

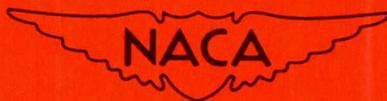
# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

**TECHNICAL MEMORANDUM 1301**

THE FLOW OF GASES IN NARROW CHANNELS

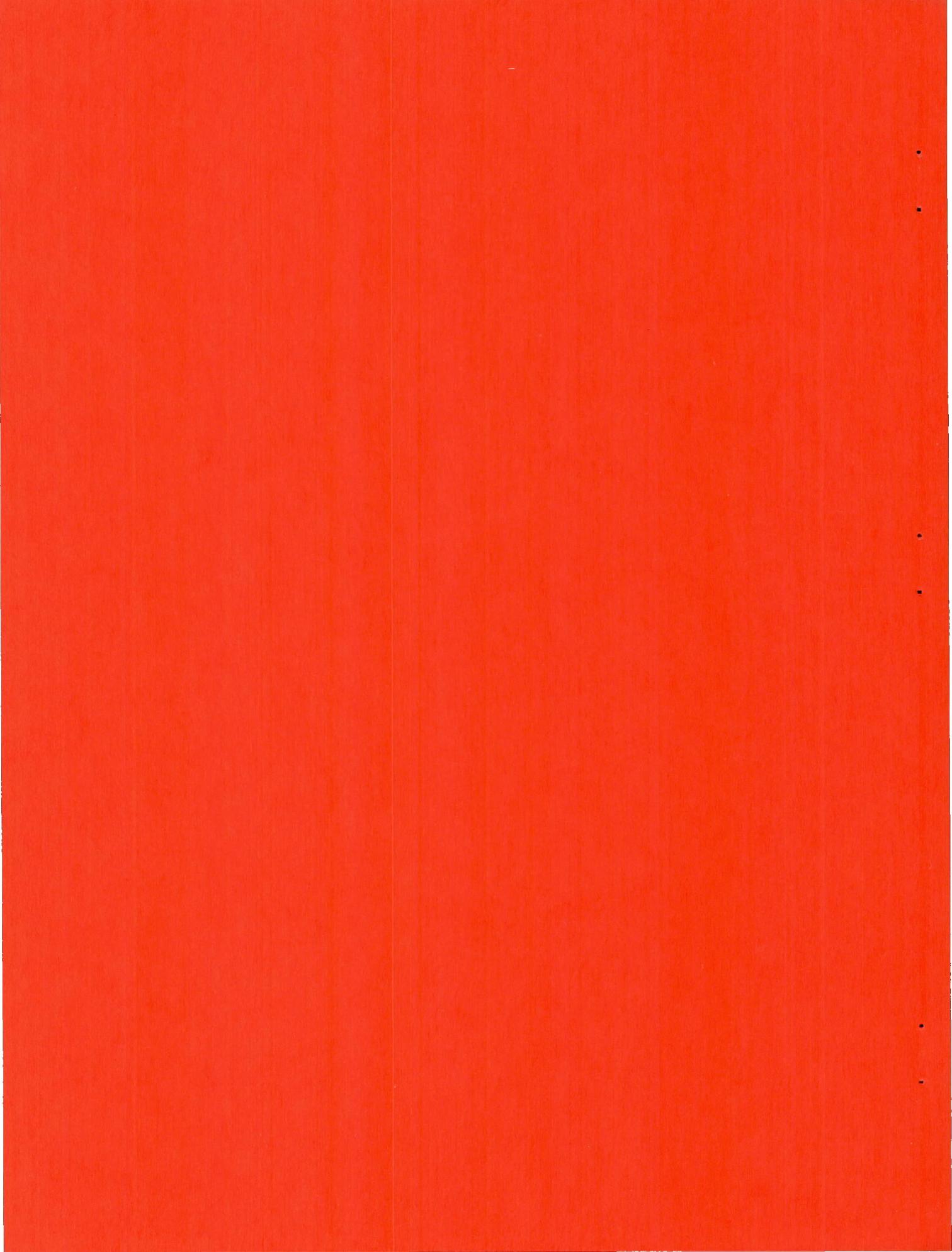
By R. E. H. Rasmussen

Translation of "Über die Strömung von Gasen in engen Kanälen."  
Annalen der Physik, Band 29, Heft 8, August 1937



Washington

August 1951



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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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THE FLOW OF GASES IN NARROW CHANNELS\*

By R. E. H. Rasmussen

SUMMARY

This report deals with the measurements of the air flow  $T$  per second per unit of pressure difference through various channels at average pressures of from 0.00003 to 40 cm Hg. Hydrogen, oxygen, carbon dioxide, argon, and air were utilized.

The flow channels consisted of:

- (1) Narrow annular slits between optically plane glass plates in Christiansen prismatic devices
- (2) A rectangular slit between ground and soot-blackened glass plates
- (3) A cylindrical slit between coaxial cylindrical surfaces of brass
- (4) A porous plate (filter plate) of sintered glass

It was demonstrated that the flow rate  $T$  at high pressure increases linearly with the mean pressure in the channel in agreement with the laminar-flow theory. The width of the annular slits, of from about 3 to  $10\mu$ , was measured according to Christiansen's data by means of Herschel interferences; the optically obtained slit width was about  $0.2\mu$  larger than that obtained from the flow data.

At decreasing pressure,  $T$  assumes a minimum, if the mean path length  $\lambda$  is approximately equal to the slit width  $a$ ; the minimum value  $T_{\min}$  is approximately equal to the value obtained by Knudsen's molecule flow formula. Hence,  $\lambda_{\min} \approx a$ ,  $T_{\min} \approx T_{Kn}$ ; this holds for all channels with well-defined slit width and for all gases.

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\*"Über die Strömung von Gasen in engen Kanälen." Annalen der Physik, Band 29, Heft 8, August 1937, pp. 665-697.

At further decreasing pressure,  $T$  increases again and ultimately assumes a constant value  $T_0$ , when the mean path length has become substantially greater than the length of the channel. The most accurately determined test values of  $T_0$  are tolerably agreeable with the values obtained from Clausius's formulas by an only approximately correct application.

It was shown that the quantity  $T\sqrt{M} = f_{(\lambda)}$  is approximately the same function of the mean path length for all gases for a particular channel, hence, independent of the gas. This rule may be of practical significance for determining the flow resistance of a channel for different gases within a random pressure zone.

The effect of the divergence from the cosine law on the molecule flow, identified by Knauer and Stern, was investigated. It was found that it amounts to only a few percent of the total flow.

The decrease of  $T$  from the value  $T_0$  to  $T_{\min}$  is a consequence of the collisions of the molecules.

## 1. INTRODUCTION

The principal subject of the present report is the quantitative study of the flow of various gases in well-defined annular slits formed in Christiansen's prismatic apparatuses. In the course of the investigation, it was further found necessary to study the flow in different channels also; a part of these measurements is included in the description.

The best known earlier investigations by Graham, Kundt, Warburg, and Christiansen are path-breaking in this phase of research. Christiansen investigated the flow in a narrow annular slit between the plane surfaces of two rectangular glass prisms, the spacing of which was measured by an interferometer. It was found that the flow velocity is governed by the internal friction of the gases, when this spacing is great in comparison to the mean free path length. For very narrow slits, Graham's law is applicable: The rate of diffusion of a gas is inversely proportional to the square root of its density.

In 1909, Martin Knudsen<sup>1</sup> made a theoretical and experimental study of the flow in long cylindrical capillary tubes and gave a complete

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<sup>1</sup>M. Knudsen, Ann. d. Phys. 28, p. 75, 1909.  
M. Knudsen, Ann. d. Phys. 35, p. 389, 1911.

explanation of this flow for the entire pressure range of from  $p = 1$  atm. to  $p = 0$ . At high pressure - in the laminar-flow region - where the mean path length  $\lambda$  is small compared to the tube radius, Poiseuille's law with slip correction is applicable. For the lowest pressures - in the molecule region of flow - where  $\lambda \gg r$ , Knudsen's formula

$$T_0 \sqrt{M} = \frac{4}{3} \sqrt{2\pi} \sqrt{R \Theta} \frac{r^3}{L}$$

is applicable.

$T_0$  = the gas volume measured by the product of volume and pressure flowing through the tube in unit time at unit pressure difference (Gaede used the letter G).  $M$  = molecule number of the gas;  $R$  = gas constant;  $\Theta$  = absolute temperature;  $r$  = radius;  $L$  = length of tube. In the boundary zone, where  $\lambda$  and  $r$  are of equal order of magnitude,  $T$  passes through a minimum so that  $T_{\min}$  is about 5 percent less than  $T_0$ .

Gaede<sup>2</sup> measured the flow of hydrogen in a slit (about  $4\mu$  in width) and found a very low minimum so that  $T_0$  (at the lowest pressure) was about 112 percent greater than  $T_{\min}$ , obtained at pressure  $p = 2.3$  cm Hg. Since the width of the slit was not measured accurately, the absolute values of the flow could not be compared with the theoretical values.

Smoluchowski<sup>3</sup> solved the molecule flow problem for infinitely long cylindrical channels of constant cross section on the assumption of the cosine law. This result can be written

$$T \sqrt{M} = \frac{1}{4} \sqrt{\frac{2}{\pi}} \sqrt{R \Theta} \frac{J}{L}$$

where  $J$  signifies a definite integral

$$J = \iiint dS \cos \varphi \, ds = \iiint dS \, r \, d\alpha$$

$dS$  = an element of the cross section  $S$ ,  $ds$  = an element of the cross-section periphery,  $\varphi$  = the angle between the normal of the element  $ds$  and the distance  $r$  from  $dS$  to  $ds$ ,  $d\alpha$  = the angle at which  $ds$  is seen from  $dS$ .

<sup>2</sup>W. Gaede, Ann. d. Phys. 41, p. 289, 1913

<sup>3</sup>M. v. Smoluchowski, Ann. d. Phys. 33, p. 1559, 1910.

Clausing<sup>4</sup> calculated the molecule flow in special short channels, using the nonuniform molecule flow for predicting the absorption periods of gas molecules.

## 2. EQUIPMENT AND EXPERIMENTAL METHOD

The flow measurement was patterned after Knudsen's method: When a flow passes in  $\tau$  seconds from a tank of volume  $V'$  and pressure  $p'$  through a channel to another tank of volume  $V''$  and pressure  $p''$ , and the flow may be regarded as steady in each time element  $dt$ , so that

$$-d(V'p') = d(V''p'') = T(p' - p'')dt$$

where  $T$  is constant for every definite mean pressure and for small value of the pressure difference, then

$$T = \frac{1}{\tau} \frac{V'V''}{V' + V''} \ln \frac{p_1' - p_1''}{p_2' - p_2''} \quad (1)$$

Subscripts 1 and 2 refer to the measurement before and after the flow.

The description of the apparatus falls, obviously, into two parts: the set of pressure gages, and the flow channels. The pressure-gage assembly is represented in figure 1. The volume  $V'$  consists of the flask  $K'$ , the two McLeod gages  $l_v$  and  $l_k$ , the pressure gage  $M'$ , and the necessary connections of glass tubes. The flow channel indicated at  $K$  is discussed later.

The volume  $V''$  is completely symmetrical with  $V'$  and just about as great. The pressure gage  $M_4$  is mounted for direct reading of the pressure difference  $p' - p''$ . Two small flasks  $Ph$  contain phosphorus pentoxide as drying agents. The apparatus is connected with a mercury diffusion pump through the cock  $H_4$ . The tube  $L$  leads to the gas generator or gas tank. With the small pipette  $k$  and the manometer  $M$ , appropriate quantities of gas can be measured and fed into the apparatus.

Pressure measurements in the 3 to 40 cm Hg range are made with the gages  $M'$  and  $M''$ , and the pressure difference  $p' - p''$  is read on gage  $M_4$  with a cathetometer. In the 0.1 to 3 cm Hg pressure range,

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<sup>4</sup>P. Clausing: Over den Verblijftijd van Moleculen ..., Amsterdam, 1932.

the McLeod gages  $1_v$  and  $2_v$ , whose pressure tubes had a cross-section area of about  $0.4 \text{ cm}^2$ , were used. The lowest pressures were recorded with the McLeod gages  $1_k$  and  $2_k$ . The cross section of the pressure tubes was about  $0.015 \text{ cm}^2$  and the manometer volume about  $250 \text{ cm}^3$ . The pressure readings of the gages were then compared direct several times, and the equation  $p_1'V' + p_1''V'' = p_2'V' + p_2''V''$  checked at each flow measurement. It ensures a very effective check of the pressure measurements. The uncertainty of these measurements is usually less than  $1/2$  percent, but naturally it is much greater for the lowest pressures.

