

$$\frac{\partial\Phi}{\partial x} + ik\Phi = M(x,y,z) \tag{2.12}$$

there is obtained

$$\Phi(x,y,z) = e^{-ikx}\int_0^x e^{ikx}\,M(x,y,z)\,dx + \Phi(0,y,z)\,e^{-ikx} \tag{2.13}$$

whence it is clear that both the function Φ and its derivative with respect to x remain finite near the rear semicircumference BCD. The derivatives of M with respect to y and z will be of the order $\delta^{-1/2}$ near BCD, as follows from a consideration analogous to that which was adduced previously for determining the behavior of the function $\Phi(x,y,z)$ near the forward edge of the wing DAB. Since

$$\frac{\partial\Phi}{\partial y} = e^{-ikx}\int_0^x e^{ikx}\,\frac{\partial M}{\partial y}\,dx + \frac{\partial\Phi(0,y,z)}{\partial y}\,e^{-ikx}$$

it is clear that the derivative $\partial\Phi/\partial y$, and similarly $\partial\Phi/\partial z$, remain finite near the rear edge of the wing BCD.

The function (2.7) thus satisfies all the imposed conditons. The only condition not utilized was condition (1.18)

$$\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} = Z(x,y) \qquad \text{on } S \tag{2.14}$$

When the following formulas are employed:

$$\lim_{z \to +0} \iint_S \frac{\partial K}{\partial z} f(\xi,\eta)\, d\xi\, d\eta = -2\pi f(x,y)$$

$$\lim_{z \to +0} \frac{\partial}{\partial z}\sqrt{a^2 - x^2 - y^2 - z^2 + R} = \begin{cases} 0 & \text{for } x^2 + y^2 < a^2 \\ \dfrac{a\sqrt{2}}{\sqrt{x^2 + y^2 - a^2}} & \text{for } x^2 + y^2 > a^2 \end{cases}$$

it is found without difficulty that on S

$$\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} = -f(x,y) + g(y)\, e^{-ikx} \tag{2.15}$$

where

$$g(y) = \frac{a}{2\pi^3} \iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}\,(x^2 + y^2 - a^2)^{-1/2}\,(a^2 - \xi^2 - \eta^2)^{1/2} \cos\gamma\, f(\xi,\eta)\, d\gamma\, dx\, d\xi\, d\eta}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{2.16}$$

The following equation is thus obtained:

$$-f(x,y) + g(y)\, e^{-ikx} = Z(x,y) \tag{2.17}$$

whence

$$f(x,y) = -Z(x,y) + g(y)\, e^{-ikx} \tag{2.18}$$

Substitution of this value of the function $f(x,y)$ in equation (2.16) yields, for the determination of the function $g(y)$, an integral equation of Fredholm

$$g(y) = N(y) + \int_{-a}^{a} H(y,\eta)\, g(\eta)\, d\eta \tag{2.19}$$

where

$$N(y) = -\frac{a}{2\pi^3} \iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}\sqrt{a^2 - \xi^2 - \eta^2}\, G(x,y,z,\gamma) \cos\gamma\, Z(\xi,\eta)\, d\gamma\, dx\, d\xi\, d\eta}{\sqrt{x^2 + y^2 - a^2}\,(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{2.20}$$

with $G(x,y,z,\gamma)$ according to equations (2.2) and

$$H(y,\eta) =$$

$$\frac{a}{2\pi^3} \times \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ik(x-\xi)}(x^2+y^2-a^2)^{-1/2}(a^2-\xi^2-\eta^2)^{1/2}\cos\gamma \, d\gamma \, dx \, d\xi}{(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)(\xi^2+\eta^2+a^2-2a\xi\cos\gamma-2a\eta\sin\gamma)} \tag{2.21}$$

3. Computation of forces

The pressure p may be determined from formula (1.6), which with the notation (1.14) may be written in the following form:

$$p = \rho c\left\{\frac{\partial\varphi_0}{\partial x} + \mathrm{Re}\left[\left(\frac{\partial\Phi}{\partial x} + ik\Phi\right)e^{-i\omega t}\right]\right\} \tag{3.1}$$

For the computation of the forces acting on the wing, it is necessary to know the pressure on the circle S.

Because of equation (1.10), the pressures above and below the wing differ only in sign:

$$p_- = -p_+ \tag{3.2}$$

For clarity, the signs of the functions on the wing will henceforth be assumed to be the limiting values in approaching the wing from above, that is, for $z \to +0$.

For the lift force P the following expression is obtained

$$P = \iint_S (p_- - p_+)\,dx\,dy = -2\iint_S p_+\,dx\,dy =$$

$$-2\rho c\iint_S\left\{\frac{\partial\varphi_0}{\partial x} + \mathrm{Re}\left[\left(\frac{\partial\Phi}{\partial x} + ik\Phi\right)e^{-i\omega t}\right]\right\}dx\,dy \tag{3.3}$$

But by formulas (2.9) and (2.11), the following equation applies on the upper side of the circle S:

$$\frac{\partial\Phi(x,y,0)}{\partial x} + i\,k\,\Phi(x,y,0) = \frac{1}{2\pi}\iint_S \frac{\partial f}{\partial \xi} K(x,y,0,\xi,\eta)\,d\xi\,d\eta \; +$$

$$\frac{1}{2\pi}\iint_S f(\xi,\eta)\Bigg[i\,k\,K(x,y,0,\xi,\eta) - \tag{3.4}$$

$$\frac{1}{\pi^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sqrt{a^2 - x^2 - y^2}\,\sqrt{a^2 - \xi^2 - \eta^2}\,\cos\gamma\,d\gamma}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}\Bigg]d\xi\,d\eta$$

This expression is integrated over the entire area of the circle S. The order of integration is interchanged and the two integrals must be computed first of all by formula (4.13) of reference 1

$$\iint_S \frac{\sqrt{a^2 - x^2 - y^2}}{x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma}\,dx\,dy = 2\pi a \tag{3.5}$$

It will be proven further that

$$\iint_S K(x,y,0,\xi,\eta) = 4\sqrt{a^2 - \xi^2 - \eta^2} \tag{3.6}$$

For this proof, the following function is considered:

$$F(x,y,z) = \iint_S K(x,y,z,\xi,\eta)\,d\xi\,d\eta \tag{3.7}$$

Because of the definition of the function K, the function F(x,y,z) is a harmonic function over the entire space outside the circle S. By formula (2.35) of reference 1, the following condition is satisfied on the surface of this circle:

$$\frac{\partial F}{\partial z} = -\,2\pi \qquad \text{on } S \tag{3.8}$$

and therefore the function (3.7) is the potential of the nonvortical motion of a fluid corresponding to the translational motion of a circular disk with velocity $+2\pi$ along the negative z-axis normal to the plane of the disk. This motion, however, belongs to those that have been studied in classical hydrodynamics, from which can be taken the corresponding expression of the function.

$$
\begin{aligned}
F(x,y,z) &= \iint_S K(x,y,z,\xi,\eta)\, d\xi\, d\eta \\
&= 2\sqrt{2}\sqrt{R + a^2 - x^2 - y^2 - z^2}\left\{1 - \right. \\
&\left.\sqrt{\frac{R + x^2 + y^2 + z^2 - a^2}{2a^2}}\ \operatorname{arc\,ctn}\sqrt{\frac{R + x^2 + y^2 + z^2 - a^2}{2a^2}}\right\}
\end{aligned}
\tag{3.9}
$$

Passing to the limit $z \to +0$ yields the formula

$$
\iint_S K(x,y,0,\xi,\eta)\, d\xi\, d\eta = 4\sqrt{a^2 - x^2 - y^2} \quad \text{on} \quad S
$$

which is equivalent to equation (3.6), since $K(x,y,0,\xi,\eta)$ is a symmetrical function with respect to the points $M(x,y)$ and $N(\xi,\eta)$.

The following formula is thus obtained:

$$
\begin{aligned}
\iint_S \left[\frac{\partial\Phi(x,y,0)}{\partial x} + i\,k\,\Phi(x,y,0)\right] dx\, dy &= \frac{2}{\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\frac{\partial f}{\partial \xi} + ikf\right\} d\xi\, dn\ - \\
\frac{a}{\pi^2}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta) &\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\,\xi\cos\gamma - 2a\eta\sin\gamma}\, d\xi\, d\eta
\end{aligned}
\tag{3.10}
$$

If this expression and similar expressions are substituted for the function φ_0, obtained from equation (3.10) for $k = 0$, the final expression of the lift force acting on the wing is obtained:

$$P = -\frac{4\rho c}{\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\frac{\partial f_0}{\partial \xi} + \mathrm{Re}\left[e^{-i\omega t}\left(\frac{\partial f}{\partial \xi} + ikf\right)\right] - \right.$$

$$\left.\frac{a}{2\pi}\left[f_0 + \mathrm{Re}\left(e^{-i\omega t}\right)f\right]\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\}d\xi\, d\eta \tag{3.11}$$

By integration by parts and with the aid of the following formula

$$\int_0^{2\pi}\frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} = \frac{2\pi\xi}{a(a^2 - \xi^2 - \eta^2)} \tag{3.12}$$

equation (3.11) may be rewritten in the form:

$$P = -\frac{4\rho c}{\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\mathrm{Re}\left(ikfe^{-i\omega t}\right) + \right.$$

$$\left.\frac{a}{2\pi}\left[f_0 + \mathrm{Re}\left(fe^{-i\omega t}\right)\right]\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\}d\xi\, d\eta \tag{3.13}$$

In a similar manner, the formulas for the moments of the forces about the x- and y-axes are obtained.

For the moment of the pressure forces about the x-axis

$$M_x = \iint_S y(p_- - p_+)\, dx\, dy = -2\iint_S yp_+\, dx\, dy \tag{3.14}$$

there is obtained

$$M_x = - 2\rho c \iint_S y\left\{\frac{\partial\varphi_0}{\partial x} + \mathrm{Re}\left[\left(\frac{\partial\Phi}{\partial x} + i\ k\ \Phi\right) e^{-i\omega t}\right]\right\} dx\ dy \qquad (3.15)$$

The order of integration is interchanged by use of equation (3.4). It is here necessary to compute two integrals. By formula (4.44) of reference 1,

$$\iint_S \frac{y\sqrt{a^2 - x^2 - y^2}\ dx\ dy}{x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma} = \frac{4}{3} \pi a^2 \sin \gamma \qquad (3.16)$$

It will now be shown that

$$\iint_S y\ K(x,y,0,\xi,\eta)\ dx\ dy = \frac{8}{3} \eta\sqrt{a^2 - \xi^2 - \eta^2} \qquad (3.17)$$

For this derivation, the following function is considered:

$$F_1(x,y,z) = \iint_S \eta\ K(x,y,z,\xi,\eta)\ d\xi\ d\eta$$

By formula (2.35) of reference 1, the following equation applies on the circle S:

$$\frac{\partial F_1}{\partial z} = - 2\pi y \qquad (3.18)$$

and therefore $F_1(x,y,z)$ is the potential of the motion of a fluid corresponding to the rotation of a disk about the x-axis with angular velocity -2π, a case studied in classical hydrodynamics:

$$F_1(x,y,z) = \iint_S \eta\ K(x,y,z,\xi,\eta)\ d\xi\ d\eta$$

$$= 2\sqrt{2}\ y\sqrt{R + a^2 - x^2 - y^2 - z^2}\left\{1 - \frac{2a^2}{3(R + x^2 + y^2 + z^2 + a^2)} - \sqrt{\frac{R + x^2 + y^2 + z^2 - a^2}{2a^2}}\ \mathrm{arc\ ctn}\sqrt{\frac{R + x^2 + y^2 + z^2 - a^2}{2a^2}}\right\} \tag{3.19}$$

Passing to the limit $z \to +0$ yields the formula

$$\iint_S \eta\ K(x,y,0,\xi,\eta)\ d\xi\ d\eta = \frac{8}{3}\ y\sqrt{a^2 - x^2 - y^2} \quad \text{on} \quad S$$

equivalent to equation (3.17).

As a result, the following formula is obtained

$$\iint_S y\left[\frac{\partial\Phi}{\partial x} + i\ k\ \Phi\right] dx\ dy = \frac{4}{3\pi}\iint_S \eta\sqrt{a^2 - \xi^2 - \eta^2}\left\{\frac{\partial f}{\partial\xi} + ikf\right\} d\xi\ d\eta -$$

$$\frac{2a^2}{3\pi^2}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\ f(\xi,\eta)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin\gamma\ \cos\gamma\ d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\ \cos\gamma - 2a\eta\ \sin\gamma}\ d\xi\ d\eta \tag{3.20}$$

Hence, for the moment of the pressure forces about the x-axis, the following expression is obtained:

$$M_x = -\frac{8\rho c}{3\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\eta\left[\frac{\partial f_0}{\partial\xi} + \mathrm{Re}\left\{e^{-i\omega t}\left(\frac{\partial f}{\partial\xi} + ikf\right)\right\}\right] - \frac{a}{2\pi}\left[f_0 + \mathrm{Re}(e^{-i\omega t}\ f)\right]\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\sin\gamma\ \cos\gamma\ d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\ \cos\gamma - 2a\eta\ \sin\gamma}\right\} d\xi\ d\eta \tag{3.21}$$

or, on account of the formula

$$\int_0^{2\pi} \frac{\sin \gamma \cos \gamma \, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2a\eta \sin \gamma} = \frac{2\pi\xi\eta}{a^2(a^2 - \xi^2 - \eta^2)} \tag{3.22}$$

the equivalent expression

$$M_x = -\frac{8\rho c}{3\pi} \iint_S \sqrt{a^2 - \xi^2 - \eta^2} \left\{ \eta \operatorname{Re}(ikfe^{-i\omega t}) + \frac{a^2}{2\pi} \left[f_0 + \operatorname{Re}(fe^{-i\omega t}) \right] \times \right.$$

$$\left. \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\sin \gamma \cos \gamma \, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2a\eta \sin \gamma} \right\} d\xi \, d\eta \tag{3.23}$$

In the same way for the moment of the pressure forces about the y-axis

$$M_y = -\iint_S x(p_- - p_+) \, dx \, dy = -2 \iint_S xp_+ \, dx \, dy \tag{3.24}$$

there is obtained

$$M_y = 2\rho c \iint_S x \left\{ \frac{\partial \varphi_0}{\partial x} + \operatorname{Re}\left[\left(\frac{\partial \Phi}{\partial x} + i k \Phi \right) e^{-i\omega t} \right] \right\} dx \, dy \tag{3.25}$$

It is here necessary to employ the formulas

$$\iint_S \frac{x\sqrt{a^2 - x^2 - y^2}}{x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma} \, dx \, dy = \frac{4}{3} \pi a^2 \cos \gamma \tag{3.26}$$

$$\iint_S x\,K(x,y,0,\xi,\eta)\,dx\,dy = \frac{8}{3}\,\xi\sqrt{a^2 - \xi^2 - \eta^2} \tag{3.27}$$

As before, there is obtained

$$M_y = \frac{8\rho c}{3\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{\xi\left[\frac{\partial f_0}{\partial \xi} + \mathrm{Re}\left\{e^{-i\omega t}\left(\frac{\partial f}{\partial \xi} + ikf\right)\right\}\right] - \right.$$

$$\left.\frac{a^2}{2\pi}\left[f_0 + \mathrm{Re}\left(e^{-i\omega t}\ f\right)\right]\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{\cos^2\gamma\ d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\}d\xi\,d\eta \tag{3.28}$$

Integration by parts and use of the formula yields

$$\int_0^{2\pi}\frac{\cos^2\gamma\ d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} = \frac{2\pi\xi^2}{a^2(a^2 - \xi^2 - \eta^2)} + \frac{\pi}{a^2} \tag{3.29}$$

also

$$M_y = \frac{8\rho c}{3\pi}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\left\{-\frac{3}{2}f_0 - \mathrm{Re}\left[e^{-i\omega t}\left(\frac{3}{2}f - ik\xi f\right)\right] + \right.$$

$$\left.\frac{a^2}{2\pi}\left[f_0 + \mathrm{Re}(e^{-i\omega t}\ f)\right]\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{\cos^2\gamma\ d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\}d\xi\,d\eta \tag{3.30}$$

The value can now be computed for the frontal resistance W, which is composed of two parts. First, the normal force $(p_- - p_+)\,dx\,dy$ acting on an element of the wing $dx\,dy$ will have a component in the direction of the x-axis:

$$(p_- - p_+)\frac{\partial z}{\partial x}\,dx\,dy = (p_- - p_+)\left[\frac{\partial \zeta_0}{\partial x} + \mathrm{Re}\left(\frac{\partial \zeta}{\partial x}\,e^{-i\omega t}\right)\right]\,dx\,dy$$

if

$$z(x,y,t) = \zeta_0(x,y) + \mathrm{Re}\left[\zeta(x,y)\,e^{-i\omega t}\right]$$

is the equation of the surface of the wing. Integration of this expression gives the first part of the frontal resistance in the form:

$$W_1 = \iint_S (p_- - p_+)\left[\frac{\partial \zeta_0}{\partial x} + \mathrm{Re}\left(\frac{\partial \zeta}{\partial x}\,e^{-i\omega t}\right)\right]\,dx\,dy$$

$$= -\,2\rho c\iint_S \left\{\frac{\partial \varphi_0}{\partial x} + \mathrm{Re}\left[\left(\frac{\partial \Phi}{\partial x} + i\,k\,\Phi\right)\,e^{-i\omega t}\right]\right\}\left\{\frac{\partial \zeta_0}{\partial x} + \mathrm{Re}\left(\frac{\partial \zeta}{\partial x}\,e^{-i\omega t}\right)\right\}\,dx\,dy \tag{3.31}$$

In fact, the frontal resistance W will be less than W_1, since a suction force W_2 appears because of the presence of the sharp leading edge of the wing DAB; therefore,

$$W = W_1 - W_2 \tag{3.32}$$

The suction force W_2 is connected with the presence of a strong rarefaction near the edge of the wing. This rarefaction is taken into account principally by the square terms of the fundamental formulas (1.3) or (1.5) for the pressure and it is therefore unnecessary to employ these formulas here.

The suction force W_2 is computed from the law of conservation of momentum applied to a thin filament-like close region τ containing the forward semicircumference DAB of the circle S; region τ is bounded outside by surface σ and inside by part S' of the upper side of circle S adjacent to the semicircumference DAB and the part S'' of the lower side of the circle S. Figure 1 shows a section of these surfaces obtained by a passing plane through the z-axis.

The equation expressing the momentum law is projected on the x-axis:

$$-\,W_2 - \iint_\sigma p\cos(n,x)\,dS = \iiint_\tau \rho\,\frac{\partial v_x}{\partial t}\,d\tau + \rho\iint_\sigma v_n v_x\,dS + \rho\int\int_{S'+S''} v_n v_x\,dS \tag{3.33}$$

The left-hand side is the sum of the projections on the x-axis of all the forces acting on the volume of fluid considered, and on the right-hand side is the total derivative with respect to time of the component on the x-axis of the momentum of this volume; this derivative consists of two parts, a volume integral connected with the local change of velocity and a surface integral expressing the transfer of the momentum of the particles of the fluid through the bounding surfaces of the volume τ.

Equation (3.33) may be written both for the stationary system of coordinates $O_1x_1y_1z_1$ and for the moving system of coordinates Oxyz.

