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TORSION AND BENDING OF PRISMATIC RODS OF HOLLOW RECTANGULAR SECTION

By B. L. Abramyan

Translation

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TORSION AND BENDING OF PRISMATIC RODS OF HOLLOW RECTANGULAR SECTION*

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In the present paper a solution is given for the problem of the torsion and bending of prismatic rods of hollow rectangular sections.

As in the former paper (reference 1), the method given by N. Kh. Arutyunyan (reference 2) of introducing auxiliary functions was employed in the solution of this problem. This method permitted reducing the solution of the partial differential equations of the problem to the solution of linear differential equations of the second order with constant coefficients and reducing the determination of the constants of integration to the solution of an infinite completely regular system of linear equations.

The obtained formulas determine the stiffness in torsion and bending as a function of the geometric parameters of the section.

At the same time, there are indicated the limits of applicability of the semiempirical formula of Bredt (reference 3) for the determination of the stiffness in torsion of hollow thin-walled rods.

1. TORSION OF A PRISMATIC ROD OF HOLLOW RECTANGULAR SECTION

1. Statement of the problem. - The determination of the stress function $U(x,y)$ for the torsion of a rod of a doubly connected cross section reduces, as is known, to the integration of Poisson's equation

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -2 \tag{1.1}$$

when the function $U(x,y)$ becomes zero at the outer contour and assumes a constant value U_0 on the inner contour (fig. 1). From the symmetry it is sufficient to find the function $U(x,y)$ only for the part of the section ODEFBC. In order that the solution extend to the entire region of the cross section, it is necessary that on the lines DE and BC the normal derivatives of the function $U(x,y)$ be equal to zero.

*"Kruchenie i izgib prismaticheskikh sterzhnei s polym pryamougol'nym secheniem." Prikladnaya Matematika i Mekhanika, vol. 14, no. 3, 1950, pp. 265-276.

It is assumed that in the region OABC the function $U(x,y)$ assumes the value $U_1(x,y)$ and in the region ODEG the value $U_2(x,y)$.

It is ideal to obtain the function $U_i(x,y)$ in the form

$$U_i(x,y) = \Psi_i(x,y) + \Phi_i(x,y) \qquad (i = 1,2) \tag{1.2}$$

where the functions $\Phi_i(x,y)$ $(i = 1,2)$ exists only in the region OAFG, $\Psi_1(x,y)$ exists in the region OABC, and Ψ_2 exists in the region ODEG.

For the auxiliary functions $\Psi_i(x,y)$ and $\Phi_i(x,y)$, the following equations are obtained:

$$\nabla^2\Psi_i = -2 \qquad \nabla^2\Phi_i = 0 \qquad (i = 1,2) \tag{1.3}$$

The following conditions must be satisfied:

$$\Psi_1(x,0) = \left(\frac{\partial\Psi_1}{\partial x}\right)_{x=a} = \Psi_1(0,y) + \Phi_1(0,y) = 0 \qquad \Psi_1(x,d_1) = U_0 \tag{1.4}$$

$$\Psi_2(0,y) = \left(\frac{\partial\Psi_2}{\partial y}\right)_{y=b} = \Psi_2(x,0) + \Phi_2(x,0) = 0 \qquad \Psi_2(d_2,y) = U_0 \tag{1.5}$$

The boundary conditions for the determination of the functions $\Psi_1(x,y)$ and $\Psi_2(x,y)$ are nonhomogeneous; however, following G. A. Grinberg (reference 4), set

$$\Psi_1(x,y) = \sum_{k=1}^{\infty} f_k(x) \sin\frac{k\pi y}{d_1} \qquad \Psi_2(x,y) = \sum_{k=1}^{\infty} v_k(y) \sin\frac{k\pi x}{d_2} \tag{1.6}$$

For $\Phi_1(x,y)$ and $\Phi_2(x,y)$ the following conditions are then obtained:

$$\left.\begin{aligned} \Phi_1(d_2,y) &= \Phi_1(x,0) = \left(\frac{\partial\Phi_1}{dx}\right)_{x=d_2} = 0 \\ \Phi_1(x,d_1) &= \sum_{k=1}^{\infty} v_k(d_1) \sin\frac{k\pi x}{d_2} - U_0 \end{aligned}\right\} \tag{1.7}$$

$$\left.\begin{aligned} \Phi_2(x,d_1) &= \Phi_2(0,y) = \left(\frac{\partial\Phi_2}{dy}\right)_{y=d_1} = 0 \\ \Phi_2(d_2,y) &= \sum_{k=1}^{\infty} f_k(d_2)\sin\frac{k\pi y}{d_1} - U_0 \end{aligned}\right\} \quad (1.8)$$

It is ideal to obtain the functions $\Phi_1(x,y)$ and $\Phi_2(x,y)$ in the form

$$\Phi_1(x,y) = \sum_{k=1}^{\infty} \varphi_k(x)\sin\frac{k\pi y}{d_1} \qquad \Phi_2(x,y) = \sum_{k=1}^{\infty} w_k(y)\cdot\sin\frac{k\pi x}{d_2} \quad (1.9)$$

Equations (1.3) to (1.9) completely determine the function $U(x,y)$ in the region ODEFBC.

2. Solution of the equations of the problem. - Making use of equations (1.3) to (1.9), the following equations are obtained:

$$f_k(x) = A_k \operatorname{sh}\frac{k\pi x}{d_1} + B_k \operatorname{ch}\frac{k\pi x}{d_1} - (-1)^k\frac{2U_0}{k\pi} + \left[1 - (-1)^k\right]\frac{4d_1{}^2}{(k\pi)^3} \quad (2.1)$$

$$v_k(y) = M_k \operatorname{sh}\frac{k\pi y}{d_2} + N_k \operatorname{ch}\frac{k\pi y}{d_2} - (-1)^k\frac{2U_0}{k\pi} + \left[1 - (-1)^k\right]\frac{4d_2{}^2}{(k\pi)^3} \quad (2.2)$$

$$\varphi_k(x) = D_k \operatorname{sh}\frac{k\pi x}{d_1} + C_k \operatorname{ch}\frac{k\pi x}{d_1} + (-1)^k\frac{2U_0}{k\pi}\left(1 - \operatorname{ch}\frac{k\pi x}{d_1}\right) +$$

$$(-1)^k\frac{2}{\pi}\sum_{p=1}^{\infty}\frac{v_p(d_1)}{(pd_1/d_2)^2 + k^2}\left[\frac{d_1}{d_2}\,p\operatorname{sh}\frac{k\pi x}{d_1} - k\sin\frac{p\pi x}{d_2}\right] \quad (2.3)$$

$$w_k(y) = L_k \operatorname{sh} \frac{k\pi y}{d_2} + F_k \operatorname{ch} \frac{k\pi y}{d_2} + (-1)^k \frac{2U_0}{k\pi}\left(1 - \operatorname{ch} \frac{k\pi y}{d_2}\right) +$$

$$(-1)^k \frac{2}{\pi} \sum_{p=1}^{\infty} \frac{f_p(d_2)}{(pd_2/d_1)^2 + k^2}\left[\frac{d_2}{d_1} p \operatorname{sh} \frac{k\pi y}{d_2} - k \sin \frac{p\pi y}{d_1}\right] \tag{2.4}$$