The volume of the apparatus was defined by Mariotte's law, the volumes of flasks  $K'$  and  $K''$  being measured by weighing with water. Illustrative of such a measurement are the following three independent determinations of the volumes  $V_o', V_o'', (V' + V'')_o$ :

$V_o'$ $\text{cm}^3$	$V_o''$ $\text{cm}^3$	$V_o' + V_o''$ $\text{cm}^3$
1298.5	1368.0	2669.7
1301.0	1366.1	2670.2
1298.0	1370.6	2669.3
<div style="display: flex; justify-content: space-around; align-items: center;"> <span style="border-top: 1px solid black; border-left: 1px solid black; border-right: 1px solid black; padding: 0 10px;">1299.2</span> <span style="border-top: 1px solid black; border-left: 1px solid black; border-right: 1px solid black; padding: 0 10px;">1368.2</span> <span style="border-top: 1px solid black; border-left: 1px solid black; border-right: 1px solid black; padding: 0 10px;">2669.7</span> </div> <p style="text-align: center;">} averages</p>		

Since the equation  $V_o' + V_o'' = (V' + V'')_o$  is to be fulfilled, the values  $V_o' = 1300 \text{ cm}^3$ ,  $V_o'' = 1369 \text{ cm}^3$ ,  $(V' + V'')_o = 2669 \text{ cm}^3$  were involved. Subscript o indicates that the values apply when the pressure in the apparatus is zero. Owing to motion of the mercury in the gages, the volume varies with the pressure, hence  $V_p' = V_o' + \alpha p$  and analogously for  $V_p''$ . The inside diameter of the manometer tubes was around  $1.5 \text{ cm}$ , hence, the area was about  $1.8 \text{ cm}^2$  and  $\alpha = 0.9$  if  $p$  is given in  $\text{cm Hg}$ .

The formula (1) is applicable when assuming constant volumes. Gaede<sup>5</sup> has shown how the variation of the volume should be allowed for in the calculation, and that it yields a correction factor

$$k_p = 1 + \frac{p}{2} \left( \frac{\alpha}{V_o'} + \frac{\alpha}{V_o''} \right)$$

by which the quantity  $\frac{V_o' V_o''}{V_o' + V_o''}$  must be multiplied. He further called attention to the fact that the small volume  $v$  between the flow channel and the stop cock in the feed line yields a correction factor

$$k_v = 1 - \frac{v}{V'} - \frac{v}{V''} \quad \text{with which the ratio } \frac{p_1' - p_1''}{p_2' - p_2''} \text{ must be multiplied.}$$

Gaede's corrections were taken into consideration in all measurements. Every alteration of the apparatus, every exchange of the flow channels and so forth, was followed by a determination of the volumes, and the insertion of the correct values in the formula. Care was taken to keep  $v$  from getting greater than necessary, as a rule, only a few  $\text{cm}^3$  (determined by weighing out with water or mercury).

All tests were run at room temperature. The temperature  $t_K$  of the flow channel was read on a thermometer mounted as near as possible to the channel, while temperature  $t_M$  was read on a second thermometer mounted on a level with the receptacles of the McLeod gages. Because of the difference in height and the natural temperature distribution in the room,  $t_K$  was usually several tenths of a degree higher than  $t_M$ . This difference was sufficient to prevent a distillation of the mercury from the manometers up to the flow channel. All pressure measurements were, in addition, corrected for this temperature difference. Lastly, all test data were reduced to the same temperature,  $20^\circ \text{C}$ , by means of Knudsen's and Sutherland's formulas for the temperature relationship between molecule flow and internal friction. All of the temperature corrections were very small, usually less than  $1/2$  percent.

For one part of the measurements, a pair of gas traps (locks), cooled with ice or liquid air, were blown into the feed lines AK and BK (fig. 1) to prevent the mercury vapor and any other condensable impurities from passing into the flow channel. These measurements were corrected for the apparent volume increase caused by the cooling. They amounted to about  $20 \text{ cm}^3$  at the most.

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<sup>5</sup>W. Gaede, Ann. d. Phys. 41, p. 289, 1913.

All measurements had to be corrected for the flow resistance of the intake line. This resistance was unusually high for the prism apparatus I because of unsuitable design. In this case, the correction was so determined that, while widening the slit from 1 to 2 mm, the resistance was lowered to a disappearingly small value. After that,  $T$  was measured as usual; for these measurements, the resistance of the inlet line alone determines the measured values, which are indicated by  $T_R$ . The uncorrected value of the flow is indicated by  $T_{R+K}$  and the corrected value for the inlet pipe resistance by  $T_K$ . Therefore, for each pressure, according to the definition of  $T$

$$T_R(p' - p'')_R = T_K(p' - p'')_K = T_{R+K}(p' - p'')$$

together with

$$p' - p'' = (p' - p'')_R + (p' - p'')_K$$

the partial pressure decrements through the inlet lines and the channel carry the subscripts  $R$  and  $K$ . From these equations follow

$$T_K = T_{R+K} \frac{T_R}{T_R - T_{R+K}} \quad (2)$$

All measurements were corrected in the same manner.

For prism apparatus I and for the porous filter plate, the corrections at the lowest pressures amounted to more than 50 percent. For prism apparatus II and the other channels, the corrections were less than 10 percent.

#### The Flow Channels

The prism apparatus I is shown in figure 2. The top picture shows the prisms (in part in vertical section) mounted in a clamping device, the arrangement of which is readily apparent. The middle figure shows the prisms from above; one,  $P_1$ , is seen in horizontal section. The bottom picture shows the hypotenusal faces of the two prisms. In the face of the one prism  $P_1$  three concentric channels  $K_1$ ,  $K_2$ , and  $K_3$  have been ground, each one about 1 mm wide and 1 mm deep. Three conical

holes  $H_1$ ,  $H_2$ , and  $H_3$ , into which the glass tubes 1, 2, and 3 were ground, and sealed with apiezon grease, serve as leads to the channels. Two of these tubes are inlets, the third was closed and merely served as a stopper.

For the series of measurements, recorded in figure 10, channel  $K_3$  was connected with volume  $V'$  and  $K_2$  with  $V''$ . The flow was along  $K_3 \rightarrow K_2$ . The spacing of the prisms was gaged by placing a strip of tin foil 5 mm wide between the faces around the outermost channel  $K_3$ .

Correct tightening of the wing nuts  $F$  makes the ring lie flat between the faces, and become parallel. The apparatus was sealed with shellac dissolved in pure alcohol, which was made to run down in the groove between the prisms. This seal was perfectly satisfactory after the alcohol evaporated. The radii of the two annular slits were

$$R_1 = 1.1952 \text{ cm}$$

$$R_2 = 1.8897 \text{ cm}$$

in the wide annular slit between  $K_2$  and  $K_3$  and

$$R_1 = 0.8994 \text{ cm}$$

$$R_2 = 1.0919 \text{ cm}$$

in the narrow slit between  $K_1$  and  $K_2$ .

The high inlet resistance of prism apparatus I and the subsequently large corrections made it desirable to make a number of measurements with another prism apparatus of more adequate inlet design (fig. 3). This apparatus had only one annular slit between the radii  $R_1 = 0.8123 \text{ cm}$  and  $R_2 = 1.5984 \text{ cm}$ . The inlet lines were no less than 5 mm in diameter at any point.

The circular channel in the hypotenuse surface of the prism  $P_1$  had a trapezoidal cross section of about 2 mm in depth; it was 4 mm wide at the base and 8 mm on top. The clamping device had an unusual feature.

The tension was elastic; the wing nuts F did not press directly on the beam L, but on a pair of helical springs S which transmitted the pressure on L. The stiffness of the springs was measured and the compression read from scales etched in the blocks C. Thus, the pressure on the prisms was known and it was possible to obtain equal pressure in both springs.

The distance of the prism surfaces was determined by Christiansen's method by measuring the angle between the Herschel interference fringes.

When monochromatic light of wave length  $\lambda_0$  is so reflected that the angle of incidence  $i$  is very near to the boundary angle of the total reflection, the reflecting power of the surface will be very high, and the light is reflected several times (fig. 4). The effect of the interference of the reflected beams  $J_{r_1}, J_{r_2} \dots$  is such that the reflected intensity is 0, when  $\cos b = \frac{\lambda_0}{2a}m$ ; ( $a$  = distance of the surfaces,  $m$  = a whole number). Bearing in mind that  $\cos b = \sqrt{1 - \left(\frac{n_1}{n_0}\right)^2 \sin^2 i}$  and that in the case of total reflection  $m = 0, \cos b = 0, \sin i_0 = \frac{n_0}{n_1}$ , one obtains the equation

$$\sin^2 i_0 - \sin^2 i_m = \left(\frac{n_0}{n_1}\right)^2 \left(\frac{\lambda_0}{2a}\right)^2 \times m^2 \tag{3}$$

for defining the directions  $(i_m)$  of zero reflected intensity. In addition

$$i + \beta = p, n \sin \alpha = n_1 \sin \beta \tag{4}$$

As the angles between the dark fringes in the reflected light are small, it is permissible to put  $\sin^2 i_0 - \sin^2 i_m = \sin^2 i_0 di_m$ . Differentiation of the equations (4) leaves  $n \cos \alpha_0 d\alpha = -n_1 \cos \beta_0 di$ , which, when introduced in equation (3) and resolved, gives

$$a = \frac{\lambda_0}{2} \sqrt{\frac{n_0 \cos \beta_0}{2n \cos \alpha_0 \cos i_0}} \sqrt{\frac{m^2 - m_1^2}{d\alpha_m - m_1}} = K(\lambda_0) \sqrt{\frac{1}{x}}, \tag{5}$$

where  $\alpha_{m - m_1}$  is the angle between the  $m^{\text{th}}$  and the  $m_1^{\text{th}}$  dark line. If specifically  $m - m_1 = 1$ , it is seen that the quantity  $x = \frac{\Delta\alpha}{2m - 1}$ , where  $\Delta\alpha$  indicates the angle between two adjacent fringes, is to be constant. Thus, the angles  $\Delta\alpha$  act as the digits 3, 5, 7 . . . In transmitted light, the complementary image, light stripes on dark background, are visible, but the spacing of the stripes is unchanged (fig. 5).

The fringe system is observed in a telescope set to infinity, the angular distances are measured on a micrometer. As a rule, the first three to six fringes were used. Quantity  $x$  was defined as average value or else computed by a different equalization of the individual values, after which  $a$  is calculated by formula (5). Since the quantity  $K(\lambda_0)$  is constant for a certain color, it can be found when the refraction conditions and the prismatic angle are measured.

The distances were measured at several points on the prism surfaces. An illustrative example of such a measurement is given in figure 6. Decisive for the molecule flow is the average value of the second power of the distance, and for the laminar flow, the average value of the third power of the distance. In the cited example

$$\bar{a} \approx \sqrt{\bar{a}^2} \approx \sqrt[3]{\bar{a}^3} = 9.21 \pm 0.02\mu$$

The fluctuations of the room temperature had no measurable effect on the prism spacing; but  $a$  increases a little with the pressure in the apparatus so that  $a = a_0(1 + cp)$ . In the foregoing example  $c = 1.5 \times 10^{-4}$  ( $p$  in cm Hg.) Since  $p < 40$  cm Hg, it is seen that  $a$  becomes, at the most, 0.6 percent higher than  $a_0$ .

Since the annular slits of the prism apparatuses were formed between fine optical surfaces on which, according to Knauer's and Stern's investigations, at grazing incidence, a part of the molecules is mirror reflected, it was of great interest to measure the molecule flow in slits between rough surfaces for which the cosine law is rigorously applicable. For this purpose, the apparatus represented in figure 7 was resorted to. The slit was formed between the ground surfaces of two thick round pieces of plate glass  $P_1$  and  $P_2$  which were mounted in the clamping device as indicated. The lower surface of the top plate carried the three channels  $K_1, K_2, K_3$ , and the three conical openings with the glass tubes 1, 2, and 3. Tin foil with rectangular sectors 1, 2, 3, 4 was placed between the glass plates, thus forming two rectangular slits, one between  $K_2$  and  $K_3$ , the other between  $K_1$  and  $K_3$ . The width  $a$  of the slit is governed

by the thickness of the tin foil. The dimensions  $L$  - the length of the slit in flow direction - and  $b$  were measured with the cathetometer. Only the slit between  $K_1$  and  $K_3$  was used for flow measurements; its dimensions were  $L = 0.97$  cm and  $b = 1.32$  cm. Owing to the mat surface the slit width  $a$  could not be measured optically, but was determined on the basis of the laminar-flow data. A measurement of the thickness of the tin foil produced no appropriate determination of  $a$  because the large surfaces were not level enough. The grainy irregularities of the surface were of the order of magnitude of 0.01 mm. To increase this roughness still more, they were blackened in some tests with soot from a wide flame of burning turpentine oil. The apparatus was sealed with Picein, as indicated in figure 7.

