NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

TECHNICAL MEMORANDUM 1291

ON STABILITY AND TURBULENCE OF FLUID FLOWS

By Werner Heisenberg

Translation of "Über Stabilität und Turbulenz von Flüssigkeitsströmen."
Annalen der Physik, Band 74, No. 15, 1924

NACA

Washington

June 1951

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INTRODUCTION

The turbulence problem, which will quite generally form the subject of the following investigations, has been treated in the course of time in so many reports from so many different viewpoints that it is not our intention to give, as an introduction, a survey of the results obtained so far. For that purpose, we refer the reader to a report by Noether[1] on the present state of the turbulence problem, where most bibliographical data may be found as well.

For our purpose, a rough outline of the present state of the turbulence problem will be sufficient. The investigations made so far are divided into two parts; one part deals with the stability investigation of any laminar motion, the other with the turbulent motion itself.

The first-mentioned investigations led, at the beginning, to the negative result that all laminar motions investigated are stable. V. Mises[2] and L. Hopf[3] proved, on the basis of a formula by Sommerfeld[4], the stability of the linear velocity profile corresponding to Couette's arrangement. Blumenthal[5] reached the same result for a profile of the third degree, upon which Noether invited discussion. On the other hand, Noether[6] later succeeded in specifying an unstable profile - a profile which is unstable even in the case of a frictionless fluid can never be realized as a steady state of motion for actual conditions. More

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*"Über Stabilität und Turbulenz von Flüssigkeitsströmen." Annalen der Physik, Band 74, No. 15, 1924, pp. 577-627.

[1]Noether, F.: Zeitschr. f. angew. Math. u. Mech. 1, 1921, p. 125.

[2]Mises, R. v.: Beitrag z. Oszillationsprobl.: Heinr. Weber-Festchrift, 1912, p. 252.

[3]Hopf, L.: Ann. d. Phys. 44, 1914, p. 1.

[4]Sommerfeld, A.: Atti d. IV. congr. int. dei Mathem. Rom 1909.

[5]Blumenthal, O.: Sitzungsber. d. bayr. Akad. d. Wiss., 1913, p. 563.

[6]Noether, F.: Nachr. d. Ges. d. Wiss., Göttingen 1917.

recently, however, Prandtl[7] has shown that indeed profiles exist which
possess unstable characteristics only if the friction is taken into
consideration.

The other group of reports which achieved great success quite
recently by the calculations[8] of Von Kármán, Latzko, and others
investigates the turbulent motion itself proceeding by a semiempirical
method using the laws of similarity. Theoretically, the reports of
this group are based almost throughout on Prandtl's boundary-layer
theory. Their most important result for our purpose is the so-called
$y^{1/7}$-law of turbulent velocity distribution which follows from Blasius's
law of resistance (examples can be found in Schiller's report[9].)

The determination of the critical Reynolds number was always one
of the main aims of the first-mentioned reports, the stability inves-
tigations. So far, a satisfactory calculation of this number has not
been accomplished and it must be regarded as doubtful whether it could
be achieved by stability investigations. The tests of Ekman[10], Ruckes[11],
and Schiller[9], together with the negative results of Hopf concerning
the linear velocity profile, rather suggest the notion that the critical
Reynolds number does not indicate the point where the laminar motion
starts to become unstable, but the point where, for the first time, the
turbulent motion is possible as steady state. From the viewpoint of
theory, we must thus be prepared to find eventually two critical
Reynolds numbers, one corresponding to the beginning of turbulence, the
other to the breaking down of the laminar motion.

The present investigation also will be divided into two different
parts, the treatment of the stability problem on the one hand, that of
the turbulent motion on the other.

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[7]Prandtl, L.: Physik. Zeitschr. 23, 1922, p. 19, and Tietjens, O.,
Dissert. Göttingen, 1922.

[8]See Zeitschr. f. angew. Math. u. Mech., 1, 1921, p. 233f.

[9]Schiller, L.: Rauhigkeit und kritische Zahl. Physik. Zeitschr. 3,
1920, p. 412.

[10]Ekman, V.: Turbulent Motions of Liquids. Arch. f. Mat. och
fysik 6, 1919. p. 12.

[11]Ruckes, W.: Dissert. Würzburg 1907. See also Lecture by W. Wien,
Über turbul. Bewegungen. Phys. Zeitschr. 8., 1904, and Verh. d. deutsch.
phys. Gesellsch. 9, 1907.

The aim of the first part is to summarize all previous investigations under a unified point of view, that is, to set up as generally as possible the conditions under which a profile possesses unstable or stable characteristics, and to indicate the methods for solution of the stability equation for any arbitrary velocity profile and for calculation of the critical Reynolds number for unstable profiles. This aim can, of course, be attained only imperfectly by the use of approximation methods. Nevertheless, we hope to be able to clarify by such calculations the qualitatively essential viewpoints. At first, the investigation of any arbitrary profile seems physically meaningless since only certain profiles actually occur; however, since we may interpret any profile as finite disturbance of another, as for instance Noether has done elsewhere, and since we must, on the other hand, later extend the investigations to the (at first unknown) basic profile of the turbulent motion, the investigation of an arbitrary profile seems, after all, to be of great importance.

As application of the methods, the parabola profile will be calculated completely.

In the second part, we shall attempt to derive, under certain greatly idealizing assumptions, differential equations for the turbulent motions and to obtain from them qualitative information about several properties of the turbulent velocity distribution.

PART I: THE STABILITY EQUATION

1. Statement of the Mathematical Problem

The most essential limitation we impose on our calculations consists in the exclusive consideration of two-dimensional laminar motions and only two-dimensional disturbances of these motions. Taking a rectangular coordinate system X, Y, Z as basis, we therefore assume that the velocity in the Z direction is zero and all remaining quantities independent of Z. Furthermore, however, we shall only examine the stability of such laminar motions as occur between two straight parallel walls. We assume the walls to be parallel to the X axis; therefore, the laminar motion to be investigated also promises a velocity component only in the X direction. This velocity w in the X direction will, in some way, be dependent on y. Concerning the function $w = w(y)$ we reserve for later making a few assumptions about continuity, symmetry, etc.; otherwise, however, this function is to be at first quite arbitrary.

If we put $w = ay$, our formulations become exactly identical with
those investigated by Hopf in the Couette case. The problem whether
the investigated profiles $w = w(y)$ can be realized as steady motions
will not be dealt with for the present.

Before deriving once more the stability equation (already set up
elsewhere by Sommerfeld) briefly from Stokes's differential equations,
we introduce dimensionless variables in the known manner. Let h be
a characteristic length (for instance the distance between the two
walls), U a characteristic velocity of the profile, μ the viscosity,
ρ the density, and $\frac{Uh\rho}{\mu} = R$ the Reynolds number; we introduce instead
of x, y, u, v, t, and p (u, v being the velocity in x or
y direction, respectively, t the time, and p the pressure) new
variables x_0 , y_0 , u_0 , v_0 , t_0 , and p_0 , according to the relations

$$x_0 = \frac{x}{h}; \quad y_0 = \frac{y}{h}; \quad u_0 = \frac{u}{U}; \quad v_0 = \frac{v}{U}; \quad t_0 = t\frac{U}{h}; \quad p_0 = p\frac{h}{\mu U} \qquad (1)$$

If the index 0 is subsequently omitted, Stokes's equations read

$$\left.\begin{array}{c} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \dfrac{1}{R}\left(-\dfrac{\partial p}{\partial x} + \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\[4mm] \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = \dfrac{1}{R}\left(-\dfrac{\partial p}{\partial y} + \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{array}\right\} \qquad (2)$$

Since we presuppose incompressibility, we write

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \qquad (3)$$

As is well known, we obtain by the elimination of p

$$\frac{\partial}{\partial t}\Delta\psi + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\Delta\psi - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\Delta\psi = \frac{1}{R}\Delta\Delta\psi \qquad (4)$$

By $\triangle$ one understands here the differentiation symbol

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Equation (4) does not yet contain anything about our special problem, the stability investigation of a certain laminar flow. Accordingly, equation (4) will form also the basis for the calculations of part II. In order to pass over specifically to the stability investigation, we divide the motion and therewith also the vector potential ψ into a basic flow and small oscillations superimposed over it. Thus we set up the formula

$$\psi = \Phi(y) + \varphi(y)e^{i(\beta t - \alpha x)} \tag{5}$$

$$\frac{\partial \Phi}{\partial y} = w(y) = w \tag{6}$$

If we enter this formula into equation (4), omitting all terms not containing φ (since we regard equation (4) as satisfied for $\varphi = 0$), furthermore omitting all terms quadratic in φ (since we assume φ as small), the corresponding differential equation for φ reads

$$(\varphi'' - \alpha^2\varphi)\left(w - \frac{\beta}{\alpha}\right) - \varphi w'' = \frac{i}{\alpha R}(\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi) \tag{7}$$

The fact that we regard equation (4) as satisfied for $\varphi = 0$ signifies physically that we consider only such basic flows w which either, by virtue of external forces, are really steady, or show a variation with time which is slow compared to that of the small oscillations.

Equation (7) is in this generality already derived elsewhere by Noether. It is an ordinary differential equation for φ of the fourth order. It corresponds to the fact that the function φ must fulfill four boundary conditions; u and v, thus also φ and φ', must disappear at the two walls. If we put $\beta/\alpha = c$ so that c essentially signifies the wave velocity, the mathematical problem may be formulated as follows: The solutions of the equation

$$\left(\varphi'' - \alpha^2\varphi\right)(w - c) - \varphi w'' = \frac{i}{\alpha R}\left(\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi\right) \tag{7a}$$

are to be investigated with the secondary condition that at the bounding walls (for instance, $y = 1$ and $y = -1$) $\varphi = 0$ and $\varphi' = 0$. For each value of α and R the corresponding value of c and β is to be calculated; let α for reasons of simplicity always be positive. According to whether the imaginary part of β is positive, zero, or negative, we are dealing with a stable, undamped (undamped oscillations = neutrally stable oscillations), or unstable oscillation. The conditions for profile w are to be found under which equation (7a) admits only stable oscillations or, respectively, also unstable ones.

Before turning to the methods of solution we want to point out a special property of the equation (7a). In the limit of the frictionless fluid, $R = \infty$, equation (7a) is transformed into a differential equation of the second order for φ

$$\left(\varphi'' - \alpha^2\varphi\right)(w - c) - \varphi w'' = 0 \tag{8}$$

Accordingly, only two boundary conditions must now be satisfied which signify that the normal velocity component, thus v or φ but no longer φ', is to disappear at the two walls.

The conditions for the solvability of equation (8) have already been investigated in detail by Rayleigh.[12] Introducing a simple designation, one may distinguish basic flows "capable of oscillation" or "not capable of oscillation" according to whether or not equation (8) possesses a solution with real c which satisfies the boundary conditions.[13] If solutions with complex c exist, the stability problem for these oscillations has, as will be shown later, already been decided by equation (8), also in case of consideration of the friction; the oscillations are then always unstable.

One is, however, beyond this led to the conjecture that the profile w, under influence of friction, permits unstable or undamped

[12]Lord Rayleigh, Papers I., p. 361; III. pp. 575, 594; IV. p. 203.

[13]Here, however, it is by no means sufficient to approximate the profile by tangents polygons; the result with respect to possible oscillations would thereby be completely falsified.

oscillations only in one case: when it belongs to the basic flows capable of oscillation.

This supposition is the more obvious as it has been confirmed for all profiles investigated so far.[14] Nevertheless, it is by no means motivated by the fact that equation (8) results from equation (7a) in the limit R = ∞ since it has been proved, for instance, in the reports of Oseen[15] that the limiting process R = ∞ has led more than once to false results in the differential equations, particularly with respect to the boundary conditions of the frictionless fluid, and that one may therefore apply the limiting process only to the integrals of equation (7a).   Moreover, it can by no means be decided beforehand whether the friction modifies the undamped oscillations of equation (8) in the sense of a damping or an excitation.

Following, we shall attempt to prove our surmise mentioned above by showing that the systems capable of oscillation are shown to possess above a certain value of the Reynolds number and in general unstable character, whereas all systems not capable of oscillation are shown to possess a stable character.

By this principle the problem of the stability of a profile is quite considerably simplified since, as is well known, the solutions of equation (8) may be directly written down for very small values of α.

2. The Methods of Solution and the General Behavior of

the Integrals of Equation (7a)

The most important property of equation (7a) which permits an approximate representation of its solutions consists in the fact that R may be regarded as very large.   It will become evident that if a stability limit exists, this limit lies, in general, at very high values of R.   Since it is, moreover, physically quite improbable that for small values of R instability of the respective profile could occur, it is sufficient for our next purpose to regard R as very large.

This assumption makes it possible to approximate the solutions of equation (7a) by development in negative powers of R or, as will be shown, of $\sqrt{\alpha R}$.   Furthermore, we shall assume α as small and shall develop the solutions in a given case in positive powers of α^2.

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[14]Compare also Prandtl, Physikalische Zeitschrift.

[15]Compare, for instance, C. W. Oseen, Beiträge z. Hydr.   Annalen der Physik 46, pp. 231 and 623, 1915.

