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TECHNICAL MEMORANDUM 1288

THE DIFFUSION OF A HOT AIR JET IN AIR IN MOTION

By W. Szablewski

Translation of "Die Ausbreitung eines Heissluftstrahles in Bewegter Luft." GDC/2460, September 1946

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PART II. THE FLOW FIELD IN THE TRANSITION ZONE

SUMMARY

The turbulent diffusion of a hot air jet in air can be divided into two zones, the core and the transition zone. The first part of this study (reference 1) deals with the flow field in the core, the second (the present report) with that in the transitional zone.

Part A of the present report is limited to small temperature differences. The decrease in the velocity and the temperature along the jet axis, the breadth of the mixing region, as well as the asymptotic distribution functions, are determined.

The empirical constant K, a measure for the mixing length, appearing in the theory, follows closely a value of 0.010 for asymptotic conditions.

Experimental data are available only for the case of outside air at rest. The comparison with theory indicates that the asymptotic distributions are satisfactorily reproduced with exception of the boundary zone. Velocity and temperature drop are very closely reproduced by the computed functions up to a point near the boundary of the core, while a last short fraction of the region adjacent to the boundary of the core is not covered by the theory.

Part B of the present report deals with greater temperature differences. The breadth of the mixing region, as well as the velocity and temperature drop along the jet axis, is calculated. The theory is then compared with Pabst's measurements (reference 2). The ratio of interchange of temperature and velocity yielded a factor E = 2. Considering the friction loss at the nozzle wall, the agreement between the theoretical and the experimental velocity decrease along the jet axis can be regarded as satisfactory. The temperature measurement along the jet axis appears to be faulty.

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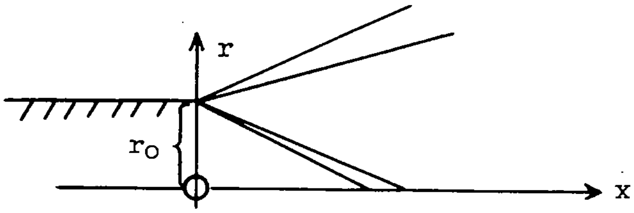
*"Die Ausbreitung eines Heissluftstrahles in Bewegter Luft." GDC/2460, September 1946.

A: SMALL TEMPERATURE DIFFERENCES BETWEEN JET AND SURROUNDING MEDIUM

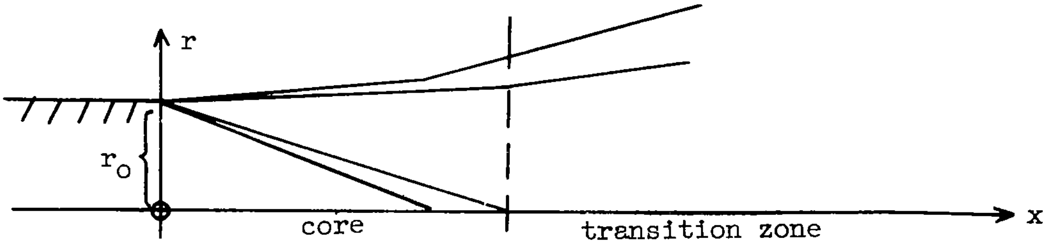
I. Method and Results

1. In the first part of the investigation (reference 1) the flow field in the core was computed.

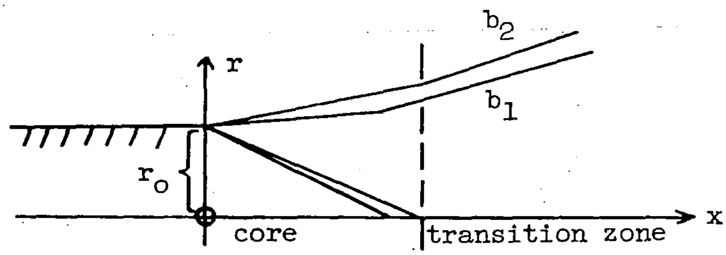


The investigation included the variation of the curves bounding the mixing zones of velocity and temperature as well as the aspect of the velocity and temperature distribution functions over the mixing zones.

In the second part, (the present report), the flow field in the zone adjoining the core is investigated.



This zone, which in the asymptote in the so-called axially symmetrical jet diffusion is characterized by the affinity of the flow processes in the cross sections of the jet, is termed the transition zone.



Variation of mixing width of the velocity and the temperature over the nozzle spacing

The following relations are involved:

b_1 the breadth of the mixing region of the velocity

b_2 the breadth of the mixing region of the temperature

In addition

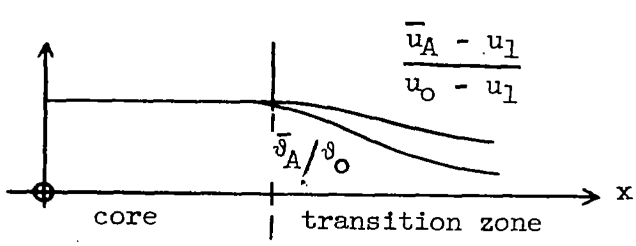
ϑ_o temperature rise of the discharging jet

$\bar{\vartheta}_A$ temperature rise of the jet on jet axis

u_o velocity of the discharging jet

u_1 velocity of the surrounding medium

$\bar{u}_A$ velocity of the jet on jet axis



Variation of jet velocity and jet temperature along the jet axis

2. In the extension of the theory of free turbulence to gases of widely variable densities, the following turbulent exchange quantities had been obtained in part I (reference 1).

$$M = El^2\left|\frac{\partial\bar{u}}{\partial y}\right|\frac{\partial\bar{\rho}}{\partial y} \qquad \text{for turbulent diffusion (1a)}$$

$$\tau = l^2\left|\frac{\partial\bar{u}}{\partial y}\right|\left(\bar{\rho}\frac{\partial\bar{u}}{\partial y} + E\bar{u}\frac{\partial\bar{\rho}}{\partial y}\right) \qquad \text{for turbulent shearing stress (1b)}$$

$$Q = El^2\left|\frac{\partial\bar{u}}{\partial y}\right|\left[\frac{\partial\left(\bar{\rho}\bar{c}_p\bar{T}\right)}{\partial y}\right] \qquad \text{for turbulent heat conduction (1c)}$$

On the assumption of constant pressure for the diffusion of a hot air jet in air in motion (axially symmetrical case) the motion equations were then obtained:

equation of continuity of mass

$$\frac{\partial(r\bar{\rho}\bar{u})}{\partial x} + \frac{\partial(r\bar{\rho}\bar{v})}{\partial r} = E\epsilon(x)\frac{\partial}{\partial r}\left(\frac{r\partial\bar{\rho}}{\partial r}\right) \qquad (2a)$$

equation of continuity of momentum

$$\frac{\partial(r\bar{\rho}\bar{u}\bar{u})}{\partial x} + \frac{\partial(r\bar{\rho}\bar{v}\bar{u})}{\partial r} = \epsilon(x)\frac{\partial}{\partial r}\left(r\bar{\rho}\frac{\partial\bar{u}}{\partial r} + Er\bar{u}\frac{\partial\bar{\rho}}{\partial r}\right) \qquad (2b)$$

equation of continuity of the heat (energy principle)

$$\frac{\partial(r\bar{\rho}\bar{u}\bar{T})}{\partial x} + \frac{\partial(r\bar{\rho}\bar{v}\bar{T})}{\partial r} = E\epsilon(x)\frac{\partial}{\partial r}\left[r\frac{\partial(\bar{\rho}\bar{T})}{\partial r}\right] \qquad (2c)$$

with the apparent kinematic viscosity

$$\epsilon(x) = Kb_1\left|\bar{u}_{max} - \bar{u}_{min}\right|$$

Owing to the continuity of mass, it further yields:

momentum

$$\frac{\partial r\bar{\rho}\bar{u}\left(\bar{u} - u_1\right)}{\partial x} + \frac{\partial r\bar{\rho}\bar{v}\left(\bar{u} - u_1\right)}{\partial r} = \epsilon(x)\frac{\partial}{\partial r}\left[r\bar{\rho}\frac{\partial\left(\bar{u} - u_1\right)}{\partial r} + Er\left(\bar{u} - u_1\right)\frac{\partial\bar{\rho}}{\partial r}\right]$$

Heat

$$\frac{\partial r\bar{\rho}\bar{u}\bar{\vartheta}}{\partial x} + \frac{\partial r\bar{\rho}\bar{v}\bar{\vartheta}}{\partial r} = E\epsilon(x)\frac{\partial}{\partial r}\left[r\frac{\partial(\bar{\rho}\bar{\vartheta})}{\partial r}\right]$$

($\bar{\vartheta}$ = temperature rise of jet.)

Integration with respect to r gives then for the transition zone:

momentum

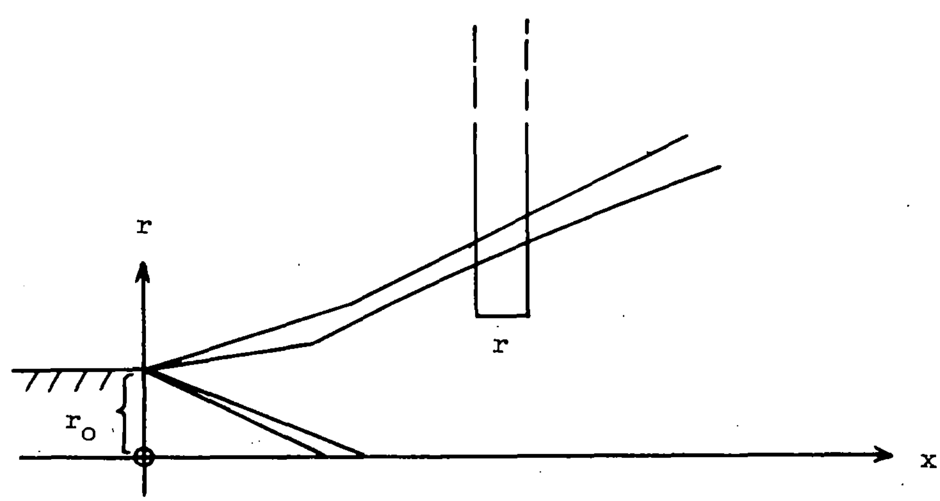
$$r\bar{\rho}\bar{v}\left(\bar{u} - u_1\right) - \frac{\partial}{\partial x}\int_r^{b_1}\bar{\rho}\bar{u}\left(\bar{u} - u_1\right)r\,dr = \epsilon(x)r\left[\rho\frac{\partial\left(\bar{u} - u_1\right)}{\partial r} + E\left(\bar{u} - u_1\right)\frac{\partial\bar{\rho}}{\partial r}\right] \quad (3a)$$

heat

$$r\bar{\rho}\bar{v}\bar{\vartheta} - \frac{\partial}{\partial x}\int_r^{b_2}\bar{\rho}\bar{u}\bar{\vartheta}r\,dr = E\epsilon(x)r\frac{\partial}{\partial r}(\bar{\rho}\bar{\vartheta}) \quad (3b)$$

with $\epsilon(x) = Kb_1\left(\bar{u}_A - u_1\right)$

Momentum and heat law can be physically interpreted as follows:



On marking off a control area in the mixing zone in the manner indicated above, the momentum theorem states that the loss of momentum, which the flow experiences on its passage through the control area, finds its equivalent in the turbulent shearing stress

$$\tau = \epsilon(x)\left[\bar{\rho}\frac{\partial\left(\bar{u} - u_1\right)}{\partial r} + E\left(\bar{u} - u_1\right)\frac{\partial\bar{\rho}}{\partial r}\right]$$

at point r. Correspondingly the heat balance goes through the control surface with consideration of the heat convection only at point r when the turbulent heat conduction $Q = E_\epsilon(x)\frac{\partial}{\partial r}(\bar{\rho}\bar{\vartheta})$ is included.

For $r = 0$ a further integration is possible which gives the formulas of the conservation of momentum

$$\int_0^{b_1} \bar{\rho}\bar{u}\left(\bar{u} - u_1\right) r \, dr = \rho_o u_o \left(u_o - u_1\right) r_o^2 / 2 \tag{4a}$$

and of heat

$$\int_0^{b_2} \bar{\rho}\bar{u}\bar{\vartheta} r \, dr = \rho_o u_o \vartheta_o \frac{r_o^2}{2} \tag{4b}$$

With the equation of state for perfect gases at constant pressure as basis

$$\bar{\rho}\,\bar{T} = \text{const} \tag{5}$$

$\bar{\rho}$ can be replaced by $\bar{\vartheta}$ in the above equations

$$\bar{\rho} = \frac{\text{const}}{\bar{T}} = \frac{\text{const}}{T_1} \frac{1}{1 + \left(\bar{\vartheta}/T_1\right)}$$

where T_1 is the temperature of the surrounding medium.

3. For the investigation of the transition zone by means of the momentum and heat equations as well as of the equations of conservation, the premise

$$\frac{\bar{u} - u_1}{u_o - u_1} = \frac{\bar{u}_A(x) - u_1}{u_o - u_1} \varphi(r,x) \tag{6a}$$

$$\bar{\vartheta}/\vartheta_o = \frac{\bar{\vartheta}_A(x)}{\vartheta_o} \quad \psi(r,x) \tag{6b}$$

is made, where $\bar{u}_A(x)$ is the jet velocity on the jet axis and $\bar{\vartheta}_A(x)$ is the temperature rise on the jet axis.

This formula, carried in the equations, gives (if a practicable assumption can be made for the variation of $\varphi(r,x)$ and $\psi(r,x)$) four equations for the four unknowns:

$$\bar{\mu}_A(x), \quad \bar{\vartheta}_A(x), \quad b_1(x), \text{ and } b_2(x)$$

The transverse component $\bar{v}$ still appearing in the equations is determined from the equation of continuity of the mass

$$\bar{v} = -\frac{1}{r}\int_0^r \frac{\partial \bar{u}}{\partial x} r\, dr \tag{7}$$

The next problem is to make a practicable premise for φ and ψ. These functions occur in the integrands of the integrals appearing in the equations and in the expressions defining the shearing stress and heat conduction at point r, whereby a mean value may be chosen for r.

In connection with the similarity of flow in the jet cross sections resulting for the asymptote it is then logical to put

$$\varphi = \varphi(\eta^*) \quad \text{with} \quad \eta^* = \frac{r}{b_1} \tag{8a}$$

$$\psi = \psi(\eta) \quad \text{with} \quad \eta = \frac{r}{b_2} \tag{8b}$$

This theorem actually seems to be confirmed by experimental results which indicate far reaching similarity of flow profiles in the transition zone (reference 2).

Recommended for the profile form is the distribution function resulting theoretically for the asymptote: for the case of air in motion (see section III) it is the function $e^{-(\sigma\eta)^2}$, where the parameter σ, defining the width of the distribution, contains characteristics of the asymptotic flow. The actual calculation by this method presents some difficulties, and besides the distribution function obtained by the differential equation does not approximate the experimental distributions in the desired measure. (See fig. 9.)

Incidentally it seems timely to comment upon a manifest inadequacy of the theory, which may be attributable to a not entirely equivalent formulation in Prandtl's concept of the mixing length. According to Prandtl's hypothesis of the mechanism of turbulent mixing, the turbulent exchange is effectuated by turbulence balls, which for the duration of their own existence transfer their properties invariantly from one layer to the other. Accordingly the momentary fluctuation u' with the momentary mixing length l should be put as

$$u' = \bar{u}(y + l) - \bar{u}(y) = l\frac{\partial \bar{u}}{\partial y} + \frac{l^2}{2}\frac{\partial^2 \bar{u}}{\partial y^2} + \ldots$$

In the original mathematical wording u' was put equal to $l\frac{\partial \bar{u}}{\partial y}$ which, in general, represents a permissible approximation considering the smallness of l, but at particular points such as in the profile center, for example, where $\frac{\partial \bar{u}}{\partial y} = 0$ and therefore u' should be put $u' = 0$, the results are inadequate, although fluctuations are certain to occur. In consequence, Prandtl (reference 3) suggested the average value formation

$$\epsilon = Kb\left|\bar{u}_{max} - \bar{u}_{min}\right|$$

(wherein K = empirical constant, b = breadth of mixing region of the velocity) for the apparent kinematic viscosity

$$\epsilon = l^2\left|\frac{\partial \bar{u}}{\partial y}\right|\frac{\partial \bar{u}}{\partial y}$$

in the original draft. This theorem has then no longer the differential character of the original formulation and avoids the said inadequacy; however, it then results in inadequacies at the edge of the profile where the fluctuations and, with them, the apparent kinematic viscosity cancel out, whereas by the new theorem an amount of ϵ constant over the entire width of the profile is involved. On the other hand, the new theorem has the implicit advantage of being substantially simpler mathematically and hence of simplifying the calculations considerably.

A very practical representation of the distributions, which gives a very good approximation, is presented by the function

$$\left(1 - \eta^{*3/2}\right)^2 \qquad \text{and} \qquad \left(1 - \eta^{3/2}\right)^2$$

already used by other workers (reference 4).

Thus

$$\psi = \left(1 - \eta^{3/2}\right)^2 \tag{9a}$$

$$\varphi = \left(1 - \eta^{*3/2}\right)^2 = \left[1 - \left(\eta \cdot b_2/b_1\right)^{3/2}\right]^2 \tag{9b}$$

with $\eta = r/b_2$. To simplify the calculation for b_2/b_1 for the case of outside air in motion $\left(u_1 \neq 0\right)$, the ratio of the widths resulting from the theory of the asymptotic distribution functions for outside air in motion is approximately put as

$$b_2/b_1 = \sqrt{E} \tag{10}$$

(See section III.)

The case of quiescent outside air $\left(u_1 = 0\right)$ is treated later.

4. Part A of this report is restricted to small temperature differences. This case is amenable to calculation and represents the conditions accompanying any temperature differences in first approximation.

The velocity field is computed by the momentum equations.

The equation of conservation of the momentum gives the breadth of the mixing region of the velocity b_1 as a function of the axial velocity $\bar{u}_A$

$$b_1/r_o = \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}} \frac{1}{\left[d_1 \frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + e_1 \frac{u_1}{u_o}\right]^{1/2}} \tag{11}$$

with the constants $d_1 = 0.133$ and $e_1 = 0.257$, and the subsequent calculation gives the axial velocity $\bar{u}_A$ as the describing variable of the flow field rather than the nozzle distance x. The momentum equation is specialized to $r = r_o$.

The use of the foregoing result gives a linear differential equation of the first order for the nozzle distance x as function of the velocity on the jet axis $\bar{u}_A$.

When the integration constant is fixed by the postulate $\frac{\bar{u}_A - u_1}{u_o - u_1} = 1$ for the boundary of the core x_K,

$$\frac{x}{r_o} = \frac{1}{K}\left(\frac{u_o - u_1}{u_o}\right)^{1/2} \frac{u_o}{u_1}\left\{\alpha\left\langle\zeta^{1/2} - \epsilon_o^{1/2}\right\rangle + \beta\left\langle\frac{1}{\zeta^{1/2}} - \frac{1}{\epsilon_o^{1/2}}\right\rangle + \gamma\left\langle\frac{1}{\zeta^{3/2}} - \frac{1}{\epsilon_o^{3/2}}\right\rangle\right\} + \frac{x_K}{r_o} \tag{12}$$

with

$$\zeta = \frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \mu\frac{u_1}{u_o - u_1}\right]}, \qquad \epsilon_o = \frac{1}{\left(1 + \mu\frac{u_1}{u_o - u_1}\right)}$$

and the constants

$\alpha = 0.0275,\qquad \beta = 0.0549,\qquad \gamma = 0.0375,\qquad \mu = 1.9259$

The length of core x_K/r_o is to be taken from the theory of the core; K is an empirical constant.