The measurements indicated that the magnitude  $T\sqrt{M} = f(\lambda)$  for a given channel is solely a function of the mean path length and independent of the gas. For a more accurate check of this rule, flow channels were used which first remained geometrically constant during the time interval of the test series with different gases. Two channels, both satisfying the cited demand, were used for this purpose: a cylindrical slit between two coaxial cylindrical surfaces and a porous body, a filter plate of sintered glass. The cylindrical slit was obtained by fitting an accurately milled brass plug (compare fig. 8) in an accurately drilled hole of a solid brass block. Figure 8 represents a cut through the axis of the apparatus. The cylindrical surfaces  $A$  and  $B$  fit the holes exactly, while the diameter of the surface  $C$  is about 0.03 mm less. The tubes  $R_1$  and  $R_2$  serve as inlets; so the flow in the cylindrical slit is parallel with the axis. The slit was  $L = 0.487$  cm in length, the diameter of the surface  $C$  was  $1.9997 \pm 0.0001$  cm. The diameter of the hole itself was not measured, but that of the employed drill was 2.0019 cm. The apparatus was sealed by means of stiff stop-cock grease applied to the groove  $R$ . The channels  $h$  were drilled to prevent an almost closed dead-air space from forming below the part  $B$ .

The porous filter plate was fused in a glass tube of the form depicted in figure 9. The diameter of the plate was about 1.9 cm, its thickness about 0.2 cm. The size of the pores was not known from the start. An upper limit is obtained, according to Weber<sup>6</sup> by recording the pressure required to force air bubbles through the filter when it is wet and coated with a layer of water. At around 13 cm Hg, the air started to penetrate at a certain point, but an increase of the pressure to about 17 cm already forced air through at many places. This proved that the filter was homogeneous to some extent. On computing the surface tension of the water at 73 dynes/cm and assuming complete wetting, it is found

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<sup>6</sup>S. Weber, Teknisk Tidsskrift 1917, Nr. 37.

that the narrowest spot of the widest pore of the filter has a diameter of about  $17\mu$  when the pore is regarded as circular.

### 3. TEST DATA

Flow measurements were made with hydrogen, oxygen, carbon dioxide, and argon.

Hydrogen was produced in the Kipp apparatus from zinc and dilute hydrochloric or sulfuric acid. It was purified and dried in a solution of potassium hydroxide, a saturated alkaline solution of potassium permanganate, concentrated sulphuric acid, and lastly, in phosphorous anhydride.

Oxygen was taken from a commercial tank and then dried in sulfuric acid and phosphorous anhydride.

Carbon monoxide was manufactured in Kipp's apparatus from marble and dilute hydrochloric acid, washed in distilled water and then dried like the other gases. An assay with absorption in potassium solution indicated 99.9 percent purity.

Argon was obtained from the factory guaranteed 99.5 percent pure; it was used without being treated.

#### The Measurements with the Annular Slits

The principal test data are correlated in table 1. The lines 2 to 4 give the slit dimensions with the optically defined values of the slit width  $a$ . Line 6 gives the minimum values of  $T\sqrt{M}$  ( $M$  = molecule number of gas). Line 7 gives the values of  $T\sqrt{M}$  at the lowest pressures at which tolerably reliable measurements could be made. The other lines are discussed below. The accuracy of the data is characterized by the digit number, which is such that the uncertainty is one or several units of the the last digit.

The several measurements are represented in figures 10, 11, 12, and 13. The quantity  $T\sqrt{M}$  is plotted against the  $\log_{10} \frac{1}{\lambda} = \log_{10} \frac{p}{\lambda_1}$ ;  $\lambda_1$  = mean path length measured at 1 cm Hg,  $20^\circ$  in units  $\mu$ . The reason for this choice of representation is found in the following paragraphs.

At high pressures - in the laminar-flow region - the internal friction of the gas governs the flow. Assuming that the speed of the gas along the surface is zero

$$T = \frac{\pi a^3}{6\eta l \frac{R_2}{R_1}} p$$

where  $\eta$  = coefficient of internal friction,  $l$  = natural log.

Bearing in mind the slip, one obtains the boundary condition

$u_0 \xi = \eta \frac{du}{dN}$ , with  $u_0$  = velocity along the surface,  $\xi$  the coefficient of external friction, and  $\frac{du}{dN}$  the velocity gradient. With  $\zeta = \frac{\eta}{\xi}$  = slip coefficient

$$T = \frac{\pi(a^3 + 6a^2\zeta)}{6\eta l \frac{R_2}{R_1}} p = A_1 p + B_1$$

where  $A_1$  and  $B_1$  are constants inside of a pressure area in which  $\zeta_p$  is to be regarded as constant.

By kinetic theory,  $\eta = c\rho\Omega\lambda$ , with  $\rho$  = gas density,  $\Omega$  = average value of thermal velocities, and  $c$  = numerical factor.

The definition of  $\lambda$  is that given in Landolt-Börnstein's tables<sup>7</sup>

$$\lambda_1 (0.76 \text{ cm Hg}) = \frac{\pi}{8} \frac{1}{0.49} \frac{\eta_0 \Omega_0}{1,013,250} \text{ cm} \quad (7)$$

where

$$\Omega_0^2 = \frac{\pi}{8} 273 \frac{R}{M}; \quad R = 83.15 \times 10^6 \frac{\text{erg}}{\text{degree mol}} = \text{gas constant}$$

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<sup>7</sup>Landolt-Börnstein, Tables I, 1923, p. 119.

The employed numerical values were:

Gas	$\lambda_2$ (0°76 cm Hg)	$\lambda_1$ (20°1 cm Hg.)	$\eta_{20^\circ}$	C
H <sub>2</sub>	$1123 \times 10^{-8}$ cm	8.65 $\mu$	$0.881 \times 10^{-4}$	72
O <sub>2</sub>	$647 \times 10^{-8}$ cm	5.03 $\mu$	$2.040 \times 10^{-4}$	128
CO <sub>2</sub>	$397 \times 10^{-8}$ cm	3.12 $\mu$	$1.485 \times 10^{-4}$	274
A	$635 \times 10^{-8}$ cm	4.96 $\mu$	$2.247 \times 10^{-4}$	170

C = Sutherland's Constant

With these numerical values

$$\eta_{20^\circ} = 0.717 \times 10^{-5} \times \lambda_1 \times \sqrt{M} \quad (7a)$$

for all gases.

Inserting (7a) in (6) and bearing in mind that 1 cm Hg = 13,296 dynes per cm<sup>2</sup> one obtains

$$T \sqrt{M} = 0.971 \times 10^9 \times \frac{a^3}{\frac{R_2}{R_1}} \frac{p}{\lambda_1} + 5.83 \times 10^9 \frac{a^2}{\frac{R_2}{R_1}} \frac{\zeta p}{\lambda_1} = A \frac{1}{\lambda} + B, \quad (8)$$

if  $p$  is measured in cm Hg and  $\lambda_1$  in  $\mu$ . From it, it is seen that  $T \sqrt{M}$  at high pressure becomes the same function of the mean path length for all gases as a result of the definition of  $\lambda$ .

From figure 14, where the measurements with the annular slit in prism apparatus I are reproduced, it then also follows that  $T \sqrt{M}$  at high pressure is linearly dependent on  $p/\lambda_1$ , and that the dependence is the same for all gases, which simply implies that the measurements are in relative agreement with the employed values  $\eta_{20}$ .

Starting from the concepts of the molecule flow, it must be assumed that the rule  $T \sqrt{M} = f(\lambda)$  must hold also at the lowest pressures, where

the internal friction has either only little or else no significance at all for the flow.

And so figure 10 actually indicates that the dependence of  $T\sqrt{M}$  on  $p/\lambda_1$  is nearly the same for all gases, not only at the high pressures, but also in the entire pressure range explored. The discrepancies lie at the limit of instrumental accuracy.

The foregoing indicates that the minimum of  $T$  lies at the pressure where the mean path length  $\lambda_m$  is equal to the slot width  $a$ .

The height of the minimum can also be approximated quite simply in view of the validity of the relation  $T_{\min} \approx T_{Kn}$ ,  $T_{Kn}$  being calculated from Knudsen's general formula<sup>8</sup>

$$T_{Kn} = \sqrt{\frac{R\Theta}{M}} \frac{1}{w}, \quad w = \frac{3}{8} 2\pi \int_0^L \frac{a}{s^2} dl \quad (9)$$

which, applied to a circular slit, gives

$$T_{Kn} = \frac{16}{3} \sqrt{\frac{\pi}{2}} \sqrt{\frac{R\Theta}{M}} \frac{a^2}{R_2} \frac{1}{R_1} \quad (10)$$

The values  $T_{Kn}\sqrt{M}$  computed from equation (10) are given on line 8 of table 1.

At the lowest pressures,  $T\sqrt{M}$  becomes constant independent of the pressure, as is plainly seen in figure 13 (hydrogen in prism apparatus II), where the measurements have been carried out to about  $p = 3 \times 10^{-5}$  cm Hg. Throughout the entire pressure zone in which the mean path length was relatively large compared to the slit width  $R_2 - R_1$  (marked on all graphs),  $T$  is practically constant ( $T_0$ ). In these measurements, the inlet lines were cooled by liquid air, so that the flowing hydrogen was not contaminated by mercury vapor. Another effect of the cooling was that the McLeod measurements were more reliable and this made it possible to extend the measurements so far into the low pressure zone. In the other test series, the boundary values  $T_0$  were less accurately determined.

<sup>8</sup>Ann. d. Phys. 28, p. 76, 1909.

A calculation of the free molecule flow in slits of the form employed here has never been attempted, to the writer's knowledge. An attempt at an accurate treatment of this problem resulted in difficult calculations and will not be mentioned. Professor N. Bohr pointed out that an approximate solution could be obtained by appropriate use of Clausing's formula for the molecule flow in a rectangular slot. This formula, which is valid for  $b \gg L \gg a$ , can be written as

$$T_{C1} \sqrt{M} = \frac{1}{2} \sqrt{\frac{2}{\pi}} \sqrt{R^{\ominus}} \left( \frac{1}{2} + \sqrt{\frac{L}{a}} \right) \frac{a^2 b}{L} \quad (11)$$

For example, putting  $L = R_2 - R_1$ ,  $b = \pi(R_2 + R_1)$  gives for the flow in the circular slot

$$T_{C1} \sqrt{M} = \frac{1}{2} \sqrt{2\pi} \sqrt{R^{\ominus}} \left( \frac{1}{2} + \sqrt{\frac{R_2 - R_1}{a}} \right) \frac{a^2 (R_2 + R_1)}{R_2 - R_1} \quad (12)$$

and this expression must become exact when  $(R_2 + R_1) \gg R_2 - R_1 \gg a$ .

The values of line  $T_{C1} \sqrt{M}$  in table 1 are computed from formula (12); the ratio  $\frac{R_2 + R_1}{R_2 - R_1}$  is also given in the table for the various circular

slits. It is seen that the computed values  $T_{C1} \sqrt{M}$  are greater everywhere than the observed  $T_O \sqrt{M}$ . This is due, in part, to the fact that the measurements were not extended to sufficiently low pressures in all test series, and in part to the fact that the mercury vapor was frozen in only one of the aforementioned series and therefore had a retarding effect on the flow in the other series; lastly, the condition  $R_2 + R_1 \gg R_2 - R_1$  is not fulfilled and it is easily seen that the very noncompliance with this condition must result in a divergence in the

direction of computed value > observed value. The ratio  $\frac{R_2 + R_1}{R_2 - R_1}$

is greatest for the small circular slit, namely, 10.4, and here also the agreement between observation (1.6) and calculation (1.72) is fairly good.

It was found that all test data can be represented in close approximation by

$$T\sqrt{M} = A \frac{1}{\lambda} + B + C \log \frac{\frac{\lambda}{a} + D}{\frac{\lambda}{R_2 - R_1} + D} \quad (13)$$

The values of the constants A, B, C, and D, which are dependent on the slit, but not on the nature of the gas, were determined graphically and entered in lines 9 to 12 of table 1. The curves of figures 10 to 14 are computed from equation (13) with these constants.