Both methods of development in $(\alpha R)^{-1/2}$ and α^2, respectively, seem contradictory insofar as in the first case αR is assumed large, in the second case α^2 small; however, the contradiction is eliminated by the fact that R may be regarded, in general, as extraordinarily large so that for instance for $R = 2000$, $\alpha = 1/10$, αR becomes equal to 200, $\alpha^2 = 1/100$ which is fully sufficient for a satisfactory convergence of the two developments. However, the convergence properties of these approximation methods must be considered more exactly. The investigation shows that the series in $(\alpha R)^{-1/2}$ are generally divergent, yet show the well-known characteristics of the semiconvergent series, that is, that the terms first decrease, then again increase, and that one obtains the optimum approximation if one breaks off the series with the smallest term. Our approximation method has, therefore, convergence properties similar to those of the series of the perturbation theory used in astronomy, the behavior of which is described in detail by Poincaré, Meth. nouv. d. l. mec. cel. II.

The use of the semiconvergent developments is rendered considerably difficult by the fact that they lose their validity in the neighborhood of a certain point so that it cannot be immediately decided in what manner the approximate solutions on both sides of the point must be joined in order to approximate a certain integral of the equation (7a) on both sides. This question will be discussed in detail in section 3.

The development in positive power series of α^2 seems, in general, to be actually convergent. For special profiles this development may be strictly proved (for instance, for the linear profile); however, we have not carried out an investigation of the problem under what conditions for the profile w this convergence actually occurs.

We start with the derivation of the approximate solutions of equation (7a)

$$\left(\varphi'' - \alpha^2\varphi\right)(w - c) - w''\varphi = \frac{i}{\alpha R}\left(\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi\right) \tag{7a}$$

For this purpose we first put

$$\varphi = e^{\int g\,dy}, \quad g = \sqrt{\alpha R}\,g_0 + g_1 + \frac{1}{\sqrt{\alpha R}}\,g_2 + \cdots \tag{9}$$

We shall limit the development to the two highest terms in $\sqrt{\alpha R}$. There follows

$$\alpha R g_0{}^2(w - c) + \sqrt{\alpha R}\left(g_0{}' + 2g_0 g_1\right)(w - c) = i\alpha R g_0{}^4 + i\sqrt{\alpha R}\left(4g_0{}^3 g_1 + 6g_0{}^2 g_0{}'\right)$$

α^2 and w'' are presupposed to be of the order of magnitude 1 or, at any rate, $\ll \sqrt{\alpha R}$. By means of simple calculation there now results

$$g_0 = \sqrt{-i(w - c)}, \quad g_1 = -\frac{5}{2}\frac{g_0{}'}{g_0}, \quad \int g_1 \, dy = -\frac{5}{2}\log g_0 \qquad (10)$$

Thus we obtain two particular integrals of the equation (7a)

$$\varphi_{1,2} = (w - c)^{-5/4} e^{\pm\int_{y_0}^{y}\sqrt{-i\alpha R(w-c)}} \qquad (11)$$

The point y_0 is to be determined by w being $= c$ for $y = y_0$. Thus y_0 may, under certain conditions, be complex. The sign of the root is to be chosen so that for

$$w - c = -ae^{i\omega}, \quad -i\alpha R(w - c) = \alpha Rae^{i\left(\omega + \frac{\pi}{2}\right)}$$

the root becomes

$$(\alpha Ra)^{1/2} e^{i\left(\frac{\omega}{2} + \frac{\pi}{4}\right)}$$

A remarkable fact about these two integrals is that α^2 in φ does not appear in this approximation (that is, only in the combination αR which, as follows from equation (7a), could in a certain sense, be called the true Reynolds number).

As we shall see later, the two integrals (11) determine the behavior of φ in the boundary layer, and the nonoccurrence of α^2 in equation (11) signifies physically that we consider only oscillations

the wave length of which is large compared to the boundary-layer thick-
ness, which is certainly the case for the empirically observed unstable
oscillations.

For a complete system of solution of equation (7a), however, we
need two further integrals; naturally, we shall choose those which
result from the integrals of equation (8) by the development of
$(\alpha R)^{-1}$ in a power series.

For this purpose we first solve equation (8) by the development
of α^2 in a power series. Thus we put

$$\left(\varphi'' - \alpha^2\varphi\right)(w - c) - \varphi w'' = 0 \tag{8}$$

$$\varphi = \varphi^{(0)} + \alpha^2\varphi^{(1)} + \alpha^4\varphi^{(2)} + \ldots \tag{12}$$

Hence follows

$$\varphi^{(0)''}(w - c) - \varphi^{(0)}w'' = 0$$

$$\varphi^{(1)''}(w - c) - \varphi^{(1)}w'' = \varphi^{(0)}(w - c)$$

$$\varphi^{(2)''}(w - c) - \varphi^{(2)}w'' = \varphi^{(1)}(w - c)$$

By the method of variation of the constants the two integrals

$$\left.\begin{aligned}
\varphi_{3(R=\infty)} &= (w - c)\left(1 + \alpha^2 \int \frac{dy}{(w - c)^2} \int dy(w - c)^2 + \ldots\right) \\
\varphi_{4(R=\infty)} &= (w - c)\int \frac{dy}{(w - c)^2}\left(1 + \right. \\
&\quad \left. \alpha^2 \int dy(w - c)^2 \int \frac{dy}{(w - c)^2} + \ldots\right)
\end{aligned}\right\} \tag{13}$$

result.   In addition, these integrals have now to be corrected by
quantities of the order $(\alpha R)^{-1}$, . . . etc., if they are to satisfy
equation (7a).

Without writing the corresponding series development down in
detail, we give as result φ with the quantities of the order $(\alpha R)^{-1}$

$$
\left.
\begin{aligned}
\varphi_3 &= (w - c)\left\{1 + \alpha^2 \int \frac{dy}{(w - c)^2} \int dy(w - c)^2 + \alpha^4 \ldots + \right. \\
&\qquad \frac{i}{\alpha R}\int \frac{dy}{(w - c)^2} \frac{d^3}{dy^3}(w - c) + \ldots \bigg\} \\[2em]
\varphi_4 &= (w - c)\int \frac{dy}{(w - c)^2}\left\{1 + \alpha^2 \int dy(w - c)^2 \int \frac{dy}{(w - c)^2} + \right. \\
&\qquad \alpha^4 \ldots + \frac{i}{\alpha R} \frac{d^3}{dy^3}(w - c)\int \frac{dy}{(w - c)^2} + \ldots \bigg\}
\end{aligned}
\right\} \quad (14)
$$

With equations (11) and (14), a complete solution system of the equa-
tion (7a) has been found approximately.

Before applying this system of solution for satisfying the boun-
dary conditions, it will be useful to clarify the physical significance
of the four integrals φ_1, φ_2, φ_3, and φ_4, and to anticipate a few
results which we cannot establish until later.

The integrals φ_1, φ_2 are very rapidly variable for the high
values of R which are of interest to us, as can be seen from the
exponent of the order $\sqrt{\alpha R}$. If, therefore, for instance φ_1 is at
one wall of the order of magnitude 1, it will vanish exponentially at
some distance from the wall. (In itself, it also could become extraor-
dinarily large; however, this is naturally prevented by the boundary
conditions.) Consequently, φ is composed, except for the immediate
neighborhood of the wall, of φ_3 and φ_4 alone, that is, is
very similar to the behavior of φ in the frictionless fluid.

The fact that α^2 does not explicitly occur in φ_1, φ_2, (compare equation (11)), but does appear in φ_3, φ_4 (compare equation 14)) must evidently be interpreted physically to signify that, if α^2 is assumed to be not very large $\left(\alpha^2 < \sqrt{\alpha R}\text{, compare equations (7a) and (9)}\right)$, the wave length may be considered as infinitely large compared to the boundary-layer thickness, but not compared to the width of the channel. The characteristic difference between the "boundary-layer integrals" φ_1, φ_2, on one hand, and the integrals corresponding to the frictionless fluid φ_3, φ_4, on the other, is therefore significantly expressed in the occurrence or nonoccurrence of α^2.

Concerning the convergence of the development in power series of α^2, we may hope that for values of α^2 of the order of magnitude 1 it is still amply sufficient to enable a good approximation, since for a linear profile the series for φ_3, φ_4 become series of the type of power development of $\cos\alpha$ which in the neighborhood $\alpha = 1$ still converges very rapidly.

The flow pattern to be expected after all these conclusions corresponds to the formulations made in Prandtl's boundary-layer theory. Except for the immediate neighborhood of the walls, the motion obeys very nearly the differential equation of the frictionless fluids. To the walls themselves, however, adheres a boundary layer the thickness of which is of the order of magnitude $(\alpha R)^{-1/2}$. In this boundary layer the velocity u decreases toward the wall rapidly toward zero whereas v is almost zero even outside of the boundary layer.

3. The Connecting Substitutions

If we want to study the course of the integrals of equation (7a) from one bounding wall to the other, we must take into account that, at a point $y = y_0$ in the channel $w - c = 0$ (or that at least the the real part of $w - c$ is zero), therefore the wave velocity there agrees with the velocity of the basic flow. At this point, the approximation formulas (11) and (14) for the integrals of equation (7a) cease to be valid.

Thus it is necessary to know the connecting substitutions for φ_1, φ_2, φ_3, and φ_4 which have to be applied for the transition from $\mathrm{Re}(w - c) > 0$ to $\mathrm{Re}(w - c) < 0$. For this purpose, we develop w

and φ in the neighborhood of the critical point y_0 in the power series of $(\alpha R)^{-1/3}$ and put therefore

$$y - y_0 = \eta(\alpha R)^{-1/3}$$

Furthermore we assume the imaginary part of y_0 to be of smaller order of magnitude than $(\alpha R)^{-1/3}$. If it is of higher order of magnitude, the connecting substititions are self evident because then nowhere in the entire range of real y does a "critical point" appear. If the imaginary part is of the same order of magnitude, the behavior of φ and w may be easily interpolated from the two limiting cases just mentioned. At first we may even put

$$\mathrm{Im}\big(y_0\big) = 0$$

since φ in our case

$$\mathrm{Im}\big(y_0\big) \ll (\alpha R)^{-1/3}$$

may be developed in power series of $\mathrm{Im}\big(y_0\big)$ and at first only the behavior of φ for $\mathrm{Im}\big(y_0\big) = 0$ is needed.

Thus we now put

$$\varphi = \varphi_0 + \epsilon\varphi_1 + \epsilon^2\varphi_2 + \ldots$$

$$\epsilon = (\alpha R)^{-1/3}$$

$$w - c = \epsilon a\eta + \epsilon^2 b\eta^2 + \ldots$$

Then there results from equation (7a)

$$\varphi_0{}'''' + \epsilon\varphi_1{}'''' + \ldots = -i\left[(\varphi_0{}'' + \epsilon\varphi_1{}'' + \ldots)a\eta + \epsilon\varphi_0{}''b\eta^2 - 2\epsilon\varphi_0 b\right] + \ldots$$

Thus in first approximation

$$\varphi_0{}'''' = -i\varphi_0{}''\eta a \tag{15}$$

in second approximation

$$\varphi_1{}'''' = -i\left[\varphi_1{}''\eta a + \varphi_0{}''b\eta^2 - 2\varphi_0 b\right] \tag{15a}$$

For the integrals φ_1, φ_2, equation (11), we infer from equation (15) that they behave in the critical range (η order of magnitude 1) like the integrals found by Hopf for the linear profile, that is, like certain cylindrical functions. Thus, we may conclude at this point that the connecting substitutions for φ_1, φ_2 from equation (11) except for quantities of the order $(\alpha R)^{-1/3}$ must be the same as for the linear profile

$$\left.\begin{array}{c} \varphi_1 \longrightarrow \varphi_1 + i\varphi_2 \\[2ex] \varphi_1 - i\varphi_2 \longrightarrow \varphi_1 \end{array}\right\} \tag{16}$$

corresponding to a transition of

$$\mathrm{Re}(w - c) < 0 \longrightarrow \mathrm{Re}(w - c) > 0$$

However, for the study of the integrals φ_3, φ_4 in the neighborhood of $y - y_0 = 0$, the simple calculations made so far are not sufficient since for the latter the approximate solution (15) would read $\varphi'' = 0$; however, we know from equation (14) that, in the limit $R \longrightarrow \infty$, φ_4', in general, becomes logarithmically infinite at the point $y - y_0 = 0$. Equations (15) and (15a) are therefore in this form unsuitable for expressing this singularity.

Instead, we now set $\alpha = 0$ and $w''' = 0$ (that is, we break off the development of w with the second term); otherwise, however, integrate equation (7a) exactly at first. In doing so, we notice that $\varphi = w - c$ must be a particular integral of this simplified equation (7a) and we make, therefore, for φ the statement familiar from the theory of linear differential equations

$$\varphi = (w - c) \int \psi \, dy$$

Then there follows from

$$\varphi'''' = -i\alpha R\left(\varphi''(w - c) - w''\varphi\right)$$

for

$$\varphi = (w - c) \int \psi \, dy$$

$$\psi'''(w - c) + 4\psi''w' + 6\psi'w'' = -i\alpha R\left(2w'(w - c)\psi + (w - c)^2\psi'\right)$$

which after repeated integration becomes

$$\psi''(w - c) + 3\psi'w' + 3\psi w'' = -i\alpha R\left((w - c)^2\psi - c\right) \tag{17}$$

C is an integration constant. If one now again introduces

$$\eta = \left(y - y_0\right)(\alpha R)^{1/3}, \quad \epsilon = (\alpha R)^{-1/3},$$

$$w - c = \epsilon a\eta + \epsilon^2 b\eta^2, \quad \psi = \psi_0 + \epsilon\psi_1 + \ldots$$

there results

$$\left.\begin{array}{l} \psi_0{}''a\eta + 3\psi_0{}'a = -i\left(a^2\eta^2\psi_0 - c\right) \\[2ex] \psi_1{}''a\eta + 3\psi_1{}'a + \psi_0{}''b\eta^2 + 6\psi_0{}'b\eta + 6\psi_0 b = \\[2ex] \qquad -i\left(2ab\psi_0\eta + a^2\eta^2\psi_1\right) \end{array}\right\} \qquad (17a)$$

Of course, these differential equations still contain all solutions of equation (7a). We intend to study particularly φ_4 (φ_3 shows for $\alpha = 0$ at the point $y = y_0$ regular behavior); therefore, we select the one solution of equation (17a) which behaves at some distance from y_0, thus for large $(w - c)\alpha R$, like $\dfrac{1}{(w - c)^2}$, since we know from equation (14) that φ_4 at some distance from y_0 is given by

$$(w - c) \int \frac{dy}{(w - c)^2}$$

Thus we obtain according to equation (17a)

$$\psi_0 = \frac{c}{a^2\eta^2}$$

and

$$\psi_1{}''\eta + 3\psi_1{}' = -i\left(\frac{2b}{a^2} c\eta + a\eta^2\psi_1\right) \qquad (18)$$

ψ_1 is again fully determined by the fact that it is to behave "at infinity" like $-\dfrac{2bC}{a^3\eta}$.