In the asymptote the formulas are obtained

$$b_1/r_o \approx \left(\frac{u_o}{u_1}\right)^{1/2} A_1 \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}} \tag{13}$$

with the constant $A_1 = 1.972$

and

$$\frac{x}{r_o} \approx \frac{1}{K}\left(\frac{u_o}{u_o - u_1}\right)\left(\frac{u_1}{u_o}\right)^{1/2} A_2 \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{3/2}} \tag{14}$$

with $A_2 = 0.1002$.

With the nozzle distance as independent variable the asymptotic formulas read

$$b_1/r_o \approx K^{1/3} \frac{\left(\frac{u_o - u_1}{u_o}\right)^{1/3}}{\left(u_1/u_o\right)^{2/3}} B_1 \left(x/r_o\right)^{1/3} \tag{15}$$

with $B_1 = 4.246$ and

$$\frac{\bar{u}_A - u_1}{u_o - u_1} \approx \frac{1}{K^{2/3}} \frac{\left(u_1/u_o\right)^{1/3}}{\left(\frac{u_o - u_1}{u_o}\right)^{2/3}} B_2 \frac{1}{\left(x/r_o\right)^{2/3}} \tag{16}$$

with $B_2 = 0.216$.

In figures 1, 2, and 6 the functions of the velocity mixing field are shown for the parameter values

$$\frac{u_o - u_1}{u_o} = 1.0, \quad 0.75, \text{ and } 0.5$$

The temperature field is computed by means of the heat equations.

By the equation of conservation of heat, the breadth of the mixing region of the temperature b_2 is

$$b_2/r_o = \frac{1}{\left(\vartheta_A/\vartheta_o\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)d_2 + e_1 \frac{u_1}{u_o}\right]^{1/2}} \tag{17}$$

with $d_2 = 0.0786$ and $e_1 = 0.257$.

The heat formula contains the empirical constant E, the ratio of interchange of temperature and velocity; according to the measurements by Pabst (reference 2) and others it is equal to 2.

From the heat formula a Bernoulli differential equation is obtained for $\bar{\vartheta}_A/\vartheta_o$ as a function of $\frac{\bar{u}_A - u_1}{u_o - u_1}$. Unfortunately the quadrature cannot be carried out. Fixing the integration constant by the initial condition $\bar{\vartheta}_A/\vartheta_o = 1$ for $\frac{\bar{u}_A - u_1}{u_o - u_1} = 1$, the result is

$$\bar{\vartheta}_A/\vartheta_o = \frac{\zeta^{2C}}{\left[D\left(\frac{u_o - u_1}{u_o}\right)^{1/2} \int_{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}^{1} \zeta^{C-1/2}\eta \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{3/2}} d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \zeta^{(1)}\right]^2} \tag{18}$$

with the functions

$$\zeta = \frac{\mu_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o}\mu_2}{\mu_3\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o - u_1}\mu_4}$$

$$\eta = \frac{\nu_1\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 + \nu_2\frac{u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \nu_3\frac{u_1}{u_o}\frac{u_1}{u_o - u_1}}{\left\langle\nu_4\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \nu_5\frac{u_1}{u_o}\right\rangle^{3/2}\left[\nu_6\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \nu_7\frac{u_1}{u_o - u_1}\right]^{1/2}}$$

and the constants

$C = -0.6111$ $D = 0.4848$

$\mu_1 = 0.0241$ $\mu_2 = 0.0522$ $\mu_3 = 0.0876$ $\mu_4 = 0.2571$

$\nu_1 = 0.0066$ $\nu_2 = 0.0253$ $\nu_3 = 0.0196$

$\nu_4 = 0.1335$ $\nu_5 = \mu_4$ $\nu_6 = \mu_1$ $\nu_7 = \mu_2$.

In the asymptote the formulas

$$b_2/r_o \approx \sqrt{2}\, b_1/r_o \tag{19}$$

are obtained in agreement with the mathematical assumption

$$b_2/b_1 = \sqrt{E} = \sqrt{2}$$

In addition

$$\bar{\vartheta}_A/\vartheta_o \approx \frac{1}{2}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \tag{20}$$

In figures 3, 4, 5, and 6 the functions of the temperature mixing field are represented for the parameter values

$$\frac{u_o - u_1}{u_o} = 1.0 \qquad\qquad 0.5$$

The integral occurring in formula (18) was obtained by numerical integration.

The case of quiescent outside air $(u_1 = 0)$ presents a singular behavior as evidenced by the fact that the breadth of the mixing region in the asymptote is represented by a linear function $b_1(x) \sim x$, while with outside air in motion $b_1(x) \sim x^{1/3}$. It is found further that the ratio of the asymptotic mixing width obtained for outside air in motion is not applicable here. The theory of the asymptotic distribution functions produces, in this instance, an impracticable result $(b_2/b_1 \rightarrow \infty)$. The asymptotic ratio

$$b_2/b_1 = 1.33 \quad \text{for} \quad u_1 = 0 \tag{21}$$

is obtained from the law of conservation of heat by an approximation method.

The functions of the turbulent diffusion for the singular case $u_1 = 0$ are as follows:

velocity field

$$b_1/r_o = F_1 \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} \tag{22}$$

with $F_1 = 2.737$

$$x/r_o = \frac{1}{K} F_2 \left[\frac{1}{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)} - 1\right] + x_K/r_o \tag{23}$$

with $F_2 = 0.1347$.

With the nozzle distance x/r_o as independent variable

$$\frac{b_1}{r_o} = G_1 + KG_2\left(\frac{x}{r_o} - \frac{x_K}{r_o}\right) \tag{24}$$

$$\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) = \frac{1}{1 + K\dfrac{G_2}{G_1}\left(\dfrac{x}{r_o} - \dfrac{x_K}{r_o}\right)} \tag{25}$$

with $G_1 = 2.737$, $G_2 = 20.321$

$$\left(G_2/G_1 = 7.425\right)$$

Temperature field

$$b_2/r_o = H_1 \frac{\left[H_2 + (1 - H_2)\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}\right]}{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)} \tag{26}$$

$$\bar{\vartheta}_A/\vartheta_o = \frac{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[H_2 + \left(1 - H_2\right)\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}\right]^2} \tag{27}$$

with

$H_1 = 3.245$ $\qquad$ $H_2 = 1.123$

For infinitely great nozzle distances, the formulas are

$$b_1/r_o \approx KG_2 x/r_o \tag{28}$$

$$\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \approx \frac{1}{K}\frac{G_1}{G_2}\frac{1}{(x/r_o)} \tag{29}$$

and

$$b_2/r_o \approx K_1 \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} \tag{30}$$

with $K_1 = 3.644$

$$\bar{\vartheta}_A/\vartheta_o \approx K_2\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \tag{31}$$

with $K_2 = 0.793$.

All formulas still exhibit the empirical constant K. It is embodied in the apparent kinematic viscosity

$$\epsilon(x) = Kb_1\left(\bar{u}_A - u_1\right)$$

and represents a measure for l/b, the ratio of mixing length to mixing width, which is regarded as constant for the individual jet sections; K and l/b are to be considered functions of the characteristic length x/r_o. But in view of the far-reaching similarity of the profiles, the dependence on x/r_o throughout almost the entire transition zone is expected to be slight, as the measurements seem to confirm. For example, Tollmien's investigations (reference 5) at the plane jet boundary (these conditions prevail in the immediate vicinity of the nozzle mouth!) give $l/b = 0.068$; for the axially symmetrical jet diffusion (the conditions encountered at very great distance from the mouth of the nozzle), he obtained $l/b = 0.073$ (where, for reasons of continuity at transition from the core to the transition zone, b in the latter, as done here, is to be put equal to the jet radius).

Also of interest are the distribution functions (section III). As already pointed out, the empirically obtained function $\left(1 - \eta^{3/2}\right)^2$ gives a very close approximation of the experimental distributions. The theory gives the following asymptotic distributions for $u_1 \neq 0$:

Velocity distribution

$$\varphi = e^{-(\sigma_o \eta)^2}, \quad \eta = r/b_1 \tag{32}$$

whereby with

$$\frac{\bar{u}_A - u_1}{u_o - u_1} \approx u_\infty \; x^{-2/3}, \quad b_1 \approx b_\infty \; x^{1/3}$$

$$\sigma_o = \sqrt{\frac{1}{6} \frac{1}{K} \frac{u_1}{u_o - u_1} \frac{b_\infty}{u_\infty}}$$

As reflected by the previously obtained results concerning the structure of the mixing zones

$$\sigma_o = 1.811 \tag{33}$$

for $\eta = r/b_1$.

Utilizing in correspondence with the asymptotic behavior of the mixing width $\left(b_1 \approx b_\infty \times x^{1/3}\right)$ the coordinate $\eta^* = \frac{r}{x^{1/3}}$

$$\varphi = e^{-\left(\sigma_o^* \eta^*\right)^2} \tag{34}$$

with

$$\sigma_o^* = 0.427 \frac{1}{K^{1/3}} \frac{\left(\frac{u_1}{u_o}\right)^{2/3}}{\left(\frac{u_o - u_1}{u_o}\right)^{1/3}}$$

In the case of quiescent outside air $\left(u_1 = 0\right)$ the asymptotic velocity distribution function is

$$\varphi = \frac{1}{\left[1 + \left(\sigma_o \eta\right)^2\right]^2}, \quad \eta = r/x \tag{35}$$

with

$$\sigma_0 = \frac{1}{2} \frac{1}{\sqrt{2Kb_\infty}}$$

and

$$\sigma_0 = (1/K) \times 0.0784$$

For the temperature distribution the important relation

$$\psi = \varphi^{1/E} \quad \text{and} \quad \psi = \varphi^{1/2} \tag{36}$$

is obtained. This result is already indicated in Reichardt's report (reference 6).

The asymptotic distribution functions are represented in figures 7 and 8.

Now, the theory is compared with the experimental results. The former contains two empirical constants E and K, E being equal to 2.

In the first part of the investigation (reference 1) the constant K for the core had been determined on the basis of Tollmien's study at the plane jet boundary (reference 5). The result was $K = 0.0106$ on the basis of the heat equation, but the thus defined value is somewhat uncertain owing to the doubtful flow losses as a result of the friction at the nozzle wall and the assumption – valid exact only for small velocity and temperature difference – that the breadth of the mixing regions of temperature and velocity which served as basis of the calculations act as $\sqrt{E} : 1$.

Quantity K is determined next for asymptotic conditions. Tollmien (reference 5) obtained $b = 0.214x$ for the diffusion of a jet issuing from point source, as against $b_1 = K \times 20.321 \times x$ according to the calculations by equation (28). The comparison gives $K = 0.0105$. When the determination of K is based upon experimental results of other structures of the diffusion field, such as the gradient of the asymptotic distribution function or the decrease of velocity and temperature along the jet axis, the foregoing value of K is almost exactly reproduced. The calculations have been based on the value

$$K = 0.010 \tag{37}$$

For asymptotic conditions, no incidental flow loss due to friction at the nozzle wall needs to be considered. (The flow loss can be allowed for by introducing an effective nozzle radius (see part I, (reference 1)); for infinitely great distance from the nozzle, however, the size of the nozzle radius is of no influence.) The ratio of the breadth of the mixing region of temperature and velocity serving as a basis of the calculations is rigorously valid for the asymptote. The above value of K should therefore represent a safe value for asymptotic conditions.

Experimental data for small temperature differences as treated here are few and limited to the case of outside air at rest $\left(u_1 = 0\right)$. For greater temperature differences (to be discussed in part B of this report), Pabst's comprehensive measurements (reference 2) are available. For the asymptotic velocity distribution Reichardt's measurements (reference 7) are available. The comparison (fig. 9) shows practicable agreement up to the boundary zone where the divergence is fairly great. This difference is, as stated above, attributable to the nature of the calculation method.

The decrease of velocity and temperature along the jet axis was measured by Ruden (reference 8). Unfortunately the published report dealt only with the test curves without giving the test points or any further details of the measurements. The comparison reproduced in figures 10 and 11 indicates very good agreement between the test curves and the computed functions up to a short transition arc.

This transition arc still defies interpretation at the present state of turbulence research. The calculation by integral formulas contains as essential premise the assumption of profile similarity for the individual jet sections. The result is a hyperbolic variation of the distance function up to the boundary of the core, while the experiment and also the differential equations (2) of the mixing process indicate zero tangent in the boundary of the core. It is in this short transition arc that the transition from the asymptotic profile form to that of the plane jet boundary is largely effectuated. The profiles throughout the entire core are very similar to the profile forming in direct proximity of the nozzle mouth, which corresponds to the mixing of two plane jets (compare part I (reference 1)). For the calculation of the transition arc, a return to the differential equation (2) would be necessary, while the determination of the breadth of the mixing region involved in the apparent kinematic viscosity $\epsilon(x) = Kb_1\left(\bar{u}_A - u_1\right)$, would call for the conservation formula. Even so, the variation of K within the arc would remain, whereby it is to be noted that the assumption of similar profiles is equivalent to assuming a constant K, so that the existing dependence of K on x/r_0 would essentially have to occur inside the arc. But every hypothesis for this dependence of K and of l/b (assumed as constant in the individual jet sections) fails at present.

According to the experiment, the transition arc joining on the common core boundary terminates in the hyperbolic branch of the distance function; the velocity decrease in the arc is at first very slow compared to the temperature. In the comparison (figs. 10 and 11) the starting point of the computed hyperbolic branch was chosen accordingly.

In the calculation, the integration constant was so defined that $\bar{\vartheta}_A/\vartheta_o = 1$ for $\dfrac{\bar{u}_A - u_1}{u_o - u_1} = 1$, i.e., that the hyperbolic branches of the velocity and temperature decrease started at a common boundary. In these conditions, a displacement of the experimental correlation is to be expected relative to the theoretical correlation of the related velocity and temperature, which would have to disappear for greater nozzle distances. This displacement is manifest in the comparison, figure 12.

II. Calculation of the Variations of the Mixing Width and of the Axial Functions

Momentum equation (3a)

$$r\bar{\rho}\bar{v}\left(\bar{u} - u_1\right) - \frac{\partial}{\partial x}\int_r^{b_1} \bar{\rho}\bar{u}\left(\bar{u} - u_1\right) r\ dr = \epsilon(x) r\left[\rho\frac{\partial\left(\bar{u} - u_1\right)}{\partial r} + E\left(\bar{u} - u_1\right)\frac{\partial\bar{\rho}}{\partial r}\right]$$

Heat equation (3b)

$$r\bar{\rho}\bar{v}\bar{\vartheta} - \frac{\partial}{\partial x}\int_r^{b_2} \bar{\rho}\bar{u}\bar{\vartheta} r\ dr = E\epsilon(x) r\ \frac{\partial}{\partial r}\left(\bar{\rho}\bar{\vartheta}\right)$$

where

$$\epsilon(x) = Kb_1\left(\bar{u}_A(x) - u_1\right)$$

Equations of conservation (4a) and (4b)

$$\int_0^{b_1} \bar{\rho}\bar{u}\left(\bar{u} - u_1\right) r\ dr = \rho_o u_o\left(u_o - u_1\right) r_o^2/2$$

$$\int_0^{b_2} \bar{\rho}\bar{u}\bar{\vartheta} r\ dr = \rho_o u_o \vartheta_o\ r_o^2/2$$

In correspondence with the equation $\bar{\rho} \times \bar{T} = \text{constant}$ $\bar{\rho}$ is replaced by

$$\bar{\rho} = \frac{\text{const}}{T_1} \frac{1}{1 + \bar{\vartheta}/T_1}$$

The transverse component is obtained from the equation of continuity

$$\frac{\partial(r\bar{u})}{\partial x} + \frac{\partial(r\bar{v})}{\partial r} = 0 \qquad :$$

$$\bar{v} = -\frac{1}{r}\int_0^r \frac{\partial \bar{u}}{\partial x} r \, dr$$

Hence, for the momentum:

$$\begin{aligned} &-\left(\int_0^r \frac{\partial \bar{u}}{\partial x} r \, dr\right)\left(\bar{u} - u_1\right) \frac{1}{1 + \bar{\vartheta}/T_1} - \frac{\partial}{\partial x} \int_r^{b_1} \frac{\bar{u}\left(\bar{u} - u_1\right)}{1 + \vartheta/T_1} r \, dr \\ &= Kb_1\left(\bar{u}_A - u_1\right) r \left[\frac{1}{1 + \bar{\vartheta}/T_1} \frac{\partial\left(\bar{u} - u_1\right)}{\partial r} + E\left(\bar{u} - u_1\right) \frac{\partial \frac{1}{1 + \vartheta/T_1}}{\partial r}\right] \end{aligned} \tag{38a}$$

and

$$\int_0^{b_1} \frac{\bar{u}\left(\bar{u} - u_1\right)}{1 + \bar{\vartheta}/T_1} r \, dr = \frac{u_o\left(u_o - u_1\right)}{1 + \vartheta_o/T_1} r_o^2/2 \tag{38b}$$

for the heat:

$$\begin{aligned} &-\left(\int_0^r \frac{\partial \bar{u}}{\partial x} r \, dr\right) \frac{\bar{\vartheta}}{1 + \bar{\vartheta}/T_1} - \frac{\partial}{\partial x} \int_r^{b_2} \bar{u} \frac{\bar{\vartheta}}{1 + \bar{\vartheta}/T_1} r \, dr \\ &= EKb_1\left(\bar{u}_A - u_1\right) r \frac{\partial}{\partial r} \frac{\bar{\vartheta}}{1 + \bar{\vartheta}/T_1} \end{aligned} \tag{39a}$$

and

$$\int_0^{b_2} \bar{u}\frac{\bar{\vartheta}}{1 + \bar{\vartheta}/T_1} r \, dr = u_o \frac{\vartheta_o}{1 + \vartheta_o/T_1} r_o^2/2 \tag{39b}$$

If we limit ourselves to small temperature differences $\left(\vartheta_o/T_1\right) \ll 1$, we get: momentum

$$-\left(\int_0^r \frac{\partial \bar{u}/u_o}{\partial x} r \, dr\right)\left(\frac{\bar{u} - u_1}{u_o - u_1}\right) - \frac{\partial}{\partial x}\int_r^{b_1} \frac{\bar{u}}{u_o}\left(\frac{\bar{u} - u_1}{u_o - u_1}\right) r \, dr$$

$$= K b_1\left(\frac{\bar{u}_A - u_1}{u_o}\right) r \frac{\partial\left(\frac{\bar{u} - u_1}{u_o - u_1}\right)}{\partial r} \tag{40a}$$

and

$$\int_0^{b_1} \frac{\bar{u}}{u_o}\left(\frac{\bar{u} - u_1}{u_o - u_1}\right) r \, dr = r_o^2/2 \tag{40b}$$

heat

$$-\left(\int_0^r \frac{\partial\left(\bar{u}/u_o\right)}{\partial x} r \, dr\right)\left(\bar{\vartheta}/\vartheta_o\right) - \frac{\partial}{\partial x}\int_r^{b_2}\left(\bar{u}/u_o\right)\frac{\bar{\vartheta}}{\vartheta_o} r \, dr$$

$$= EKb_1\frac{\left(\bar{u}_A - u_1\right)}{u_o} r \frac{\partial}{\partial r}\left(\bar{\vartheta}/\vartheta_o\right) \tag{41a}$$

and

$$\int_0^{b_2}\left(\bar{u}/u_o\right)\frac{\bar{\vartheta}}{\vartheta_o} r \, dr = r_o^2/2 \tag{41b}$$