At high pressures, the last term (C) in equation (13) is disappearingly small and by comparison with equation (8) it is seen that

$$A = 0.971 \times 10^9 \frac{a^3}{R_2 R_1}$$

or  $a$  can be computed when the other is known. Computing  $a$  from the A values gives the values indicated with  $a_{str}$  in table 1; they are all smaller than those defined optically. The difference  $\Delta a = a_{opt} - a_{str}$  is in all cases about  $0.15 - 0.2\mu$ . No satisfactory explanation of this discrepancy could be found. It was possible to attribute it to the inertia forces which were not allowed for in the calculation of equations (6) and (8); but the calculation indicated that they were only about  $1/10^6$  of the friction forces and hence were altogether insignificant. On computing the coefficient  $A_1$  of formula (6) for each gas and then  $a_{str}$  by means of the value  $\eta_{20^\circ}$ , almost the same value is obtained for all three gases  $H_2$ ,  $O_2$ , and  $CO_2$ , which, at about  $9.0\mu$ , is in good agreement with the value  $9.02\mu$  obtained in this case from equation (8). This implies that the employed gases were sufficiently pure. To be completely sure in this respect, a series of measurements with atmospheric air were carried out. The air was dried in sulphuric acid and phosphorous anhydride. The inlet tubes were cooled with ice in order to reduce the mercury-vapor pressure in the slit to a low value. The measurements were carried out in such a way that for each value of the pressure  $p$  (average values) several (2, 3, or 4) independent measurements of the quantity  $T$  were made at decreasing values of the pressure difference. The value of  $T$  for the same  $p$  never indicated a systematic course, which confirms that the inertia forces are disappearingly small; hence, the use of the average value of  $T$  for each  $p$ . The measurements were limited to the high-pressure range. The results are represented in the following table:

Atmospheric Air in Prism Apparatus II;  $a_{opt} = 8.15\mu$ 

p	$T_{obs}$	$C'$	$(T - C')_{obs}$	$(T - C')_{ber}$	$\Delta_{o-b}$
2.15	0.2251	0.0085	0.2166	0.2167	-0.0001
5.21	.3061	.0037	.3024	.3045	-.0021
9.36	.4271	.0020	.4251	.4236	+0.0015
13.79	.5545	.0014	.5531	.5508	+0.0023
20.08	.7304	.0010	.7294	.7313	-.0019
24.82	.8616	.0007	.8609	.8673	-.0064
28.59	.9828	.0007	.9821	.9755	+0.0066
32.39	1.0810	.0006	1.0804	1.0845	-.0041
36.50	1.2150	.0005	1.2145	1.2026	+0.0119

In order to be able to use all the measurements for determining the slope coefficient  $A_1$ , the curvature of the  $T, p$  curves must be taken into account to some extent. For this reason, the  $C$  term of formula (13) was computed and subtracted from  $T_{obs}$ . The values of the constants

$$D = 1.5, \quad \frac{C}{\sqrt{M}} = \frac{0.67}{\sqrt{29}} = 0.12; \quad \lambda_1 = 4.7\mu$$

were utilized. The correction term is indicated with  $C'$  in the table. The corrected term  $(T - C')_{obs}$  is to be linearly dependent on  $p$ . Smoothing resulted in  $T - C' = 0.0287p + 0.1550$ ; the values computed from it along with the differences  $\Delta_{o-b}$  are reproduced in the table. Inserting  $A_1 = \frac{0.0287}{13,300}$  and  $\eta_{20} = 1.820 \times 10^{-4}$  in equation (6) gives  $a_{str} = 7.98\mu$ .

An optical measurement of the slit width accompanied the flow measurements. The results were  $a_{opt} = 8.17\mu$  measured with green mercury light and  $a_{opt} = 8.13\mu$  measured with yellow mercury light; average value  $a_{opt} = 8.15 \pm 0.02\mu$ . The difference  $a_{opt} - a_{str} = 0.17\mu$ . This example proves without a doubt that the nonagreement of the two methods is real, and that it cannot be explained by insufficient purity of gas and subsequent uncertainty of the value of  $\eta$ . The optical test method is not gone into any further. Unless stated otherwise, the optically determined values of the slit widths are used in this report.

Returning to the discussion of formula (13), it is noted that, by comparing the expressions (12) and (13) for  $\lambda = \infty$

$$B = b \sqrt{RT} \frac{a^2 (R_2 + R_1)}{R_2 - R_1} = \frac{b}{c} C$$

where  $\frac{c}{B} = \frac{b}{C}$  and  $b$  are pure numbers. From this, it is seen that the ratio  $\frac{B}{C} = \frac{b}{c}$  should be constant, which (compare table 1) is not altogether true. The discrepancies are largely due to the previously cited insufficient fulfillment of the condition  $R_2 + R_1 \gg R_2 - R_1$ .

As stated before

$$\lambda_{\min} \simeq a, \quad T_{\min} \simeq T_{\text{Kn}} = \frac{16}{3} \sqrt{\frac{\pi}{2}} \sqrt{\frac{R\Theta}{M}} \frac{d^2}{R_1}$$

is valid for the minimum.

Knudsen's general formula

$$T_{\text{Kn}} = \sqrt{\frac{R\Theta}{M}} \frac{1}{w}, \quad w = \frac{3}{8} \sqrt{2\pi} \int_0^L \frac{v}{S^2} dz$$

is obtained by computing the tangential motion quantity  $B_1$  given off on the channel wall per unit time per unit area, and by putting the total motion quantity  $\int_0^L B_1 v dz$  equal to the driving force  $(p' - p'')S$ .

A premise of this calculation is that  $B_1$  can be proportional to the average value of the flow velocities over the channel cross section. In consequence, the calculation can be valid only for channels with somewhat homogeneous cross sections, for example, slits of constant width, circular-cylindrical pipes, and so forth, as already noted by v. Smoluchowski and Clausing. Furthermore, these researchers have indicated that an exact solution on the assumption of the cosine law for  $\lambda = \infty$  produces different and greater values of the quantity  $T$  than Knudsen's calculation. Then it is readily apparent (for instance, on examining the Smoluchowski-Clausing calculation, or in the calculation of the pure effusion in a slit; compare on the next page) that a very large part of the free

molecule flow in the slit is due to molecules which fly approximately parallel to the slit walls and therefore cover great distances between two collisions against the walls; this holds for infinitely small pressures.

If the pressure has such a value that a (small) number of collisions occur (in the zone  $L > \lambda > a$ ), these far-reaching molecules will contribute much less by reason of the mutual collisions. Against it is a contribution of molecules which have participated on mutual collisions but which at the beginning (that is, when  $\lambda$  is still great compared to  $a$ ) is very much smaller than the decrease in flow caused by the collisions. The total effect of the collisions is a decrease in flow in the zone  $\lambda = \infty$  to  $\lambda \simeq a$ .

In the minimum zone  $\lambda \simeq a$ , the assumptions of Knudsen's calculation, particularly the assumptions of constant flow velocity over the channel cross section, appear to be fulfilled to some extent, hence  $T_{\min} \simeq T_{\text{Kn}}$ .

The discussion of the circular-slit measurements is concluded with a few remarks about the pure effusion and the effect of the mirror reflection.

The effusion, that is, the contribution to the flow due to molecules which pass the slit without impinging on the wall, is easily computed. The result is discussed because it is a fine example of the importance of the cosine law.

The flow in the annular slit is assumed to be from the inside toward the outside:

I. Assumption: The molecules are emitted from the cylindrical surface  $2\pi R_1 \times a$  according to the cosine law, so that

$$T_{\text{Ef}}^I \sqrt{M} = 2 \sqrt{\frac{2}{\pi}} \sqrt{R\Theta} a^2 J$$

where J is the defined integral

$$J = \int_0^1 \frac{dx}{\sqrt{\left(\frac{R_2}{R_1}\right)^2 - x^2} \sqrt{1 - x^2}}$$

$$= \frac{1}{\left(\frac{R_2}{R_1}\right)^2 - 1} \left( \frac{1}{2} \arcsin \frac{R_1}{R_2} + 2 \frac{R_1^2}{R_2^2} \sqrt{\left(\frac{R_2}{R_1}\right)^2 - 1} + \frac{\pi}{4} \right)$$

II. Assuming, by way of contrast, that the molecules proceed from the plane circular areas  $2\pi R_1^2$  according to the cosine law

$$T_{Ef}^{II} \sqrt{M} = \sqrt{2\pi} \sqrt{R_0} a^2 \frac{R_1^2}{R_2^2 - R_1^2}$$

The results are, like the assumptions, not identical. To indicate the numerical significance, the ratio

$$\frac{T_{Ef}^I}{T_{Ef}^{II}} = \frac{2J}{\frac{\pi R_1^2}{R_2^2 - R_1^2}}$$

was computed for several cases

	$R_2$	$R_1$	$\left(\frac{R_2}{R_1}\right)^2$	$\frac{T_{Ef}^I}{T_{Ef}^{II}}$
Prism apparatus II . . . . .	1.5984	0.8123	3.88	1.70
Prism apparatus I, wide slit . .	1.8897	1.1952	2.50	1.43
Prism apparatus I, narrow slit .	1.0919	0.8994	1.47	1.17
			1.10	1.03
			1.01	1.00

It is apparent that the ratio  $\frac{T_{Ef}^I}{T_{Ef}^{II}} \rightarrow 1$  when  $\frac{R_2}{R_1} \rightarrow 1$ , which is

also apparent from the expression for  $J$ , and for the rest is immediately clear.

It is difficult to decide which of the assumptions is preferable for the annular slits, or whether a third might not be better still; for this reason, the effusion cannot be indicated with a high degree of accuracy.

The foregoing calculations are applicable only in the absence of mutual collisions of molecules; the effusion decreases considerably with increasing pressure. An exact calculation of it is not of interest in view of the uncertainty of the first calculation and the small value of the effusion; hence, the decision to put the effusion at pressure  $p$  equal to  $T_{Ef}^I e^{-\frac{R_2-R_1}{\lambda}}$ , where  $p\lambda = \lambda_1$ .

The values of the effusion by assumption I, shown in table 1 below  $T_{Ef}^I \sqrt{M}$ , amount to a few percent of the flow.

The individual measurements  $T\sqrt{M}$  with and without correction for the effusion are plotted in figure 13. The other graphs contain only the uncorrected values, which were also used for defining the constants in formula (13).

Knauer and Stern have proved divergences from the cosine law for grazing incidence toward polished surfaces. Since precisely the molecules, moving grazingly toward the walls, contribute much to the flow, it is readily apparent that they produce a great increase in the flow when they are mirror reflected. The writer believed that up to a certain time interval, this reflection had great importance for the marked increase of flow in the zone from  $T_{min}$  to  $T_o$ . But a more accurate calculation, indicated in the following, showed that with the small values of the mirror reflection, cited by Knauer and Stern, its effect is, at the most, a few percent of the free molecule flow. This is in good agreement with the fact that the ordinary calculation of the molecule flow (Clausing's formula) explains the observed great values  $T_o$  completely.

By calculations, not repeated here, it is found that when  $T_r$  denotes the increase in flow due to the mirror reflection, for small values of the reflecting power

$$T_r \sqrt{M} < \int_0^{\alpha_m} r_\alpha (E + 2F\alpha) d\alpha \tag{14}^9$$

where  $r_\alpha$  = the reflecting power when  $\alpha$  is the grazing angle and  $\alpha_m$  = the upper limit of the grazing angle for which  $\alpha > 0$

$$E = 4 \sqrt{\frac{2}{\pi}} \sqrt{R^\ominus} R_1 \alpha,$$

$$F = \frac{1}{2} \sqrt{2\pi} \sqrt{R^\ominus} \left( \frac{1}{2} + \frac{1}{\pi} \arccos \frac{R_1}{R_2} \right) \left( \frac{R_2^2 - R_1^2}{\frac{R_2}{R_1}} - 2R_1^2 \right)$$

For the reflecting power of a polished surface (speculum metal) compared to hydrogen, Knauer and Stern have given the following values<sup>10</sup>

$\alpha = 1 \times 10^{-3}$	$1.5 \times 10^{-3}$	$2 \times 10^{-3}$	$2.25 \times 10^{-3}$ Radian
$r_{\alpha obs} = 0.05$	0.03	0.015	0.0075
$r_{\alpha ber} = 0.050$	0.033	0.016	0.0075

These results are closely approximated by the expression  $r = 0.084 - 34\alpha$ , as seen from the line  $r_{ber}$ . Introducing this expression for  $r$  along with the constants  $R_1, R_2$ , and  $a$  for the prismatic apparatus II in formula (14) gives  $T_r \sqrt{M} < 0.075$  for hydrogen. Reflection measurements on other gases have never been published to the writer's knowledge. But an explanation of the reflections by wave mechanics indicates that the reflection of a given surface diminishes when the number of molecules of the reflected gas increases because the associated wave

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<sup>9</sup>The calculation is to be found in: Om Luftarters Strömning in snaevre Kanaler. Copenhagen 1936, pp. 63-69.

<sup>10</sup>Ztschr. f. Phys. 53, 1929, p. 779.

length  $l = \frac{h}{m\Omega}$  is inversely proportional to  $\sqrt{M}$  and the condition for mirror reflection is  $\Delta \sin \alpha \leq l$ , where  $\Delta$  is the mean height of the irregularities of the surface. Therefore, if  $r$  is assumed to be

$$\sim \frac{1}{\sqrt{M}}$$

$$r_\alpha = \sqrt{\frac{2}{M}} (0.084 - 34\alpha) \quad (15)$$

for other gases.

The values  $T_r \sqrt{M}$  in table 1 were computed from equations (14) and (15); they are, as will be recalled, the upper limits.

It will be seen that whenever the reflection contributes to the flow, the rule  $T \sqrt{M} = f(\lambda)$  cannot be rigorously fulfilled; it can hold only for the quantity  $(T - T_r) \sqrt{M}$ .

