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ON THE THEORY OF THIN AND THIN-WALLED RODS*

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Through the work of V. Z. Vlasov (reference 1) a theory of thin-walled rods has been established that is widely applicable in practice. This theory was extended by A. A. Umanski (reference 2) to thin-walled rods of closed profile section. The authors based their work on the concepts of the modern theory of shells.

An attempt is made herein to construct a theory of thin- walled rods including the classical theory of deformation of thin rods by making use of a kinematic assumption.

1. Fundamental kinematic assumption. - A general assumption that may be made for the construction of a theory of thin and thin-walled rods is first presented. In the entire volume of the rod, the elongations ε_x and ε_y and the shear γ_{xy} are assumed to become zero. In the case of a thin-walled rod, this assumption is based on an analysis of the solution of the problem of Saint-Venant, Michell, and Almansi. In that case

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 0 \\ \frac{\partial v}{\partial y} &= 0 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= 0 \end{aligned}\right\} \qquad (1.1)$$

Whence, integrating the first two relations yields $u = u(y,z)$ and $v = v(x,z)$; and on the basis of $v = v(x,z)$ the following equations are obtained

*K Teorii Tonkikh i Tonkostennykh Sterzhnei. Prikl. Mat. i Mekh., Vol. XIII, Nov.-Dec. 1949, pp. 597-608.

$$\left.\begin{aligned} u &= u_0(z) - \vartheta(z)y \\ v &= v_0(z) + \vartheta(z)x \end{aligned}\right\} \tag{1.2}$$

where u_0, v_0, and $\delta(z)$ are arbitrary functions of the coordinate z.

An assumption of a more special character than equations (1.1) is made, namely that, in referring the cross sections of the rods to the principal axes of inertia

$$\left.\begin{aligned} \varepsilon_x = \varepsilon_y = \gamma_{xy} &= 0 \\ \gamma_{xz} &= \tau\left(\frac{\partial\varphi}{\partial x} - y\right) \\ \gamma_{yz} &= \tau\left(\frac{\partial\varphi}{\partial y} + x\right) \\ \varepsilon_z = \varepsilon + \varkappa_1 y - \varkappa_2 x &+ \varphi(x,y)\dot{\tau} \qquad \left(\dot{\tau} = \frac{\partial\tau}{\partial z}\right) \end{aligned}\right\} \tag{1.3}$$

where $\varkappa_1(z)$ and $\varkappa_2(z)$ are the curvatures of the deformed axis of the rod, $\tau(z)$ is the angle of torsion per unit length, ε is the relative elongation, and $\varphi(x,y)$ is a function of the torsion satisfying the conditions

$$\left.\begin{aligned} \Delta\varphi &= 0 \\ \frac{\partial\varphi}{\partial n} &= y\cos(n,x) - x\cos(n,y) \end{aligned}\right\} \tag{1.4}$$

that is, the function $\varphi(x,y)$ is a solution of the problem of Neiman [NACA note: Neumann.] and is determined with an accuracy up to an arbitrary constant. The meaning of the magnitude ε changes, depending on the choice of this constant.

If the equation $\varphi(0,0) = 0$ is assumed as a normalizing condition, ε actually represents the relative elongation ε_z at the points of the axis of the rod. For another method of normalization where the constant entering $\varphi(x,y)$ is determined by the condition

$$\iint_\Omega \varphi(x,y)\,dx\,dy = 0$$

the magnitude ε is that part of the relative elongation that is produced by the external tensile forces.

In deriving the relations of Kirchhoff, it is natural to make use of the first method of normalizing; but in considering problems in which tensile forces are absent the second method is used. The relations (1.3) differ from the classical equations by the presence of the term $\varphi(x,y)\dot{\tau}$ in the expression for ε_z. The introduction of this term is based on the fact that for variable τ the term $\varphi(x,y)\dot{\tau}$ is necessary in order that the deformations (1.3) satisfy the continuity condition of Saint-Venant. This relation can easily be established if the expressions for γ_{xz} and γ_{yz} are assumed and the form of ε_z found by integrating the equations of continuity. Moreover, as has been shown by V. V. Novozhilov (reference 3), relations (1.3) are obtained on expanding the displacements u, v, and w in series in the coordinates x and y of the points of the cross section and these relations correspond to the second approximation.

The problem of the constrained torsion of prismatic rods was approximately solved by N. V. Zvolinskii (reference 4) by giving the variables in the form $u = -yf(z)$, $v = xf(z)$, and $w = \varphi(x,y)F(z)$ and by determining the functions $f(z)$ and $F(z)$ from the variational method. This form of specifying the displacements corresponds to a more general kinematic assumption than equations (1.3).