Posing the formulas (6) and (8)

$$\bar{\vartheta}/\vartheta_o = \frac{\bar{\vartheta}_A(x)}{\vartheta_o}\psi(\eta) \quad \text{with} \quad \eta = r/b_2(x)$$

the transformation formulas read:

$$\eta = r/b_2, \quad r = \eta b_2 \tag{42}$$

$$dr = b_2\, d\eta$$

$$\left(\partial/\partial x\right)_r = \left(\partial/\partial x\right)_\eta - \left(\partial/\partial \eta\right)_x \eta \frac{b_2'}{b_2}$$

$$\left(\partial/\partial r\right)_x = \left(\partial/\partial \eta\right)_x 1/b_1$$

Correspondingly

$$\frac{\bar{u} - u_1}{u_o - u_1} = \frac{\bar{u}_A(x) - u_1}{u_o - u_1} \varphi(\xi)$$

with

$$\xi = r/b_1 = \eta\, b_2/b_1$$

giving, after effecting the transformation

momentum

$$- b_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'\left(\int_0^\xi \varphi\xi\, d\xi\right)\varphi + b_1'\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 \left(\int_0^\xi \frac{d\varphi}{d\xi} \xi^2\, d\xi\right) \varphi -$$

$$2b_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)' \left(\int_\xi^1 \varphi^2 \xi\, d\xi\right) - \frac{u_1}{u_o - u_1} b_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)' \left(\int_\xi^1 \varphi\xi\, d\xi\right) +$$

$$b_1'\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 \left(\int_\xi^1 \frac{d\left(\varphi^2\right)}{d\xi} \xi^2\, d\xi\right) + \frac{u_1}{u_o - u_1} b_1'\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \left(\int_\xi^1 \frac{d\varphi}{d\xi} \xi^2\, d\xi\right)$$

$$= K\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 \xi \frac{d\varphi}{d\xi} \tag{43a}$$

(Accents denote derivatives with respect to x)

and

$$2\left(b_1/r_o\right)^2\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\int_0^1 \varphi^2\xi\ d\xi + \frac{u_1}{u_o}\int_0^1 \varphi\xi\ d\xi\right] = 1 \qquad (43b)$$

heat

$$-\left(b_2/b_1\right)b_2\ \frac{\bar{\vartheta}_A}{\vartheta_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'\left(\int_0^\eta \varphi\eta\ d\eta\right)\psi +$$

$$\left(b_2/b_1\right)b_2'\ \frac{\bar{\vartheta}_A}{\vartheta_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\int_0^\eta \frac{d\varphi}{d\eta}\eta^2 d\eta\right) - \left(b_2/b_1\right)b_2\left[\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'\ \frac{\bar{\vartheta}_A}{\vartheta_o} +\right.$$

$$\left.\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\bar{\vartheta}_A/\vartheta_o\right)'\right]\int_\eta^1 \varphi\psi\eta\ d\eta - \left(b_2/b_1\right)b_2\frac{u_1}{u_o - u_1}\left(\bar{\vartheta}_A/\vartheta_o\right)'\left(\int_\eta^1 \psi\eta\ d\eta\right) +$$

$$\left(b_2/b_1\right)b_2'\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\ \frac{\bar{\vartheta}_A}{\vartheta_o}\left(\int_\eta^1 \frac{d(\varphi\psi)}{d\eta}\eta^2 d\eta\right) +$$

$$\left(b_2/b_1\right)b_2'\frac{u_1}{u_o - u_1}\ \frac{\bar{\vartheta}_A}{\vartheta_o}\left(\int_\eta^1 \frac{d\psi}{d\eta}\eta^2 d\eta\right) = EK\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\ \frac{\bar{\vartheta}_A}{\vartheta_o}\ \eta\frac{d\psi}{d\eta} \qquad (44a)$$

and

$$2\left(b_2/r_o\right)^2\ \frac{\bar{\vartheta}_A}{\vartheta_o}\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\int_0^1 \varphi\psi\eta\ d\eta + \frac{u_1}{u_o}\int_0^1 \psi\eta\ d\eta\right] = 1 \qquad (44b)$$

Here (9) must be inserted:

$$\psi(\eta) = \left(1 - \eta^{3/2}\right)^2,$$

$$\varphi(\xi) = \left(1 - \xi^{3/2}\right)^2 \quad \text{and} \quad \varphi(\eta) = \begin{cases} \left(1 - \left[\eta\, b_2/b_1\right]^{3/2}\right)^2 & 0 \leqq \eta \leqq b_1/b_2 \\ 0 & b_1/b_2 \leqq \eta \leqq 1 \end{cases}$$

(a) Velocity Field

I. Equation of conservation of momentum (43b)

$$2\left(b_1/r_o\right)^2\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\int_0^1 \varphi^2{}_\xi \, d\xi + \frac{u_1}{u_o}\int_0^1 \varphi\xi \, d\xi\right] = 1$$

The evaluation of the integral gives

$$\int^\xi \varphi^2\xi \, d\xi = \int^\xi \left(1 - \xi^{3/2}\right)^4 \xi \, d\xi = \frac{1}{2}\xi^2 - \frac{8}{7}\xi^{7/2} + \frac{6}{5}\xi^5 - \frac{8}{13}\xi^{13/2} + \frac{1}{8}\xi^8$$

$$\int^\xi \varphi\xi \, d\xi = \int^\xi \left(1 - \xi^{3/2}\right)^2 \xi \, d\xi = \frac{1}{2}\xi^2 - \frac{4}{7}\xi^{7/2} + \frac{1}{5}\xi^5 .$$

hence

$$\int_0^1 \varphi^2\xi \, d\xi = \frac{243}{3640}, \qquad \int_0^1 \varphi\xi \, d\xi = \frac{9}{70}$$

Then

$$2(b_1/r_0)^2\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\frac{243}{3640} + \frac{u_1}{u_o}\,\frac{9}{70}\right] = 1$$

$$(b_1/r_o)^2\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\frac{243}{1820} + \frac{u_1}{u_o}\,\frac{9}{35}\right] = 1$$

$$b_1/r_o = \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}}\,\frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)d_1 + \frac{u_1}{u_o}\,e_1\right]^{1/2}} \tag{45}$$

with $d_1 = 0.13352$ and $e_1 = 0.25714$.

In addition

$$\frac{d(b_1/r_o)}{d(x/r_o)} = -\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{3/2}}\,\frac{\frac{u_o - u_1}{u_o}\,d_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o}\,\frac{1}{2}\,e_1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)d_1 + \frac{u_1}{u_o}\,e_1\right]^{3/2}} \tag{46}$$

II. Momentum equation (43a)

$$- b_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'\left(\int_0^{\xi}\varphi\xi\,d\xi\right)\varphi + b_1'\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2\left(\int_0^{\xi}\frac{d\varphi}{d\xi}\,\xi^2\,d\xi\right)\varphi\,-$$

$$2b_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'\left(\int_{\xi}^{1}\varphi^2\xi\,d\xi\right) - \frac{u_1}{u_o - u_1}\,b_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'\left(\int_{\xi}^{1}\varphi\xi\,d\xi\right) +$$

$$b_1'\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2\left(\int_{\xi}^{1}\frac{d(\varphi^2)}{d\xi}\,\xi^2\,d\xi\right) + \frac{u_1}{u_o - u_1}\,b_1'\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\int_{\xi}^{1}\frac{d\varphi}{d\xi}\,\xi^2\,d\xi\right)$$

$$= K\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2\,\xi\,\frac{d\varphi}{d\xi}$$

Specialize to $\xi = 1/2$.

Evaluation of integral

(1)

$$\int_0^{1/2} \varphi\xi \, d\xi = \frac{21}{160} - \frac{1}{28}\sqrt{2} = 0.08074$$

$$\varphi_{\xi=1/2} = \left(1 - \xi^{3/2}\right)^2_{\xi=1/2} = \frac{9}{8} - \frac{1}{2}\sqrt{2} = 0.4179$$

$$\left(\int_0^{1/2} \varphi\xi \, d\xi\right) \varphi_{1/2} = 0.03374$$

(2)

$$\int \frac{d\varphi}{d\xi} \xi^2 \, d\xi = \int 3\left(\xi^2 - \xi^{1/2}\right)\xi^2 \, d\xi = \frac{3}{5}\xi^5 - \frac{6}{7}\xi^{7/2}$$

$$\int_0^{1/2} \frac{d\varphi}{d\xi} \xi^2 \, d\xi = \frac{3}{160} - \frac{3}{56}\sqrt{2} = -0.05701$$

$$\left(\int_0^{1/2} \frac{d\varphi}{d\xi} \xi^2 \, d\xi\right) \varphi_{1/2} = -0.02382$$

(3)

$$\int_{1/2}^{1} \varphi^2\xi \, d\xi = \frac{1}{2} - \frac{8}{7} + \frac{6}{5} - \frac{8}{13} - \frac{6}{160} - \frac{1}{8}\,\frac{1}{250} +$$

$$\sqrt{2}\left(\frac{1}{14} + \frac{1}{16}\,\frac{1}{13}\right) = 0.01158$$

(4)

$$\int_{1/2}^{1} \varphi\xi \, d\xi = \frac{1}{2} - \frac{4}{7} + \frac{1}{5} - \frac{1}{8} - \frac{1}{160} + \frac{2}{26}\sqrt{2} = 0.04783$$

(5) $$\int \frac{d(\varphi^2)}{d\xi}\xi^2\,d\xi = \int (-6)\left(\xi^{1/2} - 3\xi^2 + 3\xi^{7/2} - \xi^5\right)\xi^2\,d\xi$$

$$= -\frac{12}{7}\xi^{7/2} + \frac{18}{5}\xi^5 - \frac{36}{13}\xi^{13/2} + \frac{3}{4}\xi^8$$

$$\int_{1/2}^{1} \frac{d(\varphi^2)}{d\xi}\xi^2\,d\xi = -\frac{12}{7} + \frac{18}{5} - \frac{36}{13} + \frac{3}{4} - \frac{18}{5}\frac{1}{32} - \frac{3}{4}\frac{1}{256}$$

$$+ \sqrt{2}\left(\frac{3}{28} + \frac{9}{13}\frac{1}{32}\right) = -0.06683$$

(6)

$$\int_{1/2}^{1} \frac{d\varphi}{d\xi}\xi^2\,d\xi = \frac{3}{5} - \frac{6}{7} - \frac{3}{160} + \frac{3}{56}\sqrt{2} = -0.20013$$

(7)

$$\xi\frac{d\varphi}{d\xi} = 3\left(\xi^3 - \xi^{3/2}\right)$$

$$\left(\xi\frac{d\varphi}{d\xi}\right)_{1/2} = \frac{3}{8} - \frac{3}{4}\sqrt{2} = -0.68566$$

Abbreviating $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ to ( ) gives

$$-b_1(\,)(\,)'A + b_1'(\,)^2B - b_1(\,)(\,)'C - \frac{u_1}{u_o - u_1}b_1(\,)'D$$

$$+ b_1'(\,)^2F + \frac{u_1}{u_o - u_1}b_1'(\,)G = K(\,)^2H$$

with $A = 0.03374$ $\quad B = -0.02382$

$C = 0.02317$ $\quad D = 0.04783$

$F = -0.06683$ $\quad G = -0.20013$

$H = -0.68566$

Introducing (45) and (46)

$$b_1/r_o = \frac{1}{(\ )^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o} e_1\right]^{1/2}}$$

$$\frac{d\left(b_1/r_o\right)}{d\left(x/r_o\right)} = \frac{-(\ )'}{(\ )^{3/2}} \frac{\left\{\frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o} \frac{1}{2} e_1\right\}}{[\,]^{3/2}}$$

gives

$$-\frac{1}{(\ )^{1/2}} \frac{1}{[\,]^{1/2}}(\ )(\ )'A - \frac{(\ )'}{(\ )^{3/2}} \frac{\{\ \}}{[\ ]^{3/2}}(\ )^2 B - \frac{1}{(\ )^{1/2}} \frac{1}{[\ ]^{1/2}}(\ )(\ )'C$$

$$-\frac{u_1}{u_o - u_1} \frac{1}{(\ )^{1/2}} \frac{1}{[\,]^{1/2}}(\ )'D - \frac{(\ )'}{(\ )^{3/2}} \frac{\{\ \}}{[\,]^{3/2}}(\ )^2 F - \frac{u_1}{u_o - u_1} \frac{(\ )'}{(\ )^{3/2}} \frac{\{\ \}}{[\ ]^{3/2}}(\ )G$$

$$= K(\ )^2 H$$

or

$$\frac{(\ )'}{(\ )^{1/2}[\ ]^{3/2}}\left\{A(\ )\left[\frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o} e_1\right] + B(\ )\left\{\frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o} \frac{1}{2} e_1\right\}\right.$$

$$+ C(\ )\left[\frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o} e_1\right] + D\frac{u_1}{u_o - u_1}\left[\frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o} e_1\right]$$

$$\left. + F(\ )\left\{\frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o} \frac{1}{2} e_1\right\} + G\frac{u_1}{u_o - u_1}\left\{\frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o} \frac{1}{2} e_1\right\}\right\}$$

$$= -K(\ )^2 H$$

or

$$\frac{(\ )'}{(\ )^{1/2}[\ ]^{3/2}}\left\{\frac{u_o - u_1}{u_o} d_1\left\langle A + B + C + F\right\rangle (\ )^2 + \frac{u_1}{u_o}\left\langle e_1\left(A + \frac{B}{2} + C + \frac{F}{2}\right) + \right.\right.$$

$$\left.\left. d_1(D + G)\right\rangle (\ ) + \frac{u_1}{u_o - u_1}\frac{u_1}{u_o}\left\langle e_1\left(D + \frac{G}{2}\right)\right\rangle\right\} = -K(\ )^2 H$$

hence

$$\frac{(\ )'}{(\ )^{1/2}}\frac{1}{[\ ]^{3/2}}\left\{\alpha\frac{u_o - u_1}{u_o}(\ )^2 + \beta\frac{u_1}{u_o}(\ ) + \gamma\frac{u_1}{u_o - u_1}\frac{u_1}{u_o}\right\} = -K(\ )^2 H$$

with $\alpha = -0.004505$, $\beta = -0.01736$, $\gamma = -0.01343$, $H = -0.68566$

and finally the differential equation

$$-d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left\{\frac{\alpha_1\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 + \beta_1\frac{u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \gamma_1\frac{u_1}{u_o}\frac{u_1}{u_o - u_1}}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{5/2}\left[\frac{u_o - u_1}{u_o}d_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o}e_1\right]^{3/2}}\right\}$$

$$= K\ d\left(x/r_o\right) \tag{47}$$

with $\alpha_1 = 0.00657$, $\beta_1 = 0.02531$, $\gamma_1 = 0.01959$, $d_1 = 0.13352$, $e_1 = 0.25714$

Writing y instead of $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ and x in place of x/r_o involves the equation

$$-dy\left\{\frac{Ay^2 + By + C}{y^2(Ey + F)\sqrt{Ey^2 + Fy}}\right\} = K\ dx$$

or

$$-dy\left\{\frac{Ay^2 + By + C}{Ey^3 + Fy^2}\frac{1}{\sqrt{E}}\frac{1}{\sqrt{y^2 + \frac{F}{E}y}}\right\} = K\ dx$$

where

$$A = \alpha_1 \frac{u_o - u_1}{u_o} \qquad B = \beta_1 \frac{u_1}{u_o} \qquad C = \gamma_1 \frac{u_1}{u_o} \frac{u_1}{u_o - u_1}$$

$$E = d_1 \frac{u_o - u_1}{u_o} \qquad F = e_1 \frac{u_1}{u_o}$$

Transformation of

$$y = \frac{-F/E}{1 - \xi^2} \qquad \left(\xi = (1/y)\sqrt{y^2 + F/Ey}\right)$$

$$\frac{dy}{d\xi} = 2\frac{F}{E}\frac{\xi}{\left(1 - \xi^2\right)^2} \qquad y^2 = \frac{(F/E)^2}{\left(1 - \xi^2\right)^2} \qquad y^3 = \frac{-(F/E)^3}{\left(1 - \xi^2\right)^3}$$

gives

$$Kx = -\frac{1}{\sqrt{E}}\int \frac{A\frac{(F/E)^2}{\left(1 - \xi^2\right)^2} - B\frac{(F/E)}{\left(1 - \xi^2\right)} + C}{-E\frac{(F/E)^3}{\left(1 - \xi^2\right)^3} + F\frac{(F/E)^2}{\left(1 - \xi^2\right)^2}} \frac{1}{(F/E)\frac{\xi}{\left(1 - \xi^2\right)}} 2(F/E)\frac{\xi}{\left(1 - \xi^2\right)^2} d\xi$$

$$= -\frac{2}{\sqrt{E}}\int \frac{A(F/E)^2 - B(F/E)\left(1 - \xi^2\right) + C\left(1 - \xi^2\right)}{-E(F/E)^3 + F(F/E)^2\left(1 - \xi^2\right)} d\xi$$

$$= -\frac{2}{\sqrt{E}}\int \left[\frac{C}{F(F/E)^2}\left(1 - \xi^2\right) + \frac{\frac{C}{F(F/E)^2}E(F/E)^3 - B(F/E)}{F(F/E)^2}\right] d\xi -$$

$$\frac{2}{\sqrt{E}}\int \frac{A + \left(\frac{CE}{F} - B\right)\frac{E}{F}}{-E(F/E) + F\left(1 - \xi^2\right)} d\xi$$

$$= -\frac{2}{\sqrt{E}}\left\{\frac{C}{F}\frac{1}{(F/E)^2}\int\left(1 - \xi^2\right)d\xi + \frac{\left(\frac{CE}{F} - B\right)}{F(F/E)}\int d\xi - \left[\frac{A}{F} + \left(\frac{CE}{F} - B\right)(E/F)(1/F)\right]\int\frac{d\xi}{\xi^2}\right\}$$

$$Kx = \frac{-2}{\sqrt{E}} \frac{1}{F} \left\{ \left[\frac{2C - B(F/E)}{(F/E)^2}\right]\xi + \left[- \frac{1}{3} \frac{C}{(F/E)^2}\right]\xi^3 + \left[A + \frac{\left(\frac{CE}{F} - B\right)}{(F/E)}\right] 1/\xi \right\} + \text{const.}$$

By retaining the old variables and inserting the original values for the constants, we obtain

$$x/r_o = \frac{1}{K}\left(\frac{u_o - u_1}{u_o}\right)^{1/2}\frac{u_o}{u_1}\left\{\alpha\left\langle \zeta^{1/2} - \zeta(1)^{1/2}\right\rangle + \beta\left\langle\frac{1}{\zeta^{1/2}} - \frac{1}{\zeta(1)^{1/2}}\right\rangle + \gamma\left\langle\frac{1}{\zeta^{3/2}} - \frac{1}{\zeta(1)^{3/2}}\right\rangle\right\} + x_K/r_o \tag{48}$$

with

$$\zeta = \frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \mu \frac{u_1}{u_o - u_1}\right]} \qquad \zeta(1) = \frac{1}{\left(1 + \mu \frac{u_1}{u_o - u_1}\right)}$$

and the constants

$\alpha = 0.0275$ $\beta = 0.0549$ $\gamma = 0.0375$ $\mu = 1.9259$

The integration constant was so determined that the value $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) = 1$ was obtained for the core boundary x_K/r_o, x_K/r_o to be taken from the theory of the central zone.