We now ask for the transformation substitutions for φ_4 (and φ_3), meaning thereby the following: In the asymptotically valid representations for φ_3, φ_4 (equation (14)), we always find the integral $\displaystyle\int \frac{dy}{(w-c)^2}$ which loses its sense if it is to be extended beyond the point $y = y_0$ ($w - c = 0$). Actually, $\dfrac{1}{(w-c)^2}$ is, near the critical point, replaced by the function ψ. Thus the behavior of ψ (particularly ψ_1) in the critical neighborhood is the solely decisive factor. If this behavior and therein the magnitude of the integral $\displaystyle\int \psi\, dy$ (extended beyond the critical point) be known, this knowledge is equivalent to knowing the transformation substitutions for φ_3, φ_4.

The solution ψ_1 characterized by equation (18) and the boundary condition at infinity reads:

$$\psi_1(\eta) = -\frac{\pi bC}{3a^2\eta}\left\{ H_{2/3}^{(1)}\left[\frac{2}{3}\left(-i\alpha_0\eta\right)^{3/2}\right] \int_{+\infty}^{\eta} H_{2/3}^{(2)}\eta^2 d\eta \right.$$

$$\left. H_{2/3}^{(2)} \int_{-\infty}^{\eta} H_{2/3}^{(1)}\eta^2 d\eta \right\} \tag{19}$$

Therein Hankel's cylinder functions of the index $2/3$ and the argument $\dfrac{2}{3}\left(-i\alpha_0\eta\right)^{3/2}$ appear $\left(\alpha_0 = a^{1/3}\right)$. The sign of $\left(-i\alpha_0\eta\right)^{3/2}$ is to be taken so that $\left(-i\alpha_0\eta\right)^{3/2}$ becomes positive for $\eta = \dfrac{re^{(\pi i)/2}}{\alpha_0}$. A closer investigation of equation (19) shows that ψ_1 behaves in the entire upper semiplane, and partially, even in the lower one, namely, for $\eta = re^{i\xi}$ within the limits $-\dfrac{i\pi}{6} < \xi < \dfrac{7i\pi}{6}$ at infinity like

$-\dfrac{2bC}{a^3\eta}$, if α_0 or a is positive. If a is negative, the upper and

lower semiplane are interchanged

$$\lim_{r \to \infty} \psi_1\left(re^{i\xi}\right) = -\frac{2bC}{a^3\eta} \quad \text{is valid, if} \quad \begin{cases} \dfrac{5i\pi}{6} < \xi < \dfrac{13\pi i}{6} \\ \\ a < 0 \end{cases}$$

$$\text{or} \quad \begin{cases} -\dfrac{i\pi}{6} < \xi < \dfrac{7\pi i}{6} \\ \\ a > 0 \end{cases} \tag{20}$$

Hence we infer the important result:

$$\int_{-\infty}^{\infty} \psi_1 \, d\eta = \begin{cases} \dfrac{2bC}{a^3} \, i\pi & a > 0 \\ \\ -\dfrac{2bC}{a^3} \, i\pi & a < 0 \end{cases} \tag{21}$$

Thus the transformation substitutions for φ_3, φ_4, accurate up to the magnitudes of the order $(\alpha R)^{-1/3}$, are now found for finite values of α also; we now know - and that is sufficient - what, according to equation (21), the integral $\displaystyle\int \frac{dy}{(w - c)^2}$, extended from $w - c < 0$ to $w - c > 0$, signifies.

The formulas (16) also may be derived once more from equation (17); to the asymptotic solutions (11) of equation (7a) correspond the integrals

$$\frac{1}{\eta} H_{2/3}^{(1),(2)}\left[\frac{2}{3}(-i\alpha_0\eta)^{3/2}\right] \tag{19a}$$

of the homogeneous equation (18) $(C = 0)$. The problem of finding the transformation substitutions of the "asymptotic" integrals (11) and (14) is therewith completed with the required accuracy $\left(\text{except for quantities of the order } (\alpha R)^{-1/3}\right).$

4. Fulfillment of the Boundary Conditions and the

Stability of the Oscillations Corresponding

to the Solution System I

Our considerations so far have been quite independent of the type of profile except for a few limitations concerning the singular points which had to be imposed on the profile. In order not to lose ourselves in an excessive number of different possibilities, we shall further specialize the character of the basic flow. The considerations, however, have much more general validity. We thus assume that the bounding walls are represented by the equations $y = +1$ and $y = -1$, that, furthermore, the wall $y = 1$ possesses, with respect to the other, a relative velocity in the positive X direction (of the magnitude $w(+1) - w(-1)$) and that the laminar flow adheres to the walls (which corresponds to Couette's test arrangement); finally, we assume that in the range $-1 < y < +1$, that is, in the fluid, $Re(w - c)$ once and only once is zero. Moreover, we shall presuppose in the entire region continuity for w and the derivatives of w and, beyond this, make the additional assumption that the functions w, w', w'', etc., always are of normal magnitude, that is, that they do not, for instance, at certain points, assume a magnitude of the order $(\alpha R)^{1/2}$.

Furthermore, for the following calculations, we at first regard α as so small and αR as so large that we may put with sufficient accuracy

$$\left.\begin{aligned}
\varphi_3 &= w - c \\[2ex]
\varphi_4 &= (w - c) \int_{-1}^{y} \frac{dy}{(w - c)^2}
\end{aligned}\right\} \tag{14a}$$

The fixing of the lower limit of the integral in φ_4 obviously does not signify a limitation of the generality of our solutions. Rather we determine thereby φ_4 as that linear combination of φ_3 and φ_4 which disappears at the point $y = -1$. In case $w - c$ should disappear there also, φ_4 obviously is replaced by the function $\varphi_3 = w - c$ which now for $y = -1$ becomes zero.

In order to satisfy the boundary conditions first at the wall $y = -1$, we form two aggregates f_1, f_2 from φ_1, φ_2, φ_3, φ_4 for which really $\varphi = \varphi' = 0$ for $y = -1$

$$f_1 = \varphi_4 + \frac{1}{g_0(-1)\left[w(-1) - c\right] - \frac{9}{4} w'(-1)} \left[\frac{\varphi_3}{w(-1) - c} - \frac{\varphi_1}{\varphi_1(-1)}\right]$$

$$f_2 = \varphi_4 - \frac{1}{g_0(-1)\left[w(-1) - c\right] + \frac{9}{4} w'(-1)} \left[\frac{\varphi_3}{w(-1) - c} - \frac{\varphi_1}{\varphi_1(-1)}\right]$$

$$\tag{22}$$

Therein we understand from now on by g_0 the root $\sqrt{-i\alpha R(w - c)}$, not as in equation (10), $\sqrt{-i(w - c)}$, in order to save writing down the factor $\sqrt{\alpha R}$.

In order to satisfy the boundary conditions at the other wall as well, one must attempt to determine two constants A and B so that

$$A f_1(+1) + B f_2(+1) = 0$$

$$A f_1'(+1) + B f_2'(+1) = 0$$

The condition for the possibility of such a determination is

$$\begin{vmatrix} f_1(+1) & f_2(+1) \\ \\ f_1'(+1) & f_2'(+1) \end{vmatrix} = 0 \tag{23}$$

By this condition c or β, respectively, is determined if R and α
are given. Thus it is now a question of solving equation (23) for c
and of determining the sign of the imaginary part of β. Equation (23)
forms the perfect analogue to Sommerfeld's turbulence equation for the
linear profile.

From equation (16) we infer

$$
f_1(+1) = \varphi_4(+1) + \frac{1}{g_0(-1)\left[w(-1) - c\right] - \frac{9}{4}w'(-1)}\left[\frac{\varphi_3(+1)}{w(-1) - c} - \right.
$$

$$
\left. \frac{\varphi_1(+1) + i\varphi_2(+1)}{\varphi_1(-1)}\right]
$$

$$
f_1'(+1) = \varphi_4'(+1) + \frac{1}{g_0(-1)\left[w(-1) - c\right] - \frac{9}{4}w'(-1)}\left[\frac{\varphi_3'(+1)}{w(-1) - c} - \right.
$$

$$
\left. \frac{\varphi_1'(+1) + i\varphi_2'(+1)}{\varphi_1(-1)}\right]
$$

$$
f_2(+1) = \varphi_4(+1) - \frac{1}{g_0(-1)\left[w(-1) - c\right] + \frac{9}{4}w'(-1)}\left[\frac{\varphi_3(+1)}{w(-1) - c} - \right.
$$

$$
\left. \frac{\varphi_2(+1)}{\varphi_2(-1)}\right]
$$

$$
f_2'(+1) = \varphi_4'(+1) - \frac{1}{g_0(-1)\left[w(-1) - c\right] + \frac{9}{4}w'(-1)}\left[\frac{\varphi_3'(+1)}{w(-1) - c} - \right.
$$

$$
\left. \frac{\varphi_2'(+1)}{\varphi_2(-1)}\right]
$$

(24)

We insert these values of f_1, f_2, f_1', f_2' into equation (23) after having made an estimate of the magnitude of the individual terms in order to eliminate unnecessary complications of the calculation by writing down unessential terms. For this purpose we note that there will be, in general, either $\varphi_2(+1) \ll \varphi_2(-1)$ or $\varphi_2(+1) \gg \varphi_2(-1)$.

This is caused by the factor $\sqrt{\alpha R}$ in the exponent of φ_1, φ_2 in equation (11) if there does not exist the equality

$$\text{Re} \int_{y_0}^{-1} \sqrt{-i(w - c)}\,dy = \text{Re} \int_{y_0}^{1} \sqrt{-i(w - c)}\,dy$$

which we exclude.

Which one of the two cases will occur cannot be decided beforehand; generally, both are possible and yield both solutions. In the case of an obliquely symmetric profile, one case gives the solutions symmetrical to that of the other. At any rate, the two possibilities behave principally quite analogously and it is therefore sufficient to investigate one of the two. Thus we assume

$$\varphi_2(+1) \ll \varphi_2(-1)$$

that is (compare page 9), the point $w = c$ is to lie nearer to $w = w(+1)$ than to $w = w(-1)$.

Hence it follows that

$$\varphi_1(-1) \sim \frac{1}{\varphi_2(-1)}$$

is extraordinarily small, thus $\dfrac{1}{\varphi_1(-1)}$ is very large. Thus there remain in f_1 and f_1' only the terms which have $\varphi_1(-1)$ in the denominator; in f_2 and f_2' the terms containing φ_2 are eliminated.

From equation (23) we thus obtain

$$
\left[\varphi_1(+1) + i\varphi_2(+1)\right]\left[\varphi_4'(+1) - \frac{\varphi_3'(+1)}{\left[g_0(-1)\left[w(-1) - c\right] + \frac{9}{4} w'(-1)\right]\left[w(-1) - c\right]}\right] =
$$

$$
\left[\varphi_1'(+1) + i\varphi_2'(+1)\right]\left[\varphi_4(+1) - \frac{\varphi_3(+1)}{\left[g_0(-1)\left[w(-1) - c\right] + \frac{9}{4} w'(-1)\right]\left[w(-1) - c\right]}\right]
\tag{25}
$$

Even in this form the equation for c is still rather complicated.
We therefore further simplify equation (25) by cancelling now not only

quantities of the order of magnitude $e^{-\sqrt{\alpha R}}$, but also quantities of the
order $(\alpha R)^{-1/2}$.

For this purpose we determine that $g_0(+1)$ is of the order

$(\alpha R)^{1/2}$, thus at first excluding the possibility of $w(+1) - c$ being
very small, and that furthermore

$$
\varphi_1'(+1) + i\varphi_2'(+1) = - \frac{5}{4} \frac{w'(+1)}{w(+1) - c} \left[\varphi_1(+1) + i\varphi_2(+1)\right] +
$$

$$
g_0(+1) \left[\varphi_1(+1) - i\varphi_2(+1)\right]
$$

Thus we retain only those terms of equation (25) which are multiplied
by the factor $g_0(+1)$.