The expression (14) was obtained on the assumption that no mutual collisions occur. Maxwell computed the effect of the mirror reflection on the slip at higher pressures and found that

$$\zeta = \sqrt{\frac{\pi}{2}} \sqrt{\frac{R\Theta}{M}} \frac{\eta}{p} \frac{2-f}{f} \quad (16)^{11}$$

where  $f$  is the fraction of the incident molecule sent out diffused from the surface; the part  $1 - f$  is mirror-reflected. Knauer's and Stern's values for the reflection are  $1 - f = 38 \times 10^{-8}$ ,  $f$  being practically equal to unity.

With these values, the formulas (6) and (16) give

$$B_1 \sqrt{M} = \frac{\pi a^2 \zeta p}{R_2} \sqrt{M} = \pi \sqrt{\frac{\pi}{2}} \sqrt{R\Theta} \frac{2}{R_2} \frac{2}{R_1} \quad (17)$$

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<sup>11</sup>L. B. Loeb: The Kinetic Theory of Gases. 2nd edition, p. 288.

which was to be the contribution of the slip alone. The quantity  $B_1 \sqrt{M}$  is shown in table 1; it is, in all cases, smaller than the constant term  $B$  in equation (13).

Retaining the factor  $\frac{2-f}{f}$  in equation (17) and equating  $B_1 \sqrt{M}$  to the term  $B$  in equation (13) gives the values for  $f$  which are shown in table 1 and are much smaller than unity, as well as being contradictory to the afore-mentioned small values of reflection. Since it is not likely that the reflecting power of the surface at higher pressure would be so much greater, it would seem that Maxwell's value of the slip coefficient in the manner attempted here should not be employed.

We will now discuss measurements with the other gases.

The measurements with the rectangular slit between ground glass plates (fig. 7) were made for the purpose of ascertaining the flow in a channel in which no mirror reflection occurs. Three test series with hydrogen were run off:

- I. Flow between the pure ground plates
- II. Flow between the same plates blackened with soot
- III. Slit as in II, but the air traps of the inlet tubes were cooled with liquid air

The principal data of these three test series are correlated in table 2; the individual results for series II and III are represented in figures 16 and 17.

The slit width  $a$  could only be determined by means of the laminar flow data. At high pressures

$$T = \frac{(a + 2\zeta)^3 b}{12L\eta} p = \frac{a^3 b}{12L\eta} p + \frac{a^2 \zeta b p}{2L\eta}$$

or  $T = A_1 p + B_1$ ,  $A_1$  and  $B_1$  being constants within a certain pressure zone.

Figure 17 indicates the linear relationship between  $T$  and  $p$  at high pressure. It is seen that the slope  $A_1$  can be fairly accurately defined. The value of  $A_1$ , computed from the equation  $A_1 = \frac{a^3 b}{12L\eta}$ , is the average value  $\frac{a^3}{3}$ ; but the surface irregularities are of the same order of magnitude as  $a$  (about 0.01 mm).

The minimum lies at the pressure where  $\lambda \simeq a$ . Knudsen's formula, applied to the rectangular slit, gives

$$T_{Kn} = \frac{4}{3} \sqrt{\frac{2}{\pi}} \sqrt{\frac{R^\ominus}{M}} \frac{a^2 b}{L}$$

The values  $T_{Kn} \sqrt{M}$  computed from it are given in table 2. The agreement between  $T_{min}$  and  $T_{Kn}$  is less good than for the flow in the circular slits;  $T_{min}$  is about 20 percent greater than  $T_{Kn}$ . The reason for this is, in part, that the slit width is not well-defined because of the irregularities and, in part, that the flow for the same reason differs from the flow between smooth surfaces.

To compute the flow  $T_0$  at the lowest pressures the writer attempted to use Clausing's formula

$$T_{Cl} = \frac{1}{2} \sqrt{\frac{2}{\pi}} \sqrt{\frac{R^\ominus}{M}} \left( \frac{1}{2} + \lambda \frac{L}{a} \right) \frac{a^2 b}{L}$$

the formula holds for  $b \gg L \gg a$ , which is not entirely complied with because  $b$  is only 1.4 times  $L$ . The calculated values  $T_{Cl}$  are about 25 percent greater than the observed values  $T_0$ ; this discrepancy is due to the nonagreement of the geometric assumptions.

The most significant result of these measurements is that they show that the decrease of  $T$  in the zone from  $\lambda = \infty$  to  $\lambda \simeq a$  is not produced by the mirror reflection, but rather by the collisions of the molecules among one another.

The measurements with the cylindrical slit (fig. 8) and with the gas filter (fig. 9) were made for the purpose of checking the rule  $T \sqrt{M} = f(\lambda)$  in absolutely constant channels. Hydrogen and carbon dioxide were used.

The results for the cylindrical slit are represented in figure 18. At high pressures, the measurements with the two gases are in complete agreement, as indicated by the fact that when the slope  $A_1$  in the equation  $T = A_1 p + B_1$  is graphically defined and  $a$  is computed from the equation  $A_1 = \frac{\pi a^3 r}{6L\eta}$ ,  $a = 16.1\mu$  for  $CO_2$  and  $a = 15.9\mu$  for  $H_2$ , if the previous values  $\eta_{20}$  are utilized. The slit width  $a$  being very small compared to the radius  $R$  and the length  $L$ , the cylindrical slit can be regarded as a rectangular slit of width  $2\pi r$  in the calculation, and this produces the preceding formula for  $A_1$ .

The position of the minimum is, as for the other slits, determined by the fact that  $\lambda_{\min} \simeq a$ .

The application of Knudsen's formula yields

$$T_{\text{Kn}} \sqrt{M} = \frac{8}{3} \sqrt{2\pi} \sqrt{R\Theta} \frac{a^2 r}{L}$$

which, on inserting the numerical values, gives  $T_{\text{Kn}} \sqrt{M} = 5.50$ , in close agreement with the value  $T_{\min} \sqrt{M}$  (compare fig. 18).

In the range  $\lambda \simeq 20\mu$  to  $\lambda \simeq 1000\mu$ , there is a small discrepancy between  $T \sqrt{M}$  for the two gases so that  $T \sqrt{M}_{(\text{H}_2)}$  is always greater than  $T \sqrt{M}_{(\text{CO}_2)}$  for the same value of  $\frac{p}{\lambda_1}$ . The difference is about 4 percent, at the most. A plausible explanation is that the true effective path length  $\lambda_e$  in the range  $\lambda = \infty$  to  $\lambda \simeq a$  is not the same

as the path length computed from the equation  $\lambda = \frac{\lambda_1}{p}$ . This is apparent

from the remark that  $\lambda_1$  and hence  $\lambda$  is determined by the internal friction in such a manner that only the collisions, which occasion great direction changes, are regarded as cut-offs of free path lengths. But in a slit in molecule flow the collisions initiating only small direction changes are also significant because (as stated previously) a great part of the transport in a slit is due to molecules, which fly approximately parallel to the slit walls, and this contribution to the flow is greatly diminished even by small divergences. Therefore, the effective path length  $\lambda_e$  must be less than  $\lambda$  for the same value of the pressure in the low-pressure zone.

Knauer and Stern, Mais and Rabi, and others have demonstrated that the same holds true for the effective path length which is determined by scattering of molecular rays. Mais<sup>12</sup> adduces that the effective center distances for collisions between potassium atoms and hydrogen or carbon dioxide molecules, etc., are

$$\begin{aligned} \sigma_{12} &= 7.2 \text{ and } 13.3 \text{ \AA} \text{ for deviations } < 4.5^\circ \\ \sigma_{12} &= 5.7 \text{ and } 9.6 \text{ \AA} \text{ for deviations } < 1.5^\circ \\ \sigma_{12} &= 5.0 \text{ and } 5.9 \text{ \AA} \text{ for great deviations} \end{aligned}$$

The effective mean path length  $\lambda_e \sim \frac{1}{\sigma_{12}^2}$  decreases therefore much more in  $\text{CO}_2$  than in  $\text{H}_2$ .

<sup>12</sup>W. H. Mais, Phys. Rev. 45, p. 773, 1934.

On assuming that  $T\sqrt{M} = f(\lambda_e)$ , the length A - B, cut off on a horizontal line in figure 18 between the  $\left(T\sqrt{M}, \frac{P}{\lambda_1}\right)$  curves, is equal to the  $\log_{10}$  of the ratio of the effective path lengths. Hence, it follows that  $\lambda_{eH_2}$  is everywhere greater than  $\lambda_{eCO_2}$  for the same value of  $\frac{P}{\lambda_1}$ . The ratio  $\frac{\lambda_{eH_2}}{\lambda_{eCO_2}}$  is around 2, when  $\log_{10} \frac{P}{\lambda_1} = -2.5$ . This appears to be in good agreement with Mais's results.

The pure molecule flow in a cylindrical slit has been computed by Clausing<sup>13</sup>. Application of his formula gives  $T_{C1}\sqrt{M} = 12.3$ . The measured value of  $T\sqrt{M}$  at the lowest pressures was about 9; the difference is due, in part, to the fact that the mercury vapors were not frozen out and, in part, to the fact that the measurements were not extended to sufficiently low pressures.

Figure 19 represents the test data with the glass filter as flow channel. The corrections for the inlet tube resistance were rather great (about 50 percent at the lowest pressures) since the resistance in the filter was quite small. When  $\lambda >$  about  $10\mu$ ,  $T\sqrt{M}$  is practically constant. The variation of  $T\sqrt{M} = f(\lambda)$  is exactly the same for both gases throughout the entire pressure zone. According to Weber's investigations of such filters<sup>14</sup>, the flow at low pressures should be regarded as pure effusion, which agrees with the fact that no minimum exists. This is apparent by an application of Knudsen's theory of effusion through an opening in a thin plate<sup>15</sup>.

$$N_{P1} = A(v' - v'')$$

A = area of opening,  $N_{P1}$  = number of molecules per second through opening,  $v'$  and  $v''$  are collision numbers per sec  $\text{cm}^2$  at both sides of the plate, where the pressures are  $p'$  and  $p''$ .

When each pore in the porous body can be regarded as a series of chambers, between which openings of the magnitude  $A_1, A_2 \dots A_n$  are presented, the notations of figure 20 give

$$N_i = A_1(v' - v'_1) = A_2(v'_1 - v'_2) \dots A_n(v'_{n-1} - v'')$$

<sup>13</sup>Over den Verblijftijd van Moleculen . . . Amsterdam 1932, formulas 151 and 154, pp. 108-109.

<sup>14</sup>S. Weber, Teknisk Tidsskrift 1917, Nr. 37.

<sup>15</sup>Ann. D. Phys. 28, p. 999, 1909.

from which follows

$$\frac{N_1}{A_1} + \frac{N_2}{A_2} \dots \frac{N_n}{A_n} = v' - v''$$

or

$$N_i = \frac{1}{\sum_n \frac{1}{A_n}} (v' - v'')$$

If the porous plate consists of a number (i) of such pores, the total effusion is

$$N = \sum_i N_i = (v' - v'') \sum_i \left( \frac{1}{\sum_n \frac{1}{A_n}} \right)_i \quad (18)$$

Hence, it follows that  $N$  is proportional to  $v' - v''$ , or that  $T$  is constant at low pressures. Knudsen's theory is applicable to pressures where  $\lambda$  is great compared to the diameter of the opening  $A$ . At higher pressures the ratio  $\frac{N_{Pl}}{v' - v''}$  increases with  $p$ . Applied to equation 18, it indicates that  $N$  must increase also with increasing pressure in good agreement with the measurements.