The formula still contains the empirical constant K which must be defined by experiments (section I).

For the case of quiescent outside air $\left(\frac{u_1}{u_o} = 0\right)$ the differential equation reduces to

$$- d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left\{\frac{\alpha_1}{d_1^{3/2}} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2}\right\} = K\, d\left(x/r_o\right)$$

which gives

$$x/r_o = (1/K)F_2\left\{\frac{1}{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)} - 1\right\} + (x_K/r_o) \tag{49}$$

with $F_2 = 0.1347$.

(b) Temperature Field

I. Equation of conservation of heat (44b)

$$2(b_2/r_o)^2 \frac{\bar{\vartheta}_A}{\vartheta_o}\left\{\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\int_0^1 \varphi\psi\eta\, d\eta + \frac{u_1}{u_o}\int_0^1 \psi\eta\, d\eta\right\} = 1 \tag{50a}$$

By (9)

$$\psi = \left(1 - \eta^{3/2}\right)^2$$

$$\varphi(\eta) = \begin{cases} \left(1 - \left[\eta\sqrt{E}\right]^{3/2}\right)^2 & 0 \leqq \eta \leqq \dfrac{1}{\sqrt{E}} \\ 0 & \dfrac{1}{\sqrt{E}} \leqq \eta \leqq 1 \end{cases} \tag{50b}$$

Evaluation of integral:

(1)

$$\int \varphi\psi\eta\, d\eta = \int \left(1 - \left[\eta\sqrt{E}\right]^{3/2}\right)^2 \left(1 - \eta^{3/2}\right)^2 \eta\, d\eta$$

$$= \frac{\eta^2}{2} - \frac{4}{7}E^{3/4}\eta^{7/2} + \frac{1}{5}E^{3/2}\eta^5 - \frac{4}{7}\eta^{7/2} + \frac{4}{5}E^{3/4}\eta^5 - \frac{4}{13}E^{3/2}\eta^{13/2} + \frac{1}{5}\eta^5 - \frac{4}{13}E^{3/4}\eta^{13/2} + \frac{1}{8}E^{3/2}\eta^8$$

$$\int_0^1 \varphi\psi\eta\, d\eta = \int_0^{1/\sqrt{E}} \varphi\psi\eta\, d\eta = \frac{1}{E}(9/70) + \frac{1}{E^{7/4}}\left(\frac{-36}{35 \times 13}\right) + \frac{1}{E^{5/2}}\left(\frac{9}{65 \times 8}\right)$$

For $E = 2$:

$$\int_0^1 \varphi\psi\eta \, d\eta = 0.04382$$

(2) $$\int \psi\eta \, d\eta = \int \left(1 - \eta^{3/2}\right)^2 \eta \, d\eta = \frac{1}{2}\eta^2 - \frac{4}{7}\eta^{7/2} + \frac{1}{5}\eta^5$$

$$\int_0^1 \psi\eta \, d\eta = 9/70 = 0.12857^*$$

hence

$$\left(b_2/r_o\right)^2\left(\bar{\vartheta}_A/\vartheta_o\right)\left\{\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)d_2 + \frac{u_1}{u_o}e_1\right\} = 1$$

$$b_2/r_o = \frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)d_2 + \frac{u_1}{u_o}e_1\right]^{1/2}} \tag{51}$$

with $d_2 = 0.08765$ $e_1 = 0.25714$

In addition

$$\frac{d\left(b_2/r_o\right)}{d\left(x/r_o\right)} = -1/2 \times \frac{\left\{\frac{u_o - u_1}{u_o} d_2\left[\frac{\bar{\vartheta}_A}{\vartheta_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)' + \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\bar{\vartheta}_A/\vartheta_o\right)'\right] + \frac{u_1}{u_o}e_1\left(\bar{\vartheta}_A/\vartheta_o\right)'\right\}}{\left[\frac{u_o - u_1}{u_o} d_2\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o}e_1\right]^{3/2}\left(\bar{\vartheta}_A/\vartheta_o\right)^{3/2}} \tag{52}$$

*This value was incorrectly given as 0.25714 in the originial German version of this paper.

II. Heat equation (44a)

$$- b_2 \frac{\bar{\vartheta}_A}{\vartheta_o} \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)' \left(\int_0^\eta \varphi\eta \, d\eta\right)\psi + b_2' \frac{\bar{\vartheta}_A}{\vartheta_o} \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\int_0^\eta \frac{d\varphi}{d\eta}\eta^2 d\eta\right)\psi$$

$$- b_2\left[\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)' \frac{\bar{\vartheta}_A}{\vartheta_o} + \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\bar{\vartheta}_A/\vartheta_o\right)'\right]\left(\int_\eta^1 \varphi\psi\eta \, d\eta\right) -$$

$$b_2 \frac{u_1}{u_o - u_1}\left(\bar{\vartheta}_A/\vartheta_o\right)' \left(\int_\eta^1 \psi\eta \, d\eta\right) + b_2'\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \frac{\bar{\vartheta}_A}{\vartheta_o} \left(\int_\eta^1 \frac{d(\varphi\psi)}{d\eta}\eta^2 \, d\eta\right) +$$

$$b_2' \frac{u_1}{u_o - u_1} \frac{\bar{\vartheta}_A}{\vartheta_o} \left(\int_\eta^1 \frac{d\psi}{d\eta}\eta^2 \, d\eta\right) = \sqrt{E}\, K \frac{\bar{u}_A - u_1}{u_o - u_1} \frac{\bar{\vartheta}_A}{\vartheta_o} \eta \frac{d\psi}{d\eta}$$

We specialize to $\eta = 1/2$.

Evaluation of integral:

(1)

$$\int \varphi\eta \, d\eta = \int \left(1 - \left[\eta\sqrt{E}\right]^{3/2}\right)^2 \eta \, d\eta = \frac{1}{2}\eta^2 - \frac{4}{7}E^{3/4}\eta^{7/2} + \frac{1}{5}E^{3/2}\eta^5$$

$$\int_0^{1/2} \varphi\eta \, d\eta = \frac{1}{8} - \frac{1}{28}\sqrt{2}E^{3/4} + \frac{1}{160}E^{3/2}$$

For $E = 2$:
$$\int_0^{1/2} \varphi\eta \, d\eta = \frac{1}{8} - \frac{1}{14}\sqrt[4]{2} + \frac{1}{80}\sqrt{2} = 0.05773$$

$$\psi = \left(1 - \eta^{3/2}\right)^2 = 1 - 2\eta^{3/2} - \eta^3 \quad ; \quad \psi_{1/2} = \frac{9}{8} - \frac{\sqrt{2}}{2} = 0.41789$$

$$\left(\int_0^{1/2} \varphi\eta \, d\eta\right)\psi_{1/2} = 0.02413$$

(2)

$$\int \frac{d\varphi}{d\eta}\eta^2 d\eta = \int (-3)\left(1 - \left[\eta\sqrt{E}\right]^{3/2}\right)\left(\eta\sqrt{E}\right)^{1/2}\sqrt{E}\,\eta^2 d\eta$$

$$= -3E^{3/4}\int\left(\eta^{5/2} - E^{3/4}\eta^{8/2}\right)d\eta = -\frac{6}{7}E^{3/4}\eta^{7/2} + \frac{3}{5}E^{3/2}\eta^5$$

$$\int_0^{1/2} \frac{d\varphi}{d\eta}\eta^2 d\eta = \frac{3}{56}\sqrt{2}\,E^{3/4} + \frac{3}{160}E^{3/2}$$

For $E = 2$:

$$\int_0^{1/2} \frac{d\varphi}{d\eta}\eta^2 d\eta = -\frac{3}{28}\sqrt[4]{2} + \frac{3}{80}\sqrt{2} = -0.07438$$

$$\left(\int_0^{1/2} \frac{d\varphi}{d\eta}\eta^2 d\eta\right)\psi_{1/2} = -0.03108$$

(3)

$$\int_{1/2}^{1} \varphi\psi\eta\, d\eta = \int_{1/2}^{1/\sqrt{E}} \varphi\psi\eta\, d\eta = \int_0^{1/\sqrt{E}} \varphi\psi\eta\, d\eta - \int_0^{1/2} \varphi\psi\eta\, d\eta$$

$$= 0.04382 - \int_0^{1/2} \varphi\psi\eta\, d\eta \qquad (\text{for } E = 2)$$

$$\int_0^{1/2} \varphi\psi\eta\, d\eta = \left(\frac{1}{8} - \frac{1}{104} + \frac{1}{160}\right) + \sqrt{2}\left(\frac{1}{80} - \frac{1}{28} + \frac{1}{4}\,\frac{1}{256}\right) +$$

$$\sqrt[4]{2}\left(-\frac{1}{208} - \frac{1}{14}\right) + \sqrt[4]{8}\left(\frac{1}{40}\right) = 0.04157$$

$$\int_{1/2}^{1} \varphi\psi\eta\, d\eta = 0.00225$$

(4)

$$\int_{1/2}^{1} \psi\eta \, d\eta = \int_{0}^{1} \psi\eta \, d\eta - \int_{0}^{1/2} \psi\eta \, d\eta = \frac{9}{70} - \int_{0}^{1/2} \psi\eta \, d\eta$$

$$\int_{0}^{1/2} \psi\eta \, d\eta = \frac{21}{160} - \frac{1}{28}\sqrt{2} = 0.08074 \qquad \int_{1/2}^{1} \psi\eta \, d\eta = 0.04783$$

(5)

$$\int \frac{d(\varphi\psi)}{d\eta}\eta^2 d\eta = \int \frac{d\left\{\left[\left(1 - \left[\eta\sqrt{E}\right]^{3/2}\right)^2\left(1 - \eta^{3/2}\right)^2\right]\right\}}{d\eta}\eta^2 d\eta$$

$$= \int \left\{\left[\eta^{1/2}\left(-3E^{3/4} - 3\right) + \eta^2\left(3E^{3/2} + 12E^{3/4} + 3\right) + \right.\right.$$

$$\left.\left.\eta^{7/2}\left(-9E^{3/2} - 9E^{3/4}\right) + \eta^5\left(6E^{3/2}\right)\right]\right\}\eta^2 d\eta = \eta^{7/2}\left(-\frac{6}{7}E^{3/4} - \frac{6}{7}\right) +$$

$$\eta^5\left(\frac{3}{5}E^{3/2} + \frac{12}{5}E^{3/4} + \frac{3}{5}\right) + \eta^{13/2}\left(-\frac{18}{13}E^{3/2} - \frac{18}{13}E^{3/4}\right) + \eta^8\left(\frac{3}{4}E^{3/2}\right)$$

$$\int_{1/2}^{1} \frac{d(\varphi\psi)}{d\eta}\eta^2 d\eta = \int_{1/2}^{1/\sqrt{E}} \frac{d(\varphi\psi)}{d\eta}\eta^2 d\eta = \int_{0}^{1/\sqrt{E}} - \int_{0}^{1/2}$$

For $E = 2$:

$$\int_{0}^{1/\sqrt{E}} \frac{d(\varphi\psi)}{d\eta}\eta^2 d\eta = \left(-\frac{3}{7} + \frac{3}{10}\right) + \sqrt[4]{2}\left(\frac{-3}{14} + \frac{3}{5} - \frac{9}{26}\right) +$$

$$\sqrt{2}\left(\frac{3}{40} - \frac{9}{52} + \frac{3}{32}\right) = -0.08765$$

$$\int_{0}^{1/2} \frac{d(\varphi\psi)}{d\eta}\eta^2 d\eta = \left(\frac{-9}{208} + \frac{3}{160}\right) + \sqrt{2}\left(\frac{3}{80} + \frac{3}{512} - \frac{3}{56}\right) + \sqrt[4]{2}\left(-\frac{3}{28} - \frac{9}{416}\right) +$$

$$\sqrt[4]{8}\left(\frac{3}{40}\right) = -0.06597$$

$$\int_{1/2}^{1} \frac{d(\varphi\psi)}{d\eta}\eta^2 d\eta = -0.02167$$

(6)

$$\int \frac{d\psi}{d\eta}\eta^2 d\eta = \int \left(-3\eta^{1/2} + 3\eta^2\right)\eta^2 d\eta = -\frac{6}{7}\eta^{7/2} + \frac{3}{5}\eta^5$$

$$\int_{1/2}^{1} \frac{d\psi}{d\eta}\eta^2 d\eta = \left(\frac{-6}{7} + \frac{3}{5} - \frac{3}{160}\right) + \frac{3}{56}\sqrt{2} = -0.20013$$

(7)

$$\eta\frac{d\psi}{d\eta} = -3\eta^{3/2} + 3\eta^3$$

$$\left(\eta\frac{d\psi}{d\eta}\right)_{1/2} = \frac{3}{8} - \frac{3}{4}\sqrt{2} = -0.68566$$

Abbreviating $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ to $(\ )$ gives then

$$-b_2\frac{\bar{\vartheta}_A}{\vartheta_o}(\ )'A + b_2'\frac{\bar{\vartheta}_A}{\vartheta_o}(\ )B - b_2\left[(\ )'\frac{\bar{\vartheta}_A}{\vartheta_o} + (\ )\left(\bar{\vartheta}_A/\vartheta_o\right)'\right]C -$$

$$b_2\frac{u_1}{u_o - u_1}\left(\bar{\vartheta}_A/\vartheta_o\right)'D + b_2'(\ )\frac{\bar{\vartheta}_A}{\vartheta_o}F + b_2'\frac{u_1}{u_o - u_1}\frac{\bar{\vartheta}_A}{\vartheta_o}G = \sqrt{E}\,K(\ )\frac{\bar{\vartheta}_A}{\vartheta_o}H$$

with $A = 0.02413$ $\quad B = -0.03108$ $\quad C = 0.00225$

$D = 0.04783$ $\quad F = -0.02167$ $\quad G_1 = -0.20013$

$H = -0.68566$

or

$$b_2\frac{\bar{\vartheta}_A}{\vartheta_o}(\ )'\alpha + b_2'\frac{\bar{\vartheta}_A}{\vartheta_o}(\ )\beta + b_2\left(\bar{\vartheta}_A/\vartheta_o\right)'(\ )\gamma + b_2\frac{u_1}{u_o - u_1}\left(\bar{\vartheta}_A/\vartheta_o\right)'\delta +$$

$$b_2'\frac{u_1}{u_o - u_1}\frac{\bar{\vartheta}_A}{\vartheta_o}\epsilon = K\zeta\frac{\bar{\vartheta}_A}{\vartheta_o}(\ )$$

with $\alpha = 0.02638$ $\beta = 0.05276$ $\gamma = 0.00225$

$\delta = 0.04783$ $\epsilon = 0.20013$ $\zeta = 0.96967$.

The introduction of (51) and (52)

$$b_2/r_o = \frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)d_2 + \frac{u_1}{u_o}e_1\right]^{1/2}}$$

$$\frac{d\left(b_2/r_o\right)}{d\left(x/r_o\right)}$$

$$= -\frac{1}{2}\frac{\left\{\frac{u_o - u_1}{u_o} d_2\left[\frac{\bar{\vartheta}_A}{\vartheta_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)' + \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\bar{\vartheta}_A/\vartheta_o\right)'\right] + \frac{u_1}{u_o}e_1\left(\bar{\vartheta}_A/\vartheta_o\right)'\right\}}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)d_2 + \frac{u_1}{u_o}e_1\right]^{3/2}\left(\bar{\vartheta}_A/\vartheta_o\right)^{3/2}}$$

gives

$$\left(\bar{\vartheta}_A/\vartheta_o\right)'\frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}}\frac{1}{[\ ]^{1/2}}\left\{\left(\gamma_1 - \frac{\beta_1}{2}\right)(\ ) + \left(\delta_1 - \frac{\epsilon_1}{2}\right)\frac{u_1}{u_o - u_1}\right\} +$$

$$(\ )'\frac{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}}{[\ ]^{3/2}}\left\langle\alpha_1[\ ] - \frac{\beta_1}{2} d_2\frac{u_o - u_1}{u_o}(\ ) - \frac{\epsilon_1}{2} d_2\frac{u_1}{u_o}\right\rangle = K\zeta_1\frac{\bar{\vartheta}_A}{\vartheta_o}\ (\ )$$

Transforming from the variable x/r_o to the variable $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ gives

$$\left(\bar{\vartheta}_A/\vartheta_o\right)' = \frac{d\left(\bar{\vartheta}_A/\vartheta_o\right)}{d\left(\frac{\bar{u}_A - u_o}{u_o - u_1}\right)}\ (\ )' \tag{53}$$

but by (47)

$$(\ )' = \frac{d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{d\left(x/r_o\right)} = -K\left\{\frac{(\ )^{5/2}}{\alpha_1}\frac{\left[\frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o}e_1\right]^{3/2}}{\frac{u_o - u_1}{u_o}(\ )^2 + \beta_1\frac{u_1}{u_o}(\ ) + \gamma_1\frac{u_1}{u_o}\frac{u_1}{u_o - u_1}}\right\}$$

The result for $\bar{\vartheta}_A/\vartheta_o$ as a function of $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ is then the differential equation

$$\frac{d\left(\bar{\vartheta}_A/\vartheta_o\right)}{d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} = \left(\bar{\vartheta}_A/\vartheta_o\right)^{3/2}$$

$$\left\{\zeta_1 \frac{\alpha_1 \frac{u_o - u_1}{u_o}(\ )^2 + \beta_1 \frac{u_1}{u_o}(\ ) + \gamma_1 \frac{u_1}{u_o}\frac{u_1}{u_o - u_1}}{(\ )^{3/2}\left\langle \frac{u_o - u_1}{u_o} d_1(\ ) + \frac{u_1}{u_o}e_1 \right\rangle^{3/2}} \frac{\left[\frac{u_o - u_1}{u_o} d_2(\ ) + \frac{u_1}{u_o}e_1\right]^{1/2}}{\left\{\rho_1(\ ) + \rho_2 \frac{u_1}{u_o - u_1}\right\}}\right\} +$$

$$\left(\bar{\vartheta}_A/\vartheta_o\right)\left\{\frac{1}{\left[\frac{u_o - u_1}{u_o} d_2(\ ) + \frac{u_1}{u_o}e_1\right]} \frac{\rho_3 \frac{u_1}{u_o}}{\left\{\rho_1(\ ) + \rho_2 \frac{u_1}{u_o - u_1}\right\}}\right\} \tag{54}$$

with $\alpha_1 = 0.00657$ $\beta_1 = 0.02531$ $\gamma_1 = 0.01959$

$d_1 = 0.13352$ $e_1 = 0.25714$ $d_2 = 0.08765$

$\rho_1 = 0.02413$ $\rho_2 = 0.05224$ $\rho_3 = -0.00199$

$\zeta_1 = 0.96967$

This equation is of the type of a Bernoulli equation

$$\frac{dy}{dx} = y^{3/2}G(x) - yF(x)$$

and can be integrated. Unfortunately the quadrature is not feasible.