Thus the simple result is found

$$\left[\varphi_1(+1) - i\varphi_2(+1)\right]\varphi_4(+1) = 0$$

or

$$\left[e^{2\int_{y_0}^{y=+1}\sqrt{-i\alpha R(w-c)}\,dy} - i\right]\int_{-1}^{1}\frac{dy}{(w-c)^2} = 0 \tag{26}$$

This equation possesses two completely different solution systems

$$e^{2\int_{y_0}^{y=+1}\sqrt{-i\alpha R(w-c)}\,dy} = i \tag{I}$$

$$\int_{-1}^{1}\frac{dy}{(w-c)^2} = 0 \tag{II}$$

System I represents the perfect analogue to the solutions Hopf obtained for the linear profile and has discussed in detail elsewhere, section 4. Actually it is shown that the oscillations corresponding to system I always are of stable character. From

$$e^{2\int_{y_0}^{1}\sqrt{-i\alpha R(w-c)}\,dy} = i$$

follows

$$2\int_{y_0}^{1}\sqrt{-i\alpha R(w-c)}\,dy = \pi i\left(\frac{1}{2} + 2n\right) \tag{27}$$

where n is a positive (compare page 9) integer. It is easily seen
that this equation can only be satisfied when $\alpha c = \beta$ possesses a
positive imaginary part. Thus the oscillations characterized by equa-
tion (27) are actually damped, the amount of the damping being of the
order of magnitude $w(+1) - c$, and therefore by no means need be small.

5. The Solution System II and the Conditions for

Instability of a Profile

The solutions in the system II are identical with the solutions
of the Rayleigh equation (8) and satisfy the condition

$$\int_{-1}^{1} \frac{dy}{(w - c)^2} = 0 \tag{28}$$

or (compare the remark to equation (14a), page 19) quite generally

$$\varphi_4 = (w - c) \int^{y} \frac{dy}{(w - c)^2} = 0$$

for

$$y = +1$$

and

$$y = -1$$

The latter form differs from the first in certain exceptional cases
which will be discussed later; moreover, equation (28) represents, of
course, only a first approximation ($\alpha = 0$). For the solutions of
equation (28) one must now distinguish four different possibilities:
Either, (1) equation (28) has solutions with complex c, then the
profile is always unstable since the conjugate complex value of c
also always represents a solution; (2) there exist solutions of equa-
tion (28) with real c. Then we designate, as Prandtl did elsewhere,
the profile as "capable of oscillations." This can, according to
equation (21), occur only if, at the point $(w = c)$, $w'' = 0$, if,
therefore, the profile either possesses a point of inflection or is

composed of linear pieces; (3) real values of c exist which make
at least the real part of

$$\int_{-1}^{1} \frac{dy}{(w - c)^2}$$

zero; or, finally, (4) if none of these three cases occur, equation (28)
has no solution. In cases (3) and (4), we call the profile "not
capable of oscillations." We contend that case 1 always results in
instability, cases (3) and (4) always in stability, case (2) generally
in instability of the profile taken as a basis. For cases (1) and (4),
this has already been proved above. In case (3), we put $c = c_r + ic_i$
with c_r signifying that real value of c for which the real part of

$$\int_{-1}^{1} \frac{dy}{(w - c)^2}$$

disappears. Then we know from section 3 that for $c_i \lessgtr 0$, the imaginary
part of the integral becomes $\dfrac{2b}{|a^3|} \pi i$, for $c_i \gg \left|(\alpha R)^{-1/3}\right|$, however,

$-\dfrac{2b}{|a^3|} \pi i$. Thus, for reasons of continuity (compare section 3), a

point $c_i > 0$ must exist where the imaginary part of the integral (28)
also disappears. The four solutions of equation (28) thus character-
ized yield therefore a quantity c with a positive imaginary part,
thus stable oscillations.

Case (2) finally requires somewhat more detailed calculations.
Before performing them we note that to case (2) pertain two types of
solution for equation (28) which cannot be represented in the form

$$\int_{-1}^{1} \frac{dy}{(w - c)^2} = 0$$

If $w(+1) = w(-1)$ a solution of equation (28) is $\varphi = w - w(+1)$; in
fact, here $\varphi = 0$ for $y = +1$ and $y = -1$. Furthermore, it

may happen that, for instance, w' becomes infinite for $w = w(+1)$. Then

$$\varphi = \left[w - w(+1)\right] \int_{-1}^{y} \frac{dy}{\left[w - w(+1)\right]^2}$$

also is a solution of equation (3) which satisfies the boundary conditions. We shall not treat this case here in more detail since it will be discussed more thoroughly in part II; however, it must be noted as essential that the difference between cases (2) and (3) is very large and that it is, for instance, by no means sufficient to approximate, according to Rayleigh, a curved profile by a polygon.

For an illustration of this difference

$$\operatorname{Re}\left[J(c)\right] = \operatorname{Re} \int_{-1}^{1} \frac{dy}{(w - c)^2}$$

is represented qualitatively as a function of c in figure 1 where the solid curve corresponds to the curved profile, the dashed curves to the one consisting of linear pieces. One sees that every break causes a new root $\operatorname{Re}(J) = 0$ because J at the point $c = w_{break}$ for the broken profile varies like $\dfrac{1}{w - c_{break}}$. This corresponds to Rayleigh's well-known theorem that there are as many oscillation roots as breaks. Nevertheless, the curved profile does not possess an oscillation root. After this comment, we revert to our contention that the profiles capable of oscillation generally become unstable if the friction is taken into consideration.

For a proof of this instability, we return to equation (25) and to the more exact solutions in system II. Since we know that c is real, except for quantities of the order of magnitude $(\alpha R)^{-1/2}$, we may assume

$$\varphi_2(+1) \gg \varphi_1(+1)$$

If we furthermore neglect the terms of the order $(\alpha R)^{-1/2}$ in equation (25), we obtain after slight transformations

$$\int_{-1}^{1} \frac{dy}{(w - c)^2} = \frac{1}{g_0(-1)\left[w(-1) - c\right]^2} - \frac{1}{g_0(+1)\left[w(+1) - c\right]^2}$$

We put further $c = c_0 + \delta$ where c_0 is the zero of J, δ a small quantity of the order $(\alpha R)^{-1/2}$. We assume for reasons of simplicity α to be positive; then we may on the right side replace c by c_0 and may develop the left side into a Taylor series in δ. Thus there results, if we break off the Taylor series with the second term which we presuppose as sufficient approximation

$$J(c) = J\left(c_0\right) + \delta \frac{dJ}{dc}_{\left(c=c_0\right)}$$

and from equation (25) because of

$$J\left(c_0\right) = 0; \quad g_0 = \sqrt{-i\alpha R(w - c)}$$

(concerning the sign, compare page 9)

$$\delta \frac{dJ}{dc}_{\left(c=c_0\right)} \sqrt{2\alpha R} = \frac{1 - i}{\left[c_0 - w(-1)\right]^{5/2}} + \frac{1 + i}{\left[w(+1) - c_0\right]^{5/2}}$$

Hence follows, because of

$$c_0 - w(-1) > w(+1) - c_0$$

(c_0 is to lie nearer to $w(+1)$) that the imaginary part of δ and thus also that of c and of β has the same sign as

$$\frac{dJ}{dc}\left(c=c_0\right)$$

and that oscillations corresponding to a negative value of $\frac{dJ}{dc}$ have an unstable character. If therefore our partly linear profile still has the property that $\frac{dJ}{dc} < 0$ at the point $w = c$, it is unstable. This condition $\frac{dJ}{dc} < 0$, however, is satisfied very frequently, for instance, always when the point $w = c$ lies near one wall (for instance, $y = +1$) and the profile is linear from the point $w = c$ to the boundary.

Summarizing, we conclude: The instability or the stability of a profile can be decided for all profiles considered so far by their behavior in the case of frictionless fluid. Profiles which are capable of undamped oscillations in the latter case and where the friction is taken into account become, under certain presuppositions, unstable. The latter profiles must have very special properties as shown above; they must, for instance, be partly composed of linear pieces or they must have a point of inflection $w'' = 0$. (Compare above.)

At the same time, however, these profiles of type 2 are the only ones still to claim physical interest since they are the only ones whose behavior with respect to their stability corresponds approximately to Reynolds' conjectures. Following, we shall show that these profiles, in general, really have a critical Reynolds number (with the exception of the broken profiles).

6. The Reynolds Number of the Stability Limit; Numerical

Calculation on the Parabola Profile

If, therefore, a profile is prescribed which, for frictionless fluid, permits undamped oscillations and with friction is unstable, the question arises, for what minimum value of the Reynolds number does instability occur? The simplified equations (25), (26), etc., do not suffice for answering this question. We must revert to equation (23) and to the forms (11) and (14) for the integrals φ_1, φ_2, φ_3, and φ_4; however, it is, of course, quite impossible generally, for an

arbitrary profile w, to represent the critical Reynolds number as a function of w and of integrals over w; it will only be our task to indicate the way by which one arrives at the critical velocity and then to perform the calculation on a special example.

Since in our last calculations α and R had appeared only in the combination αR (because we had assumed α^2 as small), these calculations can yield at best a critical value for αR only, not for R alone. Thus we must first investigate the behavior of the roots of equation (23) for increasing α^2. Of the roots of equation (23), only those in the solution system II which satisfy the equation $\varphi_4(+1) = 0$ are of interest.

Instead of equation (23) we must therefore discuss the equation

$$\varphi = (w - c) \int \frac{dy}{(w - c)^2}\left(1 + \alpha^2 \int dy(w - c)^2 \int \frac{dy}{(w - c)^2} + \dots\right)$$

$$\varphi = 0 \quad \text{for} \quad y = -1, \quad y = +1$$

If the profile consists, as in Rayleigh's example, of linear pieces, there exists (compare page 27) always a root of equation (28a) for every break and these roots remain in existence for every value of α^2. Thus, the broken profile yields no maximum value of α^2 and therefore cannot ever lead to a critical Reynolds number.[16]

This is different if (cf. pp. 26 and 27) a solution of equations (28) or (28a), respectively, with real c is possible for the reason that either somewhere in the profile w" = 0 or that w(+1) = w(-1), $\varphi = w - w(+1)$ represents a solution of equation (28). These latter types of solution always yield a solution of equation (28a) only for a very definite value of α^2. For w" = 0, c is determined by the very fact that for w" = 0, w is to be w = c; thus the equation (28a) defines a quite definite value of α^2; however, for the case w(+1) = w(-1) a solution of equation (28a) obviously exists only for $\alpha^2 = 0$.

---

[16]It is still presupposed that R and αR are large and $\alpha \ll R$. Thus critical Reynolds numbers will possibly appear if these presuppositions are no longer valid; however, the respective Reynolds numbers R would then probably assume values so small that they certainly would be of no physical significance.

For this type of solution of equations (28) or (28a), which are characterized in the limit $R = \infty$ by a very definite value of α^2, we shall expect that, with the friction taken into consideration, α also may vary from its definite value only by small amounts. For these profiles the appearance of a maximum value (and in the case $w'' = 0$ also of a minimum value) for α is very understandable. Thus all oscillations, the wave length of which is smaller than a certain critical wave length, are in such cases damped for all values of αR.

After having found an upper limit for α^2, one will attempt to determine the approximate magnitude for the lower limit of αR. A simple investigation of equation (25) shows that essential variations in the imaginary part of c occur only after the exponent of e in the approximate representation (11) in $\varphi_1(+1)$, $\varphi_2(+1)$ has decreased to values of the order of magnitude 1; however, if this is the case, we very soon reach the critical value (for which the imaginary part of c is changed from negative to positive values) as will be shown in the numerical example. If we assume that w is essentially linear between $w = c_0$ (c_0 = real part of c) and $w(+1)$ the condition for the approximate magnitude of αR reads

$$\frac{(\alpha R)^{1/2}\left[w(+1) - c_0\right]^{3/2}}{w'(+1)} \sim 1 \tag{29}$$

or

$$(\alpha R)^{1/3} \sim \frac{w'(+1)^{2/3}}{w(+1) - c_0}$$

Since in the cases of interest to us $w(+1) - c_0$ will probably be small, we may by assumption form a conclusion as to high critical Reynolds numbers. At the same time we note that for a certain value of R there will always exist not only a maximum value but also a minimum value for α of the unstable oscillations. This follows from the fact that we did find a minimum value of αR (not R).

As numerical example for our general calculations made so far, we select the parabola profile because it is physically the most interesting one. It is to be classified as "profile capable of oscillation" of the type $w(+1) = w(-1)$.

