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GAS FLOW WITH STRAIGHT TRANSITION LINE

By L. V. Ovsiannikov

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On the basis of the solutions obtained by S. A. Chaplygin (reference 1), an investigation was made of the limiting case of a gas flow when the constant pressure in the surrounding medium is exactly equal to the critical pressure for the given initial state of the gas; the results are presented herein. For a jet flowing out of the opening in a vessel with plane walls, it is shown that equalization of the flow in the jet is attained at a finite distance from the start of the free jet, the line of transition being a straight line.

1. According to Chaplygin (reference 1), every problem on the determination of the subsonic flow that is satisfied by some conditions reduces to the solution of the system

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\theta} &= \frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\psi}{\partial\tau}\\ \frac{\partial\varphi}{\partial\tau} &= -\frac{\alpha-\tau}{2\alpha\tau(1-\tau)^{\beta+1}}\frac{\partial\psi}{\partial\theta}\end{aligned}\right\}\tag{1.1}$$

or the equivalent system

$$\left.\begin{aligned}\frac{\partial\tau}{\partial\psi} &= \frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\theta}{\partial\varphi}\\ \frac{\partial\tau}{\partial\varphi} &= -\frac{2\alpha\tau(1-\tau)^{\beta+1}}{\alpha-\tau}\frac{\partial\theta}{\partial\psi}\end{aligned}\right\}\tag{1.2}$$

for the corresponding boundary conditions.

In these equations, $\psi = \psi(\theta,\tau)$ and $\varphi = \varphi(\theta,\tau)$ are the stream function and the velocity potentia.., respectively; $\tau = V^2/V^2_{max}$, where

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*"Ob Odnom Gazovom Techenii s Priamoi Liniei Perekhoda." Prikladnaya Matematika i Mekhanika. Vol. XIII, 1949, pp. 537-542.

V is the magnitude of the velocity vector and V_{max} is the maximum value of V that satisfies the given initial state of the gas; and θ is the angle between the velocity vector and the x-axis. Furthermore, $\beta = 1/(\kappa-1)$ is the adiabatic power of the function of the density on the temperature, where for brevity,

$$\alpha = \frac{1}{2\beta+1} = \frac{\kappa-1}{\kappa+1}$$

so that the value $\tau = \alpha$ corresponds to the critical velocity

$$V_{er} = \sqrt{\frac{\kappa-1}{\kappa+1}}\, V_{max}$$

Chaplygin (reference 1) gives the solution of the problem of the flow of a gas jet out of a vessel with plane walls forming an angle of 180° into a medium with pressure p_0 = constant in the form

$$\frac{\pi}{Q}\psi = -\theta - \sum_{n=1}^{\infty} \frac{1}{n}\frac{z_n}{z_{n0}} \sin 2n\theta \tag{1.3}$$

$$\frac{\pi}{Q}\varphi = C + \frac{1}{2}\int \frac{d\tau}{\tau(1-\tau)^{\beta}} + \frac{1}{(1-\tau)^{\beta}}\left[-1 + \sum_{n=1}^{\infty} \frac{1}{n}\frac{z_n}{z_{n0}} x_n \cos 2n\theta\right] \tag{1.4}$$

where this solution satisfies the following boundary conditions (fig. 1):

$$\psi = -\frac{1}{2}Q \qquad \text{on ABC}$$

$$\psi = \frac{1}{2}Q \qquad \text{on A'B'C'}$$

In equations (1.3) and (1.4), Q is the relative quantity of flow in the gas jet and C is an arbitrary constant depending on the choice of the origin from which the values of φ are computed. The magnitudes z_n, z_{n0}, and x_n are defined by the equations

$$\left.\begin{aligned} z_n &= z_n(\tau) = \tau^n\, y_n(\tau) \\ z_{n0} &= z_n(\tau_0) \\ x_n &= x_n(\tau) = 1 + \frac{\tau}{n}\frac{y_n'(\tau)}{y_n(\tau)} = \frac{\tau}{n}\frac{z_n'(\tau)}{z_n(\tau)} \\ &(n = 1, 2, \ldots) \end{aligned}\right\} \tag{1.5}$$

where $y_n(\tau)$ is that solution of the hypergeometric equation

$$\tau(1-\tau)\, y_n'' + \left[2n + 1\ (\beta-2n-1)\tau\right] y_n' + \beta n\,(2n+1)\, y_n = 0$$

which is obtained for $\tau = 0$; τ_0 denotes the value of τ corresponding to p_0.