It gives

$$\bar{\vartheta}_A/\vartheta_o = \frac{\zeta^{2\lambda}}{\left\{\frac{\zeta_1}{2}\left(\frac{u_o - u_1}{u_o}\right)^{1/2}\int_{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}^{1} \zeta^{\lambda - 1/2} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{3/2}}\eta \, d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \zeta(1)\right\}^2} \tag{55}$$

with the functions

$$\zeta = \frac{\rho_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \rho_2\frac{u_1}{u_o - u_1}}{d_2\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + e_1\frac{u_1}{u_o - u_1}}$$

$$\eta = \frac{\alpha_1\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 + \beta_1\frac{u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \gamma_1\frac{u_1}{u_o}\,\frac{u_1}{u_o - u_1}}{\left\langle d_1\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + e_1\frac{u_1}{u_o}\right\rangle^{3/2}\left\{\rho_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \rho_2\frac{u_1}{u_o - u_1}\right\}^{1/2}}$$

and the constant

$$\lambda = +\,1/2\,\frac{\rho_3}{\rho_2\,d_2 - \rho_1 e_1} = -\,0.6111$$

The integration constant was so defined that

$$\bar{\vartheta}_A/\vartheta_o = 1 \quad \text{for} \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) = 1$$

Hence, for infinitely great nozzle distances in the case of outside air in motion $\left(u_1 \neq 0\right)$

$$\bar{\vartheta}_A/\vartheta_o \approx \left[\left(\frac{\rho_2}{e}\right)^{2\lambda}\frac{1}{\zeta_1^{\,2}}\,\frac{\rho_2 e_1^{\,3}}{\gamma_1^{\,2}}\right]\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$$

and

$$\bar{\vartheta}_A/\vartheta_o \approx \frac{1}{2}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \tag{56}$$

The asymptotic formula for b_2/r_o follows as

$$b_2/r_o \approx \sqrt{2}\;\frac{1}{\left(\frac{\bar{u}_A - u_o}{u_o - u_1}\right)^{1/2}}\;\frac{1}{\left(\frac{u_1}{u_o}e_1\right)^{1/2}}$$

or

$$b_2/r_o \approx \sqrt{2}\ b_1/r_o \tag{57}$$

By this result the assumption $b_2/b_1 = \sqrt{2}$, based upon the theory of asymptotic distribution functions, is sustained.

But this does not hold true for the case of quiescent outside air $\left(u_1 = 0\right)$. The theory of asymptotic distribution functions gives an impracticable result: $b_2/b_1 \rightarrow \infty$. The assumption of the same ratio for b_2/b_1 as for outside air in motion proves inconclusive.

In the case of quiescent outside air, the differential equation reduces to

$$\frac{d\left(\bar{\vartheta}_A/\vartheta_o\right)}{d\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)} = \gamma_2 \frac{\left(\bar{\vartheta}_A/\vartheta_o\right)^{3/2}}{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)^{3/2}}$$

with

$$\gamma_2 = \zeta_1 \frac{\alpha_1\ d_2^{1/2}}{d_1^{3/2}\ \rho_1}$$

The solution reads

$$\bar{\vartheta}_A/\vartheta_o = \frac{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[\gamma_2 + \left(1 - \gamma_2\right)\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}\right]^2}$$

For the breadth of the mixing region by

$$b_2/r_o = \frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}} \frac{1}{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}} \frac{1}{d_2^{1/2}}$$

and

$$b_2/r_o = \frac{1}{d_2^{1/2}} \frac{\left[\gamma_2 + (1 - \gamma_2)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}\right]}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}.$$

In the asymptote

$$\bar{\vartheta}_A/\vartheta_o \approx \frac{1}{\gamma_2^2}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$$

$$b_2/r_o \approx \frac{1}{d_2^{1/2}} \gamma_2 \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}$$

and

$$b_2/r_o \approx \frac{d_1^{1/2}}{d_2^{1/2}} \gamma_2 \, b_1/r_o$$

With the constants computed on the assumption $b_2/b_1 = \sqrt{2}$ the result would be $b_2/b_1 = 1.98$, hence contradictory. The assumption $b_2/b_1 = \sqrt{2}$ for the case of outside air at rest must therefore be abandoned and the correct value of b_2/b_1 obtained by a special consideration.

For this purpose the equation

$$b_2/r_o = \frac{d_1^{1/2}}{d_2^{1/2}} \gamma_2 \, b_1/r_o$$

and

$$b_2/b_1 = \frac{\zeta_1 \alpha_1}{d_1 \rho_1} \tag{58}$$

must be solved. The constants at the right-hand side contain b_2/b_1 as a parameter. By an approximation method $b_2/b_1 \approx \underline{1.33}$.

The calculation made with this value of b_2/b_1 gives for the case of outside air at rest:

$$\bar{\vartheta}_A/\vartheta_o = \frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[\gamma_2 + (1 - \gamma_2)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}\right]^2} \tag{59}$$

$$b_2/r_o = \frac{\gamma_1\left[\gamma_2 + (1 - \gamma_2)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}\right]}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} \tag{60}$$

with $\gamma_1 = 3.245$ $\gamma_2 = 1.123$

hence in the asymptote

$$b_2/r_o \approx 1.33\ b_1/r_o \tag{61}$$

In addition

$$\bar{\vartheta}_A/\vartheta_o \approx 0.79\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \tag{62}$$

as against $\bar{\vartheta}_A/\vartheta_o \approx \frac{1}{2}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ obtained for outside air in motion.

The calculated functions are represented in figures 1 to 6.

The integral in formula (55) was obtained by numerical integration.

III. Calculation of the Asymptotic Distribution Functions

At very great distances from the nozzle, the velocity and temperature differences of the jet and of the surrounding medium can be regarded as small.

Limited to small temperature differences (density approximately constant) the differential equations (part 1, reference 1, (24)) read:

momentum:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial r} = \epsilon(x)\left\{\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\,\frac{\partial \bar{u}}{\partial r}\right\} \tag{63a}$$

heat:

$$\bar{u}\frac{\partial \bar{\vartheta}}{\partial x} + \bar{v}\frac{\partial \bar{\vartheta}}{\partial r} = E\epsilon(x)\left\{\frac{\partial^2 \bar{\vartheta}}{\partial r^2} + \frac{1}{r}\,\frac{\partial \bar{\vartheta}}{\partial r}\right\} \tag{63b}$$

mass:

$$r\frac{\partial \bar{u}}{\partial x} + \bar{v} + r\frac{\partial \bar{v}}{\partial r} = 0 \tag{63c}$$

where

$$\epsilon(x) = K\, b_1(x)\left(\bar{u}_A(x) - u_1\right)$$

1. The velocity distributions are now to be calculated.

a. <u>Outside air in motion $u_1 \neq 0$</u>

The similarity formula

$$\frac{\bar{u} - u_1}{u_o - u_1} = \frac{\bar{u}_A(x) - u_1}{u_o - u_1}\,\varphi(\eta) \tag{64a}$$

$$\text{with} \quad \eta = \frac{r}{b_1(x)}$$

$$\bar{v} = \bar{v}(x)\,\chi(\eta) \tag{64b}$$

gives

$$\frac{\partial\left(\bar{u}/u_1\right)}{\partial r} = \frac{u_o - u_1}{u_1}\,\frac{\bar{u}_A(x) - u_1}{u_o - u_1}\,\frac{d\varphi}{d\eta}\,\frac{1}{b_1(x)}$$

$$\frac{\partial^2(\bar{u}/u_1)}{\partial r^2} = \frac{u_o - u_1}{u_1} \frac{\bar{u}_A(x) - u_1}{u_o - u_1} \frac{d^2\varphi}{d\eta^2} \frac{1}{b_1{}^2}$$

$$\frac{\partial(\bar{u}/u_1)}{\partial x} = \frac{u_o - u_1}{u_1}\left(\frac{\bar{u}_A(x) - u_1}{u_o - u_1}\right)' \varphi + \frac{u_o - u_1}{u_1}\left(\frac{\bar{u}_A(x) - u_1}{u_o - u_1}\right)\frac{d\varphi}{d\eta}\left(- \eta\frac{b_1{}'}{b_1}\right)$$

etc.

On the basis of computing the axial functions by the momentum and the heat equation

$$\frac{\bar{u}_A(x) - u_1}{u_o - u_1} \sim x^{-2/3} \qquad\qquad b_1(x) \sim x^{1/3}$$

is applicable to the asymptote according to (15) and (16).

By the equation of continuity of mass

$$\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)' \sim \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)b_1{}' \sim \frac{v(x)}{u_1}$$

hence

$$\frac{\bar{v}(x)}{u_1} \sim x^{-4/3} \tag{65}$$

for the asymptote.

By the equation of motion

$$\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)' \sim \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\frac{b_1{}'}{b_1} \sim \frac{\bar{v}(x)}{u_1}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)1/b_1 \sim \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 1/b_1$$

The first two terms and the last term are of the order of magnitude $x^{-5/3}$, while the term $\frac{\bar{v}(x)}{u_1}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)1/b_1$ is of the order of $x^{-7/3}$.

The term with $\bar{v}(x)$ is therefore disregarded in asymptotic considerations.

Noting further that

$$\frac{\bar{u}}{u_1} = \left(\frac{\bar{u} - u_1}{u_o - u_1}\right)\frac{u_o - u_1}{u_1} + 1 \approx 1$$

for small velocity differences gives for the case of outside air in motion the asymptotic motion equation

$$\frac{\partial(\bar{u}/u_1)}{\partial x} = Kb_1(x)\frac{\bar{u}_A(x) - u_1}{u_1}\left\{\frac{\partial^2(\bar{u}/u_1)}{\partial r^2} + (1/r)\frac{\partial(\bar{u}/u_1)}{\partial r}\right\} \tag{66}$$

the relation

$$\frac{\bar{u}_A(x) - u_1}{u_o - u_1} = u_\infty x^{-2/3} \qquad\qquad b_1(x) = b_\infty x^{1/3}$$

gives

$$-\frac{2}{3}u_\infty x^{-5/3}\varphi - u_\infty x^{-2/3}\frac{b_\infty \frac{1}{3} x^{-2/3}}{b_\infty x^{1/3}}\eta\frac{d\varphi}{d\eta}$$

$$= K\frac{u_o - u_1}{u_1}\frac{1}{b_\infty x^{1/3}}u_\infty{}^2 x^{-4/3}\left\{\frac{d^2\varphi}{d\eta^2} + \frac{1}{\eta}\frac{d\varphi}{d\eta}\right\}$$

$$-\frac{2}{3}\varphi - \frac{1}{3}\eta\frac{d\varphi}{d\eta} = K\frac{u_o - u_1}{u_1}\frac{1}{b_\infty}u_\infty\left\{\frac{d^2\varphi}{d\eta^2} + \frac{1}{\eta}\frac{d\varphi}{d\eta}\right\}$$

$$-\frac{1}{3}\left(2\eta\varphi + \eta^2\frac{d\varphi}{d\eta}\right) = K\frac{u_o - u_1}{u_1}\frac{u_\infty}{b_\infty}\left\{\eta\frac{d^2\varphi}{d\eta^2} + \frac{d\varphi}{d\eta}\right\} \qquad \text{for} \quad \eta \neq 0$$

Integrated

$$-\frac{1}{3}\eta^2\varphi = K\frac{u_o - u_1}{u_1}\frac{u_\infty}{b_\infty}\eta\frac{d\varphi}{d\eta}$$

or

$$-\frac{1}{3}\frac{b_\infty}{u_\infty}\frac{1}{K}\frac{u_1}{u_o - u_1}\eta\, d\eta = \frac{d\varphi}{\varphi}$$

Hence

$$\varphi = e^{-(\sigma_o \eta)^2} \quad \text{with} \quad \sigma_o^{\,2} = \frac{1}{6}\,\frac{1}{K}\,\frac{u_1}{u_o - u_1}\,\frac{b_\infty}{u_\infty} \tag{67}$$

But by (15) and (16)

$$b_\infty = K^{1/3}\,\frac{\left(\frac{u_o - u_1}{u_o}\right)^{1/3}}{\left(u_1/u_o\right)^{2/3}}B_1 \quad \text{with} \quad B_1 = 4.246$$

$$u_\infty = \frac{1}{K^{2/3}}\,\frac{\left(u_1/u_o\right)^{1/3}}{\left(\frac{u_o - u_1}{u_o}\right)^{2/3}}B_2 \quad \text{with} \quad B_2 = 0.216$$

Therefore

$$\sigma_o = 1.811 \tag{68}$$

Transforming correspondingly $b_1 \sim x^{1/3}$ on the variable $\eta^* = r/x^{1/3}$ gives

$$\varphi = e^{-(\sigma_o^* \eta)^2} \quad \text{with} \quad \sigma_o^* = 0.427\frac{1}{K^{1/3}}\,\frac{\left(u_1/u_o\right)^{2/3}}{\left(\frac{u_o - u_1}{u_o}\right)^{1/3}} \tag{69}$$

b. <u>Outside air at rest $(u_1 = 0)$</u>

Conformably to (28) and (29)

$$\bar{u}_A \sim 1/x$$

$$b_1 \sim x$$

we put

$$\bar{u} = \bar{u}_A(x)\varphi(\eta) \qquad \bar{\bar{v}} = \bar{v}(x)\chi(\eta) \quad \text{with} \quad \eta = r/x \tag{70}$$

A similar consideration as given under (a) yields

$$\bar{v}(x) \sim 1/x \tag{71}$$

The apparent kinematic viscosity follows as

$$\epsilon(x) = Kb_1(x)\bar{u}_A(x) = \text{constant}$$

Thus the problem appears to be reduced to that of computing the distribution function for the laminar viscosity ϵ. This has been solved by Schlichting (reference 9).

The solution is

$$\varphi = \frac{1}{\left[1 + (\sigma_o\eta)^2\right]^2} \quad \text{with} \quad \sigma_o = \frac{1}{2}\frac{1}{\sqrt{2Kb_\infty}} \tag{72}$$

But according to (28)

$$b_\infty = K\ 20.321$$

hence

$$\sigma_o = \frac{1}{K}\ 0.0784 \tag{73}$$

2. Relative to the temperature distributions Reichardt (reference 6) has shown that

$$\psi = \varphi^{A_\tau/A_Q} \tag{74}$$

with A_τ, A_Q being exchange quantities of momentum and of heat, respectively.

To put it briefly: In this representation, the ratio A_Q/A_τ corresponds to the factor E. With E = 2

$$\psi = \varphi^{1/2} \tag{75}$$

The calculated distribution functions are represented in figures 7 and 8.

B: GREATER TEMPERATURE DIFFERENCES BETWEEN JET AND SURROUNDING MEDIUM

I. Methods and Results

In view of part A of the present investigation, which dealt with the diffusion of a hot air jet for small temperature differences, we can be brief in many respects.

The differential equations of the mixing field read: (part A, equation (2))
equation of continuity of mass

$$\frac{\partial(r\bar{\rho}\bar{u})}{\partial x} + \frac{\partial(r\bar{\rho}\bar{v})}{\partial r} = E\epsilon(x)\ \frac{\partial}{\partial r}\left\{r\frac{\partial\bar{\rho}}{\partial r}\right\}$$

equation of continuity of momentum

$$\frac{\partial(r\bar{\rho}\bar{u}\bar{u})}{\partial x} + \frac{\partial(r\bar{\rho}\bar{v}\bar{u})}{\partial r} = \epsilon(x)\ \frac{\partial}{\partial r}\left\{r\bar{\rho}\frac{\partial\bar{u}}{\partial r} + Er\bar{u}\frac{\partial\bar{\rho}}{\partial r}\right\}$$

equation of continuity of heat

$$\frac{\partial(r\bar{\rho}\bar{u}\bar{T})}{\partial x} + \frac{\partial(r\bar{\rho}\bar{v}\bar{T})}{\partial r} = E\epsilon(x)\ \frac{\partial}{\partial r}\left\{r\frac{\partial(\bar{\rho}\bar{T})}{\partial r}\right\}$$

with the apparent kinematic viscosity

$$\epsilon(x) = Kb_1(x)\left|\bar{u}_{max} - \bar{u}_{min}\right|$$

Integration with respect to r gives for the transition zone (part A, equation (3)) the momentum equation

$$r\bar{\rho}\bar{v}\left(\bar{u} - u_1\right) - \frac{\partial}{\partial x}\int_r^{b_1}\bar{\rho}\bar{u}\left(\bar{u} - u_1\right)r\ dr = \epsilon(x)r\left\{\bar{\rho}\frac{\partial\left(\bar{u} - u_1\right)}{\partial r} + E\left(\bar{u} - u_1\right)\frac{\partial\bar{\rho}}{\partial r}\right\}$$

the heat equation

$$r\bar{\rho}\bar{v}\bar{\vartheta} - \frac{\partial}{\partial x}\int_r^{b_2}\bar{\rho}\bar{u}\bar{\vartheta}r\ dr = E\epsilon(x)r\ \frac{\partial}{\partial r}\left(\bar{\rho}\bar{\vartheta}\right)$$

with

$$\epsilon(x) = Kb_1(\bar{u}_A - u_1)$$

For $r = 0$ a further integration (part A, equation (4)) gives the equation of the conservation of momentum

$$\int_0^{b_1} \bar{\rho}\bar{u}(\bar{u} - u_1) r \, dr = \rho_o u_o(u_o - u_1)\frac{r_o^2}{2}$$

and of the heat

$$\int_0^{b_2} \bar{\rho}\bar{u}\bar{\vartheta} r \, dr = \rho_o u_o \vartheta_o \, r_o^2/2$$

The investigation of the mixing field was then carried out for small temperature differences (part A) by means of the cited integral equations. For the velocity and temperature distribution the "similarity" formula was used:

$$\frac{\bar{u} - u_1}{u_o - u_1} = \frac{\bar{u}_A(x) - u_1}{u_o - u_1}\varphi(\eta) \qquad \eta = \frac{r}{b_1}$$

$$\frac{\bar{\vartheta}}{\vartheta_o} = \frac{\bar{\vartheta}_A(x)}{\vartheta_o}\psi(\eta)$$

This formula expresses the following facts: For very great nozzle distances (asymptotic case) the flow proves itself similar according to theory and experience. But it also proves itself almost similar for nozzle distances extending up to a point near the boundary of the core, according to experimental data (cf. Pabst, reference 2).

In order to simplify the calculation, the asymptotic ratio

$$b_2/b_1 = \sqrt{E}$$

for the total transition zone was assumed.

The functions φ and ψ are identical with the asymptotic distribution functions (part A)

$$\varphi = e^{-(\sigma_o \eta)^2} \qquad \eta = r/b_1$$

$$\psi = \varphi^{1/E}$$

The application of the foregoing investigation method to fields with greater temperature differences involves a considerable amount of paperwork. For this reason, a different method is applied.