2. Scheme of derivation of relations between forces and kinematic characteristics. - The kinematic assumption of the preceding section is often supplemented by the assumption of the secondary character of the deformation ε. The force V_z is then obtained from the equations of statics and the equation connecting V_z and ε is dropped. For this reason in place of the complete system of equations for the theory of thin rods there is obtained from the 16 equations a system of 15 equations with 15 unknowns.

Twelve of these equations (six equations of statics and six equations of continuity) do not depend either on the form of the relation between the stresses and the deformations or on the character of the kinematic assumption. The character of the kinematic assumption shows up only in the relations between the curvature and the torsion on the one hand and the moments on the other.

The problem of the derivation of the relations between the 'generalized coordinates' $\varkappa_1$, $\varkappa_2$, τ, and ε and the 'generalized forces' M_x, M_y, M_z, and V_z is considered.

The elementary work of the generalized forces may be represented by the equation

$$\delta A = \int_0^L \left[V_z \delta\varepsilon + M_x \delta\varkappa_1 + M_y \delta\varkappa_2 + M_z \delta\tau\right] dz +$$

$$\int_0^L \iint_S p_z \delta w^* dS\, dz + \iint_\Omega \sigma_z^{\,o}\, \varphi(x,y) dx\, dy\, \delta\tau \Big|_{z=L} \qquad (2.1)$$

where L is the length of the rod, S the area of the lateral surface of the rod, $\sigma_z^{\,o}$ the normal stresses distributed along the end $z = L$, with $p_z(s,z)$ the components along the z-axis of the distributed lateral load, and w^* the difference of the displacements along the z-axis of an arbitrary point of the cross section and its center of gravity, that is, the relative displacement

$$\dot{\delta} w^* = y\delta\varkappa_1 - x\delta\varkappa_2 + \varphi\, \dot{\delta}\tau$$

Also, for simplification of the computations it is assumed that $p_z = 0$.

The equation determining the potential energy of the rod, after rejecting the terms containing the second-order stresses σ_x, σ_y, and τ_{xy}, is written in the form

$$\Pi = \frac{1}{2}\int_0^L \iint_\Omega \left[\frac{\sigma_z^{\,2}}{E} + \frac{\tau_{xz}^{\,2}}{\mu} + \frac{\tau_{yz}^{\,2}}{\mu}\right] dx\, dy\, dz \qquad (2.2)$$

where E is Young's modulus and μ is the shear molulus.

The assumption of some kinematic hypothesis permits expressing the potential energy in terms of the 'generalized coordinates'. When the variation of the potential energy $\delta\Pi$ is equated to the elementary work δA a relation is obtained from which, because of the arbitrariness of the variations of the generalized coordinates, the required relations are obtained.

3. Generalized relations of the theory of thin and thin-walled rods[1]. - The simplest form of Kirchhoff's generalized relations for an initially straight and untwisted rod is obtained on the basis of the kinematic assumption previously formulated and the assumption of the secondary character of the stresses σ_x, σ_y, and τ_{xy}.

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[1]A short presentation of the results of this section has been published in reference 5.

Assuming this hypothesis and making use of Hooke's law yields

$$\left.\begin{aligned}\sigma_z &= E\left[\varkappa_1 y - \varkappa_2 x + \varphi(x,y)\dot{\tau}\right]\\ \tau_{xz} &= \mu\left(\frac{\partial\varphi}{\partial x} - y\right)\tau\\ \tau_{yz} &= \mu\left(\frac{\partial\varphi}{\partial y} + x\right)\tau\end{aligned}\right\}\tag{3.1}$$

It has been assumed herein, that $\sigma_z = E\varepsilon_z$, that is, the problem has been restricted to establishing a relation between the principal stress σ_z and the principal strain ε_z.

When equations (3.1) are substituted in equation (2.2) an expression can be obtained for the potential energy of the rod in the form

$$\Pi = \frac{1}{2}\int_0^L\left[E\Omega\varepsilon^2 + EI_x\varkappa_1^{\ 2} + EI_y\varkappa_2^{\ 2} + EI_\varphi\dot{\tau}^2 + 2EI_{\varphi x}\varkappa_1\dot{\tau} - 2EI_{\varphi y}\varkappa_2\dot{\tau} + 2EI_{\varphi 1}\varepsilon\dot{\tau} + \mu T\tau^2\right]dz\tag{3.2}$$

where Ω is the area of the cross section, I_x and I_y are the moments of inertia, T is the geometric stiffness in free torsion, $I_{\varphi 1}$, I_φ, $I_{\varphi x}$, and $I_{\varphi y}$ are geometric characteristics defined by the equations

$$\left.\begin{aligned}I_{\varphi 1} &= \iint_\Omega \varphi\ (x,y)dx\ dy\\ I_\varphi &= \iint_\Omega \varphi^2\ (x,y)dx\ dy\\ I_{\varphi x} &= \iint_\Omega y\varphi\ (x,y)dx\ dy\\ I_{\varphi y} &= \iint_\Omega x\varphi\ (x,y)dx\ dy\end{aligned}\right\}\tag{3.3}$$

The integral I_φ was introduced by N. V. Zvolinskii (reference 4) and was computed by him for an ellipse, a rectangle, and an equilaterial triangle.