In Chaplygin's investigation, it was assumed that $\tau \leqslant \tau_0 \leqslant \alpha$. In this case, equations (1.3) and (1.4) converge everywhere in the region of flow and give the proposed solution of the problem. The case where $\tau_0 = \alpha$ is considered in the following development:

It is recalled that the functions $z_n(\tau)$ and $x_n(\tau)$ for $0 \leqslant \tau \leqslant \alpha$ satisfy the following inequality of Chaplygin:

$$\left[\frac{\tau(1-\tau)^{2\beta}}{\tau_0(1-\tau_0)^{2\beta}}\right]^n \geqslant \frac{z_n}{z_{n0}}; \quad z_n \geqslant 0 \tag{1.6}$$

$$\sqrt{\frac{\alpha-\tau}{\alpha(1-\tau)}} + \frac{q\tau}{n^{1/3}\,(1-\tau)} > x_n > \sqrt{\frac{\alpha-\tau}{\alpha(1-\tau)}}$$

$$q = \sqrt[3]{2\beta^2\,(1+2\beta)} = \text{constant} \tag{1.7}$$

2. It shall be shown that the potential of the velocity φ given by equation (1.4) for $\tau \rightarrow \tau_0$

(a) increases without limit if $\tau_0 < \alpha$

(b) remains finite if $\tau_0 = \alpha$

It shall be assumed that the limit $\tau \longrightarrow \tau_0$ is effected by moving along some streamline. The following notation is introduced:

$$\xi = \frac{\tau(1-\tau)^{2\beta}}{\tau_0(1-\tau_0)^{2\beta}}$$

$$P(\theta,\tau;\tau_0) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{z_n}{z_{n0}} x_n \cos 2n\theta$$

It is readily seen that for $0 < \tau < \tau_0 \leqslant \alpha$, then $0 < \xi < 1$.

For the potential φ at the points of the x-axis $(\theta=0)$, the following expression is obtained from equation (1.4):

$$\frac{\pi}{Q} \varphi = C + \frac{1}{2}\int \frac{d\tau}{\tau(1-\tau)^{\beta}} + \frac{1}{(1-\tau)^{\beta}}\left[-1 + P(0,\tau;\tau_0)\right] \tag{2.1}$$

On the basis of the inequalities of equations (1.6) and (1.7), for the magnitude of $P(0,\tau;\tau_0)$,

$$\sqrt{\frac{\alpha-\tau}{\alpha(1-\tau)}} \sum_{n=1}^{\infty} \frac{1}{n}\xi^n + \frac{q\tau}{1-\tau}\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}\xi^n > P(0,\tau;\tau_0) > \sqrt{\frac{\alpha-\tau}{\alpha(1-\tau)}} \sum_{n=1}^{\infty} \frac{1}{n}\frac{z_n}{z_{n0}} \tag{2.2}$$

Assertion (a) now follows in an obvious manner from the second of inequalities (2.2) because $z_n/z_{n0} \longrightarrow 1$ for $\tau \longrightarrow \tau_0$ and the sum of the series with the general term z_n/nz_{n0} therefore increases without limit; whereas the coefficient preceding the term, because $\tau_0 \neq \alpha$, approaches a positive limit.