The study is limited to fields with outside velocity $u_1 \neq 0$ different from zero, while the singular case $u_1 = 0$ (cf. part A) is disregarded.

The goal is to calculate the breadth of the mixing regions of velocity b_1 and of the temperature b_2 as well as the axial functions $\frac{\bar{u}_A(x) - u_1}{u_o - u_1}$ and $\frac{\bar{\vartheta}_A(x)}{\vartheta_o}$. For the calculation, the differential equations are used which for the jet center read

$$\left.\begin{aligned} \bar{u}\frac{\partial \bar{\rho}(\bar{u} - u_1)}{\partial x} &\doteq \epsilon(x)2\left\{\bar{\rho}\frac{\partial^2(\bar{u} - u_1)}{\partial r^2} + E(\bar{u} - u_1)\frac{\partial^2 \bar{\rho}}{\partial r^2}\right\} \\ \bar{u}\frac{\partial \bar{\rho}\bar{\vartheta}}{\partial x} &= E\epsilon(x)2\left\{\bar{\rho}\frac{\partial^2 \bar{\vartheta}}{\partial r^2} + \bar{\vartheta}\frac{\partial^2 \bar{\rho}}{\partial r^2}\right\} \end{aligned}\right\} \tag{1}$$

with

$$\epsilon(x) = Kb_1(x)\left(\bar{u}_A(x) - u_1\right)$$

also, the conservation formulas

momentum

$$\int_0^{b_1} \bar{\rho}\bar{u}\left(\bar{u} - u_1\right) r \, dr = \rho_o u_o \left(u_o - u_1\right)\frac{r_o^2}{2} \tag{2}$$

and heat

$$\int_0^{b_2} \bar{\rho}\bar{u}\bar{\vartheta} r \, dr = \rho_o u_o \vartheta_o \, r_o^2/2$$

In conformity with the almost similar behavior of the flow in the transition zone described above the following proposition is made:

$$\bar{u} - u_1 = \left(\bar{u}_A(x) - u_1\right)\varphi(\eta); \quad \eta = r/b_1$$
$$\bar{\vartheta} = \bar{\vartheta}_A(x)\varphi(\eta^*); \qquad \eta^* = r/b_2 \tag{3}$$

In other words, it is assumed that in the whole transition zone the distributions over the breadth of the mixing regions are represented by the asymptotic distribution function

$$e^{-(\sigma_o\eta)^2} \quad \text{and} \quad e^{-(\sigma_o\eta^*)^2} \tag{4}$$

The constant σ_o can be determined by experimental calibration. Approximating the experimental distributions by the function $\left(1 - \eta^{3/2}\right)^2$ gives (cf. part A).

$$\sigma = \underline{1.81} \tag{5}$$

With (3) the following differential equations

$$\frac{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)'}{\left(\dfrac{\bar{\vartheta}_A}{\vartheta_o}\right)'} = \frac{1}{E}\left(b_2/b_1\right)^2 \frac{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left(\dfrac{\bar{\vartheta}_A}{\vartheta_o}\right)}$$

and (6)

$$\frac{d\left(x/r_o\right)}{d\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)} = \frac{1}{K}\,\frac{1}{2}\,b_1 \frac{\left\langle\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right) + \dfrac{u_1}{u_o - u_1}\right\rangle}{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)^2}\,\frac{1}{\left.\dfrac{d^2\varphi}{d\eta^2}\right|_{\eta=0}}$$

are obtained from (1).

The first equation defines the mutual correlation of the axial functions $\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)$ and $\left(\bar{\vartheta}_A/\vartheta_o\right)$; the second describes the variation of the axial function $\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)$ with the nozzle distance x/r_o.

The conservation formulas (2) give by reason of (3)

$$b_1/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)(II) + \frac{u_1}{u_o}(I)\right]^{1/2}}$$

with the coefficients

$$(II) = 2\int_0^\infty \frac{\varphi^2\eta}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]}\, d\eta$$

$$(I) = 2\int_0^\infty \frac{\varphi\eta}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]}\, d\eta \tag{7a}$$

(where $\psi(\eta) = \varphi\left(\eta\frac{b_1}{b_2}\right) = \varphi(\eta^*)$)

Besides

$$b_2/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)[II] + \frac{u_1}{u_o}[I]\right]^{1/2}}$$

with the coefficients

$$[II] = 2\int_0^\infty \frac{\varphi\psi\eta^*}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\varphi\right]}\, d\eta^*$$

$$[I] = 2\int_0^\infty \frac{\varphi\eta^*}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\varphi\right]}\, d\eta^* \tag{7b}$$

(where, with $\eta^* = r/b_2$

$$\psi(\eta^*) = \varphi\left(\eta^*\, b_2/b_1\right) = \varphi(\eta)\Big)$$

In complete generality, the solution of the described equation system (6) and (7) still presents a difficult problem, which is largely due to the calculation of the integrals by which the coefficients of the breadth of the mixing regions are represented.

By contrast, the solution for the asymptote (very great nozzle distances) is readily indicated.

With

$$(I) = 2\int_0^\infty \varphi\eta \, d\eta = 1/\sigma_o^2; \quad [I] = 2\int_0^\infty \varphi\eta^* \, d\eta^* = 1/\sigma_o^2$$

we get

$$\left.\begin{aligned} b_1/r_o &\approx \frac{1}{\left(1 + \vartheta_o T_1\right)^{1/2}} \frac{1}{\sigma_o^2}\left(\frac{u_o}{u_1}\right)^{1/2} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}} \\ b_2/r_o &\approx \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\sigma_o^2}\left(\frac{u_o}{u_1}\right)^{1/2} \frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}} \end{aligned}\right\} \tag{8}$$

The differential equations (6) give then immediately

$$\left(\bar{\vartheta}_A/\vartheta_o\right) \approx \frac{1}{E}\left(\frac{u_A - u_1}{u_o - u_1}\right) \tag{9}$$

and

$$x/r_o \approx \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \quad \frac{1}{K}\frac{1}{6}\frac{1}{\sigma_o} \quad \left(\frac{u_o}{u_1}\right)^{1/2} \frac{u_1}{u_o - u_1} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{3/2}}$$

With the nozzle distance x/r_o as independent variable

$$\left.\begin{aligned} b_1/r_o &\approx \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/3}} K^{1/3} \frac{6^{1/3}}{\sigma_o^{5/3}} \frac{\left(\frac{u_o - u_1}{u_o}\right)^{1/3}}{\left(\frac{u_1}{u_o}\right)^{2/3}} \left(x/r_o\right)^{1/3} \\ b_2/b_1 &\approx \sqrt{E}; \\ \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) &\approx \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/3}} \frac{1}{K^{2/3}} \frac{1}{\left(6\sigma_o\right)^{2/3}} \frac{\left(\frac{u_1}{u_o}\right)^{1/3}}{\left(\frac{u_o - u_1}{u_o}\right)^{2/3}} \frac{1}{\left(x/r_o\right)^{2/3}} \end{aligned}\right\} \tag{10}$$

$$\left(\bar{\vartheta}_A/\vartheta_o\right) \approx \frac{1}{E}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \tag{10 con't}$$

Consequently, greater temperature differences for the asymptote make themselves felt by the factor $\dfrac{1}{\left(1 + \vartheta_o/T_1\right)^{1/3}}$. It implies that the breadth of the mixing regions increases slower at greater temperature differences and that velocity and temperature decrease considerably more.

The asymptotic solution is at the same time to be regarded as first approximation for the general solution. Comparisons with better approximations indicate, however, – for example, in the theory of small temperature differences (cf. part A) – that the asymptotic solution still represents no satisfactory approximation.

To gain a second approximation, it is logical to continue with the equation system (6) and (7) on the approximate assumption that $b_2/b_1 = \sqrt{E}$ (which holds rigorously for asymptotic conditions).

For $E = 2$ – this value is obtained by experiments as will be shown later – the coefficients of the conservation formulas can be indicated analytically:

$$(II) = \frac{2}{\sigma_o^2}\left\{\frac{\frac{1}{3}\left(\bar{\vartheta}_A/T_1\right)^3 - \frac{1}{2}\left(\bar{\vartheta}_A/T_1\right)^2 + \left(\bar{\vartheta}_A/T_1\right) - ln\left(1 + \bar{\vartheta}_A/T_1\right)}{\left(\bar{\vartheta}_A/T_1\right)^4}\right\}$$

$$(I) = \frac{2}{\sigma_o^2}\left\{\frac{\left(\bar{\vartheta}_A/T_1\right) - ln\left(1 + \bar{\vartheta}_A/T_1\right)}{\left(\bar{\vartheta}_A/T_1\right)^2}\right\}$$

$$[II] = \frac{1}{\sigma_o^2}\,\frac{\frac{1}{2}\left(\bar{\vartheta}_A/T_1\right)^2 - \left(\bar{\vartheta}_A/T_1\right) + ln\left(1 + \bar{\vartheta}_A/T_1\right)}{\left(\bar{\vartheta}_A/T_1\right)^3}$$

$$[I] = \frac{1}{\sigma_o^2}\,\frac{ln\left(1 + \bar{\vartheta}_A/T_1\right)}{\left(\bar{\vartheta}_A/T_1\right)}$$

Introduction of the functions thus obtained for b_1 and b_2 in (6) gives a system of ordinary differential equations which, however, can be solved only approximately.

But a second approximation in analytical form can be obtained, if one limits oneself to the asymptotic coefficients of the conservation formulas which read:

$$
\left.
\begin{aligned}
(\mathrm{II}) &= 2\int_0^\infty \varphi^2\eta \, d\eta = \frac{1}{2}\frac{1}{\sigma_o^2} \\
(\mathrm{I}) &= 2\int_0^\infty \varphi\eta \, d\eta = 1/\sigma_o^2 \\
[\mathrm{II}] &= 2\int_0^\infty \varphi\psi\eta^* \, d\eta^* = \frac{1}{3}\frac{1}{\sigma_o^2} \\
[\mathrm{I}] &= 2\int_0^\infty \varphi\eta^* \, d\eta^* = 1/\sigma_o^2
\end{aligned}
\right\} \tag{11}
$$

This solution seems to represent an adequate approximation. We get

$$
b_1/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)(\mathrm{II}) + \frac{u_1}{u_o}(\mathrm{I})\right]^{1/2}}
$$

$$
\text{with} \quad (\mathrm{II}) = \frac{1}{2}\frac{1}{\sigma_o^2}, \qquad (\mathrm{I}) = 1/\sigma_o^2 \tag{12}
$$

$$
b_2/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)[\mathrm{II}] + \frac{u_1}{u_o}[\mathrm{I}]\right]^{1/2}}
$$

$$
\text{with} \quad [\mathrm{II}] = \frac{1}{3}\frac{1}{\sigma_o^2}; \qquad [\mathrm{I}] = \frac{1}{\sigma_o^2}
$$

With these functions, the differential equations (6) give: For the correlation of $\left(\bar{\vartheta}_A/\vartheta_o\right)$ and $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$

$$
\left(\bar{\vartheta}_A/\vartheta_o\right) = \frac{1}{E} \frac{1}{\left(\frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} - 1\right) + \frac{1}{6}\frac{u_o - u_1}{u_1} ln\left[\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(1 + 2\frac{u_1}{u_o - u_1}\right)}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + 2\frac{u_1}{u_o - u_1}}\right] + \frac{1}{E}} \tag{13}
$$

The integration constant was therefore so determined that $\left(\bar{\vartheta}_A/\vartheta_o\right) = 1$ for $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) = 1$.

The dependence of the axial function $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ on the nozzle distance x/r_o reads

$$x/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{K} \frac{\sqrt{2}}{8} \frac{1}{\sigma_o} \left(\frac{u_o - u_1}{u_o}\right)^{1/2} \frac{u_o}{u_1} \left\{\left\langle \frac{1}{\zeta^{1/2}} - \frac{1}{\zeta(1)^{1/2}} \right\rangle + \frac{1}{3} \left\langle \frac{1}{\zeta^{3/2}} - \frac{1}{\zeta(1)^{3/2}} \right\rangle\right\} + x - K/r_o \tag{14}$$

where

$$\zeta = \frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + 2\frac{u_1}{u_o - u_1}\right]}$$

The integration constant being so defined that $x/r_o = x - K/r_o$ for $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) = 1$ (boundary of core).

The two empirical constants K and E appear in the theory; $K = 0.010$ for asymptotic conditions (cf. part A). The constant E equals 2 according to the experiment.

The structure of the mixing field for

$$\frac{u_o - u_1}{u_o} = 0.5;\ 0.75;\ 0.95$$

$$\vartheta_o/T_1 = 0;\ 0.75;\ 1.5$$

obtained with these constants is represented in figures 13 to 25.

On experimental data the measurements by Pabst (reference 2) are available. He measured the diffusion of a hot air jet of $\approx 300^\circ$ C in the transition zone and a relative speed of ≈ 400 m/sec for 18, 101, and 107 m/sec outside velocity

$$\vartheta_o/T_1 \approx 1.0;\ \frac{u_o - u_1}{u_o} \approx 0.95;\ 0.74;\ 0.53$$

Pabst's measurements of the velocity and temperature distributions over the breadth of the mixing regions were very complete. His measurements at the last three sections $(x/D = 16, 20, \text{ and } 24)$ are reproduced in figures 26 to 28, to which we have added the theoretical distribution functions for the asymptote

$$\varphi = e^{-(\sigma\eta)^2} \qquad \eta \sim r/b_1$$

and

$$\psi = \varphi^{1/E} \qquad (E = 2);$$

σ being defined accordingly. It is seen that the theoretical distribution functions reproduce the experimental distributions quite closely.

The significant result

$$E = 2$$

is quite plain.

The check of the theory on the experiment involves mainly a comparison of the theoretical and the measured axial functions

$$\frac{\bar{u}_A(x) - u_1}{u_0 - u_1} \quad \text{and} \quad \bar{\vartheta}_A(x)/\vartheta_0.$$

Pabst's average values are represented in figure 29. Figures 30 to 32 contain the comparison of the theoretical and experimental decrease in velocity along the jet axis, the constant $x - K/r_0$ of the theoretical curves being defined accordingly. Surprisingly the measurements on the whole indicate a greater decrease in velocity, although this was to be expected since, as a result of the friction at the inner and outer nozzle wall some loss of flow must be reckoned with, which stipulates an effective radius different from the geometric radius. Figures 33 to 35 show the comparison of the theoretical and experimental temperature drop along the jet axis, with the constant of the theoretical curve again properly defined. The reservations regarding any accidental loss on heat flow disappear. Even so the measurements indicate a fundamental departure in the sense that the experimental decrease is flatter; however, this difference should raise no objection to the theory since the temperature measurements along the jet axis seem unfortunately to be faulty. This is seen fairly plainly in figure 29 where, with increasing distance from the nozzle, the curves of the velocity and temperature drop approach one another and then even intersect. By contrast, the experimental distributions give $E = 2$ somewhat plainly, which, in other words, means that the temperature exchange is greater than the velocity exchange (according to theory $\left(\vartheta_A/\vartheta_o\right) \approx \frac{1}{E}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ is to be expected for greater nozzle distances).

To be sure the described behavior is due, to some extent, to the friction losses but not enough, according to preliminary calculations, to explain this difference.

II. Calculation of Mixing Region Structure

1. The differential equations describing the diffusion of a hot air jet in the transition zone read (cf. part A):

$$\frac{\partial r\bar{\rho}\bar{u}\left(\bar{u} - u_1\right)}{\partial x} + \frac{\partial r\rho\bar{v}\left(\bar{u} - u_1\right)}{\partial r} = \epsilon(x) \frac{\partial}{\partial r}\left\{r\bar{\rho}\frac{\partial\left(\bar{u} - u_1\right)}{\partial r} + Er\left(\bar{u} - u_1\right)\frac{\partial\bar{\rho}}{\partial r}\right\}$$

$$\frac{\partial r\bar{\rho}\bar{u}\bar{\vartheta}}{\partial x} + \frac{\partial r\bar{\rho}\bar{v}\bar{\vartheta}}{\partial r} = E\epsilon(x)\frac{\partial}{\partial r}\left\{r\frac{\partial\left(\bar{\rho}\bar{\vartheta}\right)}{\partial r}\right\}$$

$$\frac{\partial r\bar{u}}{\partial x} + \frac{\partial r\bar{v}}{\partial r} = 0$$

with

$$\epsilon(x) = Kb_1(x)\left(\bar{u}_A(x) - u_1\right)$$

or

$$r\bar{u}\frac{\partial\bar{\rho}\left(\bar{u} - u_1\right)}{\partial x} + r\bar{v}\frac{\partial\bar{\rho}\left(\bar{u} - u_1\right)}{\partial r} = \epsilon(x)\frac{\partial}{\partial r}\left\{r\bar{\rho}\frac{\partial\left(\bar{u} - u_1\right)}{\partial r} + r\left(\bar{u} - u_1\right)\frac{\partial\bar{\rho}}{\partial r}\right\}$$

$$r\bar{u}\frac{\partial\bar{\rho}\bar{\vartheta}}{\partial x} + r\bar{v}\frac{\partial\bar{\rho}\bar{\vartheta}}{\partial r} = E\epsilon(x)\frac{\partial}{\partial r}\left\{r\frac{\partial\bar{\rho}\bar{\vartheta}}{\partial r}\right\}$$

For the jet center $(\bar{v} = 0!)$

$$\bar{u}\frac{\partial\bar{\rho}\left(\bar{u} - u_1\right)}{\partial x} = \epsilon(x)\lim_{r\to 0}\frac{1}{r}\frac{\partial}{\partial r}\left\{r\bar{\rho}\frac{\partial\left(\bar{u} - u_1\right)}{\partial r} + Er\left(\bar{u} - u_1\right)\frac{\partial\bar{\rho}}{\partial r}\right\}$$

$$\bar{u}\frac{\partial\bar{\rho}\bar{\vartheta}}{\partial x} = E\epsilon(x)\lim_{r\to 0}\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial\bar{\rho}\bar{\vartheta}}{\partial r}\right\}.$$

Considering that the first derivatives with respect to r disappear for the jet center, one obtains

$$\bar{u}\,\frac{\partial\bar{\rho}(\bar{u} - u_1)}{\partial x} = \epsilon(x)2\left\{\bar{\rho}\,\frac{\partial^2(\bar{u} - u_1)}{\partial r^2} + E(\bar{u} - u_1)\,\frac{\partial^2\bar{\rho}}{\partial r^2}\right\}$$

$$\bar{u}\,\frac{\partial\bar{\rho}\bar{\vartheta}}{\partial x} = E\,\epsilon(x)2\left\{\bar{\rho}\,\frac{\partial^2\bar{\vartheta}}{\partial r^2} + \bar{\vartheta}\,\frac{\partial^2\bar{\rho}}{\partial r^2}\right\} \tag{15}$$

with

$$\epsilon(x) = Kb_1(x)\left(\bar{u}_A(x) - u_1\right)$$

If one puts

$$\bar{\rho} = \frac{\text{const.}}{T_1}\,\frac{1}{1 + \bar{\vartheta}/T_1}$$

according to $\bar{\rho} \times \bar{T}$ = constant, the equations read

$$\bar{u}\,\frac{\partial\dfrac{\bar{u} - u_1}{1 + \bar{\vartheta}/T_1}}{\partial x} = \epsilon(x)2\left\{\frac{1}{1 + \bar{\vartheta}/T_1}\,\frac{\partial^2(\bar{u} - u_1)}{\partial r^2} + E(\bar{u} - u_1)\frac{\partial^2\left(\dfrac{1}{1 + \bar{\vartheta}/T_1}\right)}{\partial r^2}\right\}$$

$$\bar{u}\,\frac{\partial\dfrac{\bar{\vartheta}}{1 + \bar{\vartheta}/T_1}}{\partial x} = E\epsilon(x)2\left\{\frac{\partial^2\left(\dfrac{\bar{\vartheta}}{1 + \bar{\vartheta}/T_1}\right)}{\partial r^2}\right\} \tag{16}$$