The variation of the potential energy will now be established.

$$\delta\Pi = \int_0^L \Big[E\Omega\varepsilon\delta\varepsilon + EI_x\varkappa_1\delta\varkappa_1 + EI_y\varkappa_2\delta\varkappa_2 + EI_\varphi\dot{\tau}\delta\dot{\tau} + EI_{\varphi x}\dot{\tau}\delta\varkappa_1 -$$

$$EI_{\varphi y}\dot{\tau}\delta\varkappa_2 + EI_{\varphi 1}\dot{\tau}\,\delta\varepsilon + \mu T\tau\delta\tau + EI_{\varphi x}\varkappa_1\delta\dot{\tau} - EI_{\varphi y}\varkappa_2\delta\dot{\tau} + EI_{\varphi 1}\varepsilon\delta\dot{\tau}\Big]dz$$

Integrating the terms containing $\delta\dot{\tau}$ by parts yields

$$\delta\Pi = \int_0^L \Bigg\{\Big[E\Omega\varepsilon + EI_{\varphi 1}\dot{\tau}\Big]\delta\varepsilon + \Big[EI_x\varkappa_1 + EI_{\varphi x}\dot{\tau}\Big]\delta\varkappa_1 +$$

$$\Big[EI_y\varkappa_2 - EI_{\varphi y}\dot{\tau}\Big]\delta\varkappa_2 + \Big[\mu T\tau - EI_\varphi\ddot{\tau} - EI_{\varphi x}\dot{\varkappa}_1 + EI_{\varphi y}\dot{\varkappa}_2 -$$

$$EI_{\varphi 1}\dot{\varepsilon}\Big]\delta\tau\Bigg\}dz + \Big[EI_\varphi\dot{\tau}\delta\tau + EI_{\varphi x}\varkappa_1\delta\tau - EI_{\varphi y}\varkappa_2\delta\tau + EI_{\varphi 1}\varepsilon\delta\tau\Big]_0^L \qquad (3.4)$$

According to the preceding statement $\delta\Pi = \delta A$. Equating equations (3.4) and (2.1) and comparing the coefficients therefore yields the generalized relations of Kirchhoff

$$\left.\begin{aligned} M_x &= EI_x\varkappa_1 + EI_{\varphi x}\dot{\tau} \\ M_y &= EI_y\varkappa_2 - EI_{\varphi y}\dot{\tau} \\ V_z &= E\Omega\varepsilon + EI_{\varphi 1}\dot{\tau} \\ M_z &= \mu T\tau - EI_\varphi\ddot{\tau} - EI_{\varphi x}\dot{\varkappa}_1 + EI_{\varphi y}\dot{\varkappa}_2 - EI_{\varphi 1}\dot{\varepsilon} \end{aligned}\right\} \qquad (3.5)$$

and the boundary condition at the end

$$\Big[EI_\varphi\dot{\tau} + EI_{\varphi x}\varkappa_1 - EI_{\varphi y}\varkappa_2 + EI_{\varphi 1}\varepsilon\Big]\delta\tau = 0 \qquad (3.6)$$

The supplementary terms with respect to the usual ones in equations (3.5) correspond to the following phenomena observed on rods with nonsymmetrical profile: (1) nonuniform torsion produces bending, (2) nonuniform bending produces torsion, (3) nonuniform elongation produces torsion, and (4) nonuniform torsion produces elongation.

In the expression for M_z, the term $EI_{\varphi y}\varkappa$ which is retained even in the case of a rod with symmetrical cross section, is due to the strict consideration of the loads distributed over the lateral surfaces of the rod and also could be connected with the effect of torsional constraint.

The boundary condition, equation (3.6) has an interesting structure. From equation (3.6) it follows that for arbitrary $\delta\tau$; that is, in the case where only the total forces and moments at the end are given, the following condition must be satisfied

$$B_\varphi = EI_\varphi\dot{\tau} + EI_{\varphi x}\varkappa_1 - EI_{\varphi y}\varkappa_2 + EI_{\varphi 1}\varepsilon = 0$$

Thus, at the end, the magnitude B_φ, which, following Vlasov, shall be called the bimoment, becomes zero. For other boundary conditions the bimoment at the end is not zero. In these cases, the expression for δA contains an additional term corresponding to the work of the stresses distributed at the end.