Here too we shall consider only the two-dimensional motion, that
is, not Poiseuille's flow in tubes but the flow prevailing between two
parallel walls at rest $(y = +1, \ y = -1)$ under the influence of a
constant pressure gradient. Thus we put

$$w = 1 - y^2 \tag{30}$$

The symmetry of w and $w - c$ permits the deduction that φ must
be an even function of y.[17] Thus we single out, from among the solu-
tions of equation (7a), two symmetrical particular integrals and attempt
to satisfy the boundary conditions at one of the walls, for instance,
$y = -1$. Those at the other wall then are fulfilled automatically.
Obviously we may take simply φ_3 as one of those symmetrical integrals.
For the other we choose

$$\frac{\varphi_1(y)}{\varphi_1(0)} + \frac{\varphi_2(y)}{\varphi_2(0)}$$

It follows from equation (29a) that for our profile near critical
velocity c will be small of the order $(\alpha R)^{-1/3}$; in the following
calculations we shall thus cancel terms of higher than first order
in c. Furthermore, we state that $\varphi_2(0)$ will be $\gg \varphi_1(0)$ so that
in the neighborhood of $w = 0$ and $w = c$ the second symmetrical
function φ simply is reduced to $\varphi_1(y)$. From equation (16) then
follows that we have the two integrals φ_3 and $\varphi_1 - i\varphi_2$ at disposal
for fulfillment of the boundary conditions for $y = -1$. Equation (23)
is therewith transformed into

---

[17]If one divides φ into a part even in y and a part odd in y,
each part of φ by itself must satisfy the differential equation (7a)
and the boundary conditions because of the symmetry of $w - c$ and w.
For the general stability investigation of the profile $1 - y^2$ it is
therefore sufficient to treat the two cases "φ even" and "odd"
separately and only these two cases; however, it may easily be seen
that the assumption of symmetrical oscillations, that is, "φ odd" does
not lead to a solution of equation (23) and thus not to unstable
oscillations. The assumption "φ even" therefore suffices for the
stability investigation. This is noteworthy insofar as, accordingly,
all symmetrical oscillations are stable and only unsymmetrical dis-
turbances unstable.

$$\begin{vmatrix} \varphi_3(-1) & \varphi_1(-1) - i\varphi_2(-1) \\ \varphi_3{}'(-1) & \varphi_1{}'(-1) - i\varphi_2{}'(-1) \end{vmatrix} = 0 \tag{31}$$

In φ_3 one must always take $y = 0$ as lower limit for the occurring integrals in order to guarantee the symmetry of φ_3. Furthermore, we shall develop in φ_3 only up to magnitudes of the order α^4 and in the development in $(\alpha R)^{-1}$ break-off with the terms of the order $(\alpha R)^{-1}$. We now write equation (31) in the form

$$\frac{\varphi_1{}'(-1) - i\varphi_2{}'(-1)}{\varphi_1(-1) - i\varphi_2(-1)} = \frac{\varphi_3{}'(-1)}{\varphi_3(-1)} \tag{32}$$

If one inserts equations (11) and (14), there results

$$\frac{e^{2\int_{y_0}^{-1} \sqrt{-i\alpha R(w-c)}\,dy} + i}{e^{2\int_{y_0}^{-1} \sqrt{-i\alpha R(w-c)}\,dy} - i}\sqrt{i\alpha R c} = -\frac{9}{2c} +$$

$$\frac{\dfrac{\alpha^2}{c^2}\left[\int_0^{-1} dy(w-c)^2 + \alpha^2 \ldots + \dfrac{i}{\alpha R} \ldots\right]}{1 + \alpha^2 \int_0^{-1} \dfrac{dy}{(w-c)^2}\int_0^{y} dy(w-c)^2 + \alpha^4 \ldots + \dfrac{i}{\alpha R} \ldots} \tag{33}$$

Since c becomes very small, we assume in first approximation w from 0 to c as linear; $w \sim 2(y + 1)$. Then we obtain

$$\int_{y_0}^{-1} \sqrt{c - w}\ dy = -\frac{1}{3}\, c^{3/2} \tag{34}$$

If we put

$$z = \frac{1}{3}\, c^{3/2}(2\alpha R)^{1/2}$$

there arises from equation (33)

$$\frac{e^{-(1+i)z} + i}{e^{-(1+i)z} - i}\ \frac{3z(1+i)}{2} = -\frac{9}{2} + \frac{a^2}{c}\ \frac{\int_0^{-1} dy(w-c)^2 + \dots}{1 + \alpha^2 \int \dots} \tag{35}$$

This equation is perfectly analogous to equation (26). We are interested, above all, in the limiting value R or z, respectively, for which the unstable oscillations are transformed into stable ones; thus the imaginary part of c is exactly zero. This limiting value z will, of course, also be a function of α. Thus we now assume c as real and thereby obtain the limiting value of z or R, respectively, as a function of α. The minimum value of R on this $R(\alpha)$ curve will be denoted as the characteristic Reynolds number for the parabola profile. Detailed calculation shows that one obtains by means of the form (35) of the stability equation only the upper part of the curve $R = R(\alpha)$ (solid line in fig. 2) which was to be expected according to the deliberations of section 6; however, one can calculate the lower part of the curve $R = R(\alpha)$ only by using for φ_1, φ_2, approximations other (compare equation (19a)) than the asymptotic formulas (11). The critical Reynolds number denotes just the range where the asymptotic formulas cease to be valid. Since this circumstance would lead to very complicated numerical calculations and since we cannot attach, in general, (compare section 7) any essential physical significance to the type of instability here characterized, we used rough estimates for calculation of the lower part (dashed line in fig. 1) of the curve $R = R(\alpha)$ which, of course, cannot yield quantitative results; however, the qualitative behavior of the curve is surely reproduced correctly. Thus, we conclude from figure 2:

1. There exist both a maximum value of α and a minimum value of R for instability.

2. For a definite value of R there exists a maximum as well as a minimum value of α; instability prevails within these values, stability outside.

3. The maximum value of α lies approximately at $\alpha = 0.7$ $\left(\alpha^2 = \frac{1}{2}\right)$; the magnitude of the minimum value of R is of the order of 10^3. A calculation of this minimum value with some degree of exactness is not possible from the figure.

7. Physical Discussion of the Results of Part I

Let us summarize once more in detail all results found concerning the stability problem. Above all, it became clear in the course of the calculation that the problem of the stability of a profile for a viscous fluid can generally be decided by treating like Lord Rayleigh the limiting case of frictionless fluid (equation (8)). Profiles which are unstable then (that is, for $R = \infty$) remain so for sufficiently large finite values of R (section 4) as was to be expected beforehand. Likewise profiles which, in the frictionless case, are not capable of oscillations are found to be stable (section 4) and profiles, which according to the investigation by equation (8) permit undamped oscillations, generally to be unstable (section 5). This latter case obviously is the only one which, physically, signifies something new compared to frictionless hydrodynamics; however, it should be emphasized that this case, contrary to what one might conclude at first from Rayleigh's reports, represents an exceptional case. If one disregards the possibilities $w(+1) = w(-1)$, $w'_{boundary} = \infty$ (section 5), it is a necessary condition for the occurrence of this exceptional case that somewhere in the fluid $w'' = 0$. The broken profiles consisting of linear pieces introduced by Rayleigh belong to those exceptional profiles; however, the only permissible conclusion is that curved profiles for the purpose of stability investigation may not be approximated by polygons according to Rayleigh (page 27). It is true that one may find also for profiles curved everywhere $(w'' \neq 0)$ (section 5) when using the differential equations with friction terms oscillations which are damped for every value of αR for which, however, in the limit $R = \infty$ the amount of the damping like $(\alpha R)^{-1/3}$ tends toward zero; thus one has here also undamped oscillations for $R = \infty$. However, these oscillation roots are lost if one takes the simplified differential equation (8) without friction terms as a basis. Insofar, therefore, this also is not a case of exception to the rule according to

which consideration of friction, in cases where the frictionless equation (8) permits undamped oscillations, results in an excitation; however, as said above, the possibility of undamped oscillations for equation (8) must be regarded as an exceptional case. The parabola profile belongs to these exceptional profiles (section 6). If we investigate further for the unstable profiles concerning the range of values for R within which instability occurs, it is found that generally, too, only profiles of the last class may lead to critical Reynolds numbers, that is, only profiles which permit without friction undamped oscillations; however, among the latter, Rayleigh's broken profiles nevertheless did not yield (compare also page 30, footnote 16) a critical Reynolds number. For Rayleigh's profiles a minimum value of αR does exist but no maximum value of α; therefore, no minimum value of R for the neutral stability either (section 6). Only those profiles of the last class for which in the frictionless case only a definite value of α leads to undamped oscillations (for instance, the types $w'' = 0$ for $w = c$, $w(+1) = w(-1)$) result in a maximum value of α and a minimum value of αR, thus also in a minimum value for R. For a definite value of R there exists therefore for those profiles a maximum as well as a minimum value of α for the unstable oscillations. All these results are in agreement with the stability investigations of hydrodynamic profiles made so far.[18]

The question is now how these mathematical results will manifest themselves experimentally. It seems surprising that the stable profiles (for instance, Couette's[19]) and the unstable ones (for instance, Poiseuille's) empirically show exactly the same behavior. Above a certain Reynolds number turbulence occurs in case of sufficient disturbances; if the disturbances are made as small as possible, the laminar profile may be obtained for arbitrarily high Reynolds numbers. Especially the last fact, which has been tested by Ekman (footnote 10, p. 2), on the parabola profile seems to contradict the theory for unstable profiles; however, it can easily be seen that this contradiction is only illusory:[20] The smaller the external disturbances, the

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[18]Compare the reports quoted in the introduction.

[19]However, compare the interesting investigations of Couette's motion concerning its stability against three-dimensional disturbances. G. J. Taylor, Stability of a Viscous Liquid Contained between Two Rotating Cylinders. Phil. Transact. of the Royal Society London 223., pages 289-343, 1922.

[20]This possibility of interpreting Ekman's tests as a sort of starting effect has been pointed out to me by Professor Prandtl. I should like here to express my deepest gratitude to him for this and many other valuable suggestions.

longer it takes (particularly for high Reynolds numbers since the excitation there is of the order $(\alpha R)^{-1/2}$, compare section 5) until they noticeably influence the motion. Thus it will always be possible to postpone, for flow in tubes, this moment so long that the quantity of fluid, the stability of which is dealt with, has already left the tube when its instability becomes apparent. The Reynolds number we calculated could therefore be tested only on a closed system of tubes where the same quantity of fluid always flows.[21] On the other hand, the tests by Schiller (footnote 9, p. 2) which show that below a certain Reynolds number only laminar motion exists cannot be included at all in stability investigations. The original motion here is not laminar; one rather deals with existence or nonexistence of a turbulent form of motion. At any rate, one may conclude from all these considerations only that a solution of the turbulence problem by stability considerations alone is absolutely not possible.

Still, the previous investigations may yield important qualitative results for our real purpose, the calculation of the turbulent motion. If we interpret the turbulent motion as a certain basic flow with superimposed undamped oscillations, we may conclude from our calculations that the minimum value R for which this type of motion is possible probably also lies at values of the order of magnitude 10^3; that the wave length of the undamped oscillations lies at $2\pi h/2$, namely α at values of the order 1, and that α for a prescribed R is confined to certain occasionally very narrow limits; that furthermore these oscillations have the character of a wall disturbance as may be concluded from the smallness of $w(+1) - c$. These qualitative results are quite independent of the special form of the basic flow; however, beyond such qualitative criteria the calculations made so far do not contribute anything toward the actual solution of the turbulence problem.

PART II: THE TURBULENT MOTION

1. Statement of the Mathematical Problem

The Reynolds number usually denoted as critical (which is, for instance, measured in Schiller's tests and indicates the appearance

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[21]However, the disturbances in stability observed by Ruckes (footnote 10, p. 2) for rather small Reynolds numbers are perhaps caused by instability according to section 7. This would be quite conceivable when the critical Reynolds number according to section 7 lies below the one for which turbulence (compare part II) is possible for the first time.

of turbulence in case of sufficiently large disturbances) has no con-
nection with stability problems and with the laminar flow; it is
absolutely a characteristic constant of the turbulent motion. Like-
wise, the Blasius drag law and the well-known conclusion derived from
it (that the turbulent velocity in the proximity of a wall increases
with the 1/7 power of the distance from the wall) show clearly that
the so-called turbulent motion has its own well-defined regularities
and that it represents a second possible form of motion of the viscous
fluids. Thus the only way to a solution of the turbulence problem is
to attempt to eliminate the indefiniteness of the turbulent motion
and to idealize it until it permits mathematical analysis by Stokes's
equations.

The turbulence problem of hydrodynamics is a problem of energetic,
not of dynamic stability. There exist two different forms of motion
of the viscous fluid, each of which has a certain range of values of
Reynolds numbers within which it is possible. Laminar flow is possible
from $R = 0$ to $R = \infty$ but becomes, however, under certain conditions,
above a certain value of R dynamically neutrally stable. The turbulent
motion on the other hand exists only above a certain critical value
of R and is always[22] energetically more stable than the laminar
motion. Thus one may in the range of R in which both forms of motion
are possible always let the fluid drop from the laminar to the tur-
bulent state by means of sufficiently large disturbances.

In order to make approximate mathematical treatment of the tur-
bulent motion possible, we again consider the flow between two parallel
walls and make, first, the following assumptions:[23] The flow is to
be (a) symmetrical about the X axis with the bounding walls at rest,
(b) periodic in the X direction with the period[24] $2\pi/\alpha$, and
(c) periodic with time with the period $2\pi/\beta$, and (d) all disturbances
are to propagate with a speed β/α relative to the X axis, that is
if the motion is developed into a Fourier series, only products of
$i(\beta t - \alpha x)$ are to appear in the exponents of[25] e.

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[22]Compare F. Noether, elsewhere.

[23]This statement, too, which represents a simple generalization
of Sommerfeld's stability theorem was indicated, for investigation of
the turbulent motion itself, by F. Noether without further conclusions
in his paper entitled "Zur Theorie der Turbulenz" (Concerning the
theory of turbulence), Jahresberichte des deutschen Math. Vereins 23,
page 138, 1914.

[24]The assumption of a definite α is justified by the result of
part I that α is confined between certain limits which are, partic-
ularly in the proximity of the minimum value of R, very narrow.

[25]We need not emphasize the part that the actual motions are doubt-
lessly much more complicated; nevertheless, one may well expect these
statements to permit qualitative statements concerning the turbulence.