For computing the mixing field, the conservation formulas are employed again:

momentum

$$\int_0^{b_1} \bar{\rho}\bar{u}\left(\bar{u} - u_1\right)r\;dr = \rho_o u_o\left(u_o - u_1\right)r_o^2/2$$

heat

$$\int_0^{b_2} \bar{\rho}\bar{u}\bar{\vartheta}r\;dr = \rho_o u_o \vartheta_o\; r_o^2/2 \tag{17}$$

or

$$\left.\begin{aligned}\int_0^{b_1} \frac{\bar{u}(\bar{u} - u_1)}{1 + \bar{\vartheta}/T_1} r\, dr &= \frac{u_o(u_o - u_1)}{1 + \vartheta_o/T_1} r_o^2/2 \\ \int_0^{b_2} \frac{\bar{u}\bar{\vartheta}}{1 + \bar{\vartheta}/T_1} r\, dr &= \frac{u_o \vartheta_o}{1 + \vartheta_o/T_1} r_o^2/2\end{aligned}\right\} \qquad (18)$$

2. The law of similarity is applied

$$\left.\begin{aligned}\bar{u} - u_1 &= \left(\bar{u}_A(x) - u_1\right)\varphi(\eta), \quad \eta = r/b_1(x) \\ \bar{\vartheta} &= \bar{\vartheta}_A(x)\varphi(\eta^*), \quad \eta^* = r/b_2(x)\end{aligned}\right\} \qquad (19)$$

or

$$\bar{\vartheta} = \bar{\vartheta}_A(x)\varphi\left(\eta\, b_1/b_2\right) = \bar{\vartheta}_A(x)\psi(\eta)$$

hereby it is to be noted that

$$\left(\partial/\partial x\right)_r = \left(\partial/\partial x\right)_\eta - \left(\partial/\partial \eta\right)\eta\frac{b_1'}{b_1}$$

$$\left(\partial/\partial r\right)_x = \left(\partial/\partial \eta\right)\frac{1}{b_1}$$

For the differential equations (16) we get

$$\left\langle\left(\bar{u}_A - u_1\right)\varphi + u_1\right\rangle \frac{\partial\left(\dfrac{(\bar{u}_A - u_1)\varphi}{1 + (\bar{\vartheta}_A/T_1)\psi}\right)}{\partial x}$$

$$= Kb_1\left(\bar{u}_A - u_1\right)^2\left\{\frac{1}{1 + (\bar{\vartheta}_A/T_1)\psi}\frac{\partial^2(\bar{u}_A - u_1)\varphi}{\partial r} + E(\bar{u}_A - u_1)\varphi\frac{\partial^2\left(\dfrac{1}{1 + (\bar{\vartheta}_A/T_1)\psi}\right)}{\partial r^2}\right\}$$

$$\left\langle\left(\bar{u}_A - u_1\right)\varphi + u_1\right\rangle \frac{\partial\left(\dfrac{\bar{\vartheta}_A\psi}{1 + (\bar{\vartheta}_A/T_1)\psi}\right)}{\partial x} = EKb_1\left(\bar{u}_A - u_1\right)^2\left\{\frac{\partial^2\left(\dfrac{\bar{\vartheta}_A\psi}{1 + (\bar{\vartheta}_A/T_1)\psi}\right)}{\partial r^2}\right\}$$

or, bearing in mind that $\varphi = \psi = 1$ at the jet center

$$\left\langle \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{\bar{u}_o - u_1} \right\rangle \frac{d\dfrac{\left(\dfrac{\bar{u}_A - u_1}{u_o - u_1}\right)}{1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)}}{dx}$$

$$= Kb_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 \left\{ \frac{1}{1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)} \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \frac{d^2\varphi}{d\eta^2}\bigg|_{\eta=0} \frac{1}{b_1{}^2} + \right.$$

$$\left. E\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \frac{d^2\left(\dfrac{1}{1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi}\right)}{d\eta^2}\Bigg|_{\eta=0} \frac{1}{b_1{}^2} \right\}$$

$$\left\langle \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{\bar{u}_o - u_1} \right\rangle \frac{d\dfrac{\left(\bar{\vartheta}_A/\vartheta_o\right)}{1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)}}{dx}$$

$$= EKb_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 \left\{ \left(\bar{\vartheta}_A/\vartheta_o\right) \frac{d^2\left(\dfrac{\psi}{1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi}\right)}{d\eta^2}\Bigg|_{\eta=0} \frac{1}{b_1{}^2} \right\}$$

Observing, for example, that

$$\frac{d\dfrac{1}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]}}{d\eta} = -\frac{1}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]^2}\left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\frac{d\psi}{d\eta}\frac{1}{b_1}$$

$$\frac{d^2\dfrac{1}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]}}{d\eta^2}\Bigg|_{\eta=0} = -\frac{\left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]^2}\frac{d^2\psi}{d\eta^2}\bigg|_{\eta=0}\frac{1}{b_1{}^2}$$

and

$$\frac{d\psi}{d\eta} = \frac{d\varphi\left(\eta\, b_1/b_2\right)}{d\eta} = \frac{b_1}{b_2}\frac{d\varphi}{d\eta}$$

$$\frac{d^2\varphi}{d\eta^2} = \left(\frac{b_1}{b_2}\right)^2 \frac{d^2\varphi}{d\eta^2}$$

we get

$$\left\langle\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o - u_1}\right\rangle\left\{\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]} - \frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)'}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]^2}\right\}$$

$$= 2Kb_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left\{\frac{1}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\frac{1}{b_1^2}\frac{d^2\varphi}{d\eta^2}\bigg|_{\eta=0} -\right.$$

$$\left. E\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\frac{\left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]^2}\frac{1}{b_2^2}\frac{d^2\varphi}{d\eta^2}\bigg|_{\eta=0}\right\}$$

corresponding to

$$\left\langle\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o - u_1}\right\rangle\left\{\frac{\left(\bar{\vartheta}_A/\vartheta_o\right)'}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]} - \frac{\left(\bar{\vartheta}_A/\vartheta_o\right)\left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)'}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]^2}\right\}$$

$$= 2EKb_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left\{\frac{\left(\bar{\vartheta}_A/\vartheta_o\right)}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]}\frac{1}{b_2^2}\frac{d^2\varphi}{d\eta^2}\bigg|_{\eta=0} -\right.$$

$$\left.\frac{\left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]^2}\frac{1}{b_2^2}\frac{d^2\varphi}{d\eta^2}\bigg|_{\eta=0}\right\}$$

(The accents (') signify derivatives with respect to x.)

Hence

$$\left\langle\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o - u_1}\right\rangle\left\{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)' - \frac{\left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]}\right\}$$

$$= 2K\frac{1}{b_1}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2\frac{d^2\varphi}{d\eta^2}\bigg|_{\eta=0}\left\{1 - E\frac{\left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]}\left(\frac{b_1}{b_2}\right)^2\right\} \tag{20a}$$

$$\left\langle\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o - u_1}\right\rangle\left(\bar{\vartheta}_A/\vartheta_o\right)' = 2EK\frac{1}{b_2}\left(\frac{b_1}{b_2}\right)^2\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\frac{d^2\varphi}{d\eta^2}\bigg|_{\eta=0} \tag{20b}$$

Dividing the first equation by the second gives

$$\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'}{\left(\bar{\vartheta}_A/\vartheta_o\right)'} - \frac{\left(\vartheta_o/T_1\right)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]}$$

$$= \frac{1}{E}\left(\frac{b_2}{b_1}\right)^2\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left(\bar{\vartheta}_A/\vartheta_o\right)}\left\{1 - E\frac{\left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]}\left(\frac{b_1}{b_2}\right)^2\right\}$$

or

$$\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'}{\left(\bar{\vartheta}_A/\vartheta_o\right)} - \cancel{\frac{\vartheta_o/T_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[1 + \vartheta_o/T_1\left(\bar{\vartheta}_A/\vartheta_o\right)\right]}} = \frac{1}{E}\left(\frac{b_2}{b_1}\right)^2\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left(\bar{\vartheta}_A/\vartheta_o\right)} - \cancel{\frac{\vartheta_o/T_1\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\right]}}$$

finally

$$\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'}{\left(\bar{\vartheta}_A/\vartheta_o\right)'} = \frac{1}{E}\left(\frac{b_2}{b_1}\right)^2\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left(\bar{\vartheta}_A/\vartheta_o\right)} \tag{21}$$

The elimination of $\left(\bar{\vartheta}_A/\vartheta_o\right)$ from equation (19b) by means of this equation leaves

$$\frac{d\left(x/r_o\right)}{d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} = \frac{1}{2}\,\frac{1}{K}\,b_1\frac{\left\langle\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o - u_1}\right\rangle}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2}\frac{1}{\frac{d^2\varphi}{d\eta^2}}\Bigg|_{\eta=0} \tag{22}$$

Equation (21) gives the mutual correlation of $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ and $\left(\bar{\vartheta}_A/\vartheta_o\right)$ along the jet axis.

Equation (22) contains the dependence of the axial function $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$ on the nozzle distance.

3. The application of the equation of similarity (19) to the equations of conservation (18) gives:

momentum

$$\int_0^\infty \frac{\bar{u}\left(\bar{u} - u_1\right)}{1 + \bar{\vartheta}_o/T_1} r\, dr = \frac{u_o\left(u_o - u_1\right)}{1 + \vartheta_o/T_1} r_o^2/2$$

similitude

$$\bar{u} - u_1 = \left(\bar{u}_A - u_1\right)\varphi(\eta), \quad \eta = r/b_1$$

$$\bar{\vartheta} = \bar{\vartheta}_A(x)\psi(\eta)$$

It yields

$$\int_0^\infty \frac{\left\langle \left(\bar{u}_A - u_1\right)\varphi + u_1 \right\rangle}{1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi} \left(\bar{u}_A - u_1\right)\varphi b_1{}^2\eta\, d\eta = \frac{u_o\left(u_o - u_1\right)}{1 + \vartheta_o/T_1} r_o^2/2$$

or

$$\left(b_1/r_o\right)^2 \left\{ \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 \left(\int_0^\infty \frac{\varphi^2\eta}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]} d\eta \right) + \right.$$

$$\left. \frac{u_1}{u_o - u_1}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\int_0^\infty \frac{\varphi\eta}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]} d\eta \right) \right. = \frac{u_o}{u_o - u_1} \frac{1}{1 + \vartheta_o/T_1} \frac{1}{2}$$

finally

$$\left(b_1/r_o\right)^2 \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) \left\{ \frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(\int_0^\infty \frac{\varphi^2\eta}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]} d\eta \right) + \right.$$

$$\left. \frac{u_1}{u_o}\left(\int_0^\infty \frac{\varphi\eta}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]} d\eta \right) \right. = \frac{1}{2} \frac{1}{1 + \vartheta_o/T_1}$$

From it follows

$$b_1/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)(II) + \frac{u_1}{u_o}(I)\right]^{1/2}}$$

with the coefficients

$$(II) = 2\int_0^\infty \frac{\varphi^2\eta}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]} d\eta$$

$$(I) = 2\int_0^\infty \frac{\varphi\eta}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]} d\eta \tag{23}$$

The asymptotic coefficients are

$$(II) = 2\int_0^\infty \varphi^2\eta \; d\eta; \quad (I) = 2\int_0^\infty \varphi\eta \; d\eta$$

with

$$\varphi = e^{-\left(\sigma_o\eta\right)^2}$$

Hence

$$(II) = \frac{1}{2}\frac{1}{\sigma_o^2}; \quad (I) = \frac{1}{\sigma_o^2}$$

heat

$$\int_0^\infty \frac{\bar{u}\bar{\vartheta}}{1 + \bar{\vartheta}/T_1} r \; dr = \frac{u_o\vartheta_o}{1 + \vartheta_o/T_1} r_o^2/2$$

similitude

$$\bar{\vartheta} = \bar{\vartheta}_A(x)\varphi(\eta^*); \quad \eta^* = r/b_2$$

$$\bar{u} = u_1 = \left(\bar{u}_A(x) - u_1\right)\psi(\eta^*)$$

with

$$\psi(\eta^*) = \varphi\left(\eta^* b_2/b_1\right) = \varphi(\eta)$$

The result is

$$\int_0^\infty \frac{\left\langle \left(\bar{u}_A - u_1\right)\varphi + u_1 \right\rangle \bar{\vartheta}_A \psi}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]} b_2^2 \eta^* \, d\eta^* = \frac{u_o \vartheta_o}{1 + \vartheta_o/T_1} r_o^2/2$$

or

$$\left(b_2/b_1\right)^2 \left\{ \left(\bar{\vartheta}_A/\vartheta_o\right)\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\frac{u_o - u_1}{u_o}\left(\int_0^\infty \frac{\varphi\psi\eta^*}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\varphi\right]} \, d\eta^*\right) + \right.$$

$$\left. \frac{u_1}{u_o}\left(\bar{\vartheta}_A/\vartheta_o\right)\left(\int_0^\infty \frac{\varphi\eta^*}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\varphi\right]} \, d\eta^*\right) = \frac{1}{2} \, \frac{1}{1 + \vartheta_o/T_1}\right.$$

Hence

$$b_o/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \, \frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}} \, \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)[II] + \frac{u_1}{u_o}[I]\right]^{1/2}}$$

with the coefficients

$$[II] = 2\int_0^\infty \frac{\varphi\psi\eta^*}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]} \, d\eta^*$$

$$[I] = 2\int_0^\infty \frac{\varphi\eta^*}{\left[1 + \left(\vartheta_o/T_1\right)\left(\bar{\vartheta}_A/\vartheta_o\right)\psi\right]} \, d\eta^* \tag{24}$$

The asymptotic coefficients are

$$[II] = 2\int_0^\infty \varphi\psi\eta^* \, d\eta^*; \quad [I] = 2\int_0^\infty \varphi\eta^* \, d\eta^*$$

with

$$\varphi = e^{-\left(\sigma_o\eta^*\right)^2}$$

$$\psi = e^{-E\left(\sigma_o\eta^*\right)^2}$$

which with $E = 2$ give

$$[II] = \frac{1}{3} \, \frac{1}{\sigma_o^2}; \quad [I] = \frac{1}{\sigma_o^2}$$

4. Asymptotic solution (first approximation)

The asymptote (very great nozzle distances) follows from (23) and (24) as

$$
\begin{aligned}
b_1/r_o &\approx \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}} \left(\frac{u_o}{u_1}\right)^{1/2} 1/\sigma_o \\
b_2/r_o &\approx \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}} \left(\frac{u_o}{u_1}\right)^{1/2} 1/\sigma_o
\end{aligned} \tag{25}
$$

Hence

$$
b_2/b_1 = \frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}}
$$

Insertion in the differential equation (21) gives

$$
\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)'}{\left(\bar{\vartheta}_A/\vartheta_o\right)'} = \frac{1}{E} \frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2}{\left(\bar{\vartheta}_A/\vartheta_o\right)^2} \tag{26}
$$

or

$$
\left(\bar{\vartheta}_A/\vartheta_o\right) = \frac{1}{E}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)
$$

From (22) follows further

$$
-\frac{d\left(x/r_o\right)}{d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} = \frac{1}{2} \frac{1}{K} \left(\frac{u_o}{u_1}\right)^{1/2} \frac{1}{\sigma_o} \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{\frac{u_1}{u_o - u_1}}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{3/2}} \frac{1}{\left.\frac{d^2\varphi}{d\eta^2}\right|_{\eta=0}}
$$

or

$$
x/r_o = \frac{1}{K} \frac{1}{3} \frac{1}{\sigma_o} \frac{1}{\left.\frac{d^2\varphi}{d\eta^2}\right|_{\eta=0}} \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \left(\frac{u_o}{u_1}\right)^{1/2} \frac{u_1}{u_o - u_1} \frac{-1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{3/2}}
$$

With

$$\varphi = e^{-(\sigma_o \eta)^2}$$

$$\left.\frac{d^2\varphi}{d\eta^2}\right|_{\eta=0} = -2\sigma_o{}^2$$

this results in

$$x/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \quad \frac{1}{K} \; \frac{1}{6} \; \frac{1}{\sigma_o} \left(\frac{u_o}{u_1}\right)^{1/2} \frac{u_1}{u_o - u_1} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{3/2}} \tag{27}$$

5. (Second approximation)

With the asymptotic coefficients the result is

$$b_1/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)(II) + \frac{u_1}{u_o}(I)\right]^{1/2}}$$

where

$$(II) = \frac{1}{2}\,\frac{1}{\sigma_o{}^2}; \quad (I) = \frac{1}{\sigma_o{}^2} \tag{28}$$

$$b_2/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{\left(\bar{\vartheta}_A/\vartheta_o\right)^{1/2}} \frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)[II] + \frac{u_1}{u_o}[I]\right]^{1/2}}$$

where

$$[II] = \frac{1}{3}\,\frac{1}{\sigma_o{}^2}; \quad [I] = \frac{1}{\sigma_o{}^2}$$

With these functions the differential equation (21) reads

$$\frac{d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{d\left(\bar{\vartheta}_A/\vartheta_o\right)} = \frac{1}{E} \frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2}{\left(\bar{\vartheta}_A/\vartheta_o\right)^2} \left\{\frac{\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)(II) + \frac{u_1}{u_o}(I)}{\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)[II] + \frac{u_1}{u_o}[I]}\right\}$$

Separation of the variables produces

$$\frac{1}{E}\frac{d\left(\bar{\vartheta}_A/\vartheta_o\right)}{\left(\bar{\vartheta}_A/\vartheta_o\right)^2} = \frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2}\frac{\frac{u_o - u_1}{u_o}\frac{1}{3}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o}}{\frac{u_o - u_1}{u_o}\frac{1}{2}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o}}d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)$$

When the integration constant is so defined that $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) = 1$ for $\left(\bar{\vartheta}_A/\vartheta_o\right) = 1$ then

$$\left(\bar{\vartheta}_A/\vartheta_o\right) = \frac{1}{E}\frac{1}{\left(\frac{1}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} - 1\right) + \frac{1}{6}\frac{u_o - u_1}{u_1}ln\left[\frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\left(1 + 2\frac{u_1}{u_o - u_1}\right)}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + 2\frac{u_1}{u_o - u_1}}\right] + \frac{1}{E}} \tag{29}$$