The concept of the bimoment, while it is not necessary, is nevertheless very convenient, a fact that explains its wide adoption in the literature on the theory of thin-walled rods. Hence, the bimoment may also be introduced early in the theory.

The integral of equations (3.3) can be written in the form

$$\delta\Pi = \int_0^L \left[V_z\delta\varepsilon + M_x\delta\varkappa_1 + M_y\delta\varkappa_2 + M_z{}^*\delta\tau + B_\varphi\delta\dot{\tau}\right]dz \tag{3.7}$$

where V_z, M_x, and M_y are connected with the kinematic characteristics of the previous relations and the magnitudes $M_z{}^*$ and B_φ are determined by the equations

$$\left.\begin{aligned} M_z{}^* &= \mu T\tau \\ B_\varphi &= EI_\varphi\dot{\tau} + EI_{\varphi x}\varkappa_1 - EI_{\varphi y}\varkappa_2 + EI_{\varphi 1}\varepsilon \end{aligned}\right\} \tag{3.8}$$

It is shown in equation (3.7) that the bimoment B_φ is a generalized force corresponding to the generalized coordinate $\dot{\tau}$, with $M_z{}^*$ being the generalized force corresponding to the coordinate τ (considering τ and $\dot{\tau}$ as independent parameters).

Transforming equation (3.7) by integration by parts yields

$$\delta\Pi = \int_0^L \left[V_z\delta\varepsilon + M_x\delta\varkappa_1 + M_y\delta\varkappa_2 + (M_z{}^* - \dot{B}_\varphi)\delta\tau\right]dz + \left[B_\varphi\delta\tau\right]_0^L$$

that is,

$$M_z = M_z{}^* - \frac{dB_\varphi}{dz} \tag{3.9}$$

which agrees with the corresponding equation (3.5).

4. Determination of the position of the center of stiffness. - In the consideration of the problem of the bending of a cantilever prismatic rod by the forces R_x and R_y applied at the end, $x = x_0$ and $y = y_0$. From the equations of equilibrium of the rod it follows that in the projections of the fixed axes

$$\left.\begin{aligned} V_x &= R_x \\ V_y &= R_y \\ V_z &= 0 \\ M_x &= -R_y(L - z) \\ M_y &= R_x(L - z) \\ M_z &= x_0R_y - y_0R_z \end{aligned}\right\} \tag{4.1}$$

The substitution of equations (4.1) in Kirchhoff's generalized relations yields

$$\left.\begin{aligned} EI_x\varkappa_1 + EI_{\varphi x}\dot{\tau} &= -R_y(L - z) \\ EI_y\varkappa_2 - EI_{\varphi y}\dot{\tau} &= R_x(L - z) \end{aligned}\right\} \tag{4.2}$$

$$\mu T\tau - EI_\varphi\ddot{\tau} - EI_{\varphi x}\dot{\varkappa}_1 + EI_{\varphi y}\dot{\varkappa}_2 = x_0R_y - y_0R_x \tag{4.3}$$

Solving equation (4.2) for $\varkappa_1$ and $\varkappa_2$ and substituting these values in equation (4.3) yields the differential equation for determining the angle of torsion per unit length τ:

$$\mu T\tau - E\left(I_{\varphi} - \frac{I_{\varphi x}^{2}}{I_x} - \frac{I_{\varphi y}^{2}}{I_y}\right)\ddot{\tau} = \left(x_0 + \frac{I_{\varphi x}}{I_x}\right)R_y - \left(y_0 - \frac{I_{\varphi y}}{I_y}\right)R_x \tag{4.4}$$

The right side of equation (4.4) becomes zero if the forces R_x and R_y are applied at the point with coordinates

$$\left.\begin{aligned} x_c &= -\frac{I_{\varphi x}}{I_x} \\ y_c &= \frac{I_{\varphi y}}{I_y} \end{aligned}\right\} \tag{4.4}$$

In this case the forces R_x and R_y do not produce torsion of the rod. In other words, the point x_c, y_c is the center of stiffness of the rod. It is important to note that for the determination of the coordinates of the center of stiffness, within the scope of the approximate theory, here developed, it is sufficient to know only the solution of the problem of the free torsion of a section.