From assertion (b), it is seen that

$$\sum_{n=1}^{\infty} \frac{1}{n}\xi^n = -\log(1-\xi) = -\log\left[1 - \frac{\tau(1-\tau)^{2\beta}}{\tau_0(1-\tau_0)^{2\beta}}\right]$$

by virtue of which, for $\tau_0 = \alpha$ as $\tau \to \alpha$ in equation (2.2) on the left, the first component approaches zero and the second component approaches a finite magnitude equal to

$$\frac{a\alpha}{1-\alpha} \sum_{n=1}^{\infty} n^{-4/3}$$

The same reasoning, together with the first of inequalities (2.2) is repeated for $P(\theta,\tau;\alpha)$ for $\theta \neq 0$, so that the assertion is completely proven.

3. From the previously proven boundness of the velocity potential it follows that at a finite distance from the opening BB', the jet is intersected by a certain line L along which $\tau = \alpha$, so that the velocity is equal to the velocity of sound. It will now be shown that at all points of L, $\theta = 0$.

First, it will be observed that along any fixed streamline, θ varies monotonically, as is true of the boundary streamline, and the transformation $(\theta,\tau) \to (\varphi,\psi)$ is a single sheet transformation at every interior point of the flow region. Next, by fixing some value $\psi = \bar{\psi}$ for which $|\bar{\psi}| < \frac{1}{2} Q$, the limit is approached as $\tau \to \alpha$ and $\theta_0 = \lim \theta$ is set for $\tau \to \alpha$ and $\psi = \bar{\psi}$; this limit exists because of the finiteness of θ and the monoticity of its variation.

Approaching this limit in equation (1.3) for $\psi = \bar{\psi}$ yields

$$\frac{\pi}{Q} \bar{\psi} = - \theta_0 - \sum_{n=1}^{\infty} \frac{1}{n} \sin 2n\theta_0 = \begin{cases} - \theta_0 - \left(\frac{\pi}{2} - \frac{2\theta_0}{2}\right) = - \frac{\pi}{2} & \text{if } \theta_0 > 0 \\ - \theta_0 - \left(- \frac{\pi}{2} - \frac{2\theta_0}{2}\right) = + \frac{\pi}{2} & \text{if } \theta_0 > 0 \end{cases}$$

that is, in all cases for $\theta_0 \neq 0$, $|\bar{\psi}| = \frac{1}{2} Q$, which contradicts the assumption that $|\bar{\psi}| < \frac{1}{2} Q$. Hence, the equality $\theta = 0$ must hold.

It will now be shown that the line L is straight: Along L, φ = constant; this result is readily obtained if the previously considered transition to the limit is carried out in equation (1.4).

It is sufficient to note that every displacement in the plane xy is connected with the corresponding displacement in the plane $\varphi\psi$ by the relation

$$dx = \frac{\cos\theta}{V}\,d\varphi - \frac{\sin\theta}{V(1-\tau)}\,d\psi \tag{3.1}$$

from which it follows that in a displacement along L in the xy-plane, $dx = 0$, because in a displacement along L in the $\varphi\psi$-plane, according to what was previously proven, $\theta = 0$ and $d\varphi = 0$. The line L in the xy-plane is thus a straight line perpendicular to the x-axis.

The equalization of the jet occurs along the line L. Behind this line the jet becomes uniform, flowing with constant velocity everywhere equal to the velocity of sound.

The distance of the line L from the edge of the opening will be computed. Along the boundary of the jet, ψ = constant and $\tau = \alpha$ = constant, so that at the points of this boundary, equation (3.1) assumes the form

$$dx = \frac{\cos\theta}{V_{cr}}\frac{\partial\varphi}{\partial\theta}\,d\theta$$

Substituting the expression for φ (equation (1.4)) taken for $\tau = \alpha$ yields

$$dx = -\frac{Q}{\pi V_{cr}(1-\alpha)^{\beta}}\sum_{n=1}^{\infty} 2x_n(\alpha)\sin 2n\theta\cos\theta\,d\theta \tag{3.2}$$

Inasmuch as

$$2\sin 2n\theta\cos\theta = \sin(2n+1)\theta + \sin(2n-1)\theta$$

integrating equation (3.2) from $x = x_B$, $\theta = \frac{1}{2}\pi$, to $x = x_L$, $\theta = 0$ yields

$$x_L - x_B = \frac{Q}{\pi V_{cr}(1-\alpha)^{\beta}}\sum_{n=1}^{\infty}\frac{4n}{4n^2-1}x_n(\alpha)$$