Then $f_p(d_2)$ and $v_p(d_1)$ have the values

$$f_p(d_2) = A_p \operatorname{sh} \frac{p\pi d_2}{d_1} + B_p \operatorname{ch} \frac{p\pi d_2}{d_1} - (-1)^p \frac{2U_0}{\pi p} + \frac{4{d_1}^2}{(p\pi)^3}\left[1 - (-1)^p\right]$$

(2.5)

$$v_p(d_1) = M_p \operatorname{sh} \frac{p\pi d_1}{d_2} + N_p \operatorname{ch} \frac{p\pi d_1}{d_2} - (-1)^p \frac{2U_0}{p\pi} + \frac{4{d_2}^2}{(p\pi)^3}\left[1 - (-1)^p\right]$$

3. Determination of the constants of integration. - For the functions $f_k(x)$, $v_k(y)$, $\varphi_k(x)$, and $w_k(y)$, the boundary conditions from equations (1.4), (1.5), (1.7), and (1.8) have the form

$$f_k'(a) = f_k(0) + \varphi_k(0) = 0 \qquad \varphi_k(d_2) = \varphi_k'(d_2) = 0$$

$$v_k'(b) = v_k(0) + w_k(0) = 0 \qquad w_k(d_1) = w_k'(d_1) = 0 \qquad (k = 1, 2, \ldots)$$

(3.1)

With the aid of these conditions the equations for the determination of the constants of integration are obtained from equations (2.1) to (2.4).

With the elimination of A_k, M_k, D_k, C_k, L_k, and F_k from the obtained equations, a set of two infinite systems of linear equations (reference 5) is obtained. The following notations are introduced:

$$B_k = S_k d_1 d_2 \frac{(-1)^k}{k} \operatorname{sh} \frac{k\pi d_2}{d_1} \qquad N_k = R_k d_1 d_2 \frac{(-1)^k}{k} \operatorname{sh} \frac{k\pi d_1}{d_2} \tag{3.2}$$

This set of two infinite systems of equations is then reduced to the form

$$S_k = \sum_{p=1}^{\infty} R_p a_{kp} + \beta_k \qquad R_k = \sum_{p=1}^{\infty} S_p c_{kp} + \gamma_k \qquad (k = 1,2, \ldots) \tag{3.3}$$

where

$$a_{kp} = \frac{2k}{\pi} \frac{d_1 d_2}{d_1^2 p^2 + d_2^2 k^2} \operatorname{sh} \frac{p\pi d_1}{d_2} \operatorname{sch} \frac{p\pi b}{d_2} \operatorname{ch} \frac{p\pi(b-d_1)}{d_2}$$

$$\beta_k = \frac{1}{k}\left\{\frac{2U_0}{\pi^2 d_2^2} + \frac{4}{\pi^3} \frac{1-(-1)^k}{k} \frac{d_1}{d_2} \operatorname{csch} \frac{k\pi d_2}{d_1} - \frac{2}{\pi^2}\left(1 - \frac{2d_1}{k\pi d_2} \operatorname{th} \frac{k\pi d_2}{2d_1}\right)\right\} \tag{3.4}$$

$$c_{kp} = \frac{2k}{\pi} \frac{d_1 d_2}{d_2^2 p^2 + d_1^2 k^2} \operatorname{sh} \frac{p\pi d_2}{d_1} \operatorname{sch} \frac{p\pi a}{d_1} \operatorname{ch} \frac{p\pi(a-d_2)}{d_1}$$

$$\gamma_k = \frac{1}{k}\left\{\frac{2U_0}{\pi^2 d_1^2} + \frac{4}{\pi^3} \frac{1-(-1)^k}{k} \frac{d_2}{d_1} \operatorname{csch} \frac{k\pi d_1}{d_2} - \frac{2}{\pi^2}\left(1 - \frac{2d_2}{k\pi d_1} \operatorname{th} \frac{k\pi d_1}{2d_2}\right)\right\} \tag{3.5}$$

Systems (3.3) may be written in the form

$$Z_\nu = \sum_{p=1}^{\infty} A_{\nu p} Z_p + B_\nu \qquad (\nu = 1,2, \ldots) \tag{3.6}$$

where it is necessary to set

$$\begin{aligned} &Z_{2n-1} = S_k \quad Z_{2n} = R_k \quad A_{2n,2m} = 0 \quad A_{2n,2m-1} = c_{kp} \\ &B_{2n-1} = \beta_k \quad B_{2n} = \gamma_k \quad A_{2n-1,2m-1} = 0 \quad A_{2n-1,2m} = a_{kp} \end{aligned} \tag{3.7}$$

A few cases will be considered.

1. The infinite system (3.6) is completely regular for $b \geqslant d_2$ and $a \geqslant d_1$ since from equations (3.4), (3.5), and (3.7)

$$\sum_{p=1}^{\infty} A_{2n,p} = \sum_{p=1}^{\infty} c_{kp} \leqslant \frac{k}{\pi}\frac{d_2}{d_1}\sum_{p=1}^{\infty}\frac{1}{(pd_2/d_1)^2 + k^2}$$

$$= \frac{1}{2}\left(\operatorname{cth}\frac{k\pi d_1}{d_2} - \frac{d_2}{k\pi d_1}\right) \leqslant \frac{1}{2}$$