For the stationary system of coordinates, expression (1.3) is used for the quantity p; moreover,

$$v_x = \frac{\partial\varphi}{\partial x} \qquad v_n = \frac{\partial\varphi}{\partial n} \tag{3.34}$$

By the theorem of Gauss

$$\iiint_\tau \rho \frac{\partial v_x}{\partial t_1} d\tau = \iiint_\tau \rho \frac{\partial^2\varphi}{\partial x\, \partial t_1} d\tau$$

$$= \iint_\sigma \rho \frac{\partial\varphi}{\partial t_1} \cos(n,x)\, dS + \iint_{S'+S''} \rho \frac{\partial\varphi}{\partial t_1} \cos(n,x)\, dS \tag{3.35}$$

From equation (1.3) and the equation just derived, the following expression is obtained from equation (3.33) after a number of simple transformations:

$$W_2 = \frac{\rho}{2}\iint_\sigma \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] \cos(n,x)\, dS - \rho\iint_\sigma \frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial n} dS -$$

$$\rho\iint_{S'+S''} \frac{\partial\varphi}{\partial t_1} \cos(n,x)\, dS - \rho\iint_{S'+S''} \frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial n} dS \tag{3.36}$$

Since $\partial\varphi/\partial t_1$ and $\partial\varphi/\partial x$ near the leading edge of the wing are of the order $\delta^{-1/2}$ and $\partial\varphi/\partial n$ and $\cos(n,x)$ are finite on the surface of the wing, the last integrals drop out when region τ is extended to the line DAB. The following limiting equation is therefore applicable:

$$W_2 = \lim_{\tau \to 0} \left\{ \left[\frac{\rho}{2} \int_{\sigma}\int \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 \right] \cos(n,x)\, dS - \rho \int_{\sigma}\int \frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial n}\, dS \right] \right\} \tag{3.37}$$

For computation of the suction force W_2, the expressions must be found for the components of the velocity near the leading edge of the wing DAB. The velocity of the fluid particles near the leading edge of the wing are shown to be of the order of $\delta^{-1/2}$ if δ is the distance of the particle to the contour C of the circle S. From equations (1.15) and (2.7) it is evident that

$$\varphi(x,y,z,t) = \frac{1}{2\pi} \int_S\int \left\{ f_0(\xi,\eta) + \mathrm{Re}\left[f(\xi,\eta) e^{-i\omega t} \right] \right\} K(x,y,z,\xi,\eta)\, d\xi\, d\eta + \chi(x,y,z,t) \tag{3.38}$$

where the function $\chi(x,y,z,t)$ and its derivatives remain finite near the leading edge.

The behavior of the function is now examined more closely

$$U(x,y,z) = \int_S\int f(\xi,\eta)\, K(x,y,z,\xi,\eta)\, d\xi\, d\eta \tag{3.39}$$

near the contour C of the circle S. Therefore,

$$\frac{\partial U}{\partial x} = \int_S\int \frac{\partial K}{\partial x} f(\xi,\eta)\, d\xi\, d\eta$$

Since on C the function K becomes zero, the following equation results

$$\int_S\int \frac{\partial K}{\partial \xi} f(\xi,\eta)\, d\xi\, d\eta = -\int_S\int K \frac{\partial f}{\partial \xi}\, d\xi\, d\eta$$

showing the finiteness of this integral. Therefore

$$\frac{\partial U}{\partial x} = \iint_S \left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial \xi}\right) f(\xi,\eta)\, d\xi\, d\eta + O(1)$$

where $O(1)$ denotes a magnitude which remains finite when δ approaches 0. But

$$\frac{\partial K}{\partial x} + \frac{\partial K}{\partial \xi} = - \frac{2\sqrt{2}\, a \sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\pi\left\{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right\}} \times \left\{\frac{x\sqrt{a^2 - \xi^2 - \eta^2}}{R} + \frac{\xi}{\sqrt{a^2 - \xi^2 - \eta^2}}\right\}$$

hence

$$\frac{\partial U}{\partial x} = - \frac{2\sqrt{2}}{\pi} a\sqrt{a^2-x^2-y^2-z^2+R} \iint_S \frac{f(\xi,\eta)}{2a^2r^2 + (a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)} \times \left[\frac{x\sqrt{a^2-\xi^2-\eta^2}}{R} + \frac{\xi}{\sqrt{a^2-\xi^2-\eta^2}}\right] d\xi\, d\eta + O(1) \tag{3.40}$$

The coordinates δ, θ, and α are introduced

$$x = (a + \delta \cos \alpha) \cos \theta \qquad y = (a + \delta \cos \alpha) \sin \theta \qquad z = \delta \sin \alpha \tag{3.41}$$

Then

$$\begin{aligned} a^2 - x^2 - y^2 - z^2 &= - 2a\delta \cos \alpha - \delta^2 \\ R = \delta\sqrt{4a^2 + 4a\delta \cos \alpha + \delta^2} &= 2a\delta + \ldots \\ \sqrt{a^2 - x^2 - y^2 - z^2 + R} &= 2 \sin \frac{\alpha}{2} \sqrt{a\delta} + \ldots \\ \sqrt{R - a^2 + x^2 + y^2 + z^2} &= 2 \cos \frac{\alpha}{2} \sqrt{a\delta} + \ldots \end{aligned} \tag{3.42}$$

The point with coordinates (x,y,z) is brought into correspondence with the point of the circumference C with the coordinates

$$x_0 = a \cos\theta \qquad y_0 = a \sin\theta \qquad z_0 = 0$$

and

$$r_0{}^2 = (x_0 - \xi)^2 + (y_0 - \eta)^2 = \xi^2 + \eta^2 + a^2 - 2a\xi \cos\theta - 2a\eta \sin\theta \tag{3.43}$$

Near the contour C, the principal part of the integral

$$J_1(x,y,z) = \iint_S \frac{2a^2\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} \tag{3.44}$$

is

$$N(\theta) = \iint_S f(\xi,\eta)\sqrt{a^2 - \xi^2 - \eta^2}\,\frac{d\xi\, d\eta}{r_0{}^2}$$

$$= \iint_S \frac{f(\xi,\eta)\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\theta - 2a\eta\sin\theta} \tag{3.45}$$

For this purpose, the following difference is estimated:

$$\Delta = J_1(x,y,z) - N(\theta)$$

The circle S is divided into two parts: the circle S_1 of radius $a - \varepsilon$; and the ring S_2 lying between the circumferences of radii $a - \varepsilon$ and a.

$$\Delta_1 = \iint_{S_1} f(\xi,\eta)\sqrt{a^2 - \xi^2 - \eta^2}\left\{\frac{2a^2}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} - \frac{1}{r_0{}^2}\right\} d\xi\, d\eta$$

$$= \iint_{S_1} f(\xi,\eta)\sqrt{a^2 - \xi^2 - \eta^2}\,\frac{2a^2r_0{}^2 - 2a^2r^2 - (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)}{r_0{}^2\left\{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right\}}\, d\xi\, d\eta$$

$$r_0^2 - r^2 = \xi^2 + \eta^2 + a^2 - 2a\xi \cos\theta - 2a\eta \sin\theta - (x-\xi)^2 - (y-\eta)^2 - z^2$$

$$= 2\delta \cos\alpha(\xi \cos\theta + \eta \sin\theta - a) - \delta^2$$

$$2a^2(r_0^2 - r^2) - (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)$$

$$= - 2a\delta r_0^2 \cos\alpha - (a^2 + \xi^2 + \eta^2)\,\delta^2 - (a^2 - \xi^2 - \eta^2)\,R$$

Since

$$r_0^2 \leqslant 4a^2 \qquad R \leqslant 2a\delta + \delta^2$$

therefore

$$\left|2a^2(r_0^2 - r^2) - (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right| \leqslant 2a\delta r_0^2 + 2a^2\delta^2 + (a^2 - \xi^2 - \eta^2)\,R$$

Hence if $|f(\xi,\eta)| < M$ in the circle S then

$$|\Delta_1| \leqslant 2a\delta M \iint_{S_1} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)}$$

$$2a^2\delta^2 M \iint_{S_1} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{r_0^2\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]} +$$

$$RM \iint_{S_1} \frac{\sqrt{(a^2 - \xi^2 - \eta^2)^3}\, d\xi\, d\eta}{r_0^2\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]}$$

But by equation (2.24) of reference 1

$$\iint_{S} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} \leqslant \frac{\pi}{a}$$

Since

$$2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)$$

$$= 2a^2r_0^2 + 2a\delta r_0^2 \cos\alpha + (a^2 + \xi^2 + \eta^2)\,\delta^2 + (a^2 - \xi^2 - \eta^2)\,R$$

hence for $\delta < a/2$

$$2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R) \geqslant a^2r_0^2 \tag{3.46}$$

and

$$\iint_{S_1} \frac{a^2\sqrt{a^2-\xi^2-\eta^2}\, d\xi\, d\eta}{r_0^2\left[2a^2r^2+(a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)\right]} \leqslant \iint_{S_1} \frac{1}{r_0^4}\sqrt{a^2-\xi^2-\eta^2}\, d\xi\, d\eta$$

The last integral evidently does not depend on θ; hence it may be assumed that $\theta = 0$ and therefore

$$\iint_{S_1} \frac{1}{r_0^4}\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = \int_0^{2\pi}\int_0^{a-r} \frac{\sqrt{a^2 - \rho^2}\,\rho\, d\rho\, d\vartheta}{(\rho^2 - 2a\rho\cos\vartheta + a^2)^2} = \int_0^{a-\varepsilon} \frac{2\pi(a^2 + \rho^2)\rho\, d\rho}{\sqrt{(a^2 - \rho^2)^5}}$$

$$= \left[\frac{4\pi a^2}{3\sqrt{(a^2 - \rho^2)^3}} - \frac{2\pi}{\sqrt{a^2 - \rho^2}}\right]_{\rho=0}^{\rho=a-\varepsilon} = \frac{4\pi a^2}{3\sqrt{(2a\varepsilon - \varepsilon^2)^3}} - \frac{2\pi}{\sqrt{2a\varepsilon - \varepsilon^2}} + \frac{2\pi}{3a} < \frac{4\pi\sqrt{a}}{3\sqrt{\varepsilon^3}}$$

Similarly

$$\iint_{S_1} \frac{\sqrt{(a^2 - \xi^2 - \eta^2)^3}\, d\xi\, d\eta}{r_0^2\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]} \leqslant \frac{1}{a^2}\iint_{S_1} \frac{1}{r_0^4}\sqrt{(a^2 - \xi^2 - \eta^2)^3}\, d\xi\, d\eta$$

and

$$\iint_{S_1} \frac{1}{r_0^4}\sqrt{(a^2 - \xi^2 - \eta^2)^3}\, d\xi\, d\eta = \int_0^{2\pi}\int_0^{a-\varepsilon} \frac{\sqrt{(a^2 - \rho^2)^3}\,\rho\, d\rho\, d\vartheta}{(a^2 - 2a\rho\cos\vartheta + \rho^2)^2} = \int_0^{a-\varepsilon} \frac{2\pi(a^2 + \rho^2)\,\rho\, d\rho}{\sqrt{(a^2 - \rho^2)^3}}$$

$$= \left[\frac{4\pi a^2}{\sqrt{a^2 - \rho^2}} + 2\pi\sqrt{a^2 - \rho^2}\right]_{\rho=0}^{\rho=a-\varepsilon} = \frac{4\pi a^2}{\sqrt{2a\varepsilon - \varepsilon^2}} + 2\pi\sqrt{2a\varepsilon - \varepsilon^2} - 6\pi a < \frac{4\pi\sqrt{a^3}}{\sqrt{\varepsilon}}$$

As a result, the following inequality is obtained

$$|\Delta_1| \leqslant 2\pi M\delta + \frac{8\pi M\delta^2\sqrt{a}}{3\sqrt{\varepsilon^3}} + \frac{4\pi M(2a\delta + \delta^2)}{\sqrt{a\varepsilon}}$$

The difference is estimated

$$\left|\Delta_2\right| = \iint_{S_2} \frac{2a^2 f(\xi,\eta)\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} - \iint_{S_2} \frac{1}{r_0^2} f(\xi,\eta) \sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta$$

On account of equation (3.46)

$$|\Delta_2| \leqslant 3M \iint_{S_2} \frac{1}{r_0^2} \sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta$$

But

$$\iint_{S_2} \frac{1}{r_0^2} \sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = \int_0^{2\pi} \int_{a-\varepsilon}^{a} \frac{\sqrt{a^2 - \rho^2}\, \rho\, d\rho\, d\vartheta}{\rho^2 - 2a\rho \cos\vartheta + a^2} = 2\pi \int_{a-\varepsilon}^{a} \frac{\rho\, d\rho}{\sqrt{a^2 - \rho^2}} = 2\pi\sqrt{2a\varepsilon - \varepsilon^2}$$

and therefore

$$|\Delta_2| < 6\pi M \sqrt{2a\varepsilon}$$

Thus for

$$\Delta = \Delta_1 + \Delta_2$$

the estimate is obtained

$$|\Delta| < 2\pi M \left\{\delta + \frac{4}{3} \frac{\delta^2\sqrt{a}}{\sqrt{\varepsilon^3}} + 4\delta\sqrt{\frac{a}{\varepsilon}} + \frac{2\delta^2}{\sqrt{a\varepsilon}} + 3\sqrt{2a\varepsilon}\right\}$$

Assuming

$$\varepsilon = \delta$$

yields

$$|\Delta| < 24\pi M \sqrt{a\delta}$$

Thus

$$\iint_S \frac{2a^2\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} = N(\theta) + O\left(\sqrt{\delta}\right) \tag{3.47}$$

where $O(\alpha)$ denotes a magnitude, whose ratio to α remains finite when δ approaches zero.

An estimate of the second integral entering equation (3.40) is given:

$$J_2(x,y,z) = \iint_S \frac{\xi f(\xi,\eta)\, d\xi\, d\eta}{\sqrt{a^2-\xi^2-\eta^2}\left[2a^2r^2+(a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)\right]} \tag{3.48}$$

Again assuming $\delta < \alpha/2$ yields

$$|J_2| \leqslant \frac{M}{a}\iint_S \frac{d\xi\, d\eta}{\sqrt{a^2-\xi^2-\eta^2}\,(r_0{}^2+\delta^2)} = \frac{M}{2}\int_0^a\int_0^{2\pi} \frac{\rho\, d\rho\, d\vartheta}{\sqrt{a^2-\rho^2}\,(a^2-2a\rho\cos\vartheta+\rho^2+\delta^2)}$$

$$= \frac{2\pi M}{a}\int_0^a \frac{\rho\, d\rho}{\sqrt{a^2-\rho^2}\sqrt{(a^2+\rho^2+\delta^2)^2-4a^2\rho^2}}$$

but

$$\int_0^{a-\varepsilon} \frac{\rho\, d\rho}{\sqrt{a^2-\rho^2}\sqrt{(a^2+\rho^2+\delta^2)^2-4a^2\rho^2}} < \int_0^{a-\varepsilon} \frac{\rho\, d\rho}{\sqrt{(a^2-\rho^2)^3}}$$

$$= \left[\frac{1}{\sqrt{a^2-\rho^2}}\right]_{\rho=0}^{\rho=a-\varepsilon} = \frac{1}{\sqrt{2a\varepsilon-\varepsilon^2}} - \frac{1}{a} < \frac{1}{\sqrt{a\varepsilon}}$$

$$\int_{a-\varepsilon}^{a} \frac{\rho\, d\rho}{\sqrt{a^2-\rho^2}\sqrt{(a^2+\rho^2+\delta^2)^2-4a^2\rho^2}} < \int_{a-\varepsilon}^{a} \frac{\rho\, d\rho}{2a\delta\sqrt{a^2-\rho^2}}$$

$$= \left[-\frac{\sqrt{a^2-\rho^2}}{2a\delta}\right]_{\rho=a-\varepsilon}^{\rho=a} = \frac{\sqrt{2a\varepsilon-\varepsilon^2}}{2a\delta} < \frac{\sqrt{\varepsilon}}{\delta\sqrt{2a}}$$

hence

$$|J_2| < \frac{2\pi M}{a\sqrt{a}}\left(\frac{1}{\sqrt{\varepsilon}} + \frac{\sqrt{\varepsilon}}{\delta\sqrt{2}}\right)$$

and for $\varepsilon = \delta$

$$|J_2| < \frac{4\pi M}{a\sqrt{a\delta}} \tag{3.49}$$

From equation (3.40) and equation (3.42), the following is obtained on account of the estimates (3.47) and (3.49):

$$\frac{\partial U}{\partial x} = -\frac{\sqrt{2}\,N(\theta)\,x\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\pi aR} + O(1) \tag{3.50}$$

In exactly the same way, there is obtained

$$\frac{\partial U}{\partial y} = -\frac{\sqrt{2}\,N(\theta)\,y\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\pi aR} + O(1) \tag{3.51}$$

Finally,

$$\frac{\partial U}{\partial z} = \iint_S \frac{\partial K}{\partial z} f(\xi,\eta)\, d\xi\, d\eta$$

But

$$\frac{\partial K}{\partial z} = -\frac{2z}{\pi r^3}\arc\tan A + \frac{2A}{\pi(1 + A^2)}\left[-\frac{z}{r^3} + \frac{z(a^2 + x^2 + y^2 + z^2 - R)}{rR(a^2 - x^2 - y^2 - z^2 + R)}\right]$$

where

$$A = \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{ar\sqrt{2}}$$

Assuming $z > 0$,

$$\iint_S \frac{z}{r^3}\, d\xi\, d\eta \leq 2\pi$$

hence

$$\left| \iint_S \left(\frac{2z}{\pi r^3} \text{ arc tan } A + \frac{2}{\pi} \frac{A}{1 + A^2} \frac{z}{r^3} \right) f(\xi,\eta)\, d\xi\, d\eta \right| \leq 2(\pi + 1)\, M$$

and therefore

$$\frac{\partial U}{\partial z} = \frac{2\sqrt{2}\, az(a^2 + x^2 + y^2 + z^2 - R)}{\pi R\sqrt{a^2 - x^2 - y^2 - z^2 + R}} \iint_S \frac{f(\xi,\eta)\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2 r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} + O(1)$$

Again use is made of equations (3.47) and (3.42) and the fact that for $z > 0$

$$\frac{z}{\sqrt{a^2 - x^2 - y^2 - z^2 + R}} = \frac{1}{2a}\sqrt{R - a^2 + x^2 + y^2 + z^2}$$

without difficulty:

$$\frac{\partial U}{\partial z} = \frac{a^2 + x^2 + y^2 + z^2 - R}{\pi R a^2 \sqrt{2}} N(\theta)\sqrt{R - a^2 + x^2 + y^2 + z^2} + O(1) \tag{3.52}$$

From what has been said previously about equation (3.38) it is evident that if

$$F(\xi,\eta,t) = f_0(\xi,\eta) + f_1(\xi,\eta) \cos \omega t + f_2(\xi,\eta) \sin \omega t \tag{3.53}$$

$$N(\theta,t) = \iint_S \frac{F(\xi,\eta,t)\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos \theta - 2a\eta \sin \theta} \tag{3.54}$$

the following results

$$\left.\begin{aligned}
\frac{\partial\varphi}{\partial x} &= -\frac{x\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\,\pi^2 aR} N(\theta,t) + O(1) \\
\frac{\partial\varphi}{\partial y} &= -\frac{y\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\,\pi^2 aR} N(\theta,t) + O(1) \\
\frac{\partial\varphi}{\partial z} &= \frac{1}{2\sqrt{2}\,\pi^2 a^2 R}\left(a^2 + x^2 + y^2 + z^2 - R\right) \times \\
&\qquad \sqrt{R - a^2 + x^2 + y^2 + z^2}\, N(\theta,t) + O(1)
\end{aligned}\right\} \tag{3.55}$$

or, in the coordinates δ, θ, α

$$\left.\begin{aligned}
\frac{\partial\varphi}{\partial x} &= -\frac{N(\theta,t)\cos\theta\sin\left(\frac{1}{2}\alpha\right)}{\pi^2\sqrt{2a\delta}} + O(1) \\
\frac{\partial\varphi}{\partial y} &= -\frac{N(\theta,t)\sin\theta\sin\left(\frac{1}{2}\alpha\right)}{\pi^2\sqrt{2\alpha\delta}} + O(1) \\
\frac{\partial\varphi}{\partial z} &= \frac{N(\theta,t)\cos\left(\frac{1}{2}\alpha\right)}{\pi^2\sqrt{2a\delta}} + O(1)
\end{aligned}\right\} \tag{3.56}$$

The computation of the suction force W_2 by equation (3.37) is considered. An arc D'AB' of the circumference C is taken symmetrical with respect to the x-axis with subtending angle $2\theta_0 < \pi$. For the surface σ, the part σ_0 is taken of the surface determined by equations (3.41) for constant δ_0, where θ changes from $-\theta_0$ to $+\theta_0$ and α from $-\pi$ to $+\pi$ and two bases, one of which, σ_1, corresponds to $\theta = \theta_0$ and the other, σ_2, corresponds to $\theta = -\theta_0$, where on these bases δ varies from 0 to δ_0 and α from $-\pi$ to $+\pi$.