If h denotes the width of the jet where it is uniform, then $Q = V_{cr}(1-\alpha)^{\beta} h$, so that the required distance is finally obtained in the form

$$\frac{x_L - x_B}{h} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{4n}{4n^2-1} x_n(\alpha) \tag{3.3}$$

4. If any two streamlines of the obtained flow are taken as the walls of a certain nozzle, the subsonic flow within this nozzle will be determined by equations (1.3) and (1.4). This flow becomes uniform at a certain distance and has a straight transition line. This fact corresponds entirely to the result obtained by S. A. Christianovich in his investigation (reference 2), where a general device is given for the construction of a Laval nozzle with straight transition line.

The first derivatives of θ,τ with respect to φ,ψ are estimated near the line L inasmuch as the behavior determines the possibility of continuing the obtained flow across L as a supersonic flow. The investigation of this complicated problem has been started and, it is hoped, will be presented in a forthcoming report. Remarks herein will be restricted to the following:

In the first place, from the previously noted properties of the line L, it follows that the derivatives $\partial\theta/\partial\psi$, $\partial\tau/\partial\psi$, and $\partial\theta/\partial\varphi$ are equal to zero on L.

In the second place, the derivative $\partial\tau/\partial\varphi$ is evaluated at the points of the x-axis. It is sufficient to consider the derivative $\partial\varphi/\partial\tau$, inasmuch as, on the x-axis, because $\theta = 0$ and ψ = constant,

$$\frac{\partial\tau}{\partial\varphi} = \frac{1}{\partial\varphi/\partial\tau} \tag{4.1}$$

Substituting equation (1.3) for ψ in the second of equations (1.1) for $\tau = \alpha$ and setting $z_{n\alpha} = z_n(\alpha)$ yield

$$\frac{\partial\varphi}{\partial\tau} = \frac{Q(\alpha-\tau)}{2\pi\alpha\tau(1-\tau)^{\beta+1}}\left[1 + 2\sum_{n=1}^{\infty} \frac{z_n}{z_{n\alpha}} \cos 2n\theta\right]$$

or, on the x-axis where $\theta = 0$

$$\frac{\partial\varphi}{\partial\tau} = \frac{Q(\alpha-\tau)}{\pi\alpha\tau(1-\tau)^{\beta+1}}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{z_n}{z_{n\alpha}}\right] \tag{4.2}$$

For estimating the value of the expression in brackets, which is denoted by S, the last of relations (1.5) and the inequality of Chaplygin (equation (1.7)) yield

$$\frac{n}{\tau}\sqrt{\frac{\alpha-\tau}{\alpha(1-\tau)}} + \frac{q}{1-\tau}n^{2/3} > \frac{d}{d\tau}\log z_n(\tau) > \frac{n}{\tau}\sqrt{\frac{\alpha-\tau}{\alpha(1-\tau)}} \tag{4.3}$$

Integrating equation (4.3) from $\tau \geq \tau_1 > 0$ to $\tau = \alpha$ yields

$$n\int_{\tau}^{\alpha}\frac{1}{\tau}\sqrt{\frac{\alpha-\tau}{\alpha(1-\tau)}}\,d\tau + qn^{2/3}\int_{\tau}^{\alpha}\frac{d\tau}{1-\tau} > \log\frac{z_{n\alpha}}{z_n}\quad n\int_{\tau}^{\alpha}\frac{1}{\tau}\sqrt{\frac{\alpha-\tau}{\alpha(1-\tau)}}\,d\tau$$

Raising the upper and lowering the lower limits and carrying out the integration give

$$\frac{2n}{3\tau_1\sqrt{\alpha(1-\alpha)}}(\alpha-\tau)^{3/2} + \frac{qn^{2/3}(\alpha-\tau)}{1-\alpha} > \log\frac{z_{n\alpha}}{z_n} > \frac{2n}{3\alpha\sqrt{\alpha}}(\alpha-\tau)^{3/2} \tag{4.4}$$