(3.8)

$$\sum_{p=1}^{\infty} A_{2n-1,p} = \sum_{p=1}^{\infty} a_{kp} \leqslant \frac{k}{\pi}\frac{d_1}{d_2}\sum_{p=1}^{\infty}\frac{1}{(pd_1/d_2)^2 + k^2}$$

$$= \frac{1}{2}\left(\operatorname{cth}\frac{k\pi d_2}{d_1} - \frac{d_1}{k\pi d_2}\right) \leqslant \frac{1}{2}$$

where the inequalities were used

$$\operatorname{sh}\frac{p\pi d_1}{d_2}\operatorname{sch}\frac{p\pi b}{d_2}\operatorname{ch}\frac{p\pi(b-d_1)}{d_2} \approx \operatorname{sh}\frac{p\pi d_1}{d_2}\exp\left(-\frac{p\pi d_1}{d_2}\right) \leqslant \frac{1}{2} \tag{3.9}$$

$$\operatorname{sh}\frac{p\pi d_2}{d_1}\operatorname{sch}\frac{p\pi a}{d_1}\operatorname{ch}\frac{p\pi(a-d_2)}{d_1} \approx \operatorname{sh}\frac{p\pi d_2}{d_1}\exp\left(-\frac{p\pi d_2}{d_1}\right) \leqslant \frac{1}{2} \tag{3.10}$$

$$\operatorname{cth} x - \frac{1}{x} \leqslant 1 \qquad (0 \leqslant x \leqslant \infty) \tag{3.11}$$

Hence, for any ν

$$\sum_{p=1}^{\infty} A_{\nu p} \leqslant \frac{1}{2} \tag{3.12}$$

2. The system (3.6) is regular for the particular cases where $a > d_2$, $b = d_1$ and $a = d_2$, $b > d_1$ (where a rectangular hollow section is present).

In this case

$$\sum_{p=1}^{\infty} A_{2n,p} = \sum_{p=1}^{\infty} c_{kp} < \frac{2k}{\pi}\frac{d_2}{d_1}\sum_{p=1}^{\infty}\frac{1}{(pd_2/d_1)^2 + k^2} \leqslant 1 \qquad (3.13)$$

$$\sum_{p=1}^{\infty} A_{2n-1,p} = \sum_{p=1}^{\infty} a_{kp} < \frac{2k}{\pi}\frac{d_1}{d_2}\sum_{p=1}^{\infty}\frac{1}{(pd_1/d_2)^2 + k^2} \leqslant 1 \qquad (3.14)$$

where the inequalities (3.11) were used

$$\operatorname{sh}\frac{p\pi d_1}{d_2}\operatorname{sch}\frac{p\pi d_1}{d_2}\operatorname{ch}\frac{p\pi(b-d_1)}{d_2} < 1 \qquad \operatorname{sh}\frac{p\pi d_2}{d_1}\operatorname{sch}\frac{p\pi a}{d_1}\operatorname{ch}\frac{p\pi(a-d_2)}{d_1} < 1 \qquad (3.15)$$

According to equations (3.13) and (3.14) for any ν

$$\sum_{p=1}^{\infty} A_{\nu p} < 1 \qquad (3.16)$$

3. When $a = b$ and $d_2 = d_1 = d$, a hollow square section (fig. 2), a single infinite system of linear equations which is entirely regular for $a/d \geqslant \mu > 1$, where μ is a finite number, is obtained in place of the systems (3.3). This results in

$$F_k = \sum_{p=1}^{\infty} F_p b_{kp} + \alpha_k \qquad (k = 1,2, \ldots) \qquad (3.17)$$

where

$$b_{kp} = \frac{2k}{\pi}\operatorname{sh} p\pi \operatorname{sch}\frac{p\pi a}{d}\operatorname{ch}\frac{p\pi(a-d)}{d}\frac{1}{p^2 + k^2} \qquad (3.18)$$

$$\alpha_k = \frac{1}{k}\left\{\frac{2U_0}{(\pi d)^2} + \frac{4}{k\pi^3}\,\frac{1 + (-1)^{k+1}}{\mathrm{sh}\ k\pi} - \frac{2}{\pi^2}\left(1 - \frac{2}{k\pi}\ \mathrm{th}\ \frac{k\pi}{2}\right)\right\} \quad (3.19)$$

$$\sum_{p=1}^{\infty} b_{kp} < \frac{2k}{\pi}\,\frac{1 + e^{-2\pi(\mu-1)}}{2}\sum_{p=1}^{\infty}\frac{1}{p^2 + k^2} < \frac{1 + e^{-2\pi(\mu-1)}}{2} \quad (3.20)$$

where the inequalities (3.11) were used and

$$\mathrm{sh}\ p\pi\ \mathrm{sch}\ \frac{p\pi a}{d}\ \mathrm{ch}\ \frac{p\pi(a-d)}{d} < \frac{1 + e^{-2\pi(\mu-1)}}{2} \quad (3.21)$$

As an example, the case where $a/d \geq \mu = 3/2$ is considered. From (3.20) is obtained

$$\sum_{p=1}^{\infty} b_{kp} < 1 - \theta = 0.5216$$

$$\theta = 0.4784 \quad (3.22)$$

The free term α_k of the system (3.17) satisfies the inequality

$$|\alpha_k| \leq 0.20264\ \frac{U_0}{d^2} - 0.06198 \quad (3.23)$$

The values of the unknown F_k with an excess is denoted by F_k^+ and the values with a defect by F_k^-.

Using the theory of regular and completely regular systems (reference 5) and applying limiting values yields the following estimates for F_k:

$$F_1^- = 0.27547\ \frac{U_0^-}{d^2} - 0.09629 \leq F_1 \leq 0.27588\ \frac{U_0^+}{d^2} - 0.09641 = F_1^+$$

$$F_2^- = 0.18234\ \frac{U_0^-}{d^2} - 0.11107 \leq F_2 \leq 0.18300\ \frac{U_0^+}{d^2} - 0.11128 = F_2^+$$

$$(3.24)$$

$$F_3^- = 0.14762\ \frac{U_0^-}{d^2} - 0.09749 \leq F_3 \leq 0.14845\ \frac{U_0^+}{d^2} - 0.09775 = F_3^+$$

$$0.13440\ \frac{U_0^-}{d^2} - 0.09196 \leq F_k^- \leq F_k \leq F_k^+ \leq 0.13739\ \frac{U_0^+}{d^2} - 0.09288$$

$$(k = 4, 5, \ldots)$$

where U_0^+ is the value of U_0 with an excess and U_0^- is the value with a defect.