On the toroidal surface:

$$\cos(n,x) = \cos\alpha\cos\theta \qquad \cos(n,y) = \cos\alpha\sin\theta \qquad \cos(n,z) = \sin\alpha$$

$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial x}\cos(n,x) + \frac{\partial\varphi}{\partial y}\cos(n,y) + \frac{\partial\varphi}{\partial z}\cos(n,z) = \frac{N(\theta,t)\sin\left(\frac{1}{2}\alpha\right)}{\pi^2\sqrt{2a\delta}} + O(1)$$

Hence simple computation shows that

$$\frac{1}{2}\iint_{\sigma_0}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] \cos(n,x)\ dS - \iint_{\sigma_0}\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial n}\ dS$$

$$= \frac{1}{2}\int_{-\theta_0}^{\theta_0}\int_{-\pi}^{\pi}\frac{1}{2\pi^4}N^2(\theta,t)\cos\alpha\cos\theta\ d\theta\ d\alpha + \int_{-\theta_0}^{\theta_0}\int_{-\pi}^{\pi}\frac{1}{2\pi^4}N^2(\theta,t)\cos\theta\sin^2\frac{1}{2}\alpha\ d\theta\ d\alpha +$$

$$O(\sqrt{\delta_0}) = \frac{1}{2\pi^3}\int_{-\theta_0}^{\theta_0}N^2(\theta,t)\cos\theta\ d\theta + O(\sqrt{\delta_0})$$

In the same manner, the integrals taken over the bases σ_1 and σ_2 have the order $O(\delta_0)$. Hence if δ_0 approaches zero, for the suction force developed along the arc D'AB', the following expression is obtained

$$\frac{\rho}{2\pi^3}\int_{-\theta_0}^{\theta_0}N^2(\theta,t)\cos\theta\ d\theta$$

Now when θ_0 approaches $\pi/2$, the required expression for the suction force W_2 is obtained in the following form:

$$W_2 = \frac{\rho}{2\pi^3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}N^2(\theta,t)\cos\theta\ d\theta \tag{3.57}$$

The mean value of the frontal resistance is found. Equation (3.31) shows that for the mean value of W_1

$$\overline{W}_1 = -\ 2\rho c\iint_S\left[\frac{\partial\varphi_0}{\partial x}\frac{\partial\zeta_0}{\partial x} + \frac{1}{2}\left(\frac{\partial\varphi_1}{\partial x} - k\varphi_2\right)\frac{\partial\zeta_1}{\partial x} + \frac{1}{2}\left(\frac{\partial\varphi_2}{\partial x} + k\varphi_1\right)\frac{\partial\zeta_2}{\partial x}\right] dx\ dy \tag{3.58}$$

In the same way, for the mean value of the suction force

$$\overline{W}_2 = \frac{\rho}{2\pi^3}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left[N_0^2(\theta) + \frac{1}{2}N_1^2(\theta) + \frac{1}{2}N_2^2(\theta)\right]\cos\theta\, d\theta \tag{3.59}$$

where

$$N_k(\theta) = \iint_S \frac{f_k(\xi,\eta)\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\theta - 2a\eta\sin\theta} \qquad (k = 0,1,2) \tag{3.60}$$

For the mean value of the frontal resistance

$$\overline{W} = \overline{W}_1 - \overline{W}_2 \tag{3.61}$$

4. Example

If a plane wing varies its angle of attack periodically according to the harmonic law so that the equation of its surface is

$$z = (\beta_0 + \beta_1 \cos \omega t)\, x \tag{4.1}$$

in the notation of section 1, the following is obtained

$$\zeta_0(x,y) = \beta_0 x \qquad \zeta_1(x,y) = \beta_1 x \qquad \zeta_2(x,y) = 0$$

and therefore

$$\begin{gathered} Z_0(x,y) = -c\beta_0 \qquad Z_1(x,y) = -c\beta_1 \qquad Z_2(x,y) = -ck\beta_1 x \\ Z(x,y) = Z_1 + iZ_2 = -c\beta_1(1 + i\,x) \end{gathered} \tag{4.2}$$

The function $f(x,y)$ corresponding to this value of the function $Z(x,y)$ is determined by equation (2.18) where $g(y)$ is the solution of integral equation (2.19).

Consideration is restricted to the solution of the inverse problem by assuming that

$$f_0(x,y) = A_0 \qquad f(x,y) = A + Bx$$

where A and B are constant complex numbers and A_0 is a constant real number and the shape of the wing is determined corresponding to this function. By such a method it is possible to obtain also an approximate solution of the direct problem of the nonsteady motion of a wing according to the law (4.1) for the case of small frequencies of vibration.

The forces acting on the wing are determined. For determination of the lift force P, use is made of equation (3.13). The following relations are used

$$\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = \frac{2}{3}\pi a^3 \qquad \iint_S \xi\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = 0 \tag{4.3}$$

as are equations (3.5) and (3.26), yielding without difficulty

$$P = -\frac{4\rho c}{\pi}\left\{\mathrm{Re}\left(\frac{2}{3}\, i\pi k a^3\, Ae^{-i\omega t}\right) + \right.$$

$$\left. a^2\left[A_0 + \mathrm{Re}(Ae^{-i\omega t})\right]\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\cos\gamma\, d\gamma + \frac{2}{3}a^3\,\mathrm{Re}(Be^{-i\omega t})\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\cos^2\gamma\, d\gamma\right\}$$

or

$$P = \frac{8\rho ca^2}{\pi}A_0 + \frac{8\rho ca^2}{\pi}\mathrm{Re}\left[Ae^{-i\omega t}\left(1 - \frac{ika}{3}\right)\right] - \frac{4\rho ca^3}{3}\mathrm{Re}(Be^{-i\omega t}) \tag{4.4}$$

The moment of the pressure forces about the x-axis equals zero on account of symmetry:

$$M_x = 0 \tag{4.5}$$

If the moment of the pressure forces about the y-axis is determined by equation (3.28) and, in addition to the previously mentioned formulas, use is made also of the formula

$$\iint_S \xi^2\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta = \frac{2}{15}\pi a^5$$

$$M_y = \frac{8\rho c}{3\pi}\left\{\left[\frac{2}{15}\pi a^5 \operatorname{Re}(ikBe^{-i\omega t}) - a^3\left[A_0 + \operatorname{Re}(Ae^{-i\omega t})\right]\right]\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2\gamma\, d\gamma - \frac{2}{3} a^4 \operatorname{Re}(Be^{-i\omega t})\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^3\gamma\, d\gamma\right\}$$

or

$$M_y = -\frac{4\rho ca^3}{3} A_0 - \frac{4\rho ca^3}{3} \operatorname{Re}(Ae^{-i\omega t}) - \frac{64\rho ca^4}{27\pi} \operatorname{Re}\left[Be^{-i\omega t}\left(1 - \frac{3\pi}{20} iak\right)\right] \tag{4.6}$$

The frontal resistance is computed. First the suction force is computed:

If

$$A = A_1 + iA_2 \qquad B = B_1 + iB_2$$

according to equation (3.53)

$$F(\xi,\eta,t) = A_0 + (A_1 + B_1\xi) \cos \omega t + (A_2 + B_2\xi) \sin \omega t$$

If equation (3.54) is applied and use is made of equations (3.5) and (3.26),

$$N(\theta,t) = 2\pi a(A_0+A_1 \cos \omega t+ A_2 \sin \omega t) + \frac{4}{3}\pi a^2 \cos\theta\,(B_1 \cos \omega t + B_2 \sin \omega t)$$

Equation (3.57) yields without difficulty the expression for the suction force:

$$W_2 = \frac{\rho}{2\pi^3}\left\{8\pi^2 a^2(A_0 + A_1 \cos \omega t + A_2 \sin \omega t)^2 + \frac{8}{3}\pi^3 a^3(A_0 + A_1 \cos \omega t + A_2 \sin \omega t)(B_1 \cos \omega t + B_2 \sin \omega t) + \frac{64}{27}\pi^2 a^4(B_1 \cos \omega t + B_2 \sin \omega t)^2\right\}$$

or

$$W_2 = \frac{4\rho a^2}{\pi}\Bigg\{A_0^2 + \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 + \frac{\pi}{6}aA_1B_1 + \frac{\pi}{6}aA_2B_2 + \frac{4}{27}a^2B_1^2 + \frac{4}{27}a^2B_2^2 + \left(2A_0A_1 + \frac{\pi}{3}aA_0B_1\right)\cos\omega t + \left(2A_0A_2 + \frac{\pi}{3}aA_0B_2\right)\sin\omega t + \left(\frac{1}{2}A_1^2 - \frac{1}{2}A_2^2 + \frac{\pi}{6}aA_1B_1 - \frac{\pi}{6}aA_2B_2 + \frac{4}{27}a^2B_1^2 - \frac{4}{27}a^2B_2^2\right)\cos 2\omega t + \left(A_1A_2 + \frac{\pi}{6}aA_1B_2 + \frac{\pi}{6}aA_2B_1 + \frac{8}{27}a^2B_1B_2\right)\sin 2\omega t\Bigg\} \tag{4.7}$$

The total frontal resistance is obtained by the equation

$$W = W_1 - W_2$$

where W_1 is determined by equation (3.31)

$$W_1 = -2\rho c\iint_S\left\{\frac{\partial\varphi_0}{\partial x} + \mathrm{Re}\left[\left(\frac{\partial\Phi}{\partial x} + ik\Phi\right)e^{-i\omega t}\right]\right\}\left\{\frac{\partial\zeta_0}{\partial x} + \mathrm{Re}\left(\frac{\partial\zeta}{\partial x}e^{-i\omega t}\right)\right\}dx\,dy \tag{4.8}$$

For the mean value of the frontal resistance the following is obtained:

$$\overline{W} = \overline{W}_1 - \overline{W}_2 \tag{4.9}$$

where

$$\overline{W}_2 = \frac{4\rho a^2}{\pi}\left(A_0^2 + \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 + \frac{\pi}{6}aA_1B_1 + \frac{\pi}{6}aA_2B_2 + \frac{4}{27}a^2B_1^2 + \frac{4}{27}a^2B_2^2\right) \tag{4.10}$$

$$\overline{W}_1 = -2\rho c\iint_S\left\{\frac{\partial\varphi_0}{\partial x}\frac{\partial\zeta_0}{\partial x} + \frac{1}{2}\mathrm{Re}\left[\left(\frac{\partial\Phi}{\partial x} + ik\Phi\right)\frac{\partial\zeta}{\partial x}\right]\right\}dx\,dy \tag{4.11}$$

For determination of the functions $\zeta_0(x,y)$ and $\zeta(x,y)$ characterizing the shape of the wing, equation (1.16) is used.

$$-c\frac{\partial\zeta_0}{\partial x} = Z_0(x,y) \qquad -c\left(\frac{\partial\zeta}{\partial x} + ik\zeta\right) = Z(x,y) \tag{4.12}$$

where by equation (2.17) in this case

$$Z_0(x,y) = -A_0 + g_0(y) \qquad Z(x,y) = -A - Bx + g(y)e^{-ikx} \quad (4.13)$$

and the functions $g_0(y)$ and $g(y)$ in this case according to equation (2.16) have the form:

$$g_0(y) = \frac{aA_0}{2\pi^3} \times$$

$$\iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{(x^2 + y^2 - a^2)^{-\frac{1}{2}} (a^2 - \xi^2 - \eta^2)^{\frac{1}{2}} \cos\gamma \, d\gamma \, dx \, d\xi \, d\eta}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}$$

$$g(y) = \frac{a}{2\pi^3} \times$$

$$\iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}(x^2 + y^2 - a^2)^{-\frac{1}{2}} (a^2 - \xi^2 - \eta^2)^{\frac{1}{2}} (A + B\xi) \cos\gamma \, d\gamma \, dx \, d\xi \, d\eta}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}$$

Equations (3.5) and (3.26) yield

$$\left.\begin{aligned} g_0(y) &= \frac{a^2A_0}{\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\cos\gamma \, d\gamma \, dx}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)} \\ g(y) &= \frac{a^2}{\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{e^{ikx}(A + 2/3\,aB\cos\gamma)\cos\gamma \, d\gamma \, dx}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)} \end{aligned}\right\} \quad (4.14)$$

Integration of equations (4.12) yields

$$\zeta_0(x,y) = \frac{A_0}{c} x - \frac{1}{c} g_0(y)\, x + h_0(y)$$

$$\zeta(x,y) = \left(A + \frac{iB}{k}\right) \frac{1 - e^{-ikx}}{ikc} - \frac{iBx}{kc} - \frac{1}{c}\; g(y)\; xe^{-ikx} + h(y)\; e^{-ikx} \tag{4.15}$$

where $h_0(y)$ and $h(y)$ are arbitrary functions of y.

The function $g_0(y)$ was obtained in reference 1, where, however, errors slipped into the computations. Setting

$$y = -a\cos\theta \qquad H_0(\theta) = \frac{\pi^2}{A_0}\sin\theta\; g_0(-a\cos\theta) \qquad (0 < \theta < \pi) \tag{4.16}$$

gives in place of equation (4.22) of reference 1

$$H_0(\theta) = \frac{\pi^2}{4}\sin\theta + \frac{1}{8}\sin\theta\left(\ln\frac{1 + \sin\frac{1}{2}\theta}{1 - \sin\frac{1}{2}\theta}\right)^2 + \frac{1}{8}\sin\theta\left(\ln\frac{1 - \cos\frac{1}{2}\theta}{1 + \cos\frac{1}{2}\theta}\right)^2 +$$

$$\cos\frac{1}{2}\theta\,\ln\frac{1 - \sin\frac{1}{2}\theta}{1 + \sin\frac{1}{2}\theta} + \sin\frac{1}{2}\theta\,\ln\frac{1 - \cos\frac{1}{2}\theta}{1 + \cos\frac{1}{2}\theta} \tag{4.17}$$

Hence setting $h_0(y) = 0$ and $A_0 = \alpha c$ in place of equation (4.23) of reference 1 yields

$$\zeta_0(x,y) = \alpha x\left\{\frac{3}{4} - \frac{1}{8\pi^2}\left(\ln\frac{\sqrt{2a} + \sqrt{a + y}}{\sqrt{2a} - \sqrt{a + y}}\right)^2 - \frac{1}{8\pi^2}\left(\ln\frac{\sqrt{2a} + \sqrt{a - y}}{\sqrt{2a} - \sqrt{a - y}}\right)^2 - \right.$$

$$\left.\frac{\sqrt{2a}}{2\pi^2\sqrt{a + y}}\ln\frac{\sqrt{2a} - \sqrt{a + y}}{\sqrt{2a} + \sqrt{a + y}} - \frac{\sqrt{2a}}{2\pi^2\sqrt{a - y}}\ln\frac{\sqrt{2a} - \sqrt{a - y}}{\sqrt{2a} + \sqrt{a + y}}\right\} \tag{4.18}$$

In particular for $y = 0$ and $y = \pm\, a/2$ the following values are obtained in place of those given in reference 1:

$$\zeta(x,0) = \alpha x\left[\frac{3}{4} - \frac{1}{\pi^2}\ln^2(\sqrt{2}+1) + \frac{2\sqrt{2}}{\pi^2}\ln(\sqrt{2}+1)\right] \approx 0.9263\ \alpha x$$

$$\zeta\left(x, \pm\frac{a}{2}\right) =$$

$$\alpha x\left[\frac{3}{4} - \frac{1}{2\pi^2}\ln^2(2+\sqrt{3}) - \frac{1}{8\pi^2}\ln^2 3 + \frac{2}{\pi^2\sqrt{3}}\ln(2+\sqrt{3}) + \frac{1}{\pi^2}\ln 3\right] \approx 0.9146\ \alpha x$$

In the same way, the expansion given in reference 1 of the function $H_0(\theta)$ in a trigonometric series in the interval $0 \leq \theta \leq \pi$ should be replaced by the following:

$$H_0(\theta) = \sin\theta\left(\frac{\pi^2}{2} - 4\right) + \sum_{k=1}^{\infty}\frac{\sin(2k+1)\theta}{k(k+1)}\left(\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{4k+1}\right) \tag{4.19}$$

that is,

$$H_0(\theta) = \sum_{k=0}^{\infty}\beta_{2k+1}\sin(2k+1)\theta$$

where

$$\beta_1 = 0.9348 \qquad \beta_5 = 0.1312 \qquad \beta_9 = 0.0504$$
$$\beta_3 = 0.2667 \qquad \beta_7 = 0.0796 \qquad \ldots\ldots$$

In connection with this, corrections should also be applied to the numerical values, which are given in reference 1, of the coefficients B_n of the trigonometric series for the circulation obtained by the usual theory

$$B_1 = 2.2125\ \alpha ca \qquad B_5 = -0.0296\ \alpha ca \qquad B_9 = -0.0067\ \alpha ca$$

$$B_3 = -0.0934\ \alpha ca \qquad B_7 = -0.0133\ \alpha ca \qquad \ldots\ldots$$

Hence for the lift force in place of equation (4.29) of reference 1, the following is obtained:

$$P_0 = \frac{1}{2}\pi\rho ca \qquad B_1 = 3.4755\ \rho c^2 a^2 \alpha$$

which exceeds the accurate value by 36 percent. For the induced drag, in place of equation (4.30) of reference 1, the following is obtained

$$W_0 \approx 1.9350\ \rho c^2\alpha^2 a^2$$

which exceeds the accurate value by 87 percent.

Corrections are made in the third example given in reference 1. The value of the definite integral is:

$$\int_0^1 \frac{\arctan y}{\sqrt{1 - y^2}}\, dy = \frac{\pi^2}{8} - \frac{1}{2} \ln^2(\sqrt{2} + 1)$$

Hence in equation (4.52) of reference 1 the coefficient of $\sin\theta \cos\theta$ is simplified and assumes the value $-3\pi^2/8$. In equation (4.53) the coefficient of $\sin 2\theta$ was incorrectly computed; its correct value is

$$\delta_2 = -\frac{3\pi^2}{8} + \frac{32}{9} = -0.14555$$

In this connection, the value of the coefficient B_2 should also be corrected:

$$B_2 = -0.7436\ \alpha c a^2$$

For the induced drag and the moment of the forces about the x-axis, in place of the values of equation (4.55) of reference 1, the following is obtained:

$$W = 0.4343\ \rho\alpha^2 c^2 a^4 \qquad M_x = 0.5840\ \rho\alpha^2 c^2 a^4$$

the first gives an error of 140 percent; the second of 55 percent.

The shape of the wing obtained

$$z(x,y,t) = \frac{A_0}{c}\, x - \frac{1}{c}\, g_0(y)\, x + \mathrm{Re}\left\{ e^{-i\omega t} \left[\left(A + \frac{iB}{k}\right) \frac{1 - e^{-ikx}}{ikc} - \frac{iBx}{kc} - \frac{1}{c}\, g(y)\, x e^{-ikx} \right] \right\} \qquad (4.20)$$

depends on the frequency of the vibrations and is deformed during the vibrations. The rigid wing is of greater interest.

It is possible with the aid of the results obtained to obtain an approximate solution of the problem of the vibrations of a plane circular wing for small frequencies of vibration.

The case is now considered of a wing varying its angle of attack periodically according to the harmonic law (4.1), so that equation (4.2) holds.

If

$$f_0(x,y) = A_0 \qquad f(x,y) = A + Bx$$

equation (4.2) yields

$$Z_0(x,y) = -A_0 + g(y) \qquad Z(x,y) = -A - Bx + g(y)\,e^{-ikx} \quad (4.21)$$

If

$$G_0(y) = \frac{a^2}{\pi^2}\int_{+\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{\cos\gamma\,d\gamma\,dx}{\sqrt{x^2+y^2-a^2}\,(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)}$$

$$G_1(y) = \frac{a^2}{\pi^2}\int_{+\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{e^{ikx}\cos\gamma\,d\gamma\,dx}{\sqrt{x^2+y^2-a^2}\,(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)}$$

$$G_2(y) = \frac{a^2}{\pi^2}\int_{+\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi}\frac{e^{ikx}\cos^2\gamma\,d\gamma\,dx}{\sqrt{x^2+y^2-a^2}\,(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)} \quad (4.22)$$

Then

$$g_0(y) = A_0G_0(y) \qquad g(y) = AG_1(y) + BG_2(y) \quad (4.23)$$

In place of $G_k(y)$, their mean values are taken over the area of the wing:

$$\widetilde{G}_k = \left\{\int_{-a}^{a} G_k(y)\sqrt{a^2-y^2}\,dy\right\} : \left\{\int_{-a}^{a}\sqrt{a^2-y^2}\,dy\right\} = \frac{2}{\pi a^2}\int_{-a}^{a} G_k(y)\sqrt{a^2-y^2}\,dy$$

$$(k = 0,1,2) \quad (4.24)$$

If the frequency of the vibrations is assumed small, or more accurately, the magnitude ka is assumed small, the expansion

$$e^{-ikx} = 1 - ikx - \frac{1}{2} k^2x^2 - \ . \ . \ .$$

may be limited to the first two terms.