For an accurate solution of the problem of the theory of elasticity used in the determination of the center of stiffness, the solution of the problem of rod bending is a necessity. In computing the coordinates of the center of stiffness

$$\left.\begin{aligned} x_c &= -\frac{1}{I_x}\iint_{\Omega} y\varphi \, dx \, dy \\ y_c &= \frac{1}{I_y}\iint_{\Omega} x\varphi \, dx \, dy \end{aligned}\right\} \tag{4.5}$$

the coordinates x and y must intersect at the center of stiffness of the section.

<u>5. Special formulas obtained from the general theory.</u> - The classical equations of Kirchhoff's theory are obtained by rejecting the terms containing I_{φ}, $I_{\varphi 1}$, $I_{\varphi x}$, $I_{\varphi y}$ from the fundamental equations of the general theory; thus the usual theory of thin rods corresponds to neglecting the effect of the torsional restraint and the additional effect of the loads distributed along the rod.

The fundamental equations of V. Z. Vlasov's theory (reference 1) are considered now. In order to derive these equations, the approximate expression of the torsional function for thin-walled rods, obtained by considering a curved section as an aggregate of rectangular ones, is used. If the rod is referred to the coordinates s and n (where s is the coordinate measured along the center line of the section and n along the normal to it) and solving the differential equation of torsion yields

$$\varphi(s,n) = -\omega(s) - n\,(xx' + yy') + \varphi_0 \tag{5.1}$$

where $x = x(s)$ and $y = y(s)$ are the parametric equations of the center line of the section; $\omega(s)$ is the sectorial area:

$$\omega(s) = \int_0^s (xy' - yx')ds$$

In the problem of the torsion of the rods, the constant φ_0 is determined from the condition

$$\iint_\Omega \sigma_z \, dx\, dy = 0 \tag{5.2}$$

Then

$$\iint_\Omega \sigma_z \, dx\, dy = E\dot{\tau} \iint_\Omega \varphi(s,n) \left(1 - \frac{n}{\rho}\right) ds\, dn = 0$$

whence, with an accuracy to terms of the order of n/ρ, where ρ is the radius of curvature of the center line yields

$$\varphi_0 = \frac{1}{\Omega}\int_0^l \omega(s)\, h(s)\, ds \tag{5.3}$$

where Ω is the area of the section and l the length of the center line of the section.

The principal sectorial area is introduced

$$\omega^*(s) = \omega(s) - \frac{1}{\Omega}\int_0^l \omega(s)\, h(s)\, ds$$

The term $\varphi(s,n)$ is expressed in terms of $\omega^*(s)$:

$$\varphi(s,n) = -\omega^*(s) - n(xx' + yy') \qquad (x' = dx/ds) \tag{5.4}$$

Then the computation of the integral characteristics I_φ, $I_{\varphi x}$ and $I_{\varphi y}$ is performed. Substituting equation (5.4) in expression (3.3) for I_φ yields

$$I_\varphi = \int_0^l \int_{-\frac{1}{2}h}^{\frac{1}{2}h} \left[\omega^*(s) + n(xx' + yy')\right]^2 ds\ dn$$

that is,

$$I_\varphi = \int_0^l \omega^{*2}(s)h(s)ds + \frac{1}{12}\int_0^l (xx' + yy')h^3(s)ds$$

Rejecting the second integral as a magnitude of higher order smallness yields

$$I_\varphi = \int_0^l \omega^{*2}(s)h(s)ds \tag{5.5}$$

which agrees with the expression for the sectorial moment of inertia I_ω given by Vlasov.