Translated by J. Vanier  
National Advisory Committee  
for Aeronautical

Table I

## Flow in the Annular Slit

1	Fig.	10	11	12	13	
2	$R_1$ cm	1.1952	0.8994	0.8123	0.8123	0.8123
3	$R_2$ cm	1.8897	1.0919	1.5984	1.5984	1.5984
4	$a_{opt} \mu$	9.23	3.53	8.17	8.13	8.15
5	Gas	$\overbrace{H_2 \quad O_2 \quad CO_2 \quad A}$	$H_2$	$H_2$	$H_2$	Air
6	$T_{min} \sqrt{M}$	2.00 2.01 1.92 1.98	0.70	1.09	1.04	----
7	$T_O \sqrt{M}$	$\overbrace{4.0 \quad 3.9 \quad 3.8 \quad 3.8}$	1.6	2.4	2.6	----
8	$T_{Kn} \sqrt{M}$	1.95	0.67	1.03	1.02	----
9	A	1.55	0.194	0.73	0.72	----
10	B	1.62	0.560	0.86	0.78	----
11	C	0.86	0.393	0.60	0.655	----
12	D	2.0	1.8	1.5	1.5	----
13	$a_{str} \mu$	9.02	3.38	7.98	7.94	7.98
14	$a_{opt} - a_{str} \mu$	0.21	0.15	0.19	0.19	0.17
15	$\frac{R_2 + R_1}{R_2 - R_1}$	4.44	10.4	3.07	3.07	3.07
16	$T_{C1} \sqrt{M}$	5.28	1.72	2.95	2.92	----
17	B/C	1.89	1.43	1.43	1.19	----
18	$T_{Ef} \sqrt{M}$	0.32	0.12	0.15	0.15	----
19	$T_r \sqrt{M}$	$\overbrace{0.10 \quad 0.03 \quad 0.02 \quad 0.02}$	0.02	0.07	0.07	----
20	$B_1 \sqrt{M}$	1.14	0.40	0.61	0.60	----
21	f	0.83	0.83	0.83	0.87	----

Table II

## Flow of Hydrogen in Rectangular Slits

	Size of Slit			Observed		Calculated	
	L cm	b cm	$a\mu$	$T_{\min}\sqrt{M}$	$T_o\sqrt{M}$	$T_{Kn}\sqrt{M}$	$T_{Cl}\sqrt{M}$
I	0.971	1.320	19.2	0.94	1.6	0.80	2.14
II	0.963	1.323	18.1	0.92	1.4	0.75	1.89
III	0.963	1.323	18.2	0.92	1.5	0.76	1.92

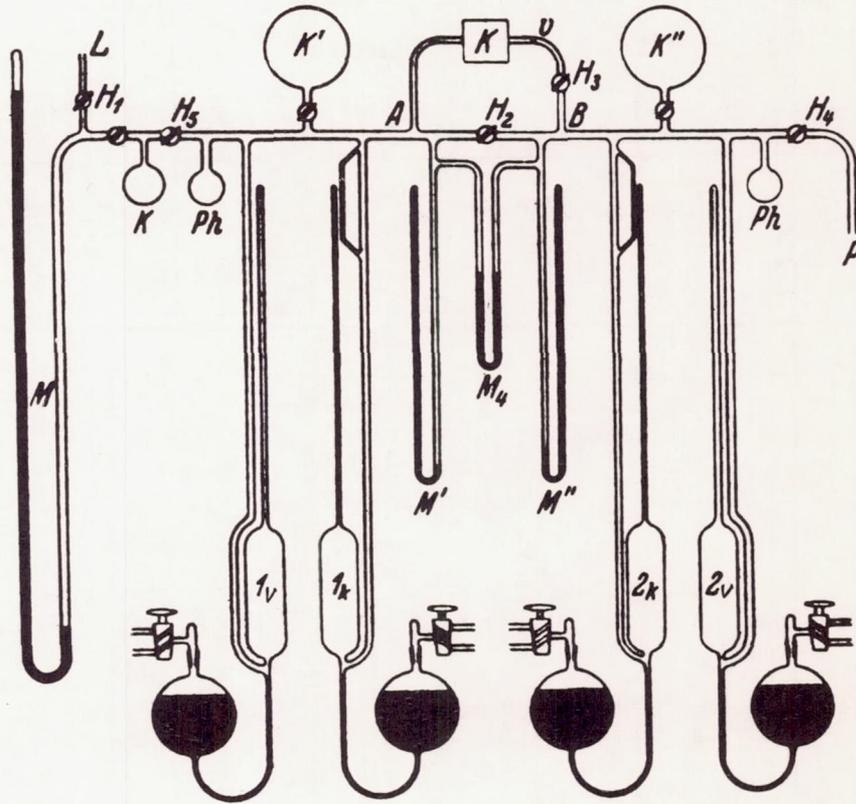


Figure 1.- Pressure-gage assembly.

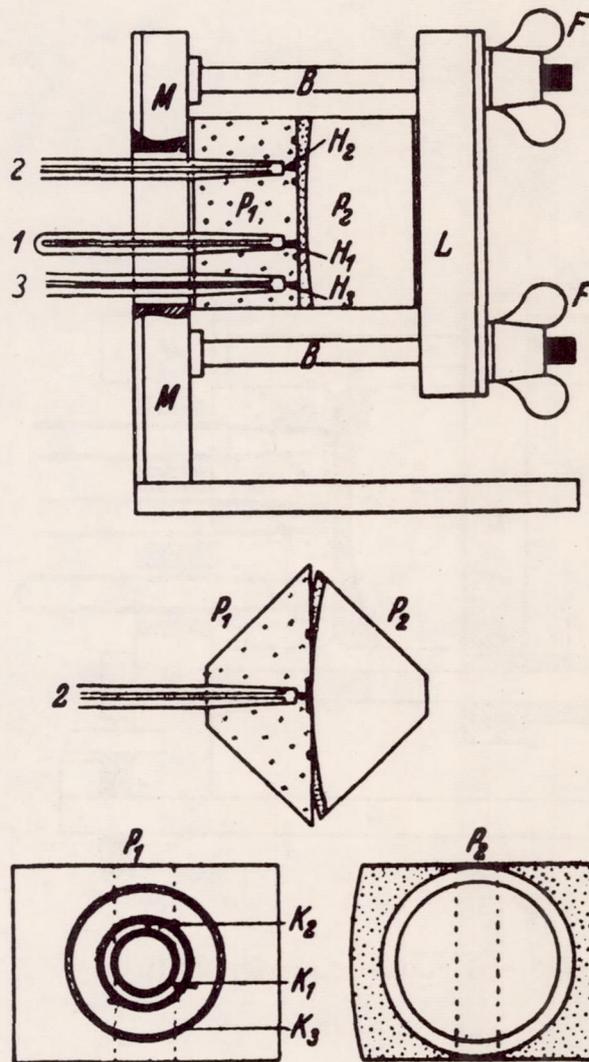


Figure 2.- Prismatic apparatus I.

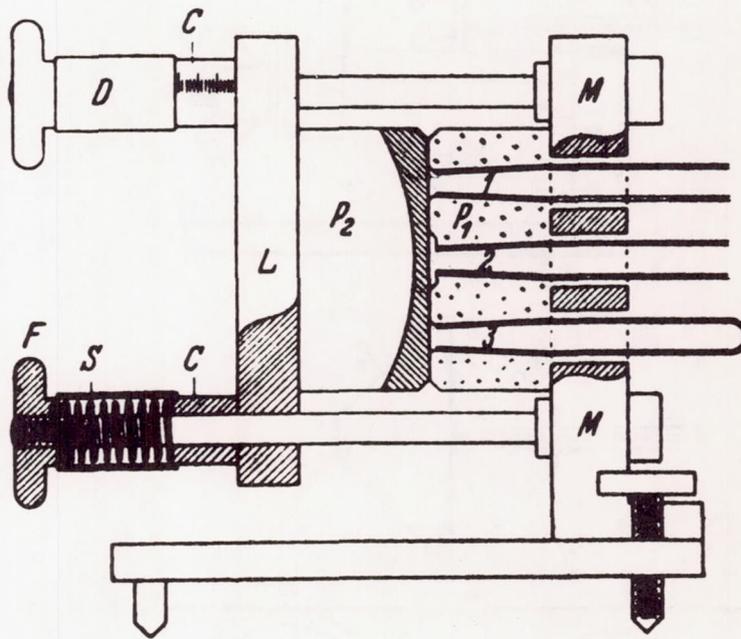


Figure 3.- Prismatic apparatus II.

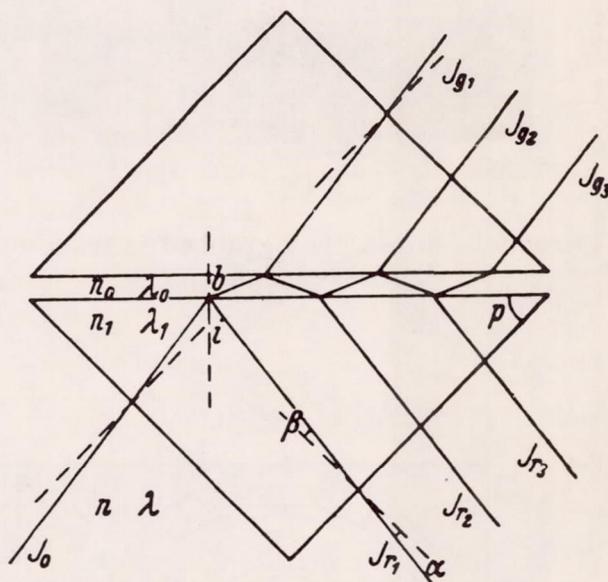


Figure 4.- Theory of Herschel's interference fringes.

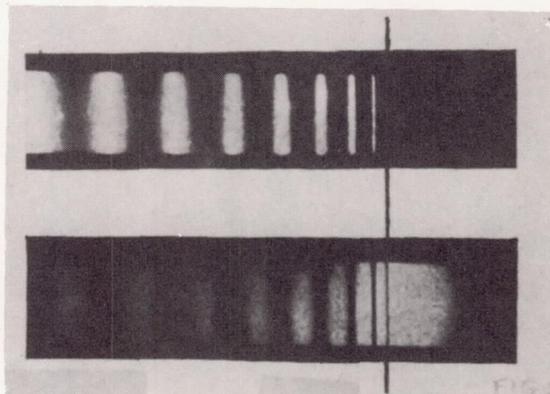


Figure 5.- Herschel fringes photographed in transmitted (top) and reflected light.

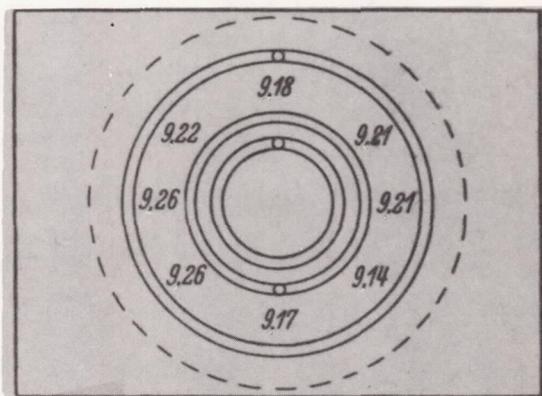


Figure 6.- The hypotenusal surface of prism  $P_1$  of the first prismatic apparatus showing measured distances in units  $\mu$ .

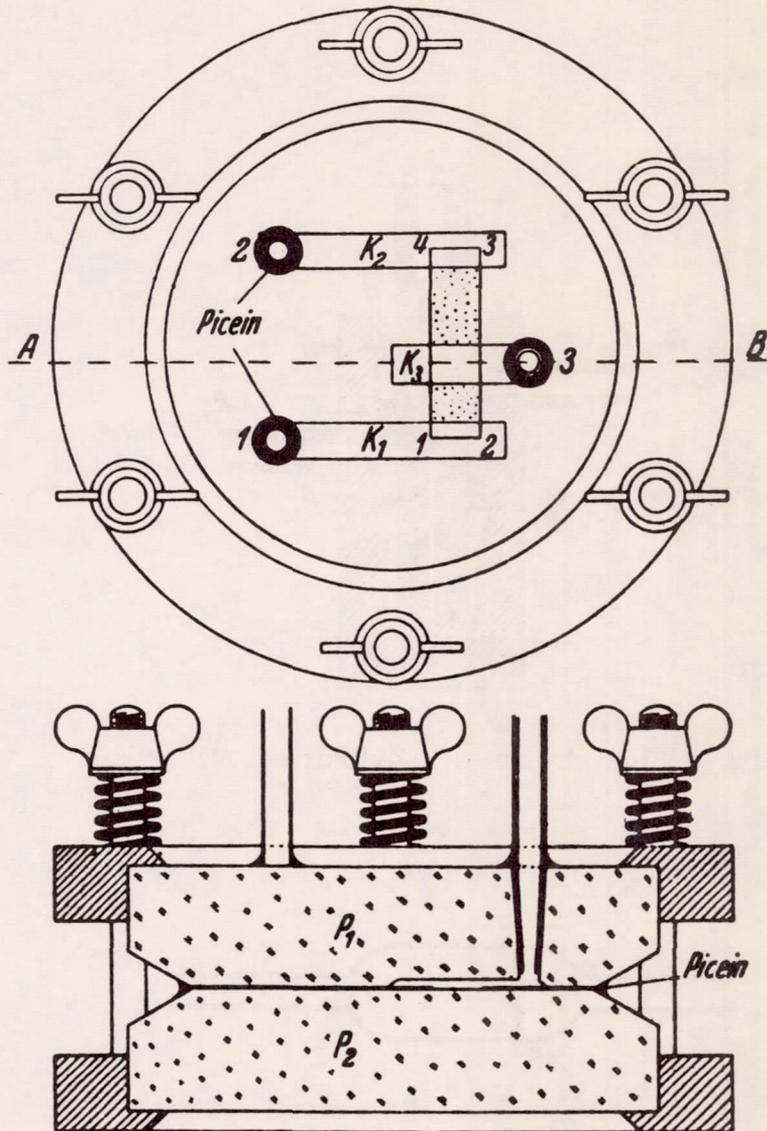


Figure 7.- Rectangular slit between ground glass plates.