From equation (4.21), the following approximate expressions were obtained

$$\begin{aligned} Z_0(x,y) &\approx - A_0 + A_0\widetilde{G}_0 \\ Z(x,y) &\approx - A - Bx + (1 - ikx)(A\widetilde{G}_1 + B\widetilde{G}_2) \end{aligned} \tag{4.25}$$

Comparison with equation (4.2) results in:

$$- c\beta_0 = - A_0 + A_0\widetilde{G}_0$$

$$- c\beta_1 = - A + A\widetilde{G}_1 + B\widetilde{G}_2$$

$$- c\beta_1 ik = - B - ik(A\widetilde{G}_1 + B\widetilde{G}_2)$$

whence

$$A_0 = \frac{c\beta_0}{1 - \widetilde{G}_0} \qquad A = \frac{c\beta_1(1 + 2ik\widetilde{G}_2)}{1 - \widetilde{G}_1 + ik\widetilde{G}_2} \qquad B = \frac{c\beta_1 ik(1 - 2\widetilde{G}_1)}{1 - \widetilde{G}_1 + ik\widetilde{G}_2} \tag{4.26}$$

The following is computed

$$\widetilde{G}_0 = \frac{2}{\pi a^2}\int_{-a}^{a} G_0(y) \sqrt{a^2 - y^2}\, dy = \frac{2}{\pi}\int_0^{\pi} G_0(- a \cos \theta)\sin^2 \theta \, d\theta$$

But by equation (4.16)

$$\sin \theta \, G_0(- a \cos \theta) = \sin \theta \, \frac{g_0(- a \cos \theta)}{A_0} = \frac{1}{\pi^2} H_0(\theta)$$

hence, expansion (4.19) is used, yielding

$$\tilde{G}_0 = \frac{2}{\pi^3}\int_0^{\pi} H_0(\theta) \sin\theta \, d\theta = \frac{2}{\pi^3}\left(\frac{\pi^2}{2} - 4\right)\frac{\pi}{2} = \frac{1}{2} - \frac{4}{\pi^2}$$

and therefore

$$\tilde{G}_0 = \frac{1}{2} - 0.4053 = 0.0947 \qquad A_0 = 1.105c\beta_0 \tag{4.27}$$

Equations (4.26) show that in computing $\tilde{G}_1$ it is sufficient to use the terms of first-order smallness relative to ka, while in computing $\tilde{G}_2$ it is sufficient to use the principal term not depending on k. For small ka the following results

$$\tilde{G}_1 \approx \tilde{G}_0 + ik\tilde{G}_{11} + O\left(k^2a^2 \ln\frac{1}{ka}\right) \qquad \tilde{G}_2 \approx \tilde{G}_{20} + O(ka^2) \tag{4.28}$$

where $\tilde{G}_{11}$ and $\tilde{G}_{20}$ are the mean values over the area of the circle S of the functions

$$G_{11}(y) = \frac{a^2}{\pi^2}\int_{\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{x\cos\gamma \, d\gamma \, dx}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)} \tag{4.29}$$

$$G_{20}(y) = \frac{2}{3}\frac{a^3}{\pi^2}\int_{+\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\cos^2\gamma \, d\gamma \, dx}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)} \tag{4.30}$$

In fact,

$$\tilde{G}_1 - \tilde{G}_0 - ik\tilde{G}_{11} = \frac{2}{\pi^3}\int_{-a}^{a}\int_{\infty}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} G^*(x,y,\gamma)\, d\gamma \, dx \, dy \tag{4.31}$$

where

$$G^*(x,y,\gamma) = \frac{(e^{ikx} - 1 - ikx)\cos\gamma\sqrt{a^2 - y^2}}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)}$$

The interval of integration with respect to x is divided into two parts: from $\sqrt{a^2 - y^2}$ to $2a$ and from $2a$ to ∞. Since for $\alpha > 0$

$$\left|e^{i\alpha} - 1 - i\alpha\right| < \alpha^2$$

in the interval $\sqrt{a^2 - y^2} \leq x \leq 2a$, $\left|e^{ikx} - 1 - ikx\right| \leq (2ka)^2$ and therefore

$$\left|\frac{2}{\pi^3}\int_{-a}^{a}\int_{2a}^{\sqrt{a^2-y^2}}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} G^*(x,y,\gamma)\, d\gamma\, dx\, dy\right| < (2ka)^2\, \tilde{G}_0 < 0.38k^2a^2 \tag{4.32}$$

On the other hand, for $x \geq 2a$, $|y| < a$, $\pi/2 \leqslant \gamma \leqslant 3\pi/2$ the inequality holds

$$x^2 + y^2 - a^2 \geqslant \frac{3}{4}x^2 \qquad (x - a\cos\gamma)^2 + (y - a\sin\gamma)^2 \geqslant x^2$$

As

$$\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \cos\gamma\, d\gamma = -2$$

$$\int_{-a}^{a}\sqrt{a^2 - y^2}\, dy = \frac{\pi a^2}{2} \quad e^{ikx} - 1 - ikx = \cos kx - 1 + i(\sin kx - kx)$$

the following inequalities are obtained when, for clarity, ka is assumed < 1,

$$\left|\frac{2}{\pi^3}\int_{-a}^{a}\int_{\infty}^{2a}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} G^*(x,y,\gamma)\, d\gamma\, dx\, dy\right| < \frac{4a^2}{\sqrt{3}\,\pi^2}\left\{\int_{2a}^{\infty}\frac{1 - \cos kx}{x^3}\, dx + \int_{2a}^{\infty}\frac{kx - \sin kx}{x^3}\, dx\right\}$$

$$= \frac{4a^2k^2}{\sqrt{3}\,\pi^2}\left\{\int_{2ak}^{\infty}\frac{1 - \cos u}{u^3}\, du + \int_{0}^{\infty}\frac{u - \sin u}{u^3}\, du\right\}$$

$$< \frac{4a^2k^2}{\sqrt{3}\,\pi^2}\left\{\int_{2ak}^{2}\frac{1}{2u}\, du + \int_{2}^{\infty}\frac{2}{u^3}\, du + \frac{\pi}{4}\right\} = \frac{4a^2k^2}{\sqrt{3}\,\pi^2}\left\{\frac{1}{2}\ln\frac{1}{ak} + \frac{1}{4} + \frac{\pi}{4}\right\}$$

$$< 0.25a^2k^2 + 0.12a^2k^2\ln\frac{1}{ak} \tag{4.33}$$

Combining inequalities (4.32) and (4.33) yields, on account of equation (4.31),

$$\left|\tilde{G}_1 - \tilde{C}_0 - ik\tilde{G}_{11}\right| < 0.63a^2k^2 + 0.12a^2k^2 \ln \frac{1}{ak}$$

from which the first of the estimates (4.30) follows.

In an entirely analogous manner, since, for $\alpha > 0$

$$\left| e^{i\alpha} - 1 \right| < \alpha$$

from the inequality

$$\tilde{G}_2 - \tilde{G}_{20} = \frac{4a}{3\pi^3} \int_{-a}^{a} \int_{\infty}^{\sqrt{a^2-y^2}} \int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\left(e^{ikx} - 1\right) \cos^2 \gamma \sqrt{a^2 - y^2}\, d\gamma\, dx\, dy}{\sqrt{x^2 + y^2 - a^2}\, \left(x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma\right)}$$

the inequality is obtained

$$\left|\tilde{G}_2 - \tilde{G}_{20}\right| < \frac{2ka}{3\pi^2} \int_{-a}^{a} \int_{\sqrt{a^2-y^2}}^{\infty} \frac{\sqrt{a^2 - y^2}\, dy\, dx}{x\sqrt{x^2 + y^2 - a^2}} = \frac{2ka}{3\pi^2} \int_{-a}^{a} \int_{1}^{\infty} \frac{dy\, dt}{t\sqrt{t^2 - 1}} = \frac{2ka^2}{3\pi}$$

which proves the correctness of the second estimate (4.28).

The integral (4.30) was considered in reference 1. The function $H_1(\theta)$ of reference 1 is obtained if

$$\frac{3\pi^2}{2a} \sin \theta\, G_{20}\left\{-a \cos \theta\right) = H_1(\theta) \qquad (0 \leqslant \theta \leqslant \pi)$$

For this function the expression was obtained (equation (4.36) of reference 1 with the correction of the error appearing therein)

$$H_1(\theta) = \frac{3\pi}{2}\left\{\sin \theta\left(1 - \sin \frac{\theta}{2} - \cos \frac{\theta}{2}\right) + \frac{\sin \theta}{12} \ln \frac{\left(1 + \cos \frac{1}{2}\theta\right)\left(1 + \sin \frac{1}{2}\theta\right)}{\left(1 - \cos \frac{1}{2}\theta\right)\left(1 - \sin \frac{1}{2}\theta\right)} + \right.$$

$$\left. \sin \theta \cos \theta \left[\ln \tan \frac{\theta}{2} + \frac{1}{4} \ln \frac{\left(1 + \cos \frac{1}{2}\theta\right)\left(1 - \sin \frac{1}{2}\theta\right)}{\left(1 - \cos \frac{1}{2}\theta\right)\left(1 + \sin \frac{1}{2}\theta\right)}\right]\right\} \qquad (4.34)$$

The expansion of this function in the interval $0 \leq \theta \leq \pi$ in a trigonometric series has the form

$$H_1(\theta) = \sum_{k=0}^{\infty} \gamma_{2k+1} \sin(2k + 1)\, \theta$$

where

$$\gamma_1 = \pi - \frac{17}{3} - \int_0^{\frac{1}{2}\pi} \ln \tan \frac{x}{2}\, dx = -0.69314$$

Hence

$$\tilde{G}_{20} = \frac{2}{\pi} \int_0^{\pi} G_{20}(-a \cos \theta) \sin^2 \theta\, d\theta = \frac{4a}{3\pi^3} \int_0^{\pi} \sin \theta\, H_1(\theta)\, d\theta = \frac{2a}{3\pi^2} \gamma_1 = -0.0468a \tag{4.35}$$

The mean value $\tilde{G}_{11}$ is computed. Integrating (4.29) with respect to γ yields

$$G_{11}(y) = \frac{a}{\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \frac{x}{\sqrt{x^2 + y^2 - a^2}} \left\{ - \frac{\pi x}{2(x^2 + y^2)} + \frac{y}{2(x^2 + y^2)} \ln \frac{x^2 + (y - a)^2}{x^2 + (y + a)^2} + \frac{x(a^2 + x^2 + y^2)}{(x^2 + y^2)(x^2 + y^2 - a^2)} \text{arc tan} \frac{x^2 + y^2 - a^2}{2ax} \right\} dx \tag{4.36}$$

If

$$x = at \qquad y = -a \cos \theta \qquad \frac{\pi^2}{a} \sin \theta\, G_{11}(-a \cos \theta) = H(\theta)$$

then

$$H(\theta) = \int_0^{\sin\theta} \frac{\sin\theta}{\sqrt{t^2 - \sin^2\theta}} \left\{ -\frac{\pi t^2}{2(t^2 + \cos^2\theta)} - \frac{t\cos\theta}{2(t^2 + \cos^2\theta)} \ln \frac{t^2 + 4\cos^4\frac{1}{2}\theta}{t^2 + 4\sin^4\frac{1}{2}\theta} + \frac{t^2(t^2 + 1 + \cos^2\theta)}{(t^2 + \cos^2\theta)(t^2 - \sin^2\theta)} \operatorname{arc\,tan} \frac{t^2 - \sin^2\theta}{2t} \right\} dt$$

Computation of this integral results in

$$H(\theta) = \pi \left\{ \sin\theta \left(1 - \sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) + \frac{1}{4}\sin\theta \ln \frac{\left(1 + \cos\frac{1}{2}\theta\right)\left(1 + \sin\frac{1}{2}\theta\right)}{\left(1 - \cos\frac{1}{2}\theta\right)\left(1 - \sin\frac{1}{2}\theta\right)} + \frac{1}{4}\sin 2\theta \left[\ln \tan\frac{\theta}{2} - \ln \frac{1 + \sin\frac{1}{2}\theta}{1 + \cos\frac{1}{2}\theta}\right] \right\} \tag{4.37}$$

Further,

$$\tilde{G}_{11} = \frac{2}{\pi} \int_0^{\pi} G_{11}(-a\cos\theta) \sin^2\theta \, d\theta = \frac{2a}{\pi^3} \int_0^{\pi} H(\theta) \sin\theta \, d\theta$$

The computation of the last integral leads to the result

$$\tilde{G}_{11} = \frac{a}{\pi^2} \left\{ \frac{2}{3}\pi - \frac{38}{9} + 2\int_0^{\frac{1}{2}\pi} \frac{u}{\sin u} du \right\} = 1.536 \frac{a}{\pi^2} = 0.1556a \tag{4.38}$$

Thus for small ka

$$\tilde{G}_1 = 0.0947 + 0.1556ika \qquad \tilde{G}_2 = -0.0468a \tag{4.39}$$

Substituting these values in (4.26) gives

$$A = c\beta_1 \frac{1 - 0.0936ika}{0.9053 - 0.202ika} \qquad B = c\beta_1 \frac{ik(0.8106 - 0.311ika)}{0.9053 - 0.202ika} \tag{4.40}$$

Thus for small frequencies of vibration, to a first approximation:

$$A_0 = 1.105c\beta_0 \qquad A = (1.105 + 0.144ika)\ c\beta_1$$

$$B = 0.895ikc\beta_1 \tag{4.41}$$

For the periodic vibrations with small frequency, in accordance with the law (4.1) of a plane circular wing, the previously derived formulas may be used for the forces where the values A_0, A, and B have the values just given. For the lift force, the approximate expression is obtained from equation (4.4)

$$P = \rho c^2 a^2 \left\{2.813\beta_0 + \beta_1(2.813 \cos \omega t - 1.766ka \sin \omega t)\right\} \tag{4.42}$$

The fluctuation in the lift force due to the vibrations of the wing thus leads the latter in phase, the maximum value of the lift force being greater than the value which was obtained in the computation for the steady motion.

In the same way, equation (4.6) leads to the following expression for the moment of the pressure forces about the y-axis:

$$M_y = -\ \rho c^2 a^3 \left\{1.473\beta_0 + \beta_1(1.473 \cos \omega t + 0.867\ ka \sin \omega t)\right\} \tag{4.43}$$

The component of the frontal resistance W_1 is determined in the given case by the evident formula

$$W_1 = P(\beta_0 + \beta_1 \cos \omega t)$$

that is,

$$W_1 = \rho c^2 a^2 \left\{2.813\beta_0^{\ 2} + 1.406\beta_1^{\ 2} + \beta_0\beta_1(5.626 \cos \omega t - 1.766ka \sin \omega t) + 1.406\beta_1^{\ 2} \cos 2\omega t - 0.883\beta_1^{\ 2}\ ka \sin 2\omega t\right\} \tag{4.44}$$

The suction force is obtained from equation (4.7), restricted to the first powers of ka,

$$W_2 = \rho c^2 a^2 \left\{1.554\beta_0^{\ 2} + 0.777\beta_1^{\ 2} + \beta_0\beta_1(3.107 \cos \omega t + 1.888ka \sin \omega t) + 0.777\beta_1^{\ 2} \cos 2\omega t + 0.944ka\ \beta_1^{\ 2} \sin 2\omega t\right\} \tag{4.45}$$

The following expression is obtained for the total frontal resistance:

$$W = W_1 - W_2 = \rho c^2 a^2 \left\{ 1.259\beta_0{}^2 + 0.630\beta_1{}^2 + \beta_0\beta_1(2.519 \cos \omega t - 3.653ka \sin \omega t) + 0.630\beta_1{}^2 \cos 2\omega t - 1.827\beta_1{}^2 \; ka \sin 2\omega t \right\} \tag{4.46}$$

For the mean value of the frontal resistance

$$\overline{W} = \rho c^2 a^2 \left\{ 1.259\beta_0{}^2 + 0.630\beta_1{}^2 \right\} \tag{4.47}$$

The flapping wing is considered such that

$$z = \beta_0 x + \beta_1 \cos \omega t \tag{4.48}$$

In this case

$$Z_0(x,y) = -c\beta_0 \qquad Z(x,y) = -ikc\beta_1 \tag{4.49}$$

Comparison of these expressions with equation (4.25) shows that in the case considered it is necessary to take

$$A_0 = \frac{c\beta_0}{1 - \tilde{G}_0} \qquad A = \frac{ikc\beta_1(1 + ik\tilde{G}_2)}{1 - \tilde{G}_1 + ik\tilde{G}_2} \qquad B = \frac{k^2 c\beta_1 \tilde{G}_1}{1 - \tilde{G}_1 + ik\tilde{G}_2} \tag{4.50}$$

that is,

$$A_0 = 1.105c\beta_0 \quad A = ikc\beta_1 \frac{1 - ika\ 0.0468}{0.9053 - 0.202ika} \quad B = \frac{k^2 c\beta_1(0.0947 + 0.156ika)}{0.9053 - 0.202ika} \tag{4.51}$$

or, by restriction to small terms of the second order with respect to k,

$$A_0 = 1.105c\beta_0 \qquad A = ikc\beta_1(1.105 + 0.195ika) \qquad B = 0.105k^2 c\beta_1 \tag{4.52}$$

For the lift force

$$P = \rho c^2 a^2 \left\{ 2.813\beta_0 + 2.813k\beta_1 \sin \omega t + 0.301k^2 a\beta_1 \cos \omega t \right\} \tag{4.53}$$

and for the moment of the pressure forces about the y-axis

$$M_y = -\rho c^2 a^3 \left\{ 1.473\beta_0 + 1.473k\beta_1 \sin \omega t - 0.181k^2 a\beta_1 \cos \omega t \right\} \tag{4.54}$$

The component of the frontal resistance

$$W_1 = P\beta_0 = \rho c^2 a^2 \left\{2.813\beta_0{}^2 + 2.813k\beta_0\beta_1 \sin \omega t + 0.301k^2 a\beta_0\beta_1 \cos \omega t\right\} \tag{4.55}$$

The suction force will be, with an accuracy up to terms of the second order with respect to ka:

$$W_2 = \rho a^2 c^2 \left\{1.554\beta_0{}^2 + 0.777k^2\beta_1{}^2 - 0.376\beta_0\beta_1 k^2 a \cos \omega t + 3.107k\beta_0\beta_1 \sin \omega t - 0.777k^2\beta_1{}^2 \cos 2\omega t\right\} \tag{4.56}$$

For the total frontal resistance

$$W = \rho a^2 c^2 \left\{1.259\beta_0{}^2 a - 0.777k^2\beta_1{}^2 - 0.294k\beta_0\beta_1 \sin \omega t + 0.677k^2 a\beta_0\beta_1 \cos \omega t + 0.777k^2\beta_1{}^2 \cos 2\omega t\right\} \tag{4.57}$$

Its mean value will be

$$\overline{W} = \rho\alpha^2 c^2 \left\{1.259\beta_0{}^2 - 0.777k^2\beta_1{}^2\right\} \tag{4.58}$$

so that a decrease is obtained in the frontal resistance as compared with the wing which does not execute a flapping motion.

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THEORY OF WING OF CIRCULAR PLAN FORM*

By N. E. Kochin

A theory is developed for a wing of circular plan form. The distribution of the bound vortices along the surface of the wing is considered in this theory, which has already been applied in a number of papers. In particular, the case of the circular wing has been examined by Kinner in reference 1.

A second method is considered herein which permits obtaining an expression in closed form for the general solution of this problem. The wing is assumed infinitely thin and slightly cambered and the problem is linearized in the usual manner.

Comparison of the results of the proposed theory with the results of the usual theory of a wing of finite span shows large divergences, which indicate the inadequacy of the usual theory of the case under consideration. For the wings generally employed in practice, which have a considerably greater aspect ratio, a more favorable relation should be obtained between the results of the usual and the more accurate theory.

1. Statement of the Problem

The forward rectilinear motion of a circular wing with constant velocity c is considered. A right-hand system of rectangular coordinates Oxyz is used and the x-axis is taken in the direction of motion of the wing. The wing is assumed thin with a slight camber and has as its projection on the xy-plane a circle ABCD of radius a with center at the origin of the coordinates (fig. 2, in which a section of the wing in the xz-plane is also shown).

Let

$$z = \zeta(x,y) \tag{1.1}$$

represent the equation of the surface of the wing, where the ratio ζ/a as well as the derivatives $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ are assumed to be small magnitudes.

*"Teoriya kryla konechnogo razmakha krugovoi formy v plane." Prikladnaya Matematika i Mekhanika, Vol. IV. No. 1, 1940, pp. 3-32.