The Fourier development of the stream function should therefore read

$$\psi = \varphi_0(y) + \varphi_1(y)e^{i(\beta t - \alpha x)} + \overline{\varphi_1(y)}e^{-i(\beta t - \alpha x)} + \varphi_2(y)e^{2i(\beta t - \alpha x)} + \ldots$$

(36)

The mathematical problem then consists in the determination of the odd (according to α) functions φ_0, φ_1, and φ_2 ($\overline{\varphi}_1$, $\overline{\varphi}_2$ . . . conjugate to φ_1, φ_2). At first, the degree of convergence of the series (36) is completely unknown; the question of convergence can be decided only after calculation of φ_0, φ_1 . . . If we want to carry accuracy so far as to the nth approximation, that is, if we want to calculate φ_0 . . . φ_n, we obviously obtain (n + 1) simultaneous differential equations for the (n + 1) unknown functions φ_0 . . . φ_n. Following, we shall need partly the first, partly the second approximation. Thus we enter equation (4) with the statement equation (36), compare the coefficients of the periodic functions on both sides, and break off with the term $e^{2i(\beta t - \alpha x)}$ and thus with φ_2, φ_2. For $\varphi_0{}'$ we write w (as in equation (6) for ϕ'). Thus three simultaneous differential equations are produced (the simultaneously obtained conjugate equations need not be written down)

$$\frac{d^2}{dy^2}\left[\left(\varphi_1{}'\overline{\varphi}_1 - \overline{\varphi}_1{}'\varphi_1\right) + 2\left(\varphi_2{}'\overline{\varphi}_2 - \overline{\varphi}_2{}'\varphi_2\right)\right] = \frac{i}{\alpha R}\,w''$$

$$\left(\varphi_1{}'' - \alpha^2\varphi_1\right)\left(w - \frac{\beta}{\alpha}\right) - \varphi_1 w'' - \varphi_2{}'\left(\overline{\varphi}_1{}'' - \alpha^2\overline{\varphi}_1\right) - 2\varphi_2\left(\overline{\varphi}_1{}''' - \alpha^2\overline{\varphi}_1{}'\right) +$$

$$2\overline{\varphi}_1{}'\left(\varphi_2{}'' - 4\alpha^2\varphi_2\right) + \overline{\varphi}_1\left(\varphi_2{}''' - 4\alpha^2\varphi_2{}'\right) =$$

$$\frac{i}{\alpha R}\left(\varphi_1{}'''' - 2\alpha^2\varphi_1{}'' + \alpha^4\varphi_1\right)$$

$$\left(\varphi_2{}'' - 4\alpha^2\varphi_2\right)\left(w - \frac{\beta}{\alpha}\right) - w''\varphi_2 + \frac{1}{2}\left(\varphi_1{}'\varphi_1{}'' - \varphi_1\varphi_1{}'''\right) =$$

$$\frac{i}{2\alpha R}\left(\varphi_2{}'''' - 8\alpha^2\varphi_2{}'' + 16\alpha^4\varphi_2\right)$$

(37)

The first of these equations may be integrated twice and yields

$$\varphi_1'\overline{\varphi}_1 - \overline{\varphi}_1'\varphi_1 + 2\left(\overline{\varphi}_2\varphi_2' - \overline{\varphi}_2'\varphi_2\right) = \frac{i}{\alpha R}\left(w' - C - C_1 y\right) \tag{37a}$$

C and C_1 signify arbitrary integration constants.

Since the left side of equation (37a) and w are, according to requirement (a), odd in y, C must disappear in our case.

If we go back from the second to the first approximation, our system of equations is reduced to two simultaneous differential equations for w and φ_1

$$\left.\begin{array}{c} \varphi_1'\overline{\varphi}_1 - \overline{\varphi}_1'\varphi_1 = \dfrac{i}{\alpha R}\left(w' - C_1 y\right) \\[3mm] \left(\varphi_1'' - \alpha^2\varphi_1\right)\left(w - \dfrac{\beta}{\alpha}\right) - w''\varphi_1 = \dfrac{i}{\alpha R}\left(\varphi_1'''' - 2\alpha^2\varphi_1'' + \alpha^4\varphi_1\right) \end{array}\right\} \tag{38}$$

By way of an interpolation we shall now reflect what replaces equation (33) if we do not consider a flow symmetrical about the X axis (requirement a), (that is, the flow of a fluid under a pressure gradient between two walls at rest), but instead a flow antisymmetrical about the X axis (that is, a flow between two walls moved relative to each other without pressure gradient as in the Couette case). Requirements (b) and (c) are to be maintained. The statement (36) will then no longer be satisfactory since φ_1, φ_2, etc., for arbitrary β/α are no longer even functions; in order to obtain the entire flow pattern in terms of odd functions, we must also include the symmetrical oscillations of the form $e^{i(-\beta t - \alpha x)}$ in the formulation for ψ, that is, ψ must start with the terms

$$\varphi_0 + \varphi_1(y)e^{i(\beta t - \alpha x)} + \varphi_1\left(-ye^{-i(\beta t - \alpha x)} + \overline{\varphi_1(y)}e^{-i(\beta t - \alpha x)} + \ldots\right)$$

As a consequence finally all the terms of the form $e^{i(m\beta t - n\alpha x)}$ appear in ψ (elimination of requirement (d)).

In place of equation (33) there results

$$
\left.
\begin{aligned}
&\overline{\varphi_1(y)\varphi_1(y)'} - \overline{\varphi_1(y)'\varphi_1(y)} + \overline{\varphi_1(-y)\varphi_1(-y)'} - \\[2mm]
&\overline{\varphi_1(-y)'\varphi_1(-y)} = \frac{i}{\alpha R}(w' - C) \\[4mm]
&\left(\varphi_1'' - \alpha^2\varphi_1\right)\left(w - \frac{\beta}{\alpha}\right) - w''\varphi_1 = \frac{i}{\alpha R}\left(\varphi_1'''' - 2\alpha^2\varphi_1'' + \alpha^4\varphi_1\right)
\end{aligned}
\right\}
\qquad (39)
$$

The two equations of the system (38) and (39), respectively, are of simple illustrative significance.

The second equation is none other than our former stability equation (7) which determines the amplitude of the oscillation superimposed on a basic flow w and which formed the basis for our investigations in part I. The first equation, however, represents the theorem of momentum. The left side of this equation essentially indicates the momentum transferred on the average by the turbulent vorticity[26], the term with w' on the right represents the laminar momentum transfer, and the constant C or $C_1 y$, respectively, is the constant of the momentum theorem.

Due to the boundary conditions at the walls $\varphi_1 = \varphi_1' = 0$. Therefore there $w' = C$ or $C_1 y_{wall}$, respectively; thus at the walls the laminar momentum transport surpasses the turbulent one, w'_{wall} will generally be very large. (Compare the next section.) At the channel center, however, that is, in the entire tunnel outside of the immediate

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[26]We are referring here to the mean momentum in the X direction which, for our problem, is transferred in the Y direction. The momentum in the X direction equals, on the whole u, the velocity of the particle transporting the momentum in the Y direction is v; thus the momentum transferred during the unit time uv, on the average $\overline{uv}$ which for the case (36) results in

$$
\overline{uv} = -i\alpha\left(\overline{\varphi_1}\varphi_1' - \overline{\varphi_1}'\varphi_1\right)
$$

wall proximity, w' is of the order of magnitude 1, thus very small
compared to C or C_1y, respectively. The turbulent momentum trans-
port will therefore here completely overbalance the laminar one.

It corresponds to the structure of the systems (38) and (39)
that we are able to give immediately a trivial solution of them,
namely $\varphi_1 = 0$, $w' = C$ or, respectively, $w' = C_1y$, that is, we thus
revert to the laminar motion.

Our problem now is to obtain definite results concerning the non-
trivial solutions of equations (38) and (39).

2. The Turbulent Motion in Wall Proximity and

the Law of Resistance

The most important result concerning the behavior of w in the
immediate proximity of the walls is the law derived by V. Kármán
(elsewhere) from the Blasius drag law by means of considerations of
similitude that w in the proximity of a wall increases with $\eta^{1/7}$
(η representing the distance from the wall). We repeat briefly
Von Kármán's train of thought since we are thereby enabled to a general
visualization of what to expect, even without knowing the Blasius law,
concerning the behavior of w of the wall.

As can easily be seen from considerations of similitude, it must
be possible to represent the shearing stress τ acting at a wall
(that is, the drag) in the form

$$\tau = \kappa\mu \frac{U}{h} f(R)$$

where κ signifies a certain dimensionless constant.

If we specially assume a power law there is

$$\tau = \kappa\mu \frac{U}{h} R^\xi = \kappa\mu \frac{U}{h}\left(\frac{Uh\rho}{\mu}\right)^\xi$$

From equation (40) follows inversely

$$U = \left[\frac{\tau}{\kappa}\left(\frac{h}{\mu}\right)^{1-\xi}\rho^{-\xi}\right]^{\frac{1}{1+\xi}}$$

The velocity distribution in wall proximity must then be represented by an equation of the form

$$w = \frac{1}{U}\left[\tau\rho^{-\xi}\left(\frac{h}{\mu}\right)^{1-\xi}\right]^{\frac{1}{1+\xi}}f\left(\frac{\eta}{h}\right)$$

(Here again (compare equations (1) and (6)) w has been selected dimensionless and therefore contains U in the denominator; η denotes the distance from the wall.)

We again assume in first approximation a power law (let σ be a dimensionless constant)

$$w = \frac{\sigma}{U}\left[\tau\rho^{-\xi}\left(\frac{h}{\mu}\right)^{1-\xi}\right]^{\frac{1}{1+\xi}}\left(\frac{\eta}{h}\right)^{\epsilon}$$

If one now requires the velocity distribution in immediate wall proximity to be a function only of τ, μ, ρ, but not of h which is physically very plausible, there follows

$$\frac{1-\xi}{1+\xi} = \epsilon \tag{41}$$

For $\xi = \frac{3}{4}$ as corresponds to Blasius' law there results $\epsilon = \frac{1}{7}$.

In order to understand the physical significance of this result we note: $w \sim \eta^{1/7}$ signifies that $w' = \frac{dw}{d\eta}$ is infinite at the

boundary, that therefore w infinitely clings to the wall; however, it is clear that actually w' at the wall cannot be infinite since w', on the contrary, essentially denotes the shearing stress at the

wall and is therefore, according to equation (40), to be equated
to R^ξ except for a numerical factor independent of R.

$$w'_{edge} \sim R^\xi \tag{42}$$

w' at the wall is therefore very large for the large values of R
which are of interest to us. Corresponding to its derivation the
velocity distribution $w \sim \eta^{1/7}$ will be strictly valid only in the
limiting case of infinitely large distance from the wall or of friction-
less fluid (R = ∞). These facts can be still more easily comprehended
if the law $w \sim \eta^{1/7}$ is written in the form $\eta \sim w^7$. From the fact
that the shearing stress is finite we know that the first term of the
power development $\eta(w)$ must be of the form $\gamma_1 w$. This term, however,
is very small, essentially equalling the reciprocal value of w' and
thus being of the order $R^{-\xi}$ (compare equation (42)).

The meaning of the derived law $w \sim \eta^{1/7}$ is thus obviously that
the series development of $\eta(w)$ is to start with the terms

$$\eta = \gamma_1 w + \gamma_7 w^7 + \ldots \tag{43}$$

where γ_1 is extraordinarily small and that therefore the first
term $\gamma_1 w$ for somewhat large values of w may be cancelled compared
with the second $\gamma_7 w^7$.

According to the explanations above we expect independent of the
validity of the 1/7-law for the basic flow of the turbulent motion
small curvatures at the centers, and in the wall proximity, clinging
of the basic flow to the walls.

For such a profile the investigations of part I do not directly
apply since there w', w'', etc., had been presupposed as finite;
however, these investigations can easily be generalized to include
profiles like the one considered here. (Compare section 5, page 26.)
Particularly, the solution of the reduced equation (3) (thus lim R = ∞)
with satisfaction of the boundary conditions becomes especially simple
here; the profile characterized just now belongs, according to sec-
tion 5, page 26, to those capable of oscillation; a solution of equa-
tion (28) with real c exists. This is extremely important because

it shows that the turbulent profiles are always unstable according to section 5, or, in other words, that it is just the deviation (42) from the laminar resistance law which makes an unstable profile and thus makes turbulence possible.

The solution of equation (28) is for $\alpha = 0$

$$\varphi = \left[w - w(+1) \right] \int_{-1}^{y} \frac{dy}{\left[w - w(+1) \right]^2} \tag{44}$$

By selection of the lower limit of the integral we made φ become zero for $y = -1$; that it becomes zero also for $y = +1$ due to selection $c = w(+1)$ can be seen easily from the following transformation

$$\varphi = \left[w - w(+1) \right] \int_{w(-1)}^{w} \frac{dw}{\left[w - w(+1) \right]^2 w'}$$

The integral of the right side becomes at the point $w = w(+1)$ infinite of lower order than $\dfrac{1}{w - w(+1)}$ since w' there (in the limit $R = \infty$) becomes infinite. Thus $\varphi_1 = 0$ for $y = +1$. By equation (44) we have represented in the limit of frictionless fluid the amplitude of the turbulent oscillations and derived from the boundary conditions the value for β/α, namely $\beta/\alpha = w(+1)$. It is, however, self-evident that the solution symmetrical to equation (44)

$$\varphi = \left[w - w(-1) \right] \int_{1}^{y} \frac{dy}{\left[w - w(-1) \right]^2} \tag{44a}$$

also satisfies the boundary conditions; thus $c = w(-1)$ is valid here. In case of the Couette arrangement we therefore conclude from equations (44) and (44a) that two mutually symmetrical oscillation systems exist, the wave velocities of which agree respectively with the velocities of the two walls ($w(+1)$ and $w(-1)$).