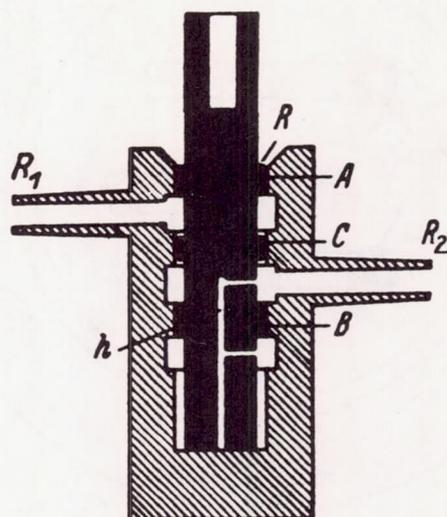


Figure 8.- Flow channel with cylindrical slit around  $C$ .

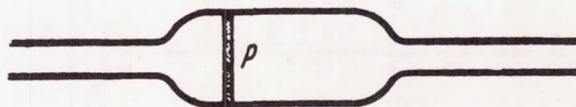


Figure 9.- Filter plate.

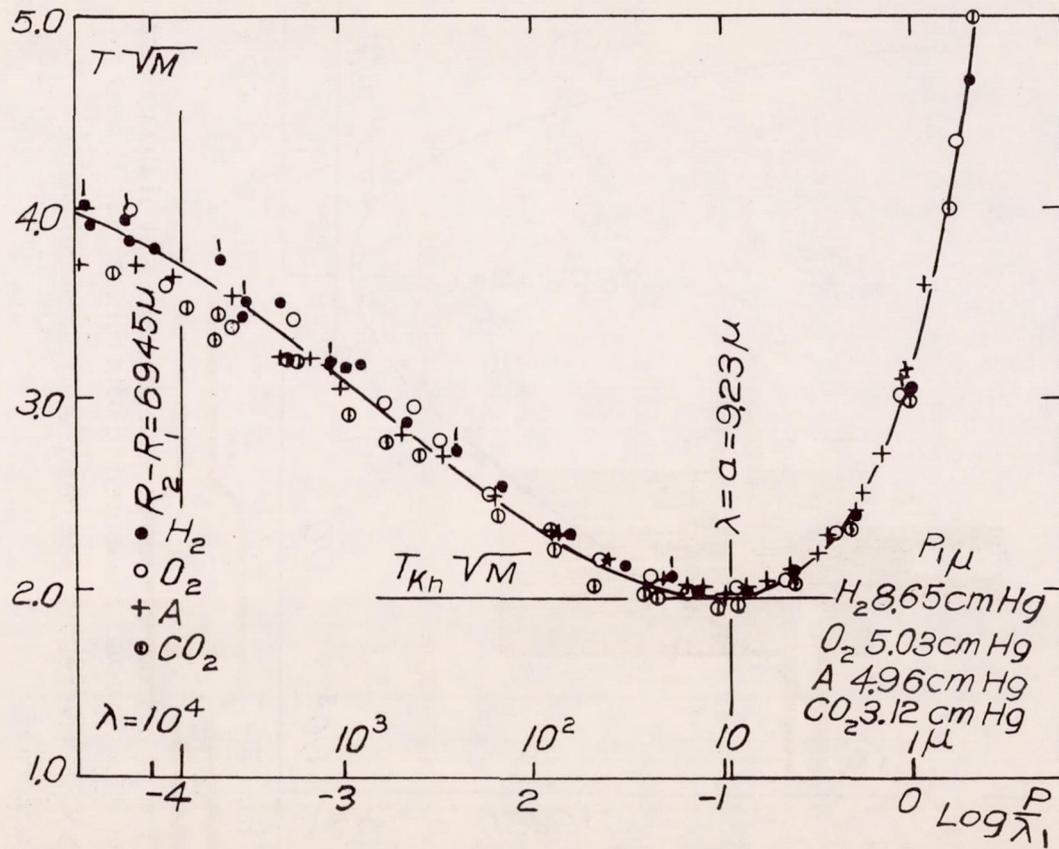


Figure 10.- Flow in prismatic apparatus I, in the wide annular slit. The hydrogen measurements marked with | were made about 2 months later than the others; in the meantime, the measurements were made with the other gases.

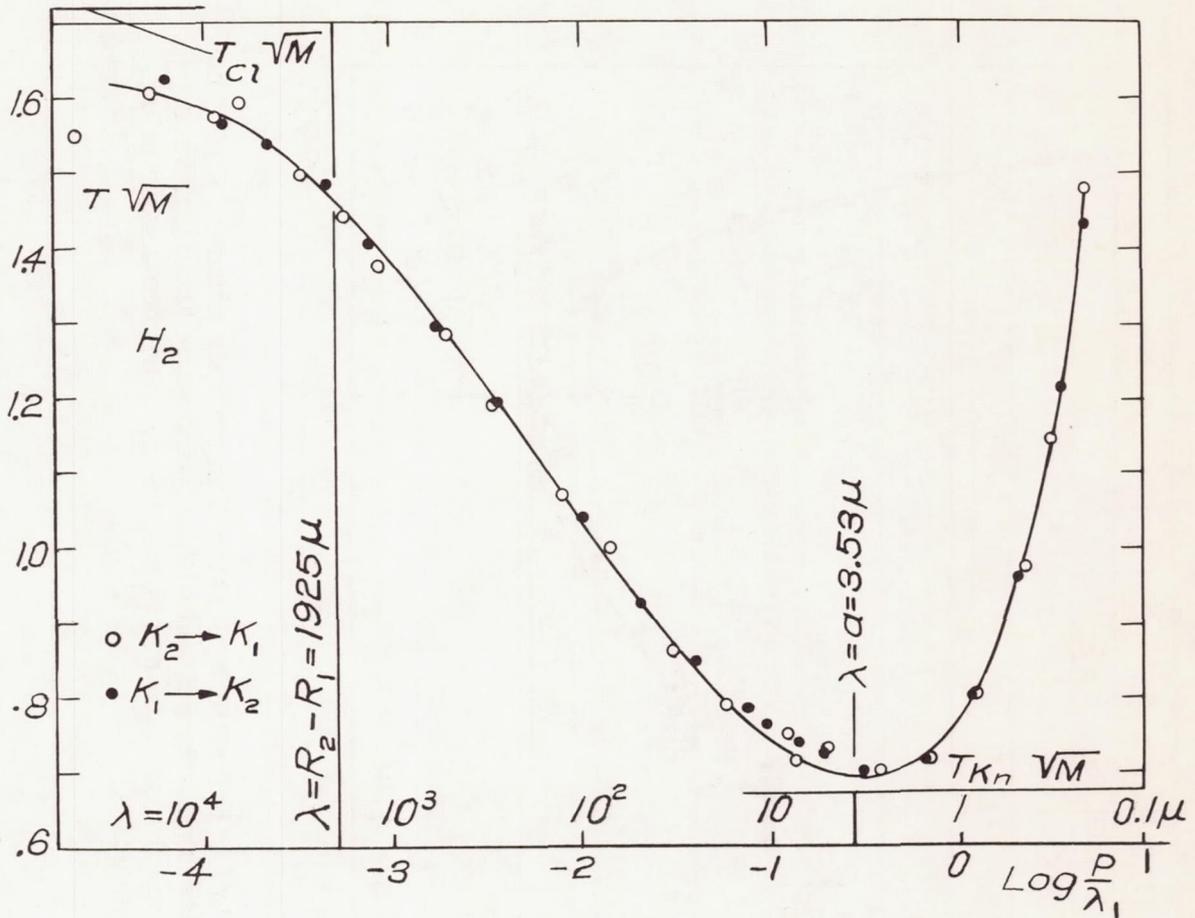


Figure 11.- Flow of hydrogen in the narrow circular slit of the prismatic apparatus I. The flow was measured in both directions  $K_1 \rightarrow K_2$  and  $K_2 \rightarrow K_1$ . The results are completely identical within measuring accuracy.

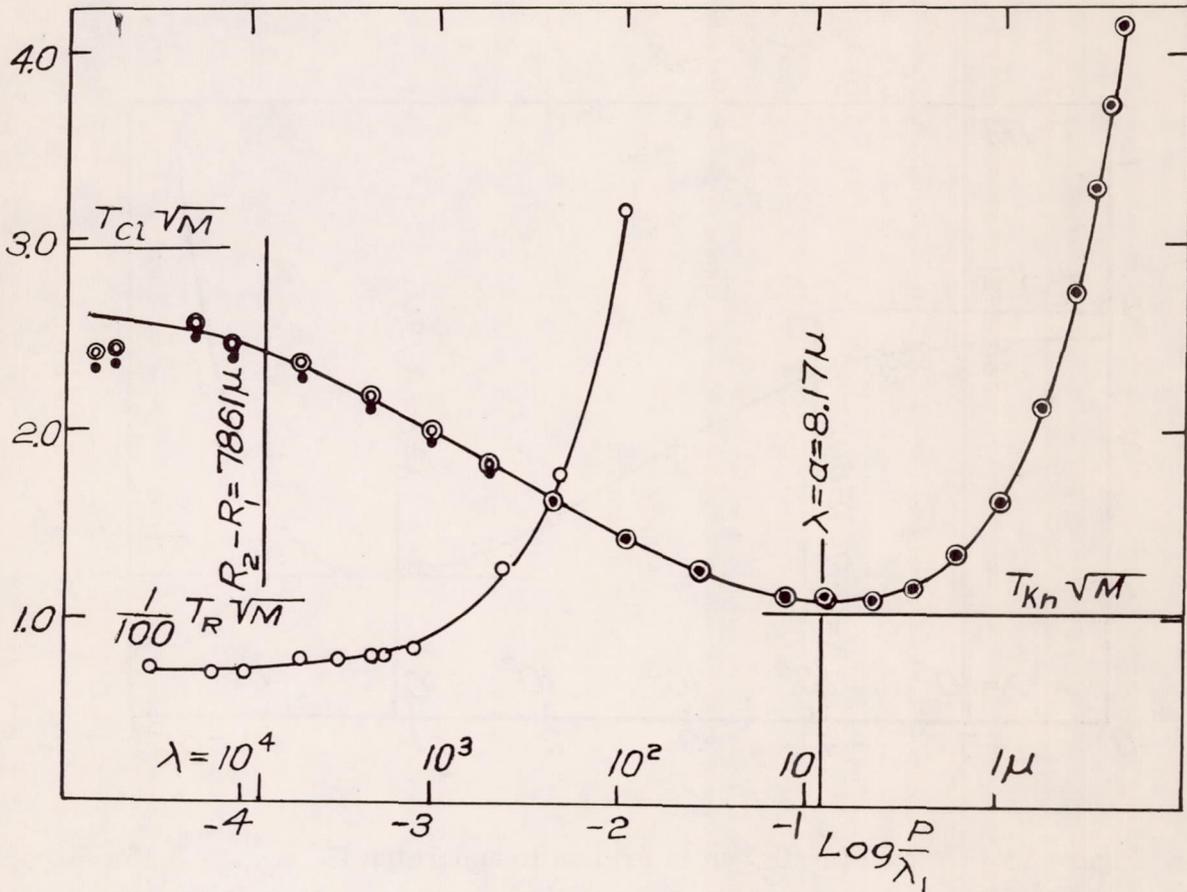


Figure 12.- Flow of hydrogen in prismatic apparatus II.  $a_{opt.} = 8.17 \mu$ .

The graph shows the uncorrected values of  $T \sqrt{M}$  ( $\cdot$ ), as well as those corrected for inlet tube resistance ( $\odot$ ) along with the measurements of the quantity  $T_R \sqrt{M}$  at 100 times smaller ordinate scale.

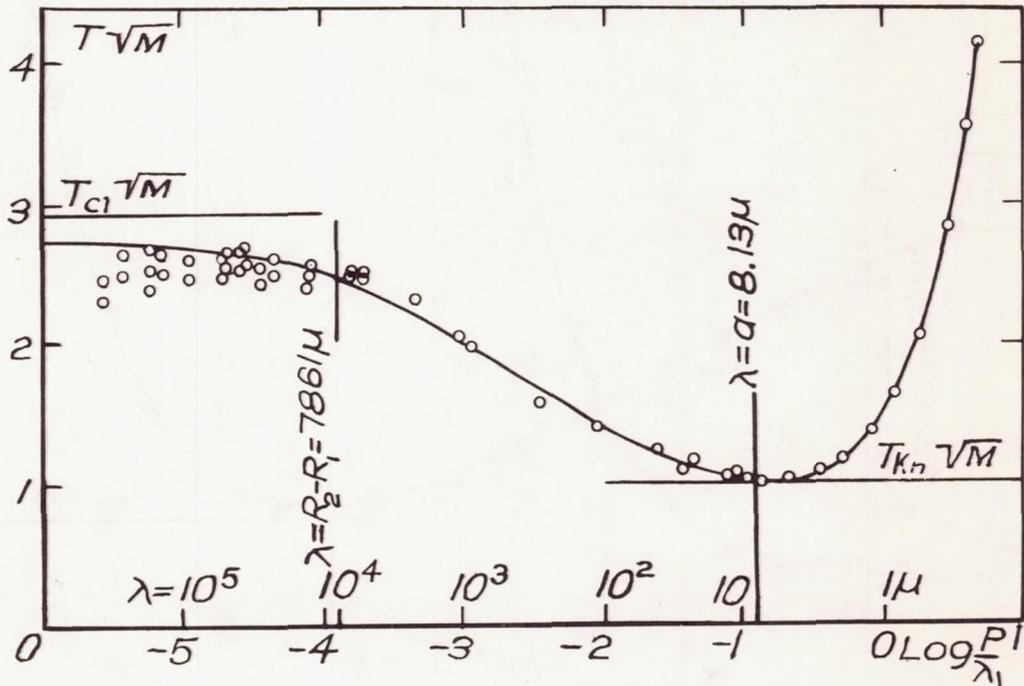


Figure 13.- Flow of hydrogen in prismatic apparatus II.  $a_{opt.} = 8.13 \mu$ .