The differential equation (22) gives

$$\frac{d(x/r_o)}{d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}}\;\frac{1}{2}\frac{1}{K}\;\frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)(II) + \frac{u_1}{u_o}(I)\right]^{1/2}}\times$$

$$\frac{\left\langle\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o - u_1}\right\rangle}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{5/2}}\left(\frac{-1}{2\sigma_o^2}\right)$$

or

$$\frac{d(x/r_o)}{d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} = -\frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}}\;\frac{1}{4}\;\frac{1}{K}\;\frac{1}{\sigma_o}\cdot\frac{\left\langle\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o - u_1}\right\rangle}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^{5/2}}\times$$

$$\frac{1}{\left[\frac{u_o - u_1}{u_o}\left(\frac{\bar{u}_o - u_1}{u_o - u_1}\right)^{1/2} + \frac{u_1}{u_o}\right]^{1/2}}$$

hence

$$x/r_o = -\frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{4} \frac{1}{K} \frac{1}{\sigma_o} \times$$

$$\frac{1}{\sqrt{\frac{1}{2}\frac{u_o - u_1}{u_o}}} \int^{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)} \frac{\left\langle \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + \frac{u_1}{u_o - u_1} \right\rangle d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 \left[\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 + 2\frac{u_1}{u_o - u_1}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)\right]^{1/2}}$$

Transforming

$$\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) = \frac{-\left(2\frac{u_1}{u_o - u_1}\right)}{1 - \xi^2}, \quad \xi = \frac{1}{\frac{\bar{u}_A - u_1}{u_o - u_1}} \sqrt{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 + 2\frac{u_1}{u_o - u_1}\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}$$

$$\frac{d\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{d\xi} = 2\left(2\frac{u_1}{u_o - u_1}\right)\frac{\xi}{\left(1 - \xi^2\right)^2}, \quad \left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)^2 = \frac{\left(2\frac{u_1}{u_o - u_1}\right)^2}{\left(1 - \xi^2\right)^2}$$

gives with

$$\alpha = 2\frac{u_1}{u_o - u_1} \quad \text{and} \quad B = -\frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}} \frac{1}{4} \frac{1}{K} \frac{1}{\sigma_o} \frac{1}{\sqrt{\frac{1}{2}\frac{u_o - u_1}{u_o}}}$$

$$x/r_o = B\int \frac{\left\langle\frac{-\alpha}{1-\xi^2} + \frac{u_1}{u_o - u_1}\right\rangle}{\frac{\alpha^2}{\left(1-\xi^2\right)^2}\xi\frac{-\alpha}{1-\xi^2}} 2\alpha\frac{\xi}{\left(1-\xi\right)^2}\,d\xi$$

$$= \frac{-2}{\alpha^2}B\int \frac{\left\langle\frac{-\alpha}{1-\xi^2} + \frac{u_1}{u_o - u_1}\right\rangle}{\frac{1}{1-\xi^2}}\,d\xi$$

$$= \frac{-2}{\alpha^2}B\int \left\langle -\alpha + \frac{u_1}{u_o - u_1}\left(1 - \xi^2\right)\right\rangle d\xi$$

$$= \frac{1}{2}\ \frac{1}{\frac{u_1}{u_o - u_1}}B\left\{\xi + \frac{1}{3}\ \xi^3\right\} + \text{const.}$$

Identifying the integration constant for $\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) = 1$ by $x/r_o = x - K/r_o$ (boundary of core) gives

$$x/r_o = \frac{1}{\left(1 + \vartheta_o/T_1\right)^{1/2}}\ \frac{1}{K}\ \frac{\sqrt{2}}{8}\ \frac{1}{\sigma_o}\ \left(\frac{u_o - u_1}{u_o}\right)^{1/2}\frac{u_o}{u_1} \times$$

$$\left\{\left\langle\frac{1}{\zeta^{1/2}} - \frac{1}{\zeta(1)^{1/2}}\right\rangle + \frac{1}{3}\ \left\langle\frac{1}{\zeta^{3/2}} - \frac{1}{\zeta(1)^{3/2}}\right\rangle\right\} + x - K/r_o \qquad (30)$$

where

$$\zeta = \frac{\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right)}{\left[\left(\frac{\bar{u}_A - u_1}{u_o - u_1}\right) + 2\frac{u_1}{u_o - u_1}\right]}$$

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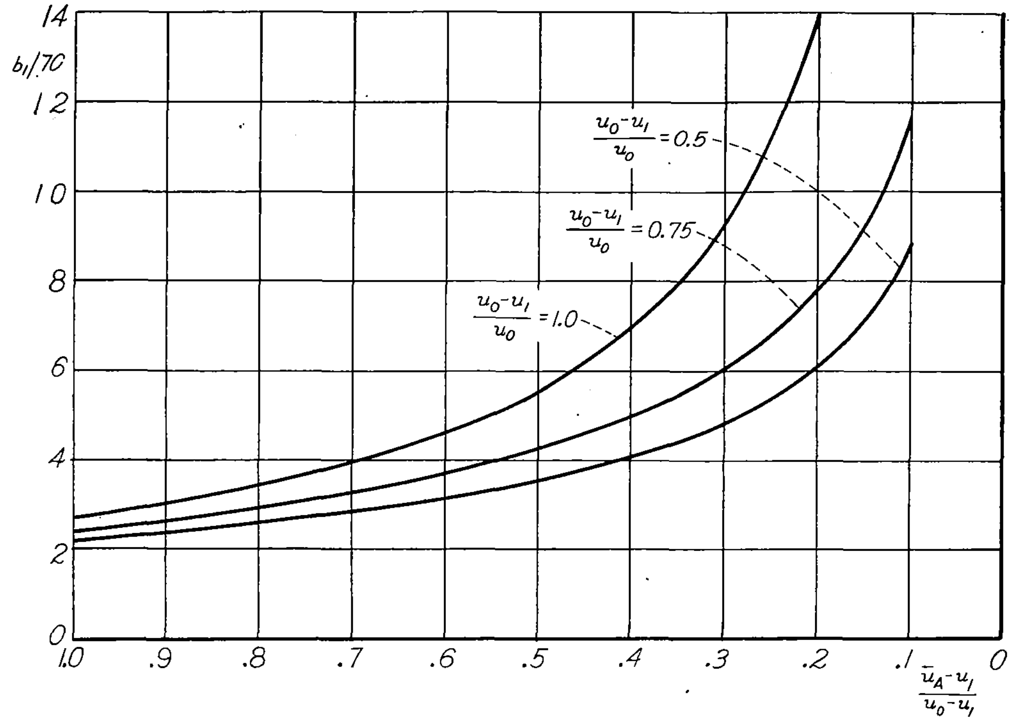

Figure 1.- Breadth of the mixing region of the velocity plotted against the velocity along the jet axis.

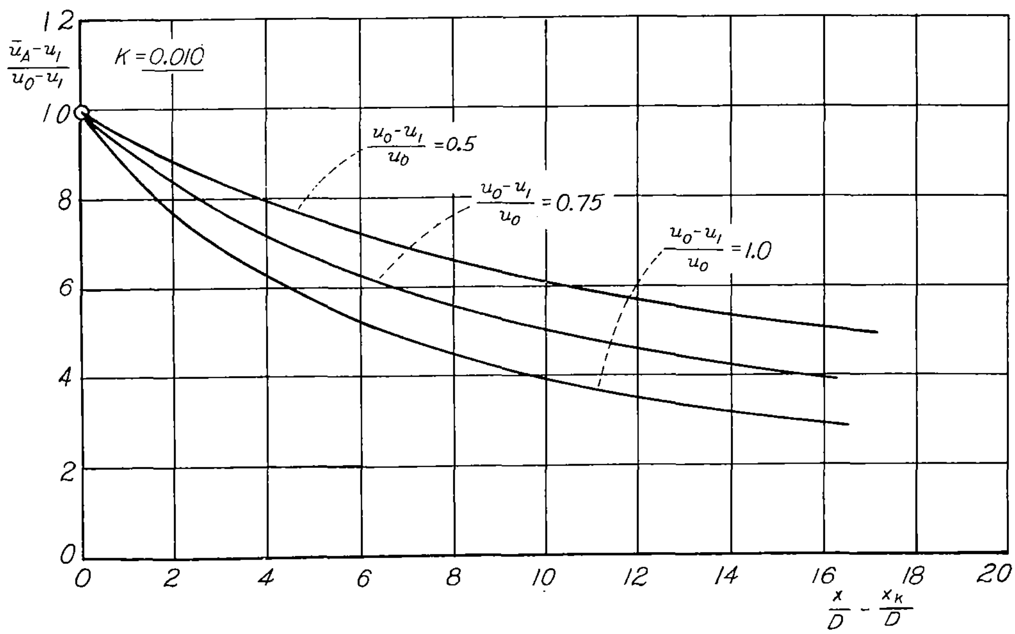

Figure 2.- Velocity decrease along jet axis.

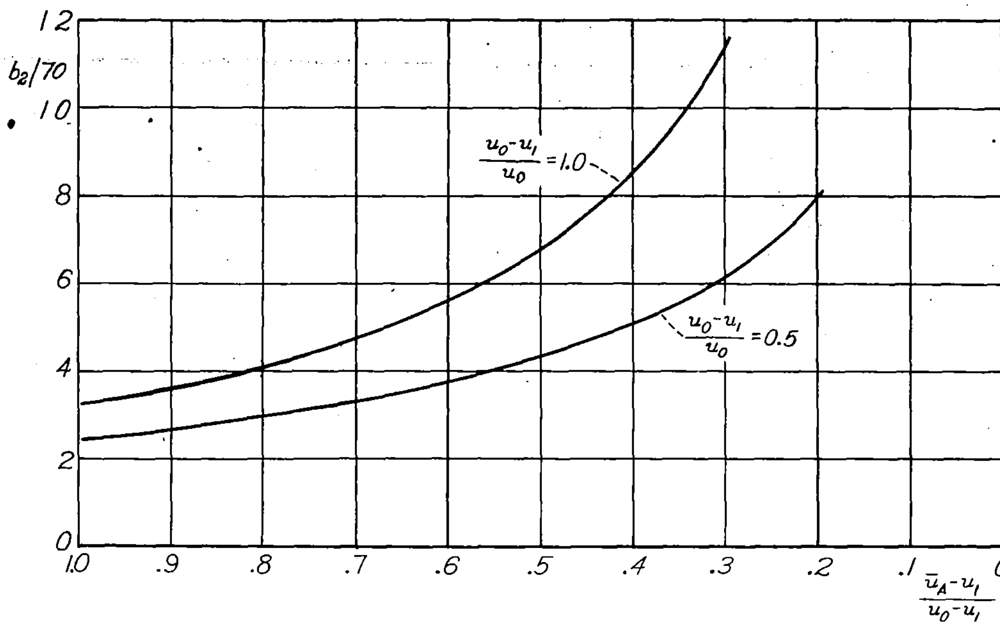

Figure 3.- Breadth of the mixing region of the temperature plotted against the velocity along the jet axis.

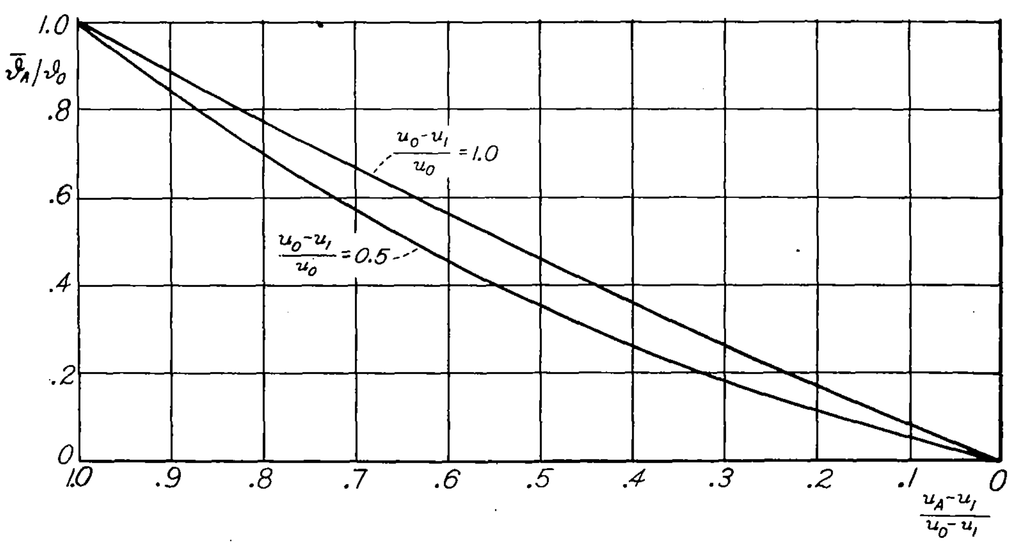

Figure 4.- Temperature drop along jet axis plotted against the velocity decrease along the jet axis.

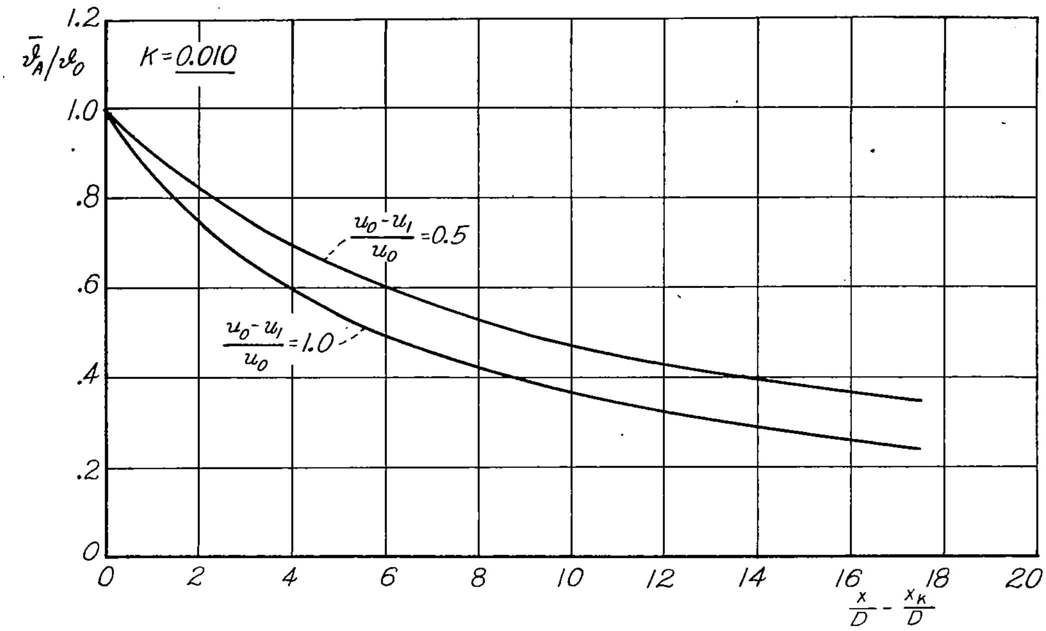

Figure 5.- Temperature drop along jet axis.

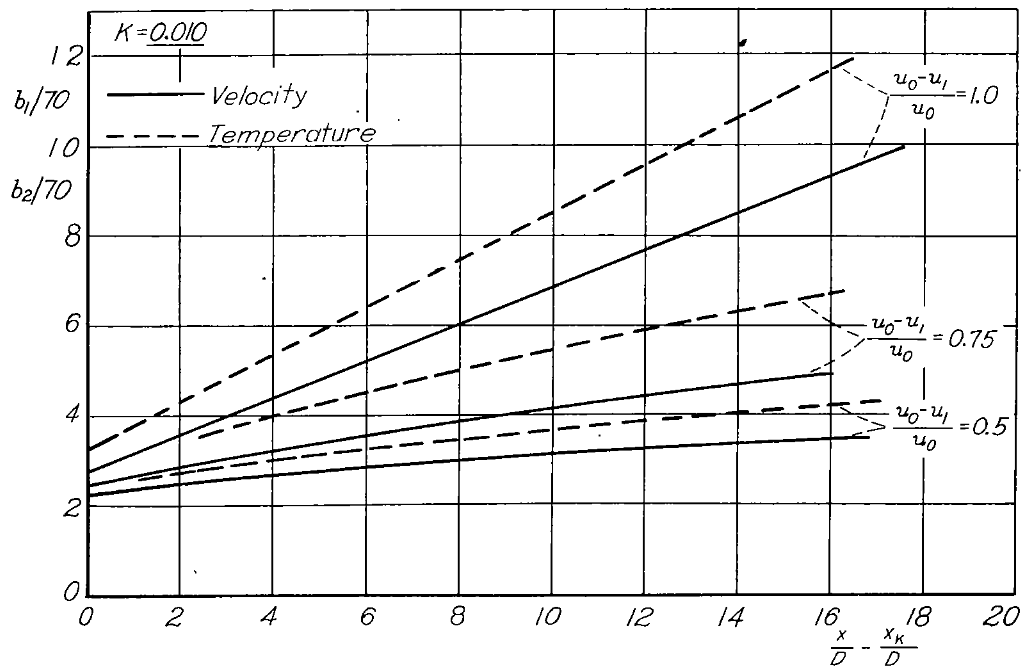

Figure 6.- Breadth of mixing regions of the velocity and the temperature.

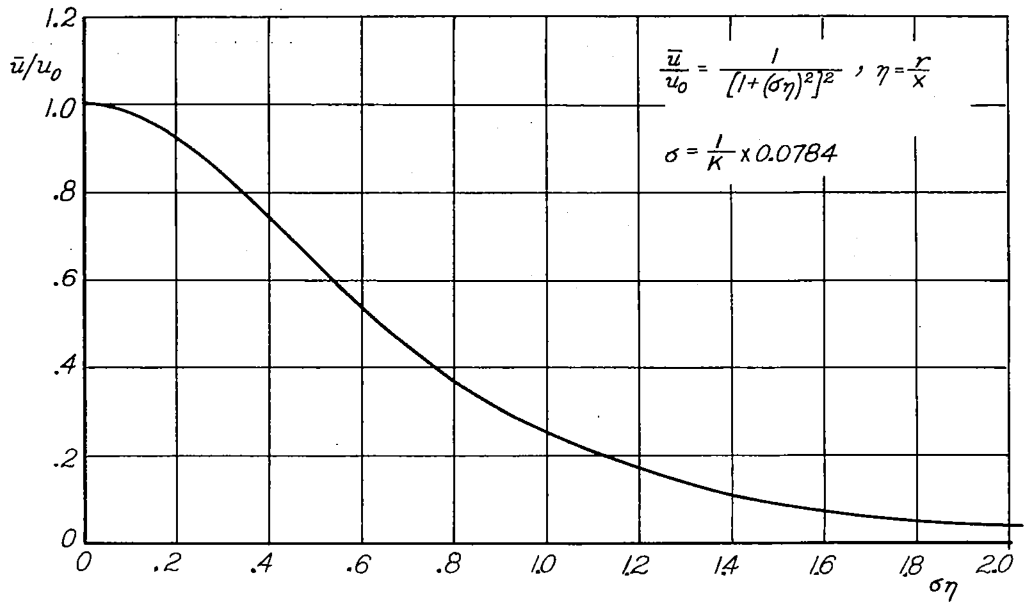

Figure 7.- Asymptotic distribution function for outside air at rest $(u_1 = 0)$.