TABLE I

General theory of solid thin and thin-walled rods	V.Z.Vlasov's theory of thin-walled rods with open profile section	A.A.Umanskii's theory of thin-walled rods with closed profile section
$\tau(z)$	$\tau(z)$	$\tau(z)$
$\varphi(x,y)$	$-\omega^*(s)$	$-\omega^o(s)$
$w = \varphi(x,y)\tau$	$w = -\omega^*(s)\tau$	$w = -\omega^o(s)\tau$
$\sigma_z = E\varphi(x,y)\dot{\tau}$	$\sigma_z = -E\omega^*(s)\dot{\tau}$	$\sigma_z = -E\omega^o(s)\dot{\tau}$
$I_\varphi = \iint_\Omega \varphi^2(x,y)dx\,dy$	$I_\omega{}^* = \int_0^l \omega^{*2}(s)h(s)ds$	$I_\omega{}^o = \oint \omega^{o2}(s)h(s)ds$
$I_{\varphi x} = \iint_\Omega y\varphi(x,y)dx\,dy$	$I_{\omega x}{}^* = \int_0^l y\omega^*(s)h(s)ds$	$I_{\omega x}{}^o = \oint y\omega^o(s)h(s)ds$
$I_{\varphi y} = \iint_\Omega x\varphi(x,y)dx\,dy$	$I_{\omega y} = \int_0^l x\omega^*(s)h(s)ds$	$I_{\omega y}{}^o = \oint x\omega^o(s)h(s)ds$
$\mu T\tau - EI_\varphi\ddot{\tau} = M_z$	$\mu T\tau - EI_\omega{}^*\ddot{\tau} = M_z$	$\mu T\tau - EI_\omega{}^o\ddot{\tau} = M_z$
$B_\varphi = EI_\varphi\dot{\tau}$	$B_\omega = -EI_\omega{}^*\dot{\tau}$	$B_\omega = -EI_\omega{}^o\dot{\tau}$
$\ddot{B}_\varphi - \frac{\mu T}{EI_\varphi}B_\varphi = -\frac{dM_z}{dz}$	$\ddot{B}_\omega - \frac{\mu T}{EI_\omega{}^*}B_\omega = -\frac{dM_z}{dz}$	$\ddot{B}_\omega - \frac{\mu T}{EI_\omega{}^o} = -\frac{dM_z}{dz}$
$x_c = -\frac{I_{\varphi x}}{I_x}$	$x_c = \frac{I_{\omega x}{}^*}{I_x}$	$x_c = \frac{I_{\omega x}{}^o}{I_x}$
$y_c = \frac{I_{\varphi y}}{I_y}$	$y_c = -\frac{I_{\omega x}{}^*}{I_y}$	$y_c = -\frac{I_{\omega y}{}^o}{I_y}$

Thus, it has been shown that the characteristic I_φ is the generalized concept of the sectorial moment of inertia.

Substituting equation (5.5) in equation (3.7) the fundamental formula of Vlasov's theory[2] is

$$M_z = \mu T\tau - I_\omega \ddot{\tau} \tag{5.6}$$

The computation of the integrals $I_{\varphi x}$ and $I_{\varphi y}$ yields

$$I_{\varphi x} = \iint_\Omega y\varphi(x,y)dx\,dy = \iint_\Omega (y + nx')\varphi(s,n)\left(1 - \frac{n}{\rho}\right) ds\,dn$$

$$I_{\varphi y} = \iint_\Omega x\varphi(x,y)dx\,dy = \iint_\Omega (x - ny')\varphi(s,n)\left(1 - \frac{n}{\rho}\right) ds\,dn$$

where ρ is the radius of curvature of the center line.

The substitution in these expressions of the value of $\varphi(s,n)$ after rejecting terms of the order of h^3 in comparison with terms of the order of h yields

$$\left.\begin{aligned} I_{\varphi x} &= -\int_0^l y\omega^*(s)h(s)ds \\ I_{\varphi y} &= -\int_0^l x\omega^*(s)h(s)ds \end{aligned}\right\} \tag{5.7}$$

The approximate expressions for the coordinates of the center of stiffness will then be

$$x_c = \frac{1}{I_x}\int_0^l y\omega^*(s)h(s)ds$$

$$y_c = -\frac{1}{I_y}\int_0^l x\omega^*(s)h(s)ds$$

[2]For simplicity in deriving certain formulas in this section it is assumed that the rod has two axes of symmetry.

These formulas likewise agree with the corresponding relations of Vlasov's theory.

In a similar manner there can be obtained also the approximate formulas in the case of thin-walled rods with closed profile section. In fact, determining the constant φ_0 in the expression for the function of the stresses $\varphi(s,n)$ for closed sections yields

$$\varphi(s,n) = -\omega(s) + \frac{s}{l}\omega(l) - n(xx' + yy') + \varphi_0 \tag{5.8}$$

from condition (5.2)

$$\varphi_0 = \frac{1}{\Omega}\left(\oint \omega(s)h(s)ds - \frac{\omega(l)}{l}\oint s\, h(s)ds\right)$$

The introducing of the principal sectorial area $\omega^o(s)$ permits writing equation (5.8) in the form

$$\varphi(s,n) = -\omega^o(s) - n(xx' + yy') \tag{5.9}$$

With the same degree of accuracy as in the case of open profile sections then

$$I_\varphi{}^o = \oint \omega^{o2}(s)h(s)ds \tag{5.10}$$

$$\left.\begin{aligned} x_c{}^o &= \frac{1}{I_x}\oint y\,\omega^o(s)h(s)ds \\ y_c{}^o &= \frac{1}{I_y}\oint x\,\omega^o(s)h(s)ds \end{aligned}\right\} \tag{5.11}$$

and equation (3.7) for M_z assumes the form:

$$M_z = \mu T\tau - EI_\varphi \ddot{\tau} \tag{5.12}$$

Equations (5.10) and (5.11) agree with the relations assumed by A. A. Umanskii in the theory of thin-walled rods with closed profile section although they correspond only to the so-called first variant of the theory of Umanskii. The second (more accurate) variant of Umanskii's theory is a particular case of a more general theory than the one here developed, namely, the theory based on a kinematic assumption containing two unknown functions $\tau(z)$ and $\zeta(z)$. The corresponding computations shall not be discussed.