The air locks (or traps) of the inlet tubes are cooled with liquid air. The effect of the effusion is indicated. The upper point of the two points linked together is not corrected, the lower one is the value of  $T\sqrt{M}$  corrected for the effusion.

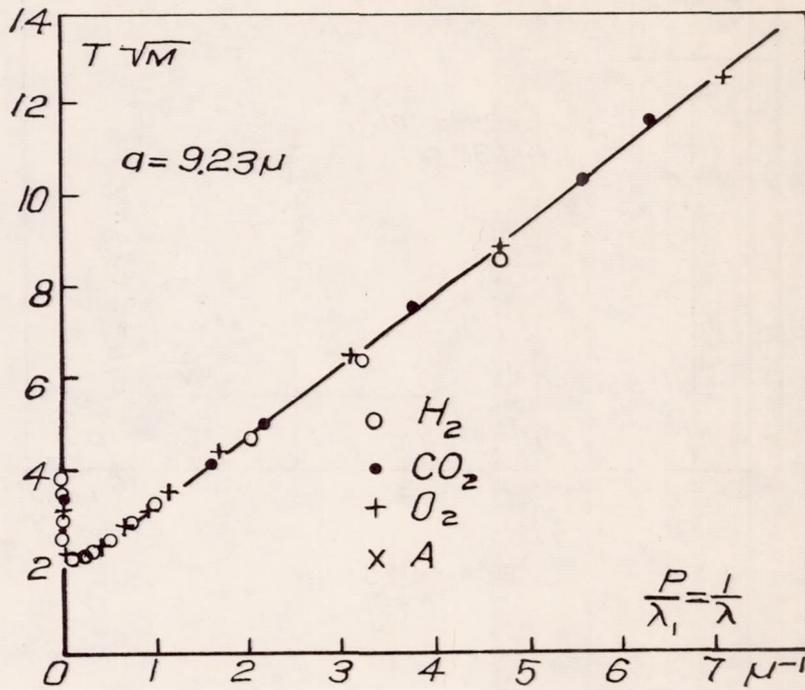


Figure 14.- Flow in the wide annular slit at high pressure.

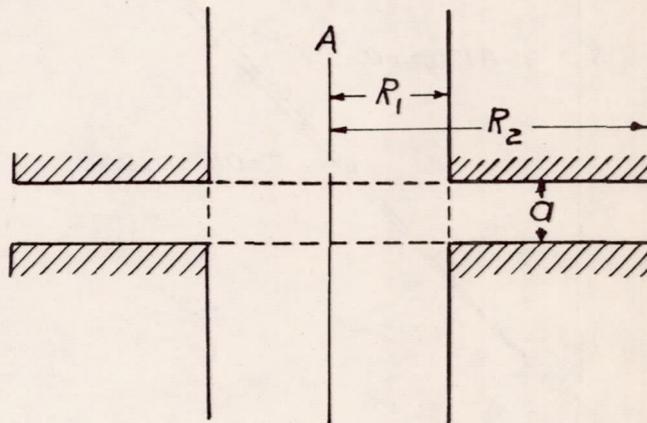


Figure 15.- Cut placed normal to the plane of the annular slit; the axis of rotation lies in the drawing plane.

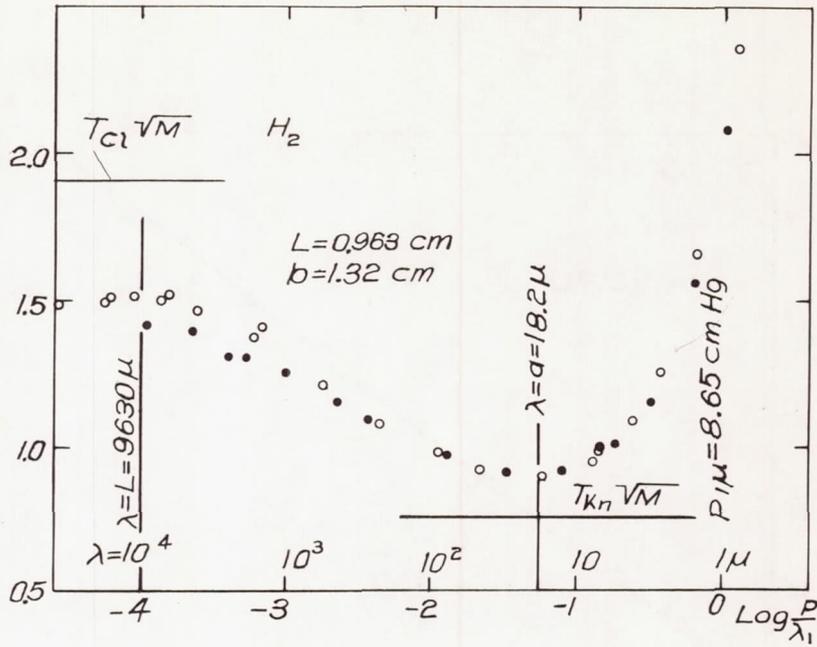


Figure 16.

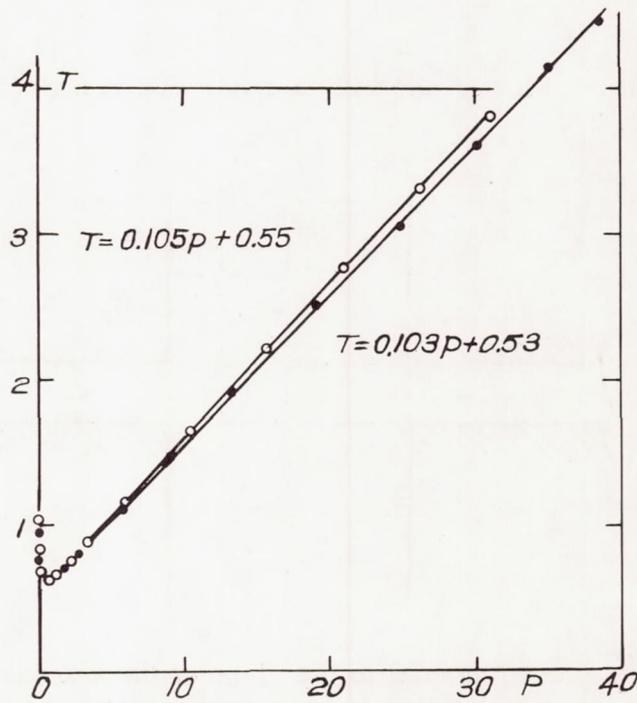


Figure 17.

Figures 16 and 17.- Flow of hydrogen in the rectangular slit.

- Indicates liquid-cooled air traps
- Indicates no cooling

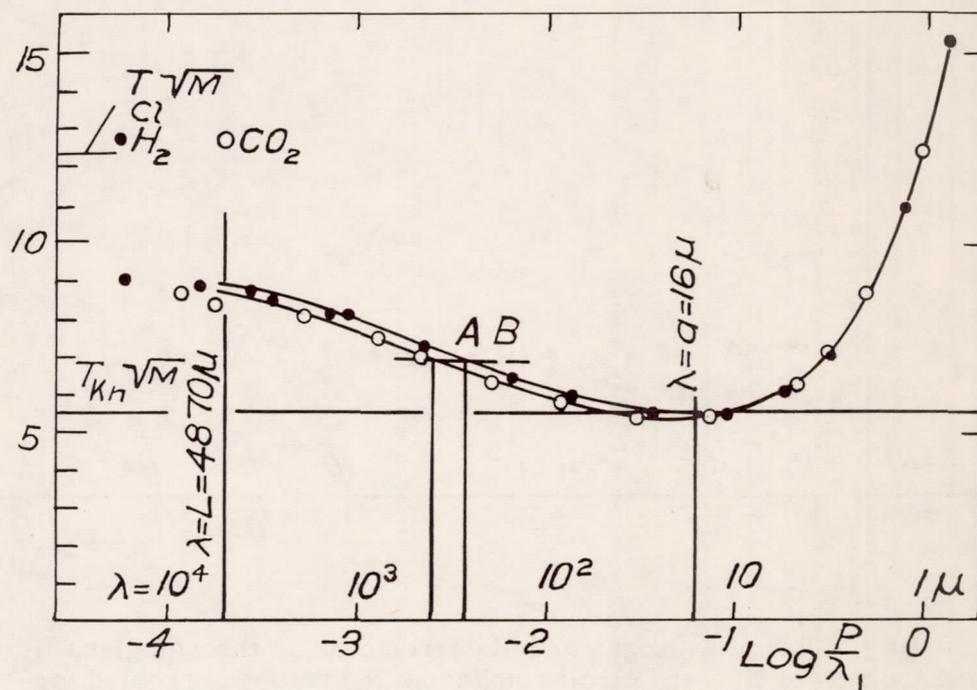


Figure 18.- Flow of hydrogen and of carbon dioxide in the cylindrical slit;  
 $r = 1.00 \text{ cm}$ ,  $L = 0.487 \text{ cm}$ ,  $a = \Delta r = 16.0\mu$ .

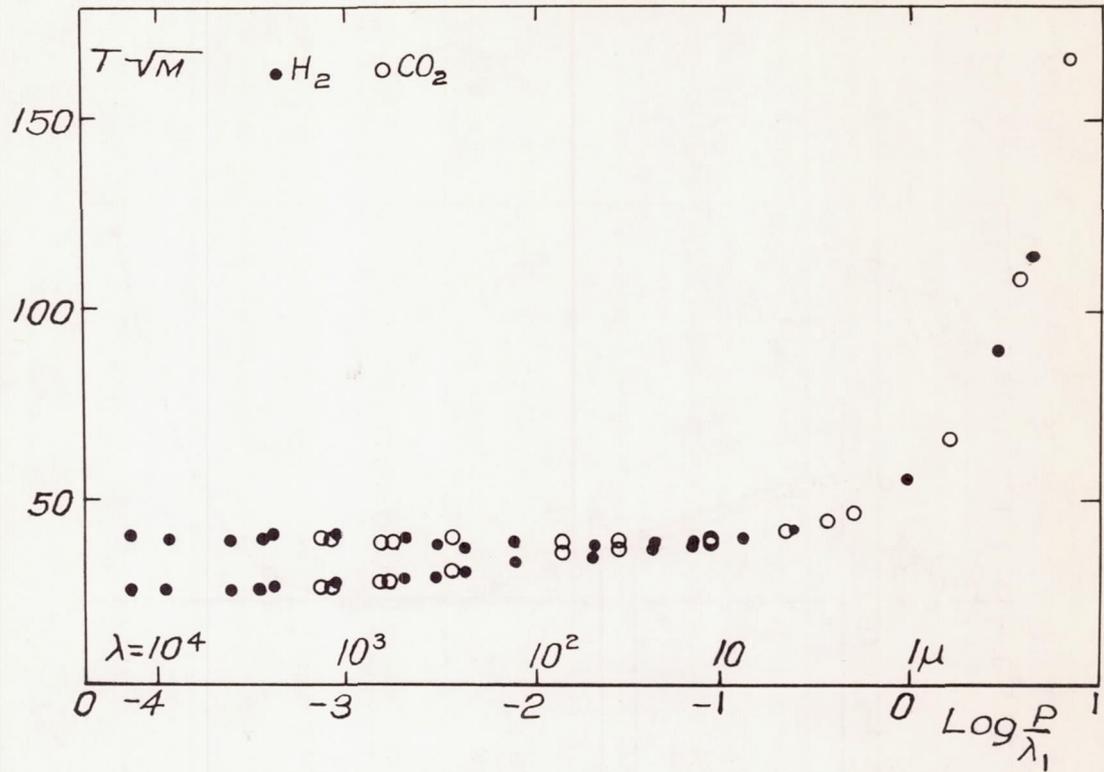


Figure 19.- Flow of hydrogen and of carbon dioxide through glass filter; the top row of dots and circles indicates the values corrected for inlet resistance.

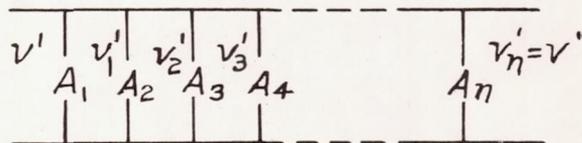


Figure 20.- Each pore equals a series of chambers.