The coordinate axes are assumed to be immovably attached to the wing. The fluid is considered incompressible and the motion nonvortical, steady, and with no acting external forces. The velocity potential of the absolute motion of the fluid will be denoted by $\varphi(x,y,z)$ so that the projection of the absolute velocity of a particle of the fluid is determined by the formulas

$$v_x = \frac{\partial \varphi}{\partial x};\ v_y = \frac{\partial \varphi}{\partial y};\ v_z = \frac{\partial \varphi}{\partial z} \tag{1.2}$$

The equation of continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

shows that the function φ must satisfy the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{1.3}$$

At the leading edge of the wing the velocity of the fluid particles is assumed to become infinite to the order of $1/\sqrt{\delta}$ where δ is the distance of the particle to the leading edge; at the trailing edge the velocity is assumed finite. From the trailing edge of the wing a surface of discontinuity is passed off on which the function φ suffers a discontinuity. The function $\varphi(x,y,z)$ and all its derivatives over the entire space bounded by the said surface of discontinuity and the surface of the wing are continuous.

The problem is linearized in the following manner. The function φ is assumed to suffer a discontinuity on an infinite half-strip Σ located in the xy-plane in the direction of the negative x-axis from the rear semicircumference BCD of the circle S to infinity. In the same manner, the condition on the surface of the wing is replaced by the condition on the surface of the circle S located in the xy-plane and in this way the function $\varphi(x,y,z)$ is assumed to be regular in the region obtained by cutting the infinite half-strip Σ and the circle S from the entire infinite space.

The boundary condition must be satisfied on the surface of the wing.

$$\frac{\partial \varphi}{\partial n} = c \cos(n,x) \tag{1.4}$$

where n is the direction of the normal to the surface of the wing. The direction of this normal, because of the assumption of small curvature of the wing, differs little from the direction of the z-axis. If small terms of the second order are rejected according to the formula

$$\cos(n,x) = - \frac{\frac{\partial\zeta}{\partial x}}{\sqrt{1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2}} \tag{1.5}$$

in place of equation (1.4),

$$\frac{\partial\varphi}{\partial z} = - c \frac{\partial\zeta}{\partial x}$$

This condition must be satisfied on the surface of the wing, but it is assumed satisfied on the surface of the circle S, that is, for $z = 0$; this again reduces to the rejection of small terms of the second order by comparison with those of the first order.

The boundary condition is obtained:

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = - c \frac{\partial\zeta(x,y)}{\partial x} \text{ for } x^2 + y^2 < a^2 \tag{1.6}$$

which must be satisfied on both the upper and lower sides of the circle S.

The boundary conditions are set up which must be satisfied on the surface of discontinuity Σ. On the surface of discontinuity at the trailing edge of the wing, the kinematic condition expresses the continuity of the normal component of the velocity, that is, the magnitude $\partial\varphi/\partial n$ must remain continuous in passing through the surface of discontinuity. Since on the surface of discontinuity the direction of the normal differs little from the direction of the z-axis, transfer of the condition on the surface of discontinuity to the half-strip Σ, gives the equation

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=+0} \left(\frac{\partial\varphi}{\partial z}\right)_{z=-0} \quad \text{for} \quad |y| < a; \quad x^2 + y^2 > a^2; \quad x < 0 \tag{1.7}$$

which expresses the continuity of $\partial\varphi/\partial z$ in passing through the surface of discontinuity Σ.

The dynamical condition expressing the continuity of the pressure in passing through the surface of discontinuity Σ is considered.

In order to determine the pressure p, the formula of Bernoulli is applied to the steady flow about a wing obtained by superposing on the flow considered, a uniform flow with velocity c in the direction of the negative x-axis. In this steady flow the velocity projections are determined by the equations

$$v_x = -c + \frac{\partial\varphi}{\partial x}; \quad v_y = \frac{\partial\varphi}{\partial y}; \quad v_z = \frac{\partial\varphi}{\partial z}$$

and therefore the formula of Bernoulli reduces to the form

$$p = -\frac{\rho}{2}\left[\left(-c + \frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] + \text{constant} \tag{1.8}$$

Rejection of small terms of the second order results in

$$p = p_0 + \rho c \frac{\partial\varphi}{\partial x} \tag{1.9}$$

where p_0 is the value of the pressure at infinity.

Since the pressure must remain continuous in passing through the surface of discontinuity at the trailing edge of the wing, the equation obtained shows that $\partial\varphi/\partial x$ does not suffer a discontinuity on the surface of discontinuity. Transfer of this condition to the surface Σ yields the condition

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=+0} = \left(\frac{\partial\varphi}{\partial x}\right)_{z=-0} \quad \text{for} \quad |y| < a; \quad x^2 + y^2 > a^2; \quad x < 0 \tag{1.10}$$

which expresses the continuity of $\partial\varphi/\partial x$ in passing through Σ.

The function φ suffers a discontinuity on the surfaces S and Σ, which means that along the surfaces S and Σ, surface vortices are located as shown in figure 2. The direction of such a surface vortex is perpendicular to the direction of the relative velocity vector of two particles of the fluid adjacent to the surface of discontinuity on its two sides. In particular, on the surface Σ, on account of equation (1.10), only $\partial\varphi/\partial y$ suffers a discontinuity and therefore the vortex lines on Σ are directed parallel to the x-axis as shown in figure 2.

Since all the vortices lie in the xy-plane, at two points symmetrical with respect to the xy-plane, the values of $\partial\varphi/\partial z$ will be the same, whereas the values of $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ will differ only in sign.

It may therefore be assumed that

$$\varphi(x,y,-z) = -\varphi(x,y,z) \tag{1.11}$$

Assuming in particular $z = 0$ yields

$$\varphi(x,y,0) = 0$$

in the entire xy-plane with the exception of the circle S and the strip Σ (on which φ suffers a discontinuity).

Since on the strip Σ both condition (1.10) and the condition derived from equation (1.11) must be satisfied

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=+0} = -\left(\frac{\partial\varphi}{\partial x}\right)_{z=-0}$$

and

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=+0} = \left(\frac{\partial\varphi}{\partial x}\right)_{z=-0} = 0 \quad \text{for} \quad |y| < a;\ x^2 + y^2 > a^2;\ x < 0 \tag{1.12}$$

Finally, since the fluid far ahead of the wing is assumed to be undisturbed, the condition at infinity is

$$\lim_{x\to+\infty} \frac{\partial\varphi}{\partial x} = \lim_{x\to+\infty} \frac{\partial\varphi}{\partial y} = \lim_{x\to+\infty} \frac{\partial\varphi}{\partial z} = 0 \tag{1.13}$$

In the hydrodynamic problem under consideration, account is taken of the distribution of the vortices along the surface of the wing. It is this circumstance which makes the treatment more accurate than the usual wing theory.

The hydrodynamic problem is thus reduced to the following mathematical problem: To find a harmonic function $\varphi(x,y,z)$ regular over the entire half-space $z > 0$, which on the circle S satisfies the condition

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = -c\,\frac{\partial\zeta}{\partial x} \tag{1.14}$$

on the strip Σ , the condition

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=0} = 0 \tag{1.15}$$

on the entire remaining part of the xy-plane, the condition

$$\varphi(x,y,0) = 0 \tag{1.16}$$

and the derivatives of which remain bounded in the neighborhood of the rear semicircumference BCD, while in the neighborhood of the forward semicircumference BAD they may approach infinity as $1/\sqrt{\delta}$ where δ is the distance of a point to the semicircumference BAD. Finally the condition at infinity (1.13) must be satisfied.

An expression for the harmonic function $\varphi(x,y,z)$ is given in closed form depending on an arbitrary function $f(x,y)$ satisfying all the imposed requirements besides equation (1.14). The function $\zeta(x,y)$ can be determined from this condition, that is, the shape of the wing corresponding to the function $f(x,y)$. An integral equation is also given, the solution of which is reduced to the determination of the function $f(x,y)$ for the given shape of the wing, that is, for a given function $\zeta(x,y)$.

2. Derivation of the Fundamental Equation

Inside the circle ABCD, the point Q with coordinates ξ, η is taken and the function $K(x,y,z,\xi,\eta)$ constructed, where x,y,z are the coordinates of the point P, according to the following conditions:

(1) The function K, considered as a function of the point P, is a harmonic function outside the circle ABCD.

(2) The function K becomes zero at the points of the plane xy lying outside the circle ABCD.

(3) The derivative $\partial K/\partial z$ becomes zero at all points of the circle ABCD, except the point Q.

(4) When the point P approaches the point Q, remaining in the upper half-space $z > 0$, the function K increases to infinity but the difference $K - (1/r)$, where

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2}$$

remains bounded.

(5) The function K remains finite and continuous in the neighborhood of the contour C of the circle ABCD.

Because of the second condition, the values of the function K at two points situated symmetrically with respect to the plane xy differ only in sign:

$$K(x,y,-z,\xi,\eta) = -K(x,y,z,\xi,\eta) \tag{2.1}$$

as follows from the principle of analytic continuation. It is then evident that if the third condition is satisfied on the upper side of the circle ABCD it will be satisfied also on the lower side, since according to equation (2.1) the derivative $\partial K/\partial z$ has the same value at two points situated symmetrically with respect to the xy-plane.

It is evident further that when the point P approaches the point Q from below so that $z < 0$ then $K(x,y,z,\xi,\eta)$ will behave as $-1/r$.

Because of the third condition, the function K can be continued into the lower half-space through the upper side of the circle ABCD as an even function of z. Thus a second branch of the function K is assumed, again determined over all the space outside the circle ABCD and differing only in sign from the initial branch of the function K. It is then evident, however, that at the points of the upper side of the circle ABCD, the values of the second branch of the function K and its derivatives coincide with the values of the first branch of the function K and its derivatives at the points of the lower side of the circle ABCD. That is, in the analytic continuation of the second branch of the function K through the upper side of the circle ABCD into the lower half-space, the initial branch of this function is again obtained.

A two-sheet Riemann space is considered for which the branching line is the circumference ABCD. In this space $K(x,y,z,\xi,\eta)$ is a single-valued harmonic function remaining finite everywhere with the exception of the two points Q having the same coordinates $(\xi,\eta,0)$, but belonging to two different sheets of space; in one sheet the function K behaves near the point Q as $1/r$ and in the other sheet as $-1/r$. Such a function $K(x,y,z,\xi,\eta)$ can readily be constructed by the method of Sommerfeld (reference 2). In this way for the case of a two-sheet Riemann space having as branch line the z-axis, a harmonic function $V(\rho,\varphi,z)$ (ρ,φ,z being the cylindrical coordinates of the point) is determined which is single-valued and continuous in the entire two-sheet space with the exception of the points Q and Q' having the cylindrical coordinates (ρ',φ',z') and $(\rho',-\varphi',z')$, where near the point Q the function V behaves as $1/r$ and near the point Q' as $-1/r$, where

$$r = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + (z - z')^2}$$

$$r' = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi + \varphi') + (z - z')^2}$$

This function V has the form:

$$V = \frac{2}{\pi}\left\{\frac{1}{r}\text{arc tan}\sqrt{\frac{\sigma + \tau}{\sigma - \tau}} - \frac{1}{r'}\text{arc tan}\sqrt{\frac{\sigma + \tau'}{\sigma - \tau'}}\right\}$$

where

$$\sigma = \frac{1}{2\sqrt{\rho\rho'}}\sqrt{(\rho + \rho')^2 + (z - z')^2};\quad \tau = \cos\frac{\varphi - \varphi'}{2};\quad \tau' = \cos\frac{\varphi + \varphi'}{2}$$

Setting, in particular,

$$\varphi' = \pi;\quad r = \sqrt{\rho^2 + \rho'^2 + 2\rho\rho'\cos\varphi + (z - z')^2}$$

yields

$$V = \frac{2}{\pi r}\left\{\text{arc tan}\sqrt{\frac{\sigma + \tau}{\sigma - \tau}} - \text{arc tan}\sqrt{\frac{\sigma - \tau}{\sigma + \tau}}\right\} = \frac{2}{\pi r}\text{arc tan}\frac{\tau}{\sqrt{\sigma^2 - \tau^2}}$$

or finally

$$V = \frac{2}{\pi r}\text{arc tan}\frac{2\sqrt{\rho\rho'}\sin\frac{\varphi}{2}}{r}$$

An inversion with respect to the point with coordinates $\rho = a$, $\varphi = 0$, $z = 0$ is carried out.

$$\rho\cos\varphi = a + \frac{2a^2(x_1 - a)}{(x_1 - a)^2 + y_1{}^2 + z_1{}^2};\quad -\rho' = a + \frac{2a^2(\xi_1 - a)}{(\xi_1 - a)^2 + \zeta_1{}^2}$$

$$\rho\sin\varphi = \frac{2a^2 y_1}{(x_1 - a)^2 + y_1{}^2 + z_1{}^2};\quad z = \frac{2a^2 z_1}{(x_1 - a)^2 + y_1{}^2 + z_1{}^2}$$

$$z' = \frac{2a^2\zeta_1}{(\xi_1 - a)^2 + \zeta_1{}^2}$$

The function

$$V_1 = \frac{2a^2 V}{\sqrt{(x_1 - a)^2 + y_1{}^2 + z_1{}^2}\sqrt{(\xi_1 - a)^2 + \zeta_1{}^2}}$$

expressed in the variables x_1, y_1, z_1 is then, as is known, a harmonic function. Computing it and replacing x_1, y_1, z_1 by y, z, x and ξ_1, ζ_1 by η, ξ yield the required expression of the function $K(x,y,z,\xi,\eta)$:

$$K(x,y,z,\xi,\eta) = \frac{2}{\pi r} \text{arc tan} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\, ar} \tag{2.2}$$

valid for $z > 0$, where

$$\begin{gathered} r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2} \\ R = \sqrt{(a^2 - x^2 - y^2 - z^2)^2 + 4a^2z^2} = \sqrt{(a^2 + x^2 + y^2 + z^2)^2 - 4a^2(x^2 + y^2)} \end{gathered} \tag{2.3}$$

That this function satisfies all the above set requirements is easily verified; the arc tangents must be taken between 0 and $\pi/2$; for $z < 0$ the value of the function K is obtained by equation (2.1).

The following function is set up:

$$\varphi_1(x,y,z) = \frac{1}{2\pi} \iint_S K(x,y,z,\xi,\eta)f(\xi,\eta)d\xi d\eta \tag{2.4}$$

where $f(x,y)$ is an arbitrary function, which is continuous together with its partial derivatives of the first and second order in the entire circle S, and the integration extends over the entire area of the circle S. Evidently, $\varphi_1(x,y,z)$ is a harmonic function in the entire space outside the circle S. Because of the first property of the function K, the function $\varphi_1(x,y,z)$ becomes zero at all points of the plane xy which are outside the circle S. Hence equations (1.15) and (1.16), which must be satisfied by the solution $\varphi(x,y,z)$ of the problem posed in section 1, will be satisfied for the function $\varphi_1(x,y,z)$. The function $\varphi_1(x,y,z)$ does not in general satisfy the condition of the finiteness of the derivatives of this function on the rear half of the contour of the circle S. For this reason, a function such that the obtained function $\varphi(x,y,z)$ also satisfies this condition is added to $\varphi_1(x,y,z)$.

The following equation is evident:

$$\frac{\partial\varphi_1}{\partial x} = \frac{1}{2\pi} \iint_S \frac{\partial K}{\partial x} f(\xi,\eta)d\xi d\eta$$

The character of the approach of the function $\partial K/\partial x$ to infinity is considered as a point approaches the contour C of the circle S. As may be easily computed

$$\frac{\partial K}{\partial x} = -\frac{2(x-\xi)}{\pi r^3} \arc\tan\frac{\sqrt{a^2-\xi^2-\eta^2}\sqrt{a^2-x^2-y^2-z^2+R}}{\sqrt{2}\,ar} -$$

$$\frac{2\sqrt{2}a}{\pi}\frac{\sqrt{a^2-\xi^2-\eta^2}\sqrt{a^2-x^2-y^2-z^2+R}}{2a^2r^2+(a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)}\left\{\frac{x-\xi}{r^2}+\frac{x}{R}\right\} \tag{2.5}$$

If a point with coordinates x,y,z is near the contour C of the circle S the distance of this point to the contour C is denoted by δ; then

$$\delta = \sqrt{a^2+x^2+y^2+z^2-2a\sqrt{x^2+y^2}} \tag{2.6}$$

Hence near the contour C, the approximate equation holds:

$$R \approx 2a\delta \tag{2.7}$$

When the fixed point ξ,η lies inside the circle S while the point with coordinates x,y,z lies near the contour C of the circle, then, as follows from equation (2.5),

$$\frac{\partial K}{\partial x} = -\frac{\sqrt{2}\,x}{\pi ar^2R}\sqrt{a^2-\xi^2-\eta^2}\sqrt{a^2-x^2-y^2-z^2+R}+O(1) \tag{2.8}$$

where the symbol $O(1)$ denotes a magnitude which remains finite when δ approaches zero. Thus $\partial K/\partial x$ has the order $1/\sqrt{\delta}$. The principal part of $\partial K/\partial x$ is not a harmonic function. It is not difficult, however, to find a harmonic function having the same infinite part near the contour C as $\partial K/\partial x$. For this, it is sufficient to form, after the analogy of equation (2.5), the derivative $\partial K/\partial \xi$; this derivative remains finite near the contour C of the wing; moreover it is easy to see that

$$\frac{\partial K}{\partial x}+\frac{\partial K}{\partial \xi} = -\frac{2\sqrt{2}\,a\sqrt{a^2-x^2-y^2-z^2+R}}{\pi\left[2a^2r^2+(a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)\right]}$$

$$\left\{\frac{x\sqrt{a^2-\xi^2-\eta^2}}{R}+\frac{\xi}{\sqrt{a^2-\xi^2-\eta^2}}\right\} \tag{2.9}$$

This function is harmonic and differs from $\partial K/\partial x$ by a quantity which remains finite near the contour C.

By computation, it is further shown that the function just described is represented in the form of the integral

$$\frac{\partial K}{\partial x} + \frac{\partial K}{\partial \xi} = -\frac{1}{\pi^2\sqrt{2}}\int_{-\pi}^{\pi} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma\, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{2.10}$$

where the function

$$\frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma}$$

is a solution of the equation of Laplace having the circumference C as the branching line and the point with coordinates $(a\cos\gamma, a\sin\gamma, 0)$ as a singular point. From this it follows that the function

$$\frac{\partial K}{\partial x} + \frac{1}{\pi^2\sqrt{2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma\, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}$$

$$= -\frac{\partial K}{\partial \xi} - \frac{1}{\pi^2\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma\, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{2.11}$$

remains finite near the points of the rear semicircumference of the circle S.

Therefore it is assumed

$$\frac{\partial \varphi}{\partial x} = \frac{1}{2\pi}\iint_S f(\xi,\eta)\Bigg\{\frac{\partial K}{\partial x} + \frac{1}{\pi^2\sqrt{2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma\, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}\Bigg\}\, d\xi\, d\eta \tag{2.12}$$

Integrating with respect to x and considering the condition at infinity (1.13) yield the final equation

$$\varphi(x,y,z) = \frac{1}{2\pi} \iint_S f(\xi,\eta) \left\{ K(x,y,z,\xi,\eta) + \right.$$

$$\left. \frac{1}{\pi^2\sqrt{2}} \int_{+\infty}^{x} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma \, d\gamma \, dx}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \right\} d\xi \, d\eta \quad (2.13)$$

This equation may be written in somewhat different form. Because of equation (2.11)

$$\frac{\partial\varphi}{\partial x} = -\frac{1}{2\pi} \iint_S \frac{\partial K}{\partial \xi} f(\xi,\eta) d\xi \, d\eta - \frac{1}{2\sqrt{2}\,\pi^3} \times$$

$$\iint_S \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma \, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} f(\xi,\eta) d\xi \, d\eta$$

Since the function K becomes zero on the contour C

$$\iint_S \frac{\partial K}{\partial \xi} f(\xi,\eta) d\xi \, d\eta = -\iint_S K \frac{\partial f}{\partial \xi} d\xi \, d\eta \quad (2.14)$$

Introduction of further notations

$$-\frac{1}{2\pi^3\sqrt{2}} \iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta) d\xi \, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} = G(\gamma); \quad \frac{\partial f}{\partial \xi} = F(\xi,\eta) \quad (2.15)$$

results in

$$\frac{\partial\varphi}{\partial x} = \frac{1}{2\pi} \iint_S K(x,y,z,\xi,\eta) F(\xi,\eta) d\xi \, d\eta +$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}\, G(\gamma)\cos\gamma}{x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma} d\gamma \quad (2.16)$$

and after integration with respect to x

$$\varphi(x,y,z) = \frac{1}{2\pi}\iint_S \int_{+\infty}^{x} K(x,y,z,\xi,\eta)dx\ F(\xi,\eta)d\xi\ d\eta +$$

$$\int_{+\infty}^{x}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}\ G(\gamma)\cos\gamma}{x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma}\,dx\,d\gamma \qquad (2.17)$$

The given functions $F(\xi,\eta)$ in the circle S and the function $G(\gamma)$ in the interval $(-\pi/2, \pi/2)$ completely determine $f(\xi,\eta)$, so that the equations (2.13) and (2.17) are equivalent.