Comparison of the differential equations of torsion of the general theory of thin and thin-walled rods with the corresponding equations of the theory of thin-walled rods permits the establishment of a far-reaching analogy between the two. This analogy becomes very clear from the comparison given in table I of the formulas of these theories. The analogy extends also to the boundary conditions and permits, for a corresponding change in the magnitudes, the application of the results obtained for thin-walled rods to the general theory of thin and thin-walled rods. Special problems may therefore be considered in detail and reference made to the monographs (reference 6).

6. Free flexural-torsional vibrations of rods. - The essential feature of the general theory of thin and thin-walled rods is that the accuracy of the determination of the coordinates of the center of stiffness is adequate for the accuracy of the equations connecting the forces and moments with the kinematic characteristics. For example, within the frame of the general theory, equations for the flexural-torsional vibrations of the rods may therefore be obtained in which the accuracy of the determination of the coordinates of the center of stiffness will correspond to the accuracy of the equations. Moreover, these equations will differ from the usual ones by the presence of additional terms corresponding to the torsional constraint and to the consideration of the distributed loads along the rod.

The derivation of such a system of equations is briefly considered herein. The principle of Ostrogradski-Hamilton is used. The expression for the kinetic energy has the form:

$$T = \frac{1}{2}\int_0^L \left\{ \rho\left(\frac{\partial u}{\partial t}\right)^2 + \rho\left(\frac{\partial v}{\partial t}\right)^2 + J\left(\frac{\partial \vartheta}{\partial t}\right)^2 \right\} dz \tag{6.1}$$

where ρ is the linear density, J the moment of inertia of a section relative to the z-axis, u and v the displacements along the x- and y-axis, respectively, ϑ the angle of rotation of a section; that is

$$\partial\vartheta/\partial z = \tau$$

For the potential energy of the rod under consideration according to equation (3.2)

$$\Pi = \frac{1}{2}\int_0^L \left\{ EI_y\left(\frac{\partial^2 u}{\partial z^2}\right)^2 + EI_x\left(\frac{\partial^2 v}{\partial z^2}\right)^2 + EI_\varphi\left(\frac{\partial^2 \vartheta}{\partial z^2}\right)^2 - 2EI_{\varphi x}\frac{\partial^2 v}{\partial z^2}\frac{\partial^2 \vartheta}{\partial z^2} - 2EI_{\varphi y}\frac{\partial^2 u}{\partial z^2}\frac{\partial^2 \vartheta}{\partial z^2} + \mu T\left(\frac{\partial \vartheta}{\partial z}\right)^2 \right\} dz \tag{6.2}$$

Substituting equations (6.1) and (6.2) in the expression for the Ostrogradski-Hamilton principle yields

$$\delta \int_{t_1}^{t_2} (T - \Pi)\, dz = 0$$

Taking the variation of the obtained expression and assuming, for simplicity of computation, u = 0 (this corresponds to the omission of the transverse vibrations along the y-axis) then

$$\int_{t_1}^{t_2} \int_0^L \left\{ \rho \frac{\partial v}{\partial t} \delta \frac{\partial v}{\partial t} + J \frac{\partial \vartheta}{\partial t} \delta \frac{\partial \vartheta}{\partial t} - EI_x \frac{\partial^2 v}{\partial z^2} \delta \frac{\partial^2 v}{\partial z^2} - EI_\varphi \frac{\partial^2 \vartheta}{\partial z^2} \delta \frac{\partial^2 \vartheta}{\partial z^2} + \right.$$

$$\left. EI_{\varphi x} \frac{\partial^2 v}{\partial z^2} \delta \frac{\partial^2 \vartheta}{\partial z^2} + EI_{\varphi x} \frac{\partial^2 \vartheta}{\partial z^2} \delta \frac{\partial^2 v}{\partial z^2} - \mu T \frac{\partial \vartheta}{\partial z} \delta \frac{\partial \vartheta}{\partial z} \right\} dz\, dt = 0 \tag{6.3}$$