4. Determination of the constant U_0. - For determining U_0, use is made of the theorem on the circulation of the tangential stress in torsion (reference 6)

$$\int_{C_0} T_s ds = 2G\tau\Omega_0 \tag{4.1}$$

where C_0 is the inner contour FHMLF of the section (fig. 1), Ω_0 is the area enclosed by this contour, G is the shear modulus, τ is the angle of torsion per unit length, and T_s is the projection of the tangential stress at any point of the contour C_0 on the direction of the tangent to the contour.

Substituting in (4.1) the value

$$T_s = \left(\frac{\partial U}{\partial y}\frac{dx}{ds} - \frac{\partial U}{\partial x}\frac{dy}{ds}\right) G\tau \tag{4.2}$$

and making a certain transformation, relation (4.1) is reduced to the form

$$\int_{d_2}^{a} \left(\frac{\partial U_1}{\partial y}\right)_{y=d_1} dx + \int_{d_1}^{b} \left(\frac{\partial U_2}{\partial x}\right)_{x=d_2} dy = 2(a-d_2)(b-d_1) \tag{4.3}$$

Use of the obtained values of $f_k(x)$ and $v_k(y)$ yields, from relations (1.2), (1.6), and (4.3),

$$U_0 = \frac{d_1 d_2}{d_2(a-d_2) + d_1(b-d_1)} \left\{ 2ab - ad_1 - bd_2 + \right.$$

$$\frac{d_2^2}{\pi} \sum_{k=1}^{\infty} \frac{R_k}{k^2} \operatorname{sh}\frac{k\pi d_1}{d_2} \operatorname{sch}\frac{k\pi b}{d_2} \operatorname{ch}\frac{k\pi(b-d_1)}{d_2} + \tag{4.4}$$

$$\left. \frac{d_1^2}{\pi} \sum_{k=1}^{\infty} \frac{S_k}{k^2} \operatorname{sh}\frac{k\pi d_2}{d_1} \operatorname{sch}\frac{k\pi a}{d_1} \operatorname{ch}\frac{k\pi(a-d_2)}{d_1} - d_1 d_2 \sum_{k=1}^{\infty} \frac{\beta_k + \gamma_k}{k} \right\}$$

where R_k and S_k are the constants of integration determined from the system (3.3); and β_k and γ_k have the values (3.4) and (3.5). If the values of the coefficients R_k and S_k with excess and with defect are substituted in (4.4) and this equation solved for U_0, the values of U_0 with excess and defect are obtained.

5. Determination of the stress function. - According to (1.2), (1.6), (1.9), and the obtained values of the functions $f_k(x)$, $v_k(y)$, $\varphi_k(x)$, and $w_k(y)$, the stress function has the form:

$$U_1(x,y) = \sum_{p=1}^{\infty} v_p(d_1) \sin\frac{p\pi x}{d_2} \operatorname{csch}\frac{p\pi d_1}{d_2} \operatorname{sh}\frac{p\pi y}{d_2} + y(d_1-y) +$$

$$d_1 d_2 \sum_{k=1}^{\infty} (-1)^k \frac{S_k}{k} \operatorname{sh}\frac{k\pi d_2}{d_1} \operatorname{sch}\frac{k\pi a}{d_1} \operatorname{ch}\frac{k\pi(a-x)}{d_1} \sin\frac{k\pi y}{d_1} +$$

$$\frac{2U_0}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \operatorname{ch}\frac{k\pi(x-d_2)}{d_1} \sin\frac{k\pi y}{d_1} + \qquad \text{for} \quad 0 \leqslant x \leqslant d_2$$

$$\frac{2}{\pi}\frac{d_1}{d_2} \sum_{k=1}^{\infty} (-1)^k \operatorname{sh}\frac{k\pi(x-d_2)}{d_1} \sin\frac{k\pi y}{d_1} \sum_{p=1}^{\infty} \frac{v_p(d_1)\, p(-1)^p}{(pd_1/d_2)^2 + k^2} \qquad (5.1)$$

$$U_1(x,y) = U_0 \frac{y}{d_1} + y(d_1-y) + \qquad \text{for} \quad d_2 \leqslant x \leqslant a$$

$$d_1 d_2 \sum_{k=1}^{\infty} (-1)^k \frac{S_k}{k} \operatorname{sh}\frac{k\pi d_2}{d_1} \operatorname{sch}\frac{k\pi a}{d_1} \operatorname{ch}\frac{k\pi(a-x)}{d_1} \sin\frac{k\pi y}{d_1} \qquad (5.2)$$

$$U_2(x,y) = U_0 \frac{x}{d_2} + x(d_2-x) + \qquad \text{for} \quad d_1 \leqslant y \leqslant b$$

$$d_1 d_2 \sum_{k=1}^{\infty} (-1)^k \frac{R_k}{k} \operatorname{sh} \frac{k\pi d_1}{d_2} \operatorname{sch} \frac{k\pi b}{d_2} \operatorname{ch} \frac{k\pi(b-y)}{d_2} \sin \frac{k\pi x}{d_2} \tag{5.3}$$

$$U_2(x,y) = \sum_{p=1}^{\infty} f_p(d_2) \sin \frac{p\pi y}{d_1} \operatorname{csch} \frac{p\pi d_2}{d_1} \operatorname{sh} \frac{p\pi x}{d_1} + x(d_2-x) +$$

$$d_1 d_2 \sum_{k=1}^{\infty} (-1)^k \frac{R_k}{k} \operatorname{sh} \frac{k\pi d_1}{d_2} \operatorname{sch} \frac{k\pi b}{d_2} \operatorname{ch} \frac{k\pi(b-y)}{d_2} \sin \frac{k\pi x}{d_2} +$$

$$\frac{2U_0}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \operatorname{ch} \frac{k\pi(y-d_1)}{d_2} \sin \frac{k\pi x}{d_2} + \qquad \text{for} \quad 0 \leqslant y \leqslant d_1$$

$$\frac{2}{\pi} \frac{d_2}{d_1} \sum_{k=1}^{\infty} (-1)^k \operatorname{sh} \frac{k\pi(y-d_1)}{d_2} \sin \frac{k\pi x}{d_2} \sum_{p=1}^{\infty} \frac{f_p(d_2)(-1)^p\, p}{(pd_2/d_1)^2 + k^2} \tag{5.4}$$

where $f_p(d_2)$ and $v_p(d_1)$ have the values

$$f_p(d_2) = \frac{(-1)^{p+1}}{p} \left\{ \frac{2U_0}{\pi} + \frac{4{d_1}^2}{p^2\pi^3} \left[1 + (-1)^{p+1}\right] - S_p d_1 d_2 \operatorname{sh} \frac{p\pi d_2}{d_1} \operatorname{sch} \frac{p\pi a}{d_1} \operatorname{ch} \frac{p\pi(a-d_2)}{d_1} \right\} \tag{5.5}$$

$$v_p(d_1) = \frac{(-1)^{p+1}}{p}\left\{\left[\frac{2U_0}{\pi} + \frac{4d_2^{\,2}}{p^2\pi^3}\left[1 + (-1)^{p+1}\right] - R_p d_1 d_2 \operatorname{sh}\frac{p\pi d_1}{d_2} \operatorname{sch}\frac{p\pi b}{d_2} \operatorname{ch}\frac{p\pi(b-d_1)}{d_2}\right]\right\} \tag{5.6}$$

By the substitution of the obtained values of the stress function $U(x,y)$ in the general formulas, the stress and stiffness in torsion are obtained.