The equation $\varphi(x,y,z)$ obtained satisfies the conditions imposed in section 1.

This function is evidently a harmonic function in the entire space exterior to the circle S and satisfies the conditions at infinity, equation (1.13). From equation (2.12) it follows, that in the plane xy for $x^2 + y^2 > a^2$ the condition is satisfied:

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=0} = 0$$

and from equation (2.13) it follows that

$$\varphi(x,y,0) = 0$$

in that part of the plane xy which lies outside the circle S and the strip Σ.

It remains to prove the finiteness of the first derivatives of the function $\varphi(x,y,z)$ at the points of the rear semicircumference C and to determine the behavior of these derivatives on approaching the points of the forward semicircumference C.

In considering the neighborhood of the rear side of the circumference C, equation (2.16) may be used. The latter shows that $\partial\varphi/\partial x$ remains continuous at the points of the rear half of the circumference C and becomes zero at these points.

The behavior of the derivatives with respect to y and z of the following function is considered:

$$\Phi(x,y,z) = \iint_S K(x,y,z,\xi,\eta)F(\xi,\eta)d\xi\, d\eta \tag{2.18}$$

near the contour C.

$$\frac{\partial\Phi}{\partial y} = \iint_S \frac{\partial K}{\partial y} F(\xi,\eta)d\xi\, d\eta \tag{2.19}$$

Similarly to equation (2.9),

$$\frac{\partial K}{\partial y} + \frac{\partial K}{\partial \eta} = - \frac{2a\sqrt{2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\pi\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]} \left\{\frac{y\sqrt{a^2 - \xi^2 - \eta^2}}{R} + \frac{\eta}{\sqrt{a^2 - \xi^2 - \eta^2}}\right\} \tag{2.20}$$

and similarly to equation (2.14),

$$\iint_S \frac{\partial K}{\partial \eta} F(\xi,\eta)d\xi\, d\eta = - \iint_S K \frac{\partial F}{\partial \eta} d\xi\, d\eta \tag{2.21}$$

where this part of the integral remains finite everywhere and on the contour C becomes zero.

In order to evaluate the remaining part of the integral equation (2.19), the following two integrals are considered:

$$\left.\begin{aligned} J_1(x,y,z) &= \iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} \\ J_2(x,y,z) &= \iint_S \frac{d\xi\, d\eta}{\sqrt{a^2 - \xi^2 - \eta^2}\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]} \end{aligned}\right\} \tag{2.22}$$

Both, on account of the symmetry, depend only on $\sqrt{x^2 + y^2}$ and z; hence without restricting the generality, it may be assumed that $y = 0$, $x > 0$. The distance δ of a point with coordinates $(x,0,z)$ is introduced to the contour C:

$$\delta = \sqrt{(a - x)^2 + z^2}$$

Since

$$R \geqslant \left|x^2 + z^2 - a^2\right|$$

the following relation will hold:

$$J_1(x,0,z) \leqslant \iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2\left[(x - \xi)^2 + \eta^2 + z^2\right]}$$

Polar coordinates are introduced

$$\xi = \rho \cos \vartheta \;; \quad \eta = \rho \sin \vartheta$$

whence

$$J_1(x,0,z) \leqslant \int_0^a \int_0^{2\pi} \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho\, d\vartheta}{2a^2\left[\rho^2 - 2\rho x \cos \vartheta + x^2 + z^2\right]}$$

Since

$$\int_0^{2\pi} \frac{d\vartheta}{\rho^2 - 2\rho x \cos \vartheta + x^2 + z^2} = \frac{2\pi}{\sqrt{(\rho^2 + x^2 + z^2)^2 - 4\rho^2 x^2}}$$

hence

$$J_1(x,0,z) \leqslant \frac{\pi}{a^2} \int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{\sqrt{(\rho^2 + x^2 + z^2)^2 - 4\rho^2 x^2}}$$

For $x \geqslant a$

$$J_1(x,0,z) \leqslant \frac{\pi}{a^2} \int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{\sqrt{(\rho^2 + x^2)^2 - 4\rho^2 x^2}}$$

$$= \frac{\pi}{a^2} \int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{x^2 - \rho^2} \leqslant \frac{\pi}{a^2} \int_0^a \frac{\rho d\rho}{\sqrt{a^2 - \rho^2}} = \frac{\pi}{a}$$

While for $x \leqslant a$, use is made of the inequality

$$R \geqslant a^2 - x^2 - z^2$$

to obtain

$$J_1(x,0,z) \leqslant \frac{1}{2} \iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{a^2\left[(x - \xi)^2 + \eta^2 + z^2\right] + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - z^2)}$$

$$= \frac{1}{2}\int_0^a \int_0^{2\pi} \frac{\rho\sqrt{a^2 - \rho^2}\, d\vartheta\, d\rho}{a^4 - 2a^2 x\rho \cos\vartheta + \rho^2(x^2 + z^2)} \tag{2.23}$$

$$= \pi\int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{\sqrt{\left[a^4 + \rho^2(x^2 + z^2)\right]^2 - 4a^4x^2\rho^2}} \leqslant \pi\int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{a^4 - x^2\rho^2} \leqslant \pi\int_0^a \frac{\rho d\rho}{a^2\sqrt{a^2 - \rho^2}} = \frac{\pi}{a}$$

The following inequality results:

$$J_1(x,y,z) \leqslant \frac{\pi}{a} \tag{2.24}$$

The second integral is considered. As before,

$$J_2(x,0,z) \leqslant \frac{\pi}{a^2}\int_0^a \frac{\rho\, d\rho}{\sqrt{a^2 - \rho^2}\sqrt{(\rho^2 + x^2 + z^2)^2 - 4\rho^2x^2}}$$

For $x \geqslant a$

$$J_2(x,0,z) \leqslant \frac{\pi}{a^2}\int_0^a \frac{\rho\, d\rho}{\sqrt{(a^2 - \rho^2)\left[(\rho + x)^2 + z^2\right]\left[(\rho - x)^2 + z^2\right]}} \leqslant \frac{\pi}{a^3}\int_0^a \frac{\rho\, d\rho}{\sqrt{a^2 - \rho^2}\sqrt{(a - x)^2 + z^2}} = \frac{\pi}{a^2\delta}$$

For $x \leqslant a$ an inequality of the type in equation (2.23) is used:

$$J_2(x,0,z) \leqslant \pi\int_0^a \frac{\rho\, d\rho}{\sqrt{(a^2 - \rho^2)\left[a^4 + \rho^2(x^2 + z^2) + 2a^2x\rho\right]\left[a^2 - x\rho)^2 + \rho^2z^2\right]}}$$

$$\leqslant \frac{\pi}{a^2}\int_0^a \frac{\rho\, d\rho}{\sqrt{(a^2 - \rho^2)\left[(a^2 - x\rho)^2 + z^2\rho^2\right]}}$$

If $z \geqslant a - x$ and therefore $\delta \leqslant z\sqrt{2}$, then

$$J_2(x,0,z) \leqslant \frac{\pi}{a^2 z} \int_0^a \frac{d\rho}{\sqrt{a^2 - \rho^2}} = \frac{\pi^2}{2a^2 z} \leqslant \frac{\pi^2}{a^2 \delta \sqrt{2}}$$

but if $0 \leqslant z \leqslant a - x$, and therefore $\delta \leqslant (a - x)\sqrt{2}$, then

$$J_2(x,0,z) \leqslant \frac{\pi}{a^2} \int_0^a \frac{\rho \, d\rho}{(a^2 - x\rho)\sqrt{a^2 - \rho^2}} < \frac{\pi}{a^3} \int_0^a \frac{\rho \, d\rho}{(a - x)\sqrt{a^2 - \rho^2}} = \frac{\pi}{a^2(a - x)} \leqslant \frac{\pi\sqrt{2}}{a^2 \delta}$$

The following approximation is obtained:

$$J_2(x,y,z) \leqslant \frac{\pi^2}{a^2 \delta \sqrt{2}} \tag{2.25}$$

where

$$\delta = \sqrt{\left(a - \sqrt{x^2 + y^2}\right)^2 + z^2} \tag{2.26}$$

Near the contour C

$$R \approx 2a\delta \tag{2.27}$$

If this relation, the evident inequality

$$\left| a^2 - x^2 - y^2 - z^2 \right| \leqslant R$$

and the obtained approximations are used, the following approximation is obtained from equation (2.20):

$$\left| \int\int_S \left(\frac{\partial K}{\partial y} + \frac{\partial K}{\partial \eta} \right) F(\xi,\eta) d\xi \, d\eta \right| = 0 \left(\frac{1}{\sqrt{\delta}} \right)$$

It is evident from equations (2.19) and (2.21) that near the contour C

$$\frac{\partial \Phi}{\partial y} = 0 \left(\frac{1}{\sqrt{\delta}} \right) \tag{2.28}$$

The following derivative is formed:

$$\frac{\partial \Phi}{\partial z} = \iint_S \frac{\partial K}{\partial z} F(\xi,\eta)d\xi\, d\eta$$

But

$$\frac{\partial K}{\partial z} = -\frac{2z}{\pi r^3}\text{arc tan } A + \frac{2}{\pi}\frac{A}{1 + A^2}\left[-\frac{z}{r^3} + \frac{z(a^2 + x^2 + y^2 + z^2 - R)}{rR(a^2 - x^2 - y^2 - z^2 + R)}\right]$$

where

$$A = \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{ar\sqrt{2}}$$

Hence if

$$|F(\xi,\eta)| < M$$

then, on account of the inequality

$$\frac{A}{1 + A^2} \leqslant \frac{1}{2}$$

for $z > 0$ the approximation results:

$$\left|\frac{\partial \Phi}{\partial z}\right| \leqslant 2M \iint_S \frac{z}{r^3}\, d\xi\, d\eta\ +$$

$$\frac{2\sqrt{2}\ azM(a^2 + x^2 + y^2 + z^2 - R)}{\pi R\sqrt{a^2 - x^2 - y^2 - z^2 + R}}\iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\ d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)}$$

Noting that

$$\iint_S \frac{z}{r^3}\, d\xi\, d\eta \leqslant 2\pi$$

and making use of approximation (2.24) yield

$$\left|\frac{\partial\Phi}{\partial z}\right| \leqslant 4\pi M + 2\sqrt{2}\,M\,\frac{z(a^2 + x^2 + y^2 + z^2 - R)}{R\sqrt{a^2 - x^2 - y^2 - z^2 + R}}$$

Since for $z > 0$

$$\frac{z}{\sqrt{a^2 - x^2 - y^2 - z^2 + R}} = \frac{z\sqrt{R - (a^2 - x^2 - y^2 - z^2)}}{\sqrt{R^2 - (a^2 - x^2 - y^2 - z^2)^2}} = \frac{\sqrt{R - a^2 + x^2 + y^2 + z^2}}{2a}$$

hence

$$\left|\frac{\partial\Phi}{\partial z}\right| \leqslant 4\pi M + \frac{\sqrt{2}\,M}{aR}(a^2 + x^2 + y^2 + z^2 - R)\sqrt{R - a^2 + x^2 + y^2 + z^2}$$

Now when the point $P(x,y,z)$ is near the contour C, then because of

$$R \approx 2a\delta\,;\quad |x^2 + y^2 + z^2 - a^2| \leqslant R$$

there is obtained

$$\left|\frac{\partial\Phi}{\partial z}\right| = 0\left(\frac{1}{\sqrt{\delta}}\right) \tag{2.29}$$

Equation (2.16) is again considered. Since the derivatives

$$\left.\begin{aligned}
&\frac{\partial}{\partial y}\sqrt{a^2 - x^2 - y^2 - z^2 + R} = -\frac{y\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{R};\\
&\frac{\partial}{\partial z}\sqrt{a^2 - x^2 - y^2 - z^2 + R} = \frac{z(a^2 + x^2 + y^2 + z^2 - R)}{R\sqrt{a^2 - x^2 - y^2 - z^2 + R}}\\
&= \frac{1}{2aR}(a^2 + x^2 + y^2 + z^2 - R)\sqrt{R - a^2 + x^2 + y^2 + z^2}
\end{aligned}\right\} \tag{2.30}$$

have near the contour C the order $1/\sqrt{\delta}$, it is clear from equation (2.16) and the obtained equations (2.28) and (2.29) that at the points of the rear semicircumference of C there is the estimate

$$\frac{\partial^2\varphi}{\partial x\partial y} = 0\left(\frac{1}{\sqrt{\delta}}\right);\quad \frac{\partial^2\varphi}{\partial x\partial z} = 0\left(\frac{1}{\sqrt{\delta}}\right) \tag{2.31}$$

But is is then evident that the derivatives $\partial\varphi/\partial y$ and $\partial\varphi/\partial z$ are finite at the points of the rear semicircumference C.

The behavior of the derivatives of the function φ near the forward semicircumference C can readily be determined, starting from equations (2.12) and (2.13).

The first of these equations may be written in the form:

$$\frac{\partial\varphi}{\partial x} = \frac{1}{2\pi}\iint_S \frac{\partial K}{\partial x} f(\xi,\eta)d\xi\, d\eta - \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}\; G(\gamma)\cos\gamma}{x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma}\, d\gamma \tag{2.32}$$

But on the one hand, the estimate

$$\iint_S \frac{\partial K}{\partial x} f(\xi,\eta)d\xi\, d\eta = 0\left(\frac{1}{\sqrt{\delta}}\right)$$

holds for the neighborhood of the entire contour C; on the other hand, on the forward semicircumference C, the second integral of equation (2.32) evidently remains finite. Hence for the forward semicircumference C the first of the estimates is obtained

$$\frac{\partial\varphi}{\partial x} = 0\left(\frac{1}{\sqrt{\delta}}\right);\quad \frac{\partial\varphi}{\partial y} = 0\left(\frac{1}{\sqrt{\delta}}\right);\quad \frac{\partial\varphi}{\partial z} = 0\left(\frac{1}{\sqrt{\delta}}\right) \tag{2.33}$$

while the latter two of these estimates are obtained in a similar manner from equation (2.13).

In this manner all the conditions which must be satisfied by the function $\varphi(x,y,z)$ are satisfied.

The shape of wing to which the obtained solution corresponds is explained. By equation (1.14)

$$c\frac{\partial\zeta}{\partial x} = -\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} \tag{2.34}$$

Hence it is necessary to find the value $\partial\varphi/\partial z$ in the plane of the circle S. Both sides of equation (2.13) are differentiated with respect to z and then z set = 0. On account of the very definition of the function K,

$$\lim_{z \to +0} \iint_S \frac{\partial K}{\partial z} f(\xi,\eta) d\xi \, d\eta = \lim_{z \to +0} \iint_S \frac{\partial \frac{1}{r}}{\partial z} f(\xi,\eta) d\xi \, d\eta = -2\pi f(x,y) \tag{2.35}$$

Moreover, on account of equation (2.30),

$$\lim_{z \to +0} \frac{\partial}{\partial z} \sqrt{a^2 - x^2 - y^2 - z^2 + R} = \begin{cases} 0 & \text{for} \quad x^2 + y^2 < a^2 \\ \dfrac{a\sqrt{2}}{\sqrt{x^2 + y^2 - a^2}} & \text{for} \quad x^2 + y^2 > a^2 \end{cases}$$

If this is taken into account,

$$\left(\frac{\partial \varphi}{\partial z}\right)_{z=0} = -f(x,y) + g(y) \tag{2.36}$$

where

$$g(y) = \frac{a}{2\pi^3} \iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2} \cos \gamma f(\xi,\eta) d\gamma \, dx \, d\xi \, d\eta}{\sqrt{x^2 + y^2 - a^2}(x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2a\eta \sin \gamma)} \tag{2.37}$$

For the function $\zeta(x,y)$ the following expression is found:

$$\zeta(x,y) = \frac{1}{c} \int_0^x f(x,y) dx - \frac{g(y)}{c} x + g_1(y) \tag{2.38}$$

where $g_1(y)$ is an arbitrary function of y.

Thus, for the assumed degree of approximation, the bending of the wing in the transverse direction produces no effect on the form of the flow.

It is assumed that the shape of the wing is given, that is, the function $\zeta(x,y)$ and therefore the following function are given:

$$c \frac{\partial \zeta}{\partial x} = M(x,y) \tag{2.39}$$

From equations (2.34) and (2.36) it is clear that

$$f(x,y) = M(x,y) + g(y) \tag{2.40}$$

Substituting this value in equation (2.37) and introducing the notations

$$N(y) = \frac{a}{2\pi^3} \times$$

$$\iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, M(\xi,\eta) \cos\gamma \, d\gamma \, dx \, d\xi \, d\eta}{\sqrt{x^2 + y^2 - a^2}(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}$$

$$H(y,\eta) = \frac{a}{2\pi^3} \times \tag{2.41}$$

$$\int_{-\sqrt{a^2-\eta^2}}^{\sqrt{a^2-\eta^2}} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2} \cos\gamma \, d\gamma \, dx \, d\xi}{\sqrt{x^2 + y^2 - a^2}(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}$$

give an integral Fredholm equation of the second kind for the determination of the function $g(y)$:

$$g(y) = N(y) + \int_{-a}^{a} H(y,\eta) g(\eta) d\eta \tag{2.42}$$

In consideration of examples, a function $f(x,y)$ shall be given and the shape of the wing then determined by equation (2.38). For the obtained shapes of the wing it is not difficult to find a solution by the usual theory, a fact which provides the possibility of evaluating the degree of accuracy of the usual theory.

3. Computation of the Forces Acting on the Wing

The fundamental equation determining the motion of the type under consideration is recalled:

$$\varphi(x,y,z) = \frac{1}{2\pi} \iint_S \Bigg\{ K(x,y,z,\xi,\eta) + \frac{1}{\pi^2\sqrt{2}} \times \tag{3.1}$$

$$\int_{+\infty}^{x} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R} \cos\gamma \, d\gamma \, dx}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \Bigg\} f(\xi,\eta) d\xi \, d\eta$$

The value of the function φ for the points of the half-strip Σ is computed. Since at the points of the half-strip Σ

$$\frac{\partial\varphi}{\partial x} = 0$$

this value is a function only of y. The notation is introduced

$$\Phi(y) = \lim_{z\to+0} \varphi(x,y,z) \quad \text{for} \quad |y| < a, \quad x^2 + y^2 > a^2, \quad x < 0 \qquad (3.2)$$

Then evidently

$$\lim_{z\to-0} \varphi(x,y,z) = -\Phi(y) \quad \text{for} \quad |y| < a, \quad x^2 + y^2 > a^2, \quad x < 0 \qquad (3.3)$$

The circulation over the contour M'NM (fig. 2) connecting the two points M and M' of which point M' lies on the lower and point M the upper side of the half-strip Σ, both points M and M' having the same coordinates $x,y,0$, is denoted by $\Gamma(y)$. It is then evident that

$$\Gamma(y) = \Phi(M) - \Phi(M') = 2\Phi(y) \qquad (3.4)$$

Since in the plane xy outside the circle S both the function K and the function

$$\sqrt{a^2 - x^2 - y^2 - z^2 + R}$$

become zero, it is clear that

$$\Phi(y) = \frac{1}{2\pi^3} \iint_S \int_{\sqrt{a^2-y^2}}^{-\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2}\, f(\xi,\eta) \cos\gamma \, d\gamma \, dx \, d\xi \, d\eta}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \qquad (3.5)$$

Computation shows that

$$\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{\sqrt{a^2 - x^2 - y^2}\, dx}{x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma} = \pi\left\{\sqrt{\frac{a(1 \pm \sin \gamma)}{|a \sin \gamma - y|}} - 1\right\} \quad (3.6)$$

where the plus sign is taken for $y < a \sin \gamma$ and the minus sign for $y > a \sin \gamma$.