If $\alpha-\tau = z(0<z\leq\alpha-\tau_1)$, where $z\rightarrow 0$ as $\tau\rightarrow\alpha$ and

$$\gamma = \frac{2}{3\tau_1\sqrt{\alpha(1-\alpha)}} > 0$$

$$q_1 = \frac{q}{1-\alpha} > 0$$

$$\delta = \frac{2}{3\alpha\sqrt{\alpha}} > 0$$

equation (4.4) assumes the form

$$\gamma\, nz^{3/2} + q_1 n^{2/3}z > \log\frac{z_{n\alpha}}{z_n} > \delta nz^{3/2}$$

Then the inequalities

$$\exp(-\gamma n z^{3/2} - q_1 n^{2/3} z) < \frac{z_n}{z_{n\alpha}} < \exp(-\delta n z^{3/2}) \tag{4.5}$$

are obtained.

Because of equation (4.5), the following inequalities are obtained for S:

$$S_1 = \sum_{n=1}^{\infty} \exp(-\gamma n z^{3/2} - q_1 n^{2/3} z) < S < \frac{1}{2} + \sum_{n=1}^{\infty} \exp(-\delta n z^{3/2}) = S_2 \tag{4.6}$$

The sum S_2 is readily computed;

$$\frac{1}{2} < S_2 = \frac{1}{2} + \frac{1}{\exp(\delta z^{3/2}) - 1} < \frac{1}{2} + \frac{1}{\delta z^{3/2}} \tag{4.7}$$

In order to estimate the sum S_1,

$$u_n = \exp(-q_1 n^{2/3} z)$$

$$v_n = \exp(-\gamma n z^{3/2})$$

$$S_n' = \sum_{k=0}^{n} v_k$$

$$\sigma_m = \sum_{n=0}^{m} u_n v_n = \sum_{n=0}^{m} \exp\left[-q_1 n^{2/3} z - \gamma n z^{3/2}\right]$$

Applying the transformation of Abel to σ_m yields

$$\sigma_m = \sum_{n=0}^{m-1} (u_n - u_{n+1}) S_n' + u_m S_m' \tag{4.8}$$

For $m^{-2/3}\,\omega \leqslant z \leqslant (m+1)^{-2/3}\,\omega$, where $\omega = \gamma^{-2/3}(\log 2)^{2/3}$, for the sum S_n'

$$S_n' = \sum_{k=0}^{n} \exp(-\gamma k z^3/3) > \frac{n+1}{2} \tag{4.9}$$

The differences $u_n - u_{n+1}$ are evaluated by the formular of finite increments. Therefore

$$u_n - u_{n+1} = \exp\left[-q_1 n^{2/3} z\right] - \exp\left[-q_1 (n+1)^{2/3} z\right]$$
$$= \frac{2q_1 z}{3}(n+\xi)^{-1/3} \exp\left[-q_1 (n+\xi')^{2/3} z\right] \qquad (0 < \xi < 1)$$

Replacing ξ by 1 yields the inequality

$$u_n - u_{n+1} > \frac{2q_1 z}{3}(n+1)^{-1/3} \exp\left[-q_1 (n+1)^{2/3} z\right] \tag{4.10}$$

Combining equations (4.9), (4.10), and (4.11) yields

$$\sigma_m > \sum_{n=0}^{m-1} \frac{1}{3} q_1 z (n+1)^{2/3} \exp\left[-q_1 (n+1)^{2/3} z\right] = \frac{1}{3} \sum_{n=1}^{m} q_1 n^{2/3} z \exp\left[-q_1 n^{2/3} z\right]$$

The function

$$f(x,y) = xy^{2/3} \exp(-xy^{2/3}) \tag{4.11}$$

is considered for $y \geqslant 0$, $x > 0$. The derivative with respect to y varies as

$$\frac{\partial f}{\partial y} = \frac{2}{3} xy^{-1/3} \exp(-xy^{2/3})(1-xy^{2/3}) \begin{cases} > 0 & \text{for } 0 \leqslant y < x^{-3/2} \\ = 0 & \text{for } y = x^{-3/2} \\ < 0 & \text{for } y > x^{-3/2} \end{cases}$$