As an example the torsional stiffness and stress of a rod with a hollow square section (fig. 2) will be determined.

6. Determination of the torsional stiffness. - The substitution in equations (5.1) to (5.6) of $b = a$ and $d_1 = d_2 = d$ and the use of the formula for the stiffness of a section

$$C = 8G\left[(a-d)^2 U_0 + \int_0^d \int_0^d U(x,y)\,dxdy + 2\int_0^d dy \int_d^a U(x,y)\,dx\right] \tag{6.1}$$

yields, after integration,

$$C = 8Gd^4\left\{\left(\frac{a}{d} - 1\right)\frac{a}{d}\frac{U_0}{d^2} + \frac{1}{3}\left(\frac{a}{d} - 1\right) + \right. \tag{6.2}$$

$$\frac{16}{\pi^4}\sum_{k=1,3,\ldots}^{\infty}\frac{1}{k^4}\left[1 - \frac{1}{k\pi}\operatorname{th}\frac{k\pi a}{d} - \frac{1}{k\pi}\operatorname{sh}\frac{k\pi(a-d)}{d}\operatorname{sch}\frac{k\pi a}{d}\right] +$$

$$\frac{4}{\pi^3}\sum_{k=1,3,\ldots}^{\infty}\frac{1}{k^2}\left[1 + \operatorname{sh} k\pi \operatorname{sch}\frac{k\pi a}{d}\operatorname{sh}\frac{k\pi(a-d)}{d} - \right.$$

$$\left.\operatorname{sch}\frac{k\pi a}{d}\operatorname{ch}\frac{k\pi(a-d)}{d}\right]\sum_{p=1}^{\infty}\frac{p(-1)^{p+1}}{p^2 + k^2}\frac{f_p(d)}{d^2} -$$

(continued on the following page)

$$\frac{4U_0}{\pi^3 d^2} \sum_{k=1,3,\ldots}^{\infty} \frac{1}{k^3}\left[\text{ch } k\pi \text{ sch } \frac{k\pi a}{d} \text{ sh } \frac{k\pi(a-d)}{d} + \text{sch } \frac{k\pi a}{d} \text{ sh } \frac{k\pi(a-d)}{d}\right] +$$

$$\left.\frac{2}{\pi^2} \sum_{p=1,3,\ldots}^{\infty} \frac{f_p(d)}{d^2} \frac{\text{th } \frac{1}{2} p\pi}{p^2}\right\} \qquad (6.2)$$

where U_0 is determined from the equation

$$\frac{U_0}{d^2} = \frac{a}{d} + \frac{1}{\pi}\frac{d}{a-d} \sum_{k=1}^{\infty} \frac{F_k}{k^2} \text{ sh } k\pi \text{ sch } \frac{k\pi a}{d} \text{ ch } \frac{k\pi(a-d)}{d} - \frac{d}{a-d} \sum_{k=1}^{\infty} \frac{\alpha_k}{k} \qquad (6.3)$$

and $f_p(d)$ has the value

$$f_p(d) = \frac{(-1)^p d^2}{p}\left\{F_p \text{ sh } p\pi \text{ sch } \frac{p\pi a}{d} \text{ ch } \frac{p\pi(a-d)}{d} - \frac{2}{\pi}\frac{U_0}{d^2} - \frac{4\left[1 + (-1)^{p+1}\right]}{p^2\pi^3}\right\} \qquad (6.4$$

the unknown constants F_k being determined by inequalities (3.24) and α_k having the value (3.19).

Substitution of the obtained values of the coefficients F_k^+ and F_k^- in (6.2) yields the upper and lower limits of stiffness. The coefficients with a defect F_k^- will correspond with the lower stiffness limit C^- and the coefficients with an excess F_k^+ will correspond with the upper stiffness limit C^+.

TABLE I

a/d	C_0^*	C_0^+	C_0^-	Δ	δ
1.5	8.0	11.052	11.051	38.4	0.009
2.0	27.0	32.952	32.949	22.0	.008
2.5	64.0	73.780	73.775	15.3	.008
3.0	125.0	139.518	139.509	11.6	.007
3.5	216.0	236.164	236.150	9.3	.007
4.0	343.0	369.716	369.697	7.8	.005
5.0	729.0	771.542	771.509	5.8	.004
10.0	6859.0	7035.066	7034.902	2.6	.002
20.0	59319.0	60034.051	60033.323	1.2	.001

In table 1 are given the relative values of the stiffness computed by formula (6.2):

$$C_0^+ = C^+/Gd^4 \qquad C_0^- = C^-/Gd^4$$

and the maximum relative error

$$\delta = (C^+ - C^+)/C^-$$

in percent for different ratios a/d.

For comparison there are also given in table 1 the values of the relative stiffness computed by the semi-empirical formula of Bredt (reference 3).

$$C_0^* = \frac{C^*}{Gd^4} = \left(2\,\frac{a}{d} - 1\right)^3 \tag{6.5}$$

and the different $\Delta = (C^+ - C^*)/C^+$ in percent.

From table 1 it is seen that the semi-empirical formula of Bredt (6.5) gives sufficiently close results only for thin-walled rods for which $a/d \geqslant 5$. For thick-walled rods, however, for which $a/d \leqslant 5$ the Bredt formula is not applicable; for $a/d = 4$ it gives an error of 8 percent which rapidly increases.

7. Determination of the stresses. - With the use of expressions (5.1) to (5.2) for the stress function, the stresses are readily obtained by the usual formulas of the theory of elasticity.