The following expression is written for the distribution of the circulation in the vortex layer formed behind the wing:

$$\Gamma(y) = -\frac{1}{\pi^2} \iint_S \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)}{(\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2a\eta \sin \gamma)}\left\{\sqrt{\frac{a(1 \pm \sin \gamma)}{|a \sin \gamma - y|}} - 1\right\} \cos \gamma \, d\gamma \, d\xi \, d\eta \quad (3.7)$$

The forces acting on the wing are computed. Denoting by p_+ the pressure at a point of the wing S on the upper side of the wing and by p_- the pressure at the same point on the lower side gives on the basis of equation (1.8)

$$p_- - p_+ = -2\rho c \frac{\partial \varphi}{\partial x} \quad (3.8)$$

where the value of $\partial\varphi/\partial x$ is taken on the upper side of the wing.

For the lift force P, the following expression is obtained:

$$P = \iint_S (p_- - p_+)\, dx\, dy = -2\rho c \iint_S \frac{\partial \varphi}{\partial x}\, dx\, dy = -2\rho c \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{\partial \varphi}{\partial x}\, dx\, dy$$

$$= -2\rho c \int_{-a}^{a} \left[\varphi(\sqrt{a^2 - y^2},y,0) - \varphi(-\sqrt{a^2 - y^2},y,0)\right] dy = 2\rho c \int_{-a}^{a} \Phi(y)\, dy$$

The following formula is obtained:

$$P = \rho c \int_{-a}^{a} \Gamma(y) dy \quad (3.9)$$

having the same form as in the usual theory of a wing of finite span. But the distribution of the circulation $\Gamma(y)$ by the present theory is somewhat different from that obtained by the usual theory. The dervivation given is not connected with the shape of the wing.

With the aid of equation (3.6) P may be directly expressed through $f(\xi,\eta)$:

$$P = -\frac{2\rho ac}{\pi^2}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta) \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\, d\xi\, d\eta \tag{3.10}$$

The expression for the induced resistance W in terms of the circulation $\Gamma(y)$ likewise has the same form as in the usual theory:

$$W = \frac{\rho}{4\pi}\int_{-a}^{a}\int_{-a}^{a} \Gamma(y)\frac{d\Gamma(y')}{dy'}\frac{1}{y - y'}\, dy\, dy' \tag{3.11}$$

because the origin of the induced resistance is due to the fact that behind the wing a region of disturbed motion of the fluid is formed; the kinetic energy of this disturbance is determined on the other hand exclusively by the distribution of the circulation at distant points from the wing.

The expression for the induced resistance is obtained from the momentum law.

A surface enclosing the wing S is denoted by B; the momentum law applied to the wing in a steady flow then leads to the expression

$$W = \iint_B p\cos(n,x)d\sigma + \iint_B \rho V_n V_x d\sigma \tag{3.12}$$

where n is the direction of the outer normal to the surface B and V_x, V_y, V_z are the components of the velocity in the relative motion of the fluid about the wing. Thus

$$V_x = -c + \frac{\partial\varphi}{\partial x};\quad V_n = -c\cos(n,x) + \frac{\partial\varphi}{\partial n}$$

$$p = p_0 + \rho c\frac{\partial\varphi}{\partial x} - \frac{\rho}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]$$

Substituting these values in the preceding formula and noting that

$$\int_B\int \cos(n,x)d\sigma = 0 \,; \quad \int_B\int \frac{\partial\varphi}{\partial n}\, d\sigma = 0$$

results in

$$W = -\frac{\rho}{2}\int_B\int \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]\cos(n,x)d\sigma + \rho\int_B\int \frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial n}\, d\sigma \quad (3.13)$$

The surface B consists of a hemisphere of large radius with center at the point $x = x_0 < -a$ of the x-axis enclosing the wing, and of the circle cut out by this hemisphere on the plane $x = x_0$. With increase in the radius of the hemisphere to infinity the corresponding parts of the integrals entering the preceding formula approach zero. On the surface $x = x_0$

$$\cos(n,x) = -1 \,; \quad \frac{\partial\varphi}{\partial n} = -\frac{\partial\varphi}{\partial x}$$

therefore

$$W = \frac{\rho}{2}\int\int \left[\left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2\right]dy\, dz \quad (3.14)$$

where the integration extends over the entire plane $x = x_0$. For $x_0 \rightarrow -\infty$ the following equation is obtained:

$$W = \frac{\rho}{2}\int\int \left[\left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right]dy\, dz \quad (3.15)$$

where $\Phi(y,z)$ denotes the velocity potential of the plane-parallel flow which is established in the transverse planes far behind the wing.

The usual transformations by Green's formula yield

$$W = -\rho\int_{-a}^{a} \Phi(y)\frac{\partial\Phi}{\partial z}\, dy \quad (3.16)$$

where the integral is taken over the upper side of the segment $(-a,a)$ in the plane yz.

Since

$$\Gamma(y) = 2\Phi(y) \,; \quad \frac{\partial\Phi}{\partial z} = \frac{1}{2\pi}\int_{-a}^{a}\frac{d\Gamma(y')}{y' - y} \quad (3.17)$$

equation (3.11) is obtained.

In order to find the center of pressure, the principal moments of the pressure forces about the Ox and Oy axes are determined.

For the moment about the Ox axis,

$$M_x = \iint_S (p_- - p_+)y \, dx \, dy = -2\rho c \iint_S \frac{\partial\varphi}{\partial x} y \, dx \, dy = 2\rho c \int_{-a}^{a} \Phi(y)y \, dy$$

from which

$$M_x = \rho c \int_{-a}^{a} y\Gamma(y)dy \tag{3.18}$$

Expressing M_x in terms of $f(x,y)$ yields

$$M_x = -\frac{4}{3}\frac{\rho c a^2}{\pi^2}\iint_S\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\sin\gamma\cos\gamma \, d\gamma \, d\xi \, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} \tag{3.19}$$

For the moment about the Oy axis,

$$M_y = -\iint_S (p_- - p_+)x \, dx \, dy = 2\rho c \iint_S x\frac{\partial\varphi}{\partial x} dx \, dy \tag{3.20}$$

Substituting the value $\partial\varphi/\partial x$ and integrating yield

$$M_y = -\frac{4\rho c}{\pi}\iint_S\left\{1 - \frac{a^2}{3\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^2\gamma \, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\}\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)d\xi \, d\eta \tag{3.21}$$

The following values are obtained for the coordinates of the center of pressure:

$$x_c = -\frac{M_y}{P}; \quad y_c = \frac{M_x}{P} \tag{3.22}$$

4. Examples

NACA comment: Errors in these examples are referred to and corrected in the paper "Steady Vibrations of Wing of Circular Plan Form".

The equations just obtained are presented again:

The velocity potential for $z > 0$ is determined by the equation

$$\varphi(x,y,z) = \frac{1}{2\pi}\iint_S \left\{K(x,y,z,\xi,\eta) + \frac{1}{\pi^2\sqrt{2}} \times \int_{+\infty}^{x}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}\sqrt{a^2 - \xi^2 - \eta^2}\cos\gamma\, d\gamma\, dx}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}\right\} f(\xi,\eta)d\xi\, d\eta \tag{4.1}$$

where

$$K(x,y,z,\xi,\eta) = \frac{2}{\pi r}\text{arc tan}\frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\, ar} \tag{4.2}$$

$$R = \sqrt{(a^2 - x^2 - y^2 - z^2)^2 + 4a^2z^2}; \quad r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2}$$

For the circulation distribution in the vortex strip formed behind the wing,

$$\Gamma(y) = -\frac{1}{\pi^3} \times \iint_S \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2}\, f(\xi,\eta)\cos\gamma\, d\gamma\, dx\, d\xi\, d\eta}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{4.3}$$

$$= -\frac{1}{\pi^2}\iint_S\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\cos\gamma}{(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}\left[\sqrt{\frac{a(1 \pm \sin\gamma)}{|a\sin\gamma - y|}} - 1\right] d\gamma\, d\xi\, d\eta$$

where the plus sign is taken for $y < a\sin\gamma$ and the minus sign for $y > a\sin\gamma$

The following expression gives the lift force:

$$P = \rho c\int_{-a}^{a}\Gamma(y)dy = -\frac{2\rho ca}{\pi^2}\iint_S\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\cos\gamma\, d\gamma\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} \tag{4.4}$$

The usual expression for the induced resistance is

$$W = \frac{\rho}{4\pi}\int_{-a}^{a}\int_{-a}^{a}\Gamma(y)\frac{d\Gamma(y')}{dy'}\frac{1}{y - y'}dy\, dy' \tag{4.5}$$

The coordinates of the center of pressure are determined by the equations

$$x_c = -\frac{M_y}{P}; \quad y_c = \frac{M_x}{P} \tag{4.6}$$

where

$$M_x = \rho c \int_{-a}^{a} y\Gamma(y)dy = -\frac{4}{3}\frac{\rho ca^2}{\pi^2}\iint_S \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\sin\gamma\cos\gamma\, d\gamma\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} \tag{4.7}$$

$$M_y = -\frac{4\rho c}{\pi}\iint_S \left\{1 - \frac{a^2}{3\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^2\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\}\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)d\xi\, d\eta \tag{4.8}$$

If y is set equal to $-a\cos\theta$ and $\Gamma(y)$ is represented in the form of a trigonometric series,

$$\Gamma(y) = A_1 \sin\theta + A_2 \sin 2\theta + \ldots \qquad (0 < \theta < \pi) \tag{4.9}$$

P, W and M_x are directly expressed in terms of the coefficients of this series by the formulas

$$P = \frac{\pi\rho ca}{2} A_1; \quad W = \frac{1}{8}\pi\rho\sum_{n=1}^{\infty} nA_n^2; \quad M_x = -\frac{1}{4}\pi\rho ca^2 A_2 \tag{4.10}$$

Finally, the shape of the wing is determined by the equation

$$\zeta(x,y) = \frac{1}{c}\int_0^x f(x,y)dx - \frac{g(y)}{c}x + g_1(y) \tag{4.11}$$

where

$$g(y) = \frac{a}{2\pi^3} \times$$

$$\iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\cos\gamma\, d\gamma\, dx\, d\xi\, d\eta}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{4.12}$$

The examples are now considered.

1. First

$$f(x,y) = c\alpha$$

where α is a small constant.

Polar coordinates are used and the following integral computed:

$$\iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos\gamma - 2a\eta \sin\gamma}$$

$$= \int_0^a \int_0^{2\pi} \frac{\sqrt{a^2 - \rho^2}\, \rho\, d\vartheta\, d\rho}{a^2 + \rho^2 - 2a\rho \cos(\vartheta - \gamma)} = \int_0^a \frac{2\pi\rho\, d\rho}{\sqrt{a^2 - \rho^2}} = 2\pi a \quad (4.13)$$

Substituting this value in equation (4.3) yields

$$\Gamma(y) = -\frac{2ac\alpha}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\gamma \left[\sqrt{\frac{a(1 \pm \sin\gamma)}{|a \sin\gamma - y|}} - 1\right] d\gamma$$

If the integral is taken,

$$\Gamma(y) = \frac{c\alpha}{\pi}\left\{-4a + 2\sqrt{2a(a - y)} + 2\sqrt{2a(a + y)} - \right.$$

$$\left.(a + y) \log \frac{\sqrt{2a} - \sqrt{a - y}}{\sqrt{2a} + \sqrt{a - y}} - (a - y) \log \frac{\sqrt{2a} - \sqrt{a + y}}{\sqrt{2a} + \sqrt{a + y}}\right\} \quad (4.14)$$

Setting $y = -a \cos\theta$ and expanding $\Gamma(-a \cos\theta)$ in a trigonometric sine series in the interval $0 < \theta < \pi$ give after simple computations

$$\Gamma(-a \cos\theta) = \frac{ac\alpha}{\pi}\left\{-4 + 4 \cos\frac{\theta}{2} + 4 \sin\frac{\theta}{2} - \right.$$

$$\left.(1 - \cos\theta) \log \frac{1 - \cos\frac{\theta}{2}}{1 + \cos\frac{\theta}{2}} - (1 + \cos\theta) \log \frac{1 - \sin\frac{\theta}{2}}{1 + \sin\frac{\theta}{2}}\right\}$$

$$= A_1 \sin\theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \ldots \qquad (0 \leq \theta \leq \pi) \quad (4.15)$$

where

$$A_1 = \frac{16ac\alpha}{\pi^2}; \; A_{2k+1} = -\frac{4ac\alpha}{\pi^2 k(k+1)(2k+1)}\left(\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{4k+1}\right)$$

$$(k = 1, 2, ..) \tag{4.16}$$

so that

$$A_3 = -\frac{16ac\alpha}{45\pi^2}; \; A_5 = -\frac{496ac\alpha}{4725\pi^2}, \ldots$$

The distribution of the circulation obtained is very near that of an elliptical distribution.

The lift force and the induced drag are obtained by application of equations (4.10).

$$\begin{aligned} P &= \frac{\pi}{2}\rho c a A_1 = \frac{8}{\pi}\rho a^2 c^2 \alpha \approx 2.5465\ \rho a^2 c^2 \alpha \\ W &= \frac{1}{8}\pi\rho\left(A_1{}^2 + 3A_3{}^2 + \ldots\right) \approx 1.034\ \rho a^2 c^2 \alpha^2 \end{aligned} \tag{4.17}$$

In order to determine the position of the center of pressure, M_y must be computed by equation (4.8).

Equation (4.13) gives

$$M_y = -\frac{4}{3}\rho c^2 a^3 \alpha; \; x_c = -\frac{M_y}{P} = \frac{\pi}{6} a \tag{4.18}$$

The distance from the center of pressure, which evidently lies on the Ox axis, to the leading edge of the wing thus constitutes about 0.238 of the diameter of the wing.

In order to determine the shape of the wing corresponding to the assumed function, it is necessary to form the function $g(y)$ by equation (4.12). If equation (4.13) is considered,

$$g(y) = \frac{a^2 c\alpha}{\pi^2}\int_{\infty}^{\sqrt{a^2-y^2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos\gamma\, d\gamma\, dx}{\sqrt{x^2+y^2-a^2}\left(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma\right)} \tag{4.19}$$

The computation shows that for $x > \sqrt{a^2 - y^2}$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos\gamma\, d\gamma}{x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma} = -\frac{\pi x}{2a(x^2 + y^2)} +$$

$$\frac{y}{2a(x^2 + y^2)} \log \frac{x^2 + (y - a)^2}{x^2 + (y + a)^2} +$$

$$\frac{x(a^2 + x^2 + y^2)}{a(x^2 + y^2)(x^2 + y^2 - a^2)} \arctan \frac{x^2 + y^2 - a^2}{2ax} \tag{4.20}$$

If

$$\begin{gathered} y = -a\cos\theta;\ \sqrt{a^2 - y^2} = a\sin\theta \\ H_0(\theta) = \frac{\pi^2}{\alpha c}\sin\theta g(-a\cos\theta) \end{gathered} \tag{4.21}$$

for $0 < \theta < \pi$

$$H_0(\theta) = \int_{+\infty}^{\sin\theta} \frac{\sin\theta}{\sqrt{t^2 - \sin^2\theta}} \left\{ -\frac{\pi t}{2(t^2 + \cos^2\theta)} - \frac{\cos\theta}{2(t^2 + \cos^2\theta)} \log \frac{t^2 + 4\cos^4\frac{\theta}{2}}{t^2 + 4\sin^4\frac{\theta}{2}} + \right.$$

$$\left. \frac{t(t^2 + 1 + \cos^2\theta)}{(t^2 + \cos^2\theta)(t^2 - \sin^2\theta)} \arctan \frac{t^2 - \sin^2\theta}{2t} \right\} dt$$

Computation of this integral gives

$$H_0(\theta) = \left(\frac{\pi^2}{2} - 2\int_0^1 \frac{\arctan y}{\sqrt{1 - y^2}} dy \right) \sin\theta +$$

$$\frac{1}{8}\sin\theta \left(\log \frac{1 + \sin\frac{\theta}{2}}{1 - \sin\frac{\theta}{2}} \right)^2 + \frac{1}{8}\sin\theta \left(\log \frac{1 + \cos\frac{\theta}{2}}{1 - \cos\frac{\theta}{2}} \right)^2 +$$

$$\cos\frac{\theta}{2} \log \frac{1 - \sin\frac{\theta}{2}}{1 + \sin\frac{\theta}{2}} + \sin\frac{\theta}{2} \log \frac{1 - \cos\frac{\theta}{2}}{1 + \cos\frac{\theta}{2}} \tag{4.22}$$

The shape of the wing is thus determined by the equation

$$\zeta(x,y) = \alpha x\left[1 - \frac{g(y)}{\alpha c}\right] = \alpha x\left\{\frac{1}{2} + \frac{2}{\pi^2}\int_0^1 \frac{\text{arc tan } y}{\sqrt{1 - y^2}}\,dy - \frac{1}{8\pi^2}\left(\log\frac{\sqrt{2a} + \sqrt{a + y}}{\sqrt{2a} - \sqrt{a + y}}\right)^2 - \frac{1}{8\pi^2}\left(\log\frac{\sqrt{2a} + \sqrt{a - y}}{\sqrt{2a} - \sqrt{a - y}}\right)^2 - \frac{\sqrt{2a}}{2\pi^2\sqrt{a + y}}\log\frac{\sqrt{2a} - \sqrt{a + y}}{\sqrt{2a} + \sqrt{a + y}} - \frac{\sqrt{2a}}{2\pi^2\sqrt{a - y}}\log\frac{\sqrt{2a} - \sqrt{a - y}}{\sqrt{2a} + \sqrt{a - y}}\right\} \quad (4.23)$$

This wing differs little from a plane wing inclined to the xy-plane by a small angle α and may be obtained from such a plane wing by twisting. The values of the function $\zeta(x,y)$ for the mean value $y = 0$ and for the values $y = \pm a/2$ are

$$\zeta(x,0) = \alpha x\left[\frac{1}{2} + \frac{2}{\pi^2}\int_0^1 \frac{\text{arc tan } y}{\sqrt{1 - y^2}}\,dy - \frac{1}{\pi^2}\log^2(\sqrt{2} + 1) + \frac{2\sqrt{2}}{\pi^2}\log(\sqrt{2} + 1)\right] \approx 0.8452\ \alpha x$$

$$\zeta\left(x, \pm\frac{a}{2}\right) = \alpha x\left[\frac{1}{2} + \frac{2}{\pi^2}\int_0^1 \frac{\text{arc tan } y}{\sqrt{1 - y^2}}\,dy - \frac{1}{2\pi^2}\log^2(2 + \sqrt{3}) - \frac{1}{8\pi^2}\log^2 3 + \frac{2}{\pi^2\sqrt{3}}\log(2 + \sqrt{3}) + \frac{1}{\pi^2}\log 3\right] \approx 0.8335\ \alpha x$$

It is of interest to consider what results for the obtained wing are given by the usual theory. The circulation obtained by this theory is denoted by $\Gamma_0(y)$; if the expansion of this circulation in a trigonometric series is

$$\Gamma_0(-a\cos\theta) = B_1\sin\theta + B_2\sin 2\theta + \ldots \qquad (0 < \theta < \pi) \quad (4.24)$$

then the usual theory gives an equation for determining the coefficients B_n, which in the case considered reduces to the form

$$\sum_{n=1}^{\infty} B_n \sin n\theta = 2\pi ca \sin \theta \left\{\alpha - \frac{g(-a \cos \theta)}{c} - \frac{1}{4ca} \sum_{n=1}^{\infty} nB_n \frac{\sin n\theta}{\sin \theta}\right\} \quad (4.25)$$

Equation (4.21) yields

$$\sum_{n=1}^{\infty} B_n\left(1 + \frac{\pi n}{2}\right) \sin n\theta = 2\pi ca\alpha \sin \theta - \frac{2a\alpha c}{\pi} H_0(\theta) \quad (4.26)$$

Expansion of the function $H_0(\theta)$ into a trigonometric series is sufficient to determine the coefficients B_n. Despite the complicated form of the function $H_0(\theta)$, it can be expanded and in the interval $0 \leqslant \theta \leqslant \pi$

$$H_0(\theta) = \sin \theta\left(\frac{9\pi^2}{16} - 4 - 2 \int_0^1 \frac{\text{arc tan } y}{\sqrt{1 - y^2}} dy\right) +$$

$$\sum_{k=1}^{\infty} \frac{\sin(2k + 1)\theta}{k(k + 1)} \left[- 1 + \frac{1}{3} + \ldots + \frac{1}{4k + 1} - \frac{2(2k + 1)^2 + 1}{(4k + 1)(4k + 3)}\right] \quad (4.27)$$

that is,

$$H_0(\theta) = \sum_{k=0}^{\infty} \beta_{2k+1} \sin(2k + 1)\theta$$

where

$$\beta_1 = - 0.1389 ; \quad \beta_3 = - 0.5048 ; \quad \beta_5 = - 0.1213$$

$$\beta_7 = - 0.0460 ; \quad \beta_9 = - 0.0212, \ldots$$

Equation (4.26) shows that

$$B_1 = \frac{4\alpha ca(\pi^2 - \beta_1)}{\pi(\pi + 2)}; \quad B_{2k} = 0 ; \quad B_{2k+1} = - \frac{4\alpha ca\beta_{2k+1}}{2 + \pi(2k + 1)} \quad (k = 1, 2, \ldots) \quad (4.28)$$

The numerical values of the first coefficients will be

$$B_1 = 2.4784 \ \alpha ca ; \quad B_3 = 0.0562 \ \alpha ca ; \quad B_5 = 0.0087 \ \alpha ca$$

$$B_7 = 0.0024 \ \alpha ca ; \quad B_9 = 0.0009 \ \alpha ca, \ldots$$

The following value is obtained for the lift force:

$$P_0 = \frac{1}{2}\pi\rho c a B_1 = 3.8932\ \rho c^2 a^2 \alpha \tag{4.29}$$

exceeding the accurate value by 53 percent.