For fixed $x > 0$, the function $f(x,y)$ increases at first from 0 to e^{-1}, then decreases and, for $y \to \infty$, approaches zero. Hence, for any $x > c$, the inequality

$$1 + \sum_{n=1}^{m} n^{2/3} x \exp(-n^{2/3}x) > \int_0^m xy^{2/3} \exp(-xy^{2/3})\, dy = g \tag{4.12}$$

is obtained.

The value of the integral g is obtained by applying the substitution $xy^{2/3} = t$. Then

$$g = \frac{3}{2x^{3/2}} \int_0^{X} t^{3/2} e^{-t}\, dt \qquad (X=m^{2/3}x) \tag{4.13}$$

Substituting equation (4.13) in equation (4.12), setting $x = q_1 z$, and noting that $m^{2/3}z \geq \omega$ yield

$$\sum_{n=1}^{\alpha} q_1 n^{2/3} z \exp(-q_1 n^{2/3} z) > \frac{3}{2q_1^{3/2}} \left[\int_0^{q_1\omega} t^{2/3} e^{-t}\, dt\right] z^{-3/2} - 1$$

hence

$$S_1 = -1 + \sigma_m > -\frac{4}{3} + \frac{1}{2q_1^{3/2}} \left[\int_0^{q_1\omega} t^{2/3} e^{-t}\, dt\right] z^{-3/2} \tag{4.14}$$

On the basis of inequalities (4.6), (4.7), and (4.14), it may be concluded that there are two positive constants δ_1 and γ_1 such that the inequalities

$$\frac{1}{\delta_1 z^{3/2}} < S < \frac{1}{\gamma_1 z^{3/2}} \tag{4.15}$$

hold over the entire interval $0 < z \leq \alpha - \tau_1$. Returning to the variable τ and comparing equation (4.2) with equation (4.15) gives the following result: Two positive constants δ_2 and $\gamma_2 (\delta_2 > \gamma_2)$ exist such that for any τ in the interval $\tau_1 < \tau < \alpha$,

$$\frac{1}{\delta_2 \sqrt{\alpha - \tau}} < \frac{\partial \varphi}{\partial \tau} < \frac{1}{\gamma_2 \sqrt{\alpha - \tau}} \tag{4.16}$$

From equations (4.1) and (4.16) there follows finally

$$\delta_2\sqrt{\alpha-\tau} > \frac{\partial\tau}{\partial\varphi} > \gamma_2\sqrt{(\alpha-\tau)} \tag{4.17}$$

which is proven for the points of the x-axis.

In a similar manner, it may be shown that on the x-axis the second derivative $\partial^2\tau/\partial\varphi^2$ remains finite as $\tau \longrightarrow \alpha$.

Translated by S. Reiss
National Advisory Committee
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REFERENCES

1. Chaplygin, S. A.: Gas Jets. NACA TM 1063. 1944.

2. Astrov, Levin, Pavlov, and Christianovich: On the Computation of Laval Nozzles. Prik. Mat. i Mek., vol. VII, no. 1, 1943.

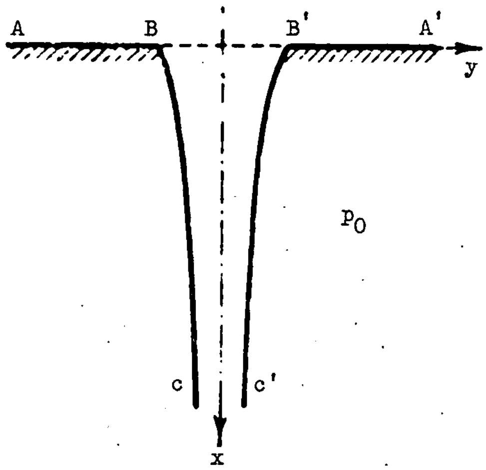

Figure 1.

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