For the symmetrical flow between two walls at rest, on the other hand, $w(+1) = w(-1) = 0$. From equations (44) and (44a) we then conclude that every integral of the form

$$\varphi = w \int \frac{dy}{w^2}$$

satisfies the boundary conditions; however, from the requirement (a) that φ is to be odd there results that we must select as lower limit of the integral $\int \frac{dy}{w^2}$ $y = 0$. Thus

$$\varphi = w \int_0^y \frac{dy}{w^2} \tag{44b}$$

In the case of symmetrical flow there is therefore, particularly for β/α

$$w(+1) = w(-1) = \frac{\beta}{\alpha} = 0 \tag{44c}$$

In the turbulent basic flow, the type of its singularity at the walls is of foremost interest to us; thus for the assumption $w \sim \eta^\epsilon$, the exponent ϵ. We shall attempt to show that from the differential equations (38) and (39) respectively in the limit $R = \infty$ at least in immediate proximity of the wall such a power law with the exponent $\epsilon = 1/7$ actually follows. It is true that the domain of convergence of the power series used is not established so that the conclusions, as far as they apply to the shape of the profile at some distance from the wall, are uncertain. We develop φ_1 and w in the neighborhood of $\eta = 0$ in integral and positive powers of η - this is possible for any finite value - and then inversely η in integral powers of w. Thus we are led directly to the formula (43) for $\eta(w)$.

We contend, and this is the most important result we shall need later, that φ_1 in first approximation may be represented by a series of the form

$$\varphi_1 = \alpha_2 \eta^2 + \alpha_5 \eta^5 + \alpha_8 \eta^3 + \ldots$$

where α_2, α_8 . . . are real, α_5, α_{11} . . . purely imaginary constants; furthermore, w is of the form

$$w = \beta_1 \eta + \beta_7 \eta^7 + \ldots \tag{45}$$

This contention may be proved for the differential equations (38) directly by expressing φ and w in undetermined coefficients if the terms α_0, α_1, α_2, α_3, and β_0, β_1 are prescribed. Thus we will, above all, attempt to determine these terms. First, φ_1 and φ_1' for $\eta = 0$ must be zero because of the boundary conditions; thus the series for φ_1 starts with $\alpha_2 \eta^2$ ($\alpha_0 = \alpha_1 = 0$). We can verify afterward that, furthermore, the following term $\alpha_3 \eta^3$ is eliminated, that is, becomes very small compared to the other terms. By way of an interpolation we shall prove here for this purpose by a single approximate integration of the second equation (38) that α_3 assumes the order of magnitude αR. For $\alpha^2 = 0$ equation (38) reads

$$\varphi_1''\left(w - \frac{\beta}{\alpha}\right) - \varphi_1 w'' = \frac{i}{\alpha R} \varphi_1''''$$

whence follows

$$\varphi_1'\left(w - \frac{\beta}{\alpha}\right) - \varphi_1 w' = \frac{i}{\alpha R} \varphi_1''' + A \tag{46}$$

The constant A is here of the same order of magnitude as the left side of equation (46) at the center of the tunnel, thus almost of the order of magnitude 1. (Compare part I, section 2.) The term φ_1''' at the edge is therefore of the order αR due to the boundary conditions. The same is valid for α_3.

Thus we shall meanwhile assume α_3 as small and later attempt to justify that assumption. Of the constants β_0, β_1, the first, β_0, equals zero because of the requirement (a), section 1.

The constants α_2 and β_1 are, at first, arbitrary[27] and there is no possibility of deriving them from the solution of the differential equations (38) and (39) in the proximity of the wall. This possibility would arise only if we should succeed in continuing the solution (45) analytically up to the other wall; however, this is an extremely complicated mathematical problem if only for the reason that, as will be seen, the simplified equations (38) and (39) are not sufficient for determining φ_1 and w at the center of the tunnel. Although we must therefore forego the solution of this problem, we may still expect to obtain, by merely developing φ_1 and w in the proximity of one wall with undetermined coefficients α_2, β_1, those qualitative characteristics of w and φ_1 in wall proximity which are, according to experience, quite independent of the behavior of the fluid at the tunnel center as, for instance, the law $w \sim \eta^{1/7}$.

We enter equation (38) with the statement

$$\varphi_1 = \alpha_2 \eta^2 + \alpha_4 \eta^4 + \alpha_5 \eta^5 + \ldots$$

$$w = \beta_1 \eta + \beta_2 \eta^2 + \beta_3 \eta^3 + \ldots$$

replacing the second equation by (46). We therefore again assume α as very small which here only signifies (compare part I, section 2) that the wave length of the oscillations is to be large compared to the boundary-layer thickness; moreover, we put, according to equation (44c)

$$\frac{\beta}{\alpha} = 0$$

For the first equation (38) we write furthermore

$$-i\alpha R\left(\varphi_1{}' \,\overline{\varphi}_1 - \overline{\varphi}_1{}' \varphi_1\right) = 2\alpha R\left(\varphi_{1i}{}' \varphi_{1r} - \varphi_{1i}\varphi_{1r}{}'\right) = w' - c_1 y$$

Therein φ_{1r} denotes the real, φ_{1i} the imaginary part of φ_1.

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[27] We shall assume α_2 as real. This does not imply a limitation of generality since φ_1 is determined only up to a factor of the form e^{iX} as the initial point of the time coordinate in equation (36) may be chosen arbitrarily.

From equations (46) and (47) now follow the recursion formulas

$$n(n - 1)(n - 2)\alpha_n = -i\alpha R \sum_{s=2}^{n-2} s(n - 2s)\, \alpha_{n-1-s}\beta_{s-1} \qquad (48)$$

$$n\beta_n = 2\alpha R \sum_{s=2}^{n-2} s(n - 2s)\alpha_{n-s}{}^i\beta_s{}^r \qquad (49)$$

in addition

$$\beta_1 = C_1 y_{edge}$$

$$2\beta_2 = C_1$$

Therein $\alpha_s{}^r$ denotes the real, $\alpha_s{}^i$ the imaginary part of α_s.

From equation (48) there follows first

$$\alpha_4 = 0$$

From equation (49) there then results

$$\beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$$

The term β_2 may also be approximately equated to zero.

From equation (49) there follows

$$\beta_2 = \frac{\beta_1}{2y_{edge}}$$

For very small η the term $\beta_2\eta^2$ is therefore to be neglected compared to the first term $\beta_1\eta$; for larger η, however, the higher terms $\beta_7\eta^7$ etc., are completely predominant.

Let us thus assume also $\beta_2 = 0$ and thus calculate the higher terms of the series for β_1 and w.

There follows

$$\alpha_5 = -i\alpha R \frac{\alpha_2\beta_1}{3 \times 4 \times 5}; \quad \alpha_6 = \alpha_7 = 0; \quad \alpha_8 = -(\alpha R)^2 \frac{\alpha_2\beta_1^2}{3 \times 5 \times 6 \times 7 \times 8}$$

$$\alpha_9 = \alpha_{10} = 0;$$

$$\alpha_{11} = -(\alpha R)^3 \left(\frac{\alpha_3^3\beta_2}{2 \times 7 \times 9 \times 10 \times 11} - \frac{\alpha_2\beta_1^3}{3 \times 5 \times 6 \times 8 \times 9 \times 10 \times 11} \right);$$

$$\alpha_{12} = \alpha_{13} = 0$$

$$\beta_7 = -(\alpha R)^2 \frac{\alpha_2^2\beta_1}{70}; \quad \beta_8 = \beta_9 = \beta_{10} = \beta_{11} = \beta_{12} = 0;$$

$$\beta_{13} = (\alpha R)^4 \left(\frac{\alpha_2^2\beta_1^3}{5 \times 5 \times 6 \times 8 \times 11 \times 13 \times 14} - \frac{\alpha_2^4\beta_1}{7 \times 10 \times 11 \times 13} \right)$$

The representation (45) for w we contended is therefore proved and it may easily be shown too that of the further terms only in every sixth term has β a finite value.

Hence follows for the representation of η as power series in w

$$\left. \begin{aligned} \eta &= \gamma_1 w + \gamma_7 w^7 + \gamma_{13} w^{13} + \ldots \\[2mm] \gamma_1 &= \frac{1}{\beta_1}, \quad \gamma_7 = (\alpha R)^2 \frac{\alpha_2^2}{70\beta_1^7} \\[2mm] \gamma_{13} &= (\alpha R)^4 \left(\frac{\alpha_2^4 \times 3 \times 3 \times 17}{7 \times 10 \times 10 \times 11 \times 13 \times \beta_1^{13}} - \frac{\alpha_2^2}{5 \times 5 \times 6 \times 8 \times 11 \times 13 \times 14 \times \beta_1^{11}} \right) \end{aligned} \right\} (50)$$

The terms γ_2 to γ_6, γ_8 to γ_{12}, γ_{14} etc., all equal zero.[28]

The development (50) now actually completely agrees with equation (43) and we seem thus to arrive, even without knowledge of the constants α_2 and β_1, to the law $\eta \sim w^7$ semiempirically derived by von Kármán. The coefficients γ_1 and γ_7, however, cannot be calculated. Inversely, we may perhaps conclude from the empirical findings for the coefficients β_1 and α_2 that γ_7 is of the order of magnitude 1, γ_1 of the order $(\alpha R)^{-3/4}$, thus $\alpha_2 \sim (\alpha R)^{13/8}$. Subsequently, we thus also confirm our former assertion $\alpha_3 \eta \ll \alpha_2$. Raising the question of what order of magnitude are the values of w for which the third term in equation (50) is small compared to the second for which therefore the w^7 profile actually is valid, one finds $w \sim \beta_1^{-1/6}$, thus $\sim R^{-1/8}$. Accordingly, the profile $w \sim \eta^{1/7}$ follows from the differential equations (38) only qualitatively at first. No information about the fact that the 1/7 profile has been observed almost up to the tunnel center is given in our calculations; however, this was not to be expected since the other constants entering the law also depend on the behavior of the fluid at the opposite wall.

As an interpolation, we shall once more briefly summarize what factors we have neglected in deriving equation (50) from equations (48) and (49) and attempt thereby to determine within what accuracy the conclusions drawn from (50) are correct. First, we used system (38) instead of (37), thus cancelled magnitudes of the order φ_2/w. Furthermore, we equated $\alpha_3 = 0$, $\beta/\alpha = 0$, $\beta_2 = 0$ and therewith neglected magnitudes of the order $\dfrac{\alpha_3 \eta}{\alpha_2}$, $\dfrac{\beta}{\alpha w}$, $\dfrac{\beta_2 \eta}{\beta_1}$, and $\dfrac{\beta_1}{\beta_7 \eta^6}$, respectively. The accuracy of our calculations will be determined by the largest of the terms here neglected. Simple considerations of the order of magnitude, not executed here, make it probable that of these terms φ_2/w is the largest but that this term, too, goes toward zero with $R \longrightarrow \infty$.

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[28]This power series $\eta(w)$ may, of course, also be derived directly from equations (46) and (47) without the detour over the series of $w(\eta)$ if w is introduced as independent variable; however, the calculations required for this purpose are somewhat more complicated.

Selection of a sufficiently large value for R will therefore make it possible to carry the accuracy of the results derived from equations (48), (49), and (50) arbitrarily far.

As to Blasius' law of resistance, it can, of course, be derived inversely according to the method described above from the law $\eta \sim w^7$ by means of consideration of similitude if one assumes, as we did, that the behavior of w in the proximity of the wall is independent of the tunnel width; however, for the reasons stated above (impossibility of analytical continuation) we must leave the question unanswered whether this latter - physically very plausible - assumption also follows from the differential equations (38) and (39), respectively.

We are, however, able to draw a noteworthy direct conclusion concerning the law of resistance from equations (38) and (39) by means of consideration of similitude. In the tunnel, except for immediate wall proximity and the point $y = 0$ (compare below equation (66)) one may write instead of the first equation (38) because of the magnitude of C_1 (compare pages 41 and 42)

$$i\alpha R\left(\varphi_1{}'\overline{\varphi}_1 - \overline{\varphi}_1{}'\varphi_1\right) = C_1 y \tag{51}$$

Since the amplitude φ_1 cannot go toward infinity with $R \longrightarrow \infty$ - this would render all our calculations devoid of physical sense - there follows that C_1 is at most of the order of magnitude αR, that therefore the exponent ξ of equation (40) must be <1 (which in a certain manner also can be seen from equation (41)). Hence follows (compare equations (42) and (40)) that the law of resistance $\tau = \text{const.}U^2$ usually assumed in hydraulics represents an upper limit for all imaginable laws of resistance of turbulence which is independent of the wall characteristics. One may conclude as an assumption that the law $\tau \sim U^{7/4}$ is valid only for smooth walls - it was for those only that we obtained $\eta \sim w^7$ - that the law of resistance for rough walls, however, more and more approaches the quadratic law.[29] For rough walls the amplitude φ_1 will be independent of R and of the magnitude of the wall disturbances; moreover, for rough walls the boundary conditions will no longer cause φ_1 to be real in first approximation as corresponds to equation (44).

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[29]Compare the more exact investigations by Von Kármán, elsewhere, and the experimental investigations by Schiller, same periodical 3, page 2, 1923.

Nothing is changed in the conclusions of this paragraph if the equations (39) are taken as a basis instead of the differential equations (38).

3. The Turbulent Motion Outside of the Immediate

Proximity of the Wall

It is essential for the motion at the tunnel center that φ_1 here is composed of those two integrals of (7a) which appear in case of frictionless fluid, thus for equation (8). (Compare part I, section 2.) The most important characteristic of φ_1 following from this fact is that it satisfies - except for magnitudes of the order φ_2 and $(\alpha R)^{-1}$ - the condition

$$\varphi_1'\overline{\varphi}_1 - \overline{\varphi}_1'\varphi_1 = \text{Const.} \tag{52}$$

This results, according to Abel's theorem, from the fact that, except for magnitudes of the order φ_2 and $(\alpha R)^{-1}$, φ_{1r}, and φ_{1i} (the real and imaginary part of φ_1) are solutions of the differential equation (8).