For the induced drag,

$$W_0 \approx 2.416\ \rho a^2 c^2 \alpha^2 \tag{4.30}$$

with an error of 134 percent.

2. If α is assumed to be small, $f(x,y) = -2c\alpha x$ is taken.

The circulation $\Gamma(y)$ is computed. First the value of the following integral is found.

$$\iint_S \frac{\xi\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos\gamma - 2a\eta \sin\gamma} = \frac{4}{3}\pi a^2 \cos\gamma \tag{4.31}$$

Equation (4.3) gives

$$\Gamma(y) = \frac{8ca^2\alpha}{3\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2\gamma \left[\sqrt{\frac{a(1 \pm \sin\gamma)}{|a \sin\gamma - y|}} - 1\right] d\gamma$$

The computation of this integral leads to the very simple expression

$$\Gamma(y) = 2c\alpha(a^2 - y^2) \tag{4.32}$$

Thus in the case considered, a parabolic distribution of the circulation was obtained. For this reason the computation of the forces can be easily carried out:

$$\left.\begin{aligned} P &= \rho c \int_{-a}^{a} \Gamma(y)dy = \frac{8}{3}\alpha\rho c^2 a^3 \approx 2.667\ \alpha\rho c^2 a^3 \\ W &= \frac{4}{\pi}\rho c^2 \alpha^2 a^4 \approx 1.2732\ \rho c^2 a^4 \alpha^2 \end{aligned}\right\} \tag{4.33}$$

Equation (4.31) is used in the computation of M_y by equation (4.8):

$$M_y = \frac{128}{27\pi}\rho c^2 a^4 \alpha \approx 1.509\ \rho c^2 a^4 \alpha\ ;\quad x_c = -\frac{M_y}{P} = -\frac{16}{9\pi}a \qquad (4.34)$$

In order to determine the shape of the wing it is necessary to compute the function $g(y)$; equation (4.12) yields

$$g(y) = -\frac{4\alpha c a^3}{3\pi^2}\int_{+\infty}^{\sqrt{a^2-y^2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^2\gamma\ d\gamma\ dx}{\sqrt{x^2+y^2-a^2}\left(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma\right)}$$

Setting

$$H_1(\theta) = -\frac{3\pi^2}{4\alpha c a}\sin\theta g(-a\cos\theta) \qquad (0 \leqslant \theta \leqslant \pi) \qquad (4.35)$$

and carrying out the integration with respect to γ yield

$$H_1(\theta) = \int_{\infty}^{\sin\theta}\frac{\sin\theta}{\left(t^2+\cos^2\theta\right)^2\sqrt{t^2-\sin^2\theta}}\left\{t\left(t^2+\cos^2\theta\right) + \right.$$

$$\frac{\pi}{4}\left(\cos^2\theta - t^2\right)\left(t^2+1+\cos^2\theta\right) - \frac{1}{2}t\cos\theta\left(t^2+1+\cos^2\theta\right)\log\frac{t^2+4\cos^4\frac{\theta}{2}}{t^2+4\sin^4\frac{\theta}{2}} +$$

$$\left.\frac{2\left(t^2+\cos^2\theta\right)^2+\left(t^2-\cos^2\theta\right)\left[1+\left(t^2+\cos^2\theta\right)^2\right]}{2\left(t^2-\sin^2\theta\right)}\arctan\frac{t^2-\sin^2\theta}{2t}\right\}dt$$

Integration yields

$$H_1(\theta) = \frac{3\pi}{2}\left\{\sin\theta\left(1-\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right) + \frac{1}{12}\log\frac{\left(1+\cos\frac{\theta}{2}\right)\left(1+\sin\frac{\theta}{2}\right)}{\left(1-\cos\frac{\theta}{2}\right)\left(1-\sin\frac{\theta}{2}\right)} + \right.$$

$$\left.\sin\theta\cos\theta\left[\log\tan\frac{\theta}{2} + \frac{1}{4}\log\frac{\left(1+\cos\frac{\theta}{2}\right)\left(1-\sin\frac{\theta}{2}\right)}{\left(1-\cos\frac{\theta}{2}\right)\left(1+\sin\frac{\theta}{2}\right)}\right]\right\} \qquad (4.36)$$

In equation (4.11), the following is taken:

$$g_1(y) = \alpha\left(a^2 - y^2\right)$$

Then for the function $\zeta(x,y)$, which determines the shape of the wing, the following expression is obtained:

$$\zeta(x,y) = \alpha(a^2 - x^2 - y^2) +$$

$$\frac{2\alpha ax}{\pi}\left\{1 - \sqrt{\frac{a+y}{2a}} - \sqrt{\frac{a-y}{2a}} + \frac{1}{12}\log\frac{(\sqrt{2a}+\sqrt{a-y})(\sqrt{2a}+\sqrt{a+y})}{(\sqrt{2a}-\sqrt{a-y})(\sqrt{2a}-\sqrt{a+y})} - \right.$$

$$\left.\frac{y}{a}\log\sqrt{\frac{a+y}{a-y}} - \frac{y}{4a}\log\frac{(\sqrt{2a}+\sqrt{a-y})(\sqrt{2a}-\sqrt{a+y})}{(\sqrt{2a}-\sqrt{a-y})(\sqrt{2a}+\sqrt{a+y})}\right\} \quad (4.37)$$

This wing is thus obtained as a deformation of the wing:

$$\zeta(x,y) = \alpha(a^2 - x^2 - y^2)$$

which for small α differs little from a segment of a sphere.

In particular, for $y = 0$,

$$\zeta(x,0) = \alpha(a^2 - x^2) + \frac{2\alpha ax}{\pi}\left[1 - \sqrt{2} + \frac{1}{3}\log(\sqrt{2}+1)\right] \approx \alpha(a^2 - x^2 - 0.0767ax)$$

In order to apply the general theory to the obtained wing $H_1(\theta)$ is expanded into a trigonometric series:

$$H_1(\theta) = \left(\pi - \frac{17}{3} - \int_0^{\frac{\pi}{2}} \log\tan\frac{x}{2}\,dx\right)\sin\theta +$$

$$\sum_{k=1}^{\infty}\frac{\sin(2k+1)\theta}{4k(k+1)(2k-1)(2k+3)}\left[-12\pi k(k+1) + \right.$$

$$\left.2(16k^2 + 16k - 3)\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{1}{4k+1}\right) + 6(2k+1)\right]$$

$$= \sum_{k=0}^{\infty}\gamma_{2k+1}\sin(2k+1)\theta \quad (4.38)$$

where

$$\gamma_1 = -0.6931 \; ; \; \gamma_3 = -0.1783 \; ; \; \gamma_5 = -0.0812$$

$$\gamma_7 = -0.0463 \; ; \; \gamma_9 = -0.0300, \; \ldots$$

For the case considered, the usual theory gives for the determination of the circulation

$$\Gamma_0(-a \cos \theta) = B_1 \sin \theta + B_2 \sin 2\theta + \ldots \qquad (0 \leqslant \theta \leqslant \pi)$$

the equation

$$\sum_{n=1}^{\infty} B_n \sin n\theta = 2\pi ca \sin \theta \left\{\alpha a \sin \theta - \frac{g(-a \cos \theta)}{c} - \frac{1}{4ca} \sum_{n=1}^{\infty} nB_n \frac{\sin n\theta}{\sin \theta}\right\} \tag{4.39}$$

Equation (4.35) and

$$\sin^2\theta = -\frac{8}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k + 1)\theta}{(2k - 1)(2k + 1)(2k + 3)} \qquad (0 \leqslant \theta \leqslant \pi) \tag{4.40}$$

give from equation (4.39) the equation

$$\sum_{n=1}^{\infty} B_n\left(1 + \frac{\pi n}{2}\right) = \sum_{k=0}^{\infty}\left[\frac{8\alpha a^2 c}{3\pi} \gamma_{2k+1} - \frac{16\alpha a^2 c}{(2k - 1)(2k + 1)(2k + 3)}\right]\sin(2k + 1)\theta \tag{4.41}$$

from which without difficulty B_n is obtained, in particular

$$B_{2k} = 0 \; ; \; B_1 = 1.8457 \; \alpha ca^2 \; ; \; B_3 = -\; 0.2132 \; \alpha ca^2$$

$$B_5 = -\; 0.0250 \; \alpha ca^2 \; ; \; B_7 = -\; 0.0075 \; \alpha ca^2 \; ; \; B_9 = -\; 0.0032 \; \alpha ca^2, \; \ldots$$

The following value is obtained for the lift force:

$$P = \frac{1}{2} \pi\rho caB_1 = 2.899 \; \alpha c^2a^3\rho \tag{4.42}$$

exceeding the accurate value by 8.7 percent.

The induced drag

$$W = 1.3927 \; \rho\alpha^2c^2a^4 \tag{4.43}$$

exceeds the accurate value by 9.4 percent.

3. In order to give an example of a nonsymmetrical wing,

$$f(x,y) = \alpha cy$$

In this case it is first necessary to compute the integral

$$\iint_S \frac{\eta\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} = \frac{4}{3}\pi a^2 \sin\gamma \qquad (4.44)$$

On account of equation (4.3),

$$\Gamma(y) = -\frac{4\alpha ca^2}{3\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin\gamma\cos\gamma\left[\sqrt{\frac{a(1 \pm \sin\gamma)}{|a\sin\gamma - y|}} - 1\right] d\gamma$$

After computing the integral,

$$\Gamma(y) = \frac{\alpha c}{\pi}\left[(a + y)\sqrt{2a(a - y)} - (a - y)\sqrt{2a(a + y)} + \right.$$
$$\frac{1}{6}(a + y)(a - 3y)\log\frac{\sqrt{2a} - \sqrt{a - y}}{\sqrt{2a} + \sqrt{a - y}} -$$
$$\left.\frac{1}{6}(a - y)(a + 3y)\log\frac{\sqrt{2a} - \sqrt{a + y}}{\sqrt{2a} + \sqrt{a + y}}\right] \qquad (4.45)$$

is obtained.

Assuming $y = -a\cos\theta$ and expanding in a trigonometric series give

$$\Gamma(-a\cos\theta) = \frac{\alpha ca^2}{\pi}\left[2(1 - \cos\theta)\cos\frac{\theta}{2} - 2(1 + \cos\theta)\sin\frac{\theta}{2} + \right.$$
$$\frac{1}{6}(1 - \cos\theta)(1 + 3\cos\theta)\log\frac{1 - \cos\frac{\theta}{2}}{1 + \cos\frac{\theta}{2}} -$$
$$\left.\frac{1}{6}(1 + \cos\theta)(1 - 3\cos\theta)\log\frac{1 - \sin\frac{\theta}{2}}{1 + \sin\frac{\theta}{2}}\right]$$
$$= A_2\sin 2\theta + A_4\sin 4\theta + \ldots \qquad (0 \leqslant \theta \leqslant \pi) \qquad (4.46)$$

where

$$A_2 = -\frac{128\ \alpha ca^2}{27\pi^2}$$

$$A_{2k} = \frac{4\alpha ca^2}{\pi^2}\left[\frac{8k^2 + 1}{6k(k^2 - 1)(4k^2 - 1)}\left(1 + \frac{1}{3} + \ldots + \frac{1}{4k - 1}\right) - \frac{2k}{(k^2 - 1)(4k^2 - 1)}\right] \tag{4.47}$$

$$(k = 2,3, \ldots)$$

so that

$$A_2 = -\ 0.4803\ \alpha ca^2\ ;\ A_4 = 0.00549\ \alpha ca^2$$

$$A_6 = 0.00234\ \alpha ca^2\ ;\ A_8 = 0.00123\ \alpha ca^2$$

Evidently there is no lift force, whereas for the induced drag the following value is obtained:

$$W = 0.1813\ \rho\alpha^2 c^2 a^4 \tag{4.48}$$

The moment of the forces about the Ox axis is:

$$M_x = -\frac{1}{4}\pi\rho ca^2 A_2 = 0.3772\ \rho\alpha c^2 a^4 \tag{4.49}$$

The moment of the forces about the Oy axis is computed with the aid of equation (4.8), where use must be made of the result (4.44), and it is found that

$$M_y = 0 \tag{4.50}$$

The following function is now computed:

$$g(y) = \frac{2a^3\alpha c}{3\pi^2}\int_{\infty}^{\sqrt{a^2-y^2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{\sin\gamma\cos\gamma\ d\gamma\ dx}{\sqrt{x^2 + y^2 - a^2}(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)}$$

Setting

$$H_2(\theta) = \frac{3\pi^2}{2a\alpha c}\sin\theta g(-a\cos\theta) \quad (0 \leqslant \theta \leqslant \pi) \tag{4.51}$$

and carrying out the integration with respect to γ give

$$H_2(\theta) = \int_{\infty}^{\sin\theta} \frac{\sin\theta}{(t^2 + \cos^2\theta)^2\sqrt{t^2 - \sin^2\theta}}\left\{-\cos\theta(t^2 + \cos^2\theta) + \frac{\pi}{2}\, t\cos\theta(t^2 + 1 + \cos^2\theta) + \frac{1}{4}(t^2 + 1 + \cos^2\theta)(\cos^2\theta - t^2)\log\frac{t^2 + 4\cos^4\frac{\theta}{2}}{t^2 + 4\sin^4\frac{\theta}{2}} - \frac{t\cos\theta\left[1 + (t^2 + \cos^2\theta)^2\right]}{t^2 - \sin^2\theta}\arctan\frac{t^2 - \sin^2\theta}{2t}\right\}dt$$

$$= \sin\theta\cos\theta\left\{\frac{3}{2}\log^2(\sqrt{2} + 1) + 3\int_0^1 \frac{\arctan y}{\sqrt{1 - y^2}}\,dy - \frac{3\pi^2}{4}\right\} + \frac{\sin\theta(1 - 3\cos\theta)}{16}\left(\log\frac{1 + \cos\frac{\theta}{2}}{1 - \cos\frac{\theta}{2}}\right)^2 - \frac{\sin\theta(1 + 3\cos\theta)}{16}\left(\log\frac{1 + \sin\frac{\theta}{2}}{1 - \sin\frac{\theta}{2}}\right)^2 + \frac{1 + 3\cos\theta}{2}\sin\frac{\theta}{2}\log\frac{1 + \cos\frac{\theta}{2}}{1 - \cos\frac{\theta}{2}} - \frac{1 - 3\cos\theta}{2}\cos\frac{\theta}{2}\log\frac{1 + \sin\frac{\theta}{2}}{1 - \sin\frac{\theta}{2}} \tag{4.52}$$

Expansion in a trigonometric series gives

$$H_2(\theta) = \sin 2\theta\left[\frac{3}{4}\log^2(\sqrt{2} + 1) + \frac{3}{2}\int_0^1 \frac{\arctan y}{\sqrt{1 - y^2}}\,dy - \frac{9\pi^2}{16} + \frac{2159}{630}\right] - \sum_{k=2}^{\infty} \frac{2(8k^2 + 1)}{(4k^2 - 1)(4k^2 - 4)}\left(1 + \frac{1}{3} + \ldots + \frac{1}{4k - 1} - \frac{12k^2}{8k^2 + 1}\right)\sin 2k\theta$$

$$= \sum_{k=1}^{\infty} \delta_{2k}\sin 2k\theta \tag{4.53}$$

where

$$\delta_2 = -0.27412 \; ; \; \delta_4 = -0.08127 \; ; \; \delta_6 = -0.05198 \; ; \; \delta_8 = -0.03641, \; \ldots$$

The usual theory for determining the circulation

$$\Gamma_0(-a \cos \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$$

gives the equation

$$\sum_{n=1}^{\infty} B_n\left(1 + \frac{\pi n}{2}\right)\sin n\theta = 2\pi ca \sin \theta \left\{-a\alpha \cos \theta - \frac{g(-a \cos \theta)}{c}\right\}$$

$$= -\pi ca^2\alpha \sin 2\theta - \frac{4\alpha ca^2}{3\pi} H_2(\theta) \tag{4.54}$$

from which without difficulty

$$B_{2k+1} = 0 \quad (k = 0,1,2, \ldots)$$

$$B_2 = -0.7304 \; \alpha ca^2 \; ; \; B_4 = 0.0047 \; \alpha ca^2 \; ; \; B_6 = 0.0021 \; \alpha ca^2 \; ; \; B_8 = 0.0011 \; \alpha ca^2, \ldots$$

The lift force is found equal to zero and the induced drag and moment of the forces about the Ox axis are

$$W = 0.4191 \; \rho\alpha^2c^2a^4 \; ; \; M_x = 0.5737 \; \rho\alpha^2c^2a^4 \tag{4.55}$$

The first gives an error of 131 percent, the second of 52 percent.

By a combination of the obtained solutions it would have been possible to obtain further examples. From the examples given it is clear that for the case of a circular wing considerable deviations are obtained between the usual and the exact theories.

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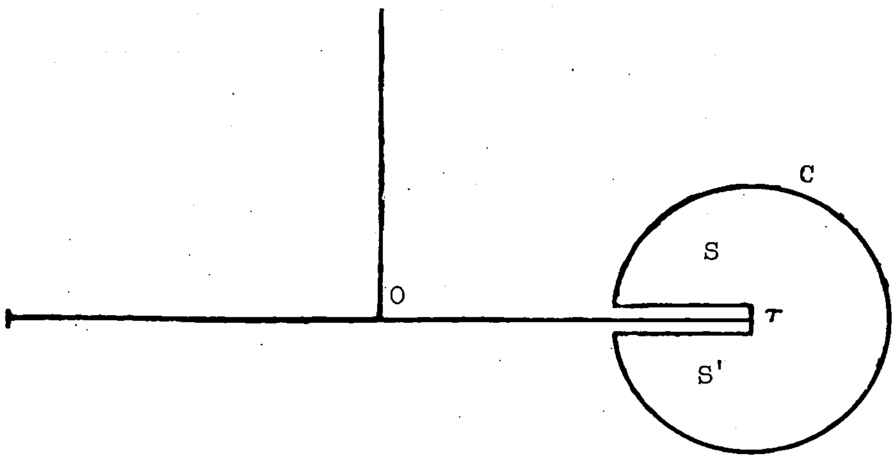

Figure 1.

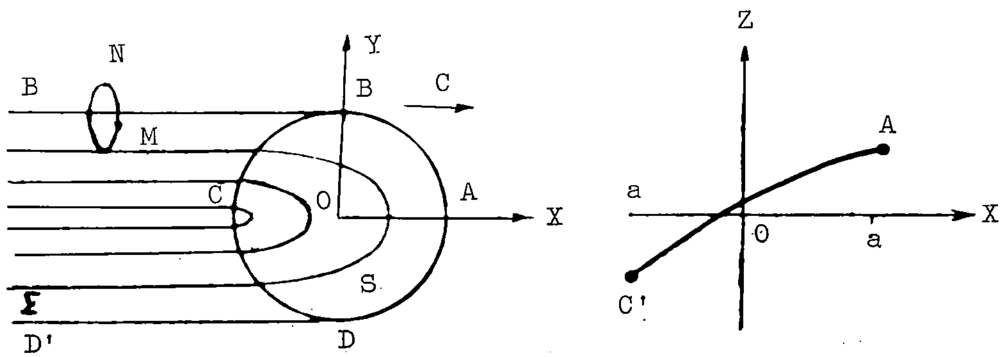

Figure 2.