Leaving out the computations gives the following expressions for the stresses at the points (a,0) and (a,d) for the rod with a hollow square section represented in figure 2:

$$X_z(a,0) = \left\{\frac{U_0}{a^2} + 1 + \pi \sum_{k=1}^{\infty} (-1)^k F_k \operatorname{sch} \frac{k\pi a}{d} \operatorname{sh} k\pi\right\} G\tau d \quad (7.1)$$

$$X_z(a,d) = \left\{\frac{U_0}{d^2} - 1 + \pi \sum_{k=1}^{\infty} F_k \operatorname{sch} \frac{k\pi a}{d} \operatorname{sh} k\pi\right\} G\tau d \quad (7.2)$$

$$Y_z(a,0) = Y_z(a,d) = 0 \quad (7.3)$$

After the substitution of the coefficients F_k^+ and F_k^-, the upper and lower limits of the stresses $X_z(a,0)$ and $X_z(a,d)$, where the upper stress limit X_z^+ will correspond with the coefficients with an excess F_k^+ $(k=1,2, \ldots)$, are determined.

In table 2 are given the computed values of the stresses $X_z(a,0)$ and $X_z(a,d)$. In the first column of each stress are given the values with an excess and in the second column, with a defect.

TABLE 2

$\frac{a}{d}$	$\frac{X_z(a,d)}{G\tau d}$		$\frac{X_z(a,0)}{G\tau d}$		$\frac{X_z{}^*{}_{max}}{G\tau d}$
1.5	0.3149	0.3143	2.0212	2.0212	2.0
2.0	0.7325	0.7322	2.6326	2.6324	2.25
2.5	1.2085	1.2082	3.1798	3.1797	2.6667
3.0	1.7050	1.7048	3.6974	3.6972	3.125
3.5	2.2068	2.2066	4.2048	4.2046	3.6
4.0	2.7092	2.7089	4.7088	4.7085	4.0833
5.0	3.7131	3.7129	5.7131	5.7129	5.0625
10.0	8.7204	8.7201	10.7204	10.7201	10.0278
20.0	18.7236	18.7234	20.7236	20.7234	20.0131

In the same table the values of the maximum stresses X_z^*, computed by the formula of Bredt, are given for comparison.

$$X_{z\ max}^* = \frac{(2a/d - 1)^2}{4(a/d - 1)} G\tau d \quad (7.4)$$

From table 2 it is seen that in the determination of the maximum stresses, the approximate formula of Bredt may be applied only for very thin-walled hollow rods.

When $a/d = 20$, the Bredt formula (7.4) gives an error of 3.5 percent.

With decrease in the ratio a/d this difference increases; for $a/d = 10$ it is equal to 7 percent, for $a/d = 5$ it is equal to 13 percent, and for $a/d = 3$ it is equal to 18 percent, etc.

II. BENDING OF A PRISMATIC ROD OF HOLLOW RECTANGULAR SECTION

8. Statement of the problem. - The stress function $F(x,y)$ in bending, as is known, satisfies the equation

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \frac{P\nu}{I(1+\nu)} (y-y_0) - \frac{P}{2I} f'(y) \qquad (8.1)$$

within the region of the section and the condition

$$\frac{\partial F}{\partial s} = \frac{P}{2I} \left[x^2 - 2xx_0 - f(y)\right] \frac{dy}{ds} \qquad (8.2)$$

on the contour of the section where P is the bending force applied to the free end of the rod at the center of gravity of the section, ν is the Poisson coefficient, x_0, y_0 are the coordinates of the center of gravity of the section, I is the axial moment of inertia of the section about the y axis; the arbitrary function $f(y)$ is to be determined from the conditions at the contour.

On account of the symmetry (fig. 3) it is sufficient to find the function $F(x,y)$ only for the part ODEFBC of the section.

In order to extend the solution over the entire region of the cross section, it is required, on the basis of the membrane analogy (reference 7), that the function $F(x,y)$ becomes zero on the vertical axis of symmetry and the derivative $\partial F/\partial x$ becomes zero along the horizontal axis of symmetry.

By condition (8.2) on the section contour, the function $F(x,y)$ is determined with an accuracy up to a constant term. For the cross section which is a doubly connected region, the number of constant terms is equal to two; and for their determination use is made of the theorem on the circulation of the tangential stresses in bending (reference 6).

It is assumed that in the region OABC the function $F(x,y)$ assumes the value $F_1(x,y)$ and in the region ODEG the value $F_2(x,y)$.

It is ideal to obtain the function $F_i(x,y)$ $(i = 1,2)$ in the form

$$F_i(x,y) = \Psi_i(x,y) + \Phi_i(x,y) \qquad (i = 1,2) \tag{8.3}$$

where the functions $\Phi_i(x,y)$ $(i = 1,2)$ exist only in the region OAFG, the functions $\Psi_1(x,y)$ exist in the region OABC, and $\Psi_2(x,y)$ in the region ODEG.

For the auxiliary functions $\Psi_i(x,y)$ and $\Phi_i(x,y)$ $(i = 1,2)$, setting $f(y) = 0$, the following equations are obtained:

$$\nabla^2\Psi_i = K(y-b) \qquad \nabla^2\Phi_i = 0 \qquad (i = 1,2) \qquad \left(K = \frac{P}{I}\frac{\nu}{1+\nu}\right) \tag{8.4}$$

where the following conditions must be satisfied:

$$\Psi_1(x,0) = \left(\frac{\partial\Psi_1}{\partial x}\right)_{x=a} = \Psi_1(0,y) + \Phi_1(0,y) = 0 \qquad \Psi_1(x,d_1) = C_1 \tag{8.5}$$

$$\Psi_2(x,b) = \Psi_2(0,y) = \Psi_2(x,0) + \Phi_2(x,0) = 0$$

$$\Psi_2(d_2,y) = \frac{P}{2I}(2a-d_2)(b-y)d_2 \tag{8.6}$$

By the method described in the first section set

$$\Psi_1(x,y) = \sum_{k=1}^{\infty} f_k(x)\sin\frac{k\pi y}{d_1} \qquad \Psi_2(x,y) = \sum_{k=1}^{\infty} v_k(y)\sin\frac{k\pi x}{d_2} \tag{8.7}$$

Then

$$\Phi_1(d_2,y) = \left(\frac{\partial\Phi_1}{\partial x}\right)_{x=d_2} = \Phi_1(x,0) = 0, \; \Phi_1(x,d_1) = \sum_{k=1}^{\infty} v_k(d_1)\sin\frac{k\pi x}{d_2} - C_1$$

$$\Phi_2(x,d_1) = \left(\frac{\partial\Phi_2}{\partial y}\right)_{y=d_1} = \Phi_2(0,y) = 0 \tag{8.8}$$

$$\Phi_2(d_2,y) = \sum_{k=1}^{\infty} f_k(d_2) \sin\frac{k\pi y}{d_1} - \frac{P}{2I}(2a-d_2)(b-y)d_2 \tag{8.9}$$

It is ideal to obtain the functions $\Phi_1(x,y)$ and $\Phi_2(x,y)$ in the form

$$\Phi_1(x,y) = \sum_{k=1}^{\infty} \Phi_k(x) \sin\frac{k\pi y}{d_1} \qquad \Phi_2(x,y) = \sum_{k=1}^{\infty} w_k(y) \sin\frac{k\pi x}{d_2} \tag{8.10}$$

Equations (8.4) to (8.10) determine $F(x,y)$ in the region ODEFBC.