Transforming equation (6.3) by integrating by parts yields an expression from which, on account of the arbitrariness of the variations, the required equations for the flextural-torsional vibrations are obtained

$$\rho \frac{\partial^2 v}{\partial t^2} + EI_x \frac{\partial^4 v}{\partial z^4} - EI_{\varphi x} \frac{\partial^4 \vartheta}{\partial z^4} = 0 \tag{6.4}$$

$$J \frac{\partial^2 \vartheta}{\partial t^2} - \mu T \frac{\partial^2 \vartheta}{\partial z^2} - EI_{\varphi x} \frac{\partial^4 v}{\partial z^4} + EI_\varphi \frac{\partial^4 \vartheta}{\partial z^4} = 0 \tag{6.5}$$

with the corresponding boundary conditions

$$\left.\begin{aligned}
\left| - \frac{\partial}{\partial z}\left(EI_x \frac{\partial^2 v}{\partial z^2}\right) + \frac{\partial}{\partial z}\left(EI_{\varphi x} \frac{\partial^2 v}{\partial z^2}\right) \right| \delta v &= 0 \\
\left| \frac{\partial}{\partial z}\left(\mu T \frac{\partial \vartheta}{\partial z}\right) - \frac{\partial}{\partial z}\left(EI_\varphi \frac{\partial^2 \vartheta}{\partial z^2}\right) + \frac{\partial}{\partial z}\left(EI_{\varphi x} \frac{\partial^2 v}{\partial z^2}\right) \right| \delta \vartheta &= 0 \\
\left| EI_x \frac{\partial^2 v}{\partial z^2} - EI_{\varphi x} \frac{\partial^2 \vartheta}{\partial z^2} \right| \delta \frac{\partial v}{\partial z} &= 0 \\
\left| EI_\varphi \frac{\partial^2 \vartheta}{\partial z^2} - EI_{\varphi x} \frac{\partial^2 v}{\partial z^2} \right| \delta \frac{\partial \vartheta}{\partial z} &= 0
\end{aligned}\right\} \tag{6.6}$$

Equations (6.4) and (6.5) differ from those usually applied not only in their structure but also in the choice of variables; that is, in the usual equations the displacement of the center of stiffness v_0 figures as the unknown variable and not the displacement of the center of gravity v as in the case in equations (6.4) and (6.5).

The relation between v and v_0 is given by the equation

$$v = v_0 - \vartheta x_c$$

where x_c is the coordinate of the center of stiffness. The replacement of v by v_0 in equations (6.4) and (6.5) leads to a system that, by the use of transformations and the relation $x_c = -I_{\varphi x}/I_x$, assumes the form

$$\rho \frac{\partial^2 v_0}{\partial t^2} - \rho x_c \frac{\partial^2 \vartheta}{\partial t^2} + EI_x \frac{\partial^4 v_0}{\partial z^4} = 0 \tag{6.7}$$

$$J_m \frac{\partial^2 \vartheta}{\partial t^2} - \mu T \frac{\partial^2 \vartheta}{\partial z^2} - \rho x_c \frac{\partial^2 v_0}{\partial t^2} + E\left[I_\varphi - x_c^2 I_x\right] \frac{\partial^4 \vartheta}{\partial z^4} = 0 \tag{6.8}$$

where $J_m = J + \rho x_c^2$ is the moment of inertia of the cross section relative to the axis passing through the center of stiffness parallel to the x-axis.

Equation (6.7) agrees entirely with the classical form but equation (6.8) differs by an additional term containing $\partial^4 \vartheta/\partial z^4$.

In conclusion, the boundary conditions in the variables v_0 and v are

$$-\frac{\partial}{\partial z}\left(EI_x \frac{\partial^2 v_0}{\partial z^2}\right)(\delta v_0 - x_c \delta \vartheta) = 0 \tag{6.9}$$

$$\left|\frac{\partial}{\partial z}\left(\mu T \frac{\partial \vartheta}{\partial z}\right) - \frac{\partial}{\partial z}\left[E(I_\varphi - x_c^2 I_x)\frac{\partial^2 \vartheta}{\partial z^2} - \frac{\partial}{\partial z}\left(EI_x x_c \frac{\partial^2 v_0}{\partial z^2}\right)\right]\right| \delta\vartheta = 0 \tag{6.10}$$

$$EI_x \frac{\partial^2 v_0}{\partial z^2} \delta\left[\frac{\partial v_0}{\partial z} - x_c \frac{\partial \vartheta}{\partial z}\right] = 0 \tag{6.11}$$

$$\left|E(I_\varphi - x_c^2 I_x)\frac{\partial^2 \vartheta}{\partial z^2} + EI_x x_c \frac{\partial^2 v_0}{\partial z^2}\right| \delta \frac{\partial \vartheta}{\partial z} = 0 \tag{6.12}$$

The solution of the equations (6.7) and (6.8) for the boundary conditions given by equations (6.9) to (6.12) is carried out by the usual exact or approximate methods.

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