Hence it can be concluded that the equations (38) and (39) are not sufficient for establishment of the motion over the entire tunnel width but that we have to go back to equation (37) and to the system of equations which corresponds to it for Couette's case.

This, in general, involves a complication of the mathematical problem. Only in Couette's case may the problem be solved comparatively easily because the first equation (39), except for magnitudes of the order w'/C, thus $(\alpha R)^{-3/4}$ and $\varphi_2{}^2$, compare equation (37), agrees with equation (52). Whereas, therefore, equation (52) in consequence of its derivation from Abel's theorem is correct only up to magnitudes of order φ_2, in Couette's case equation (39) should still be valid up to magnitudes of the order $\varphi_2{}^2$. This requirement is satisfied if we put

$$\varphi_2 = 0 \tag{53}$$

This equation is therefore to be regarded as solution for Couette's case, of the differential equation we took as a basis.

According to equation (53) it would follow for φ_1 from (37)

$$\varphi_1{}''' \varphi_1 - \varphi_1{}'' \varphi_1 = 0 \tag{54}$$

The system applying to Couette's case is not equation (37) but a more complicated one which we are not going to write down. We do, however, state about it that it leads, like (37), for $\varphi_2 = 0$ to the solution (54) and thus to the result

$$\varphi_1 = a e^{\gamma y} + b e^{-\gamma y} \tag{55}$$

Here, a, b, and γ are any complex constants. For w then follows from the second equation (39) or, respectively, from its reduced form (8)

$$w - c = a_1 e^{\gamma_2 y} + b_1 e^{\gamma_1 y} \tag{56}$$

Since at one of the walls there should be $w - c = 0$, and since, on the other hand, w should be odd in the neighborhood of $y = 0$, it follows that w, simply by the vanishing of γ_1 and a suitable increase of a_1 and b_1, must degenerate to a linear profile.

Thus, we obtain the important result that for Couette's case the basic profile w of the turbulent motion takes an essentially linear course over the entire tunnel width - however, strongly deviating from the laminar profile, it will be much flatter than the laminar one - [that, however, (compare II, section 2) at the edge it clings again like $\eta^{1/7}$ to the walls].

We shall now turn to the more complicated case of a flow between two walls at rest, thus exactly to the system (37). For a solution we must naturally be content with rough approximations. First, we

can cancel in equation (37) the right sides of all three equations, namely the friction terms; this is fully justified by the considerations of Part I, section 2. Then we equate $\beta/\alpha = 0$ (compare equation (44c)).

We thus obtain for φ_1 in the place of the second equation (37)

$$\varphi_1''w - w''\varphi_1 - \alpha^2 w\varphi_1 - \varphi_2'\left(\bar{\varphi}_1'' - \alpha^2\bar{\varphi}_1\right) - 2\varphi_2\left(\bar{\varphi}_1''' - \alpha^2\bar{\varphi}_1'\right) + 2\bar{\varphi}_1'\left(\varphi_2'' - 4\alpha^2\varphi_2\right) + \bar{\varphi}_1\left(\varphi_2''' - 4\alpha^2\varphi_2'\right) = 0 \tag{57}$$

If we develop φ_1 as solution of the equation (57) in powers of α^2 on one hand and powers of φ_2 on the other, and if we further note that φ_1 is to be odd (compare (44b)) and write $\varphi_1 = \varphi_{10} + \varphi_{11}$ there results with only the linear terms taken into consideration

$$\varphi_{10} = aw \int_0^y \frac{dy}{w^2} \tag{58}$$

$$\varphi_{11} = aw \int_0^y \frac{dy}{w^2} \int_0^y dy\left(\alpha^2 w\varphi_{10} + \varphi_2'\varphi_{10}'' + 2\varphi_2\varphi_{10}''' - 2\varphi_{10}'\varphi_2'' - \varphi_{10}\varphi_2'''\right) \tag{59}$$

Naturally φ_1 is herein not fully determined - the constant factor a assumed as real which does not signify a limitation remains undetermined.

If we substitute this value of φ_1 into the simplified first equation (37), namely

$$\varphi_1\bar{\varphi}_1'' - \bar{\varphi}_1\varphi_1'' = \text{Const.} \tag{60}$$

we obtain, with φ_{2i} denoting the imaginary part of φ_2, the result

$$\left(\varphi_{2i}{'}\varphi_{10}{''} + 2\varphi_{2i}\varphi_{10}{'''} - 2\varphi_{10}{'}\varphi_{2i}{''} - \varphi_{10}\varphi_{2i}{'''}\right)\int_0^y \frac{dy}{w^2} = \text{Const.} \quad (61)$$

Now, however, as follows from the third equation (37) and from the fact that φ_1 is real in first approximation, φ_{2i} satisfies the equation

$$\varphi_{2i}{''}w - \varphi_{2i}w{''} = 0 \quad (62)$$

thus

$$\varphi_{2i} = bw\int_0^y \frac{dy}{w^2} = c\varphi_{10} \quad (63)$$

If we substitute this value of φ_{2i} into equation (63) and if we further consider that for $y = 0$ the left side of equation (61) and therewith the constant on the right side is zero (this signifies for the constant of the right side of equation (60) only that it is in first approximation zero, that is, small of the order $\varphi_1\varphi_2{}^2$ or $\alpha^2\varphi_1\varphi_2$, respectively, or $\alpha^4\varphi_1$, we obtain

$$\varphi_{10}\varphi_{10}{'''} - \varphi_{10}{''}\varphi_{10}{'} = 0 \quad (64)$$

which fully agrees with (54).

This equation, it is true, becomes, like equation (54), trivial in the neighborhood of the point $y = 0$; it is there fulfilled identically since φ is an odd function of y. Thus it cannot permit there a determination of w. This leads for the symmetrical profile (64) to a remarkable discontinuity at the point $y = 0$. (For the odd profile such a discontinuity cannot be seen from the differential equations.) If one integrates (37a) one obtains, as shown above, after a single integration the equation

$$2\alpha R\left[\overline{\varphi}_1{''}\varphi_1 - \varphi_1{''}\overline{\varphi}_1 + 2\left(\overline{\varphi}_2{''}\varphi_2 - \varphi_2{''}\overline{\varphi}_2\right) + \ldots\right] = w{''} - c \quad (65)$$

where C (compare pp. 41 and 42), Blasius' law of resistance being valid,
is of the order of magnitude $(\alpha R)^{3/4}$, thus at any rate very large .
The left side of equation (65) disappears, however, with φ_1 and φ_2
(which, as we know, are odd functions of y) at the point y = 0. Thus

$$w_{y=0}'' = C \qquad\qquad (66)$$

must be valid there. This signifies that $w_{y=0}''$ is very large
$\left(\sim(\alpha R)^{3/4}\right)$ and that therefore w at the point y = 0 shows a sharp
break[30] $\left(\text{radius of curvature} \sim(\alpha R)^{-3/4}\right)$. At a small distance from
this point the course of w must, according to equation (64), again
be essentially linear.

We obtain the result: For the flow between two walls at rest as
well - and surely this may be applied also to the flow in the tube -
the profile is linear approximately over the entire tunnel width; at
the center, however, it shows a sharp break (it clings to the walls
with the $y^{1/7}$ law). (Compare figure 3.)

The physical cause of the sharp break is the fact that the gradient
of the turbulent momentum transfer for y = 0 disappears for reasons
of symmetry and that therefore, because the gradient of the entire
momentum transfer over the tunnel width is constant, the gradient of
the laminar momentum transfer, that is w'', must be very large there.

4. Final Remarks and Summary of the Physical Results

Our investigations still show two important gaps. First, they do
not yield the transition from the $\eta^{1/7}$ profile to the linear profile
valid in the center part. Second, they are limited to large values
of R and thus do not yield the minimum value of R, either, if such
a minimum value exists for which the turbulent motion is still possible.
The first of these two gaps is most difficult to fill in (compare
page 48); we cannot even indicate a method which would satisfactorily

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[30]Professor Prandtl was so kind as to point out this break to me
on the basis of empirical material. The break seems less sharp empiri-
cally than according to calculation results, which is easily explained by
the fact that the assumption (a), page 38, concerning the symmetry of the
vortices and disturbances also does not exactly correspond to actual
conditions.

solve this particular problem. One may attempt to piece the two
approximations together that come from the wall and from the tunnel
center. This would have to be done by means of the condition that
at the respective junction φ_1, φ_1', φ_1'', φ_1''', w, and w' are to

be continuous; however, the convergence of the developments (45) and
(50) is hardly sufficient thus to guarantee a somewhat defined approxi-
mation. At any rate the ultimate result, the profile w, is still to
a great deal dependent on the type of joining the two approximations.
Finally, it must be regarded as dubious whether such an exact carrying
out of the formulation (page 39) would yield essentially new physical
results in agreement with experience since these statements certainly
represent a very strong idealization of actual conditions.

In contrast, filling in of the second gap does not offer any
basic difficulties whatsoever; all necessary expedients are contained
in Part I and once the profile w is completely known, the methods
described in Part I are, on principle, sufficient to calculate according
to Part I, section 6, the minimum value of R for which the turbulent
motion is possible. One could, for instance, calculate the critical
Reynolds number for a profile obtained, according to the method
mentioned above, by piecing together the two approximations, or one
could base this investigation on the empirically observed profile and
thus calculate the Reynolds number in a semiempirical manner. In any
case one will - the investigations in Part I made this probable and
direct calculations, here not reproduced, confirmed it - arrive at
the same order of magnitude of the critical Reynolds number, namely
$R \sim 10^3$. The exact value of R will, it is true, still be too
dependent on the manner in which the profile was obtained to permit
comparisons with experience. For that reason we did not perform here
such a calculation of R.

Let us finally summarize what may be concluded as physical result
from our investigations concerning the turbulence problem. In Part I
we recognize that the laminar motion and its stability condition are
not of essential significance for the turbulence problem and the
critical Reynolds number. In Part II, however, we investigated the
turbulent motion itself and may hence give a few data on the turbulent
state of motion. In general, the velocity distribution over the
entire tunnel is of the simplest type; it is - according to the test
conditions - linear or constant (section 3). At the center there is,
for symmetrical flow between two walls at rest, a sharp break; at
the walls the flow clings, for the $\eta^{1/7}$ profile, to the walls
(section 2). The calculations do not disclose anything about the fact
that the 1/7 profile is valid until far into the tunnel interior. The
turbulent oscillations are for Couette's case almost harmonic in the
interior of the tunnel (section 3, equation (53)); in the proximity

of the walls all oscillations will occur. The velocity of the waves
agrees with the wall velocity (section 2, equations (44) - (44c)); for
Couette's case there exist two groups of turbulent oscillations, one
of which agrees, with respect to its velocity of propagation, with one
of the walls, whereas the other group possesses the velocity of the
other wall. Thus the turbulent disturbances show, superficially, the
character of a wall disturbance. It must, however, be emphasized that
these disturbances are capable of existence as free oscillations, inde-
pendently of wall roughness and similar influences. The amplitude of
the turbulent waves considerably increases toward the walls (this
follows from equation (44), section 2) and goes toward zero only
directly at the wall.

The wave length of the occurring oscillations (Part I, section 8)
is, with respect to order of magnitude, equal to (rather somewhat
larger than) the tunnel width. The minimum value of the Reynolds num-
ber (Part I, section 8) for which turbulence is still possible, lies -
with respect to order of magnitude - near 10^3. From the profile $\eta^{1/7}$,
Blasius' $\tau \sim u^{7/4}$ seems to result, under certain presuppositions, as
the law of resistance for smooth walls. For rough walls it probably
approaches (section 2) the hydraulic law $\tau \sim u^2$. The purpose of the
present report was not so much to establish these regularities, to a
great part known before, as it was to prove that all results obtained
so far (seemingly partly contradicting each other) can be uniformly
described mathematically with the aid of simple basic assumptions.

I wish to express here my deepest gratitude to my revered teacher,
Professor Sommerfeld, for suggesting this report and for frequent
assistance.

Translated by Mary L. Mahler
National Advisory Committee
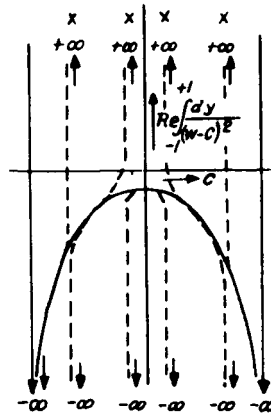for Aeronautics

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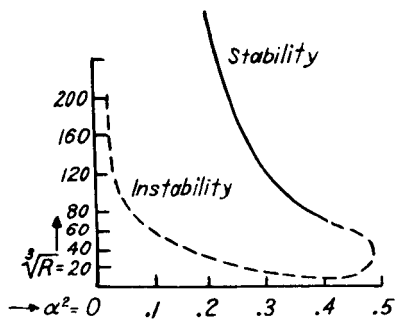--- Approximation polygon
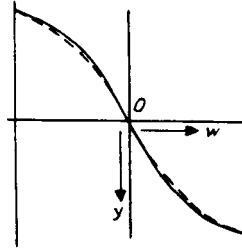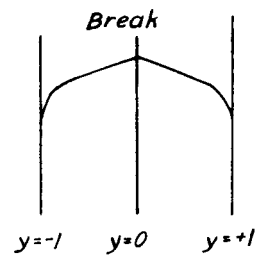
x = points where   c = w break

Figure 1

Figure 2

Figure 3