The constant C_1 is determined with the aid of the theorem on the circulation of the tangential stresses in bending for the inner contour of the section:

$$\int_{d_2}^{a} \left(\frac{\partial F_1}{\partial y}\right)_{y=d_1} dx + \int_{d_1}^{b} \left(\frac{\partial F_2}{\partial x}\right)_{x=d_2} dy = 0 \tag{8.11}$$

9. Determination of the bending stresses. - The solution of the equations of this problem is analogous to the solution of the equations of the problem of the torsion of a rod.

With the omission of the computations the values of the obtained stresses are:

for the region OAFG

$$X_z(x,y) = \frac{P}{2I}(2a-x)x + \frac{P}{6I}\frac{\nu}{1+\nu}(3y^2-d_1^{\,2}) - \frac{P}{2I}\frac{\nu}{1+\nu}(2y-d_1)b +$$

$$\frac{\pi}{d_1}\sum_{k=1}^{\infty} kB_k \operatorname{sch}\frac{k\pi a}{d_1} \operatorname{ch}\frac{k\pi(a-x)}{d_1} \cos\frac{k\pi y}{d_1} +$$

$$\frac{\pi}{d_2}\sum_{k=1}^{\infty} kv_k(d_1) \sin\frac{k\pi x}{d_2} \operatorname{csch}\frac{k\pi d_1}{d_2} \operatorname{ch}\frac{k\pi y}{d_2} -$$

(continued on following page)

$$\frac{2C_1}{d_1}\sum_{k=1}^{\infty}(-1)^k \operatorname{ch}\frac{k\pi(x-d_2)}{d_1}\cos\frac{k\pi y}{d_1} +$$

$$\frac{2}{d_2}\sum_{k=1}^{\infty}k(-1)^k \operatorname{sh}\frac{k\pi(x-d_2)}{d_1}\cos\frac{k\pi y}{d_1}\sum_{p=1}^{\infty}\frac{v_p(d_1)\ p(-1)^p}{(pd_1/d_2)^2 + k^2}$$

$$Y_z(x,y) = \frac{\pi}{d_1}\sum_{k=1}^{\infty}kB_k \operatorname{sch}\frac{k\pi a}{d_1}\operatorname{sh}\frac{k\pi(a-x)}{d_1}\sin\frac{k\pi y}{d_1} -$$

$$\frac{\pi}{d_2}\sum_{p=1}^{\infty}pv_p(d_1)\cos\frac{p\pi x}{d_2}\operatorname{csh}\frac{p\pi d_1}{d_2}\operatorname{sh}\frac{p\pi y}{d_2} -$$

$$\frac{2}{d_2}\sum_{p=1}^{\infty}k(-1)^k \operatorname{ch}\frac{k\pi(x-d_2)}{d_1}\sin\frac{k\pi y}{d_1}\sum_{k=1}^{\infty}\frac{v_p(d_1)\ p(-1)^p}{(pd_1/d_2)^2 + k^2} +$$

$$\frac{2C_1}{d_1}\sum_{k=1}^{\infty}(-1)^k \operatorname{sh}\frac{k\pi(x-d_2)}{d_1}\sin\frac{k\pi y}{d_1} \qquad (9.1)$$

for the region GFBC

$$X_z(x,y) = \frac{P}{2I}\ (2a-x)x + \frac{C_1}{d_1} + \frac{P}{6I}\frac{\nu}{1+\nu}\ (3y^2-d_1^2) -$$

$$\frac{P}{2I}\frac{\nu}{1+\nu}\ (2y-d_1)b + \frac{\pi}{d_1}\sum_{k=1}^{\infty}kB_k \operatorname{sch}\frac{k\pi a}{d_1}\operatorname{ch}\frac{k\pi(a-x)}{d_1}\cos\frac{k\pi y}{d_1}$$

$$Y_z(x,y) = \frac{\pi}{d_1} \sum_{k=1}^{\infty} kB_k \operatorname{sch} \frac{k\pi a}{d_1} \operatorname{sh} \frac{k\pi(a-x)}{d_1} \sin \frac{k\pi y}{d_1} \tag{9.2}$$

for the region AFED

$$X_z(x,y) = \frac{P}{2I} (2a-x)x - \frac{\pi}{d_2} \sum_{k=1}^{\infty} \left\{ kD_k + (-1)^k \frac{d_2 b}{\pi} \left[\frac{P}{I} (2a-d_2) + \frac{2P}{I} \frac{\nu}{1+\nu} \frac{d_2}{(k\pi)^2} \left(1 + (-1)^{k+1}\right) \right] \right\} \operatorname{csch} \frac{k\pi b}{d_2} \operatorname{ch} \frac{k\pi(b-y)}{d_2} \sin \frac{k\pi x}{d_2} - \frac{P}{2I} (2a-d_2)x - \frac{P}{2I} \frac{\nu}{1+\nu} (d_2-x)x$$

$$Y_z(x,y) = -\frac{\pi}{d_2} \sum_{k=1}^{\infty} \left\{ kD_k + (-1)^k \frac{d_2 b}{\pi} \left[\frac{P}{I} (2a-d_2) + \frac{2P}{I} \frac{\nu}{1+\nu} \frac{d_2}{(k\pi)^2} \left(1 + (-1)^{k+1}\right) \right] \right\} \operatorname{csch} \frac{k\pi b}{d_2} \operatorname{sh} \frac{k\pi(b-y)}{d_2} \cos \frac{k\pi x}{d_2} - \frac{P}{2I} (2a-d_2)(b-y) - \frac{P}{2I} \frac{\nu}{1+\nu} (b-y)(d_2-2x) \tag{9.3}$$

In these relations the constants of integration B_k and D_k are determined from the infinite completely regular system of linear equations.

Translated by S. Reiss
National Advisory Committee
for Aeronautics

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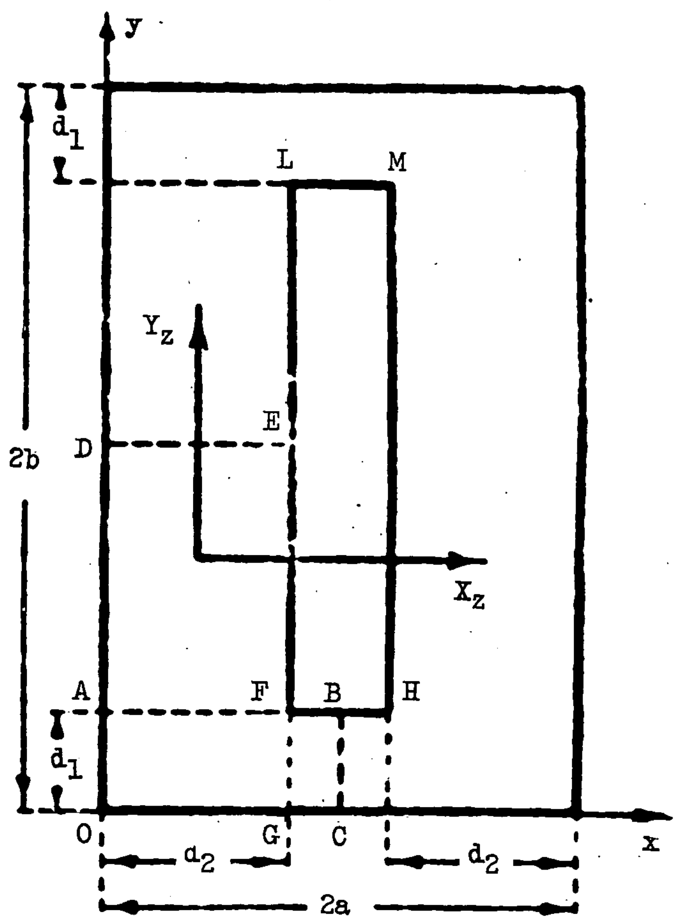

Figure 1.

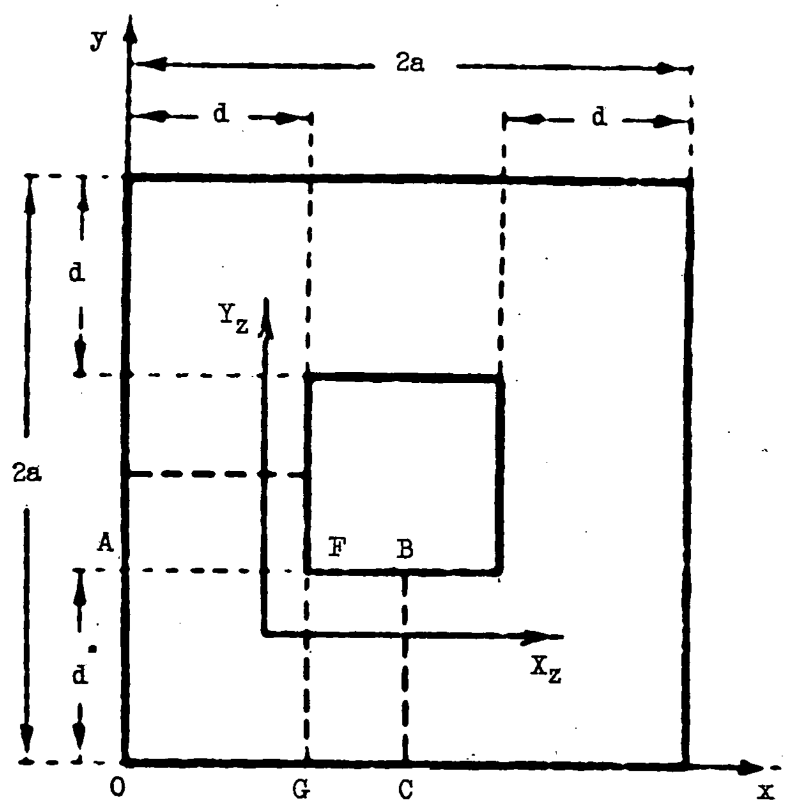

Figure 2.

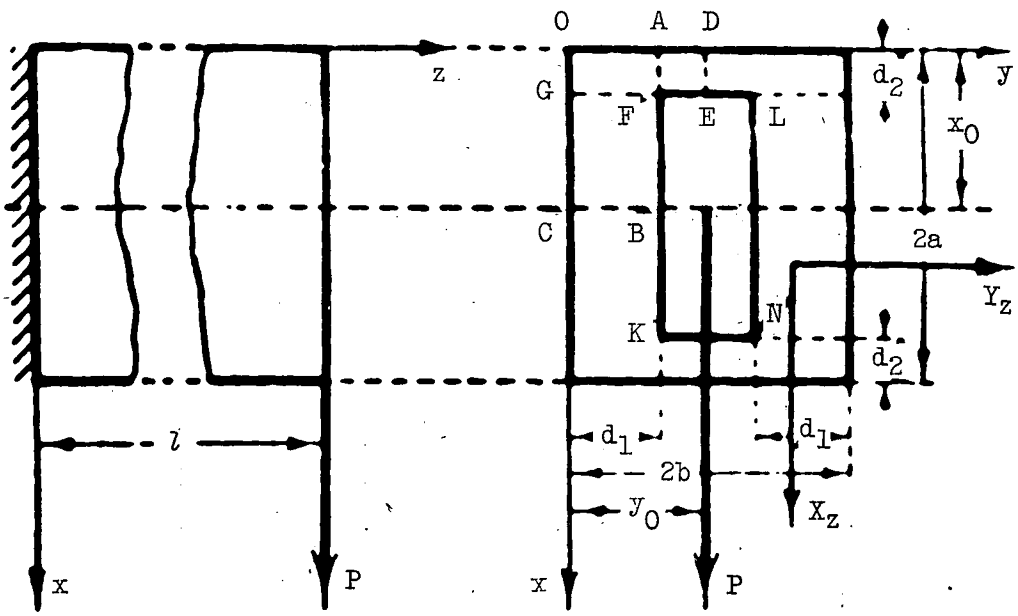

Figure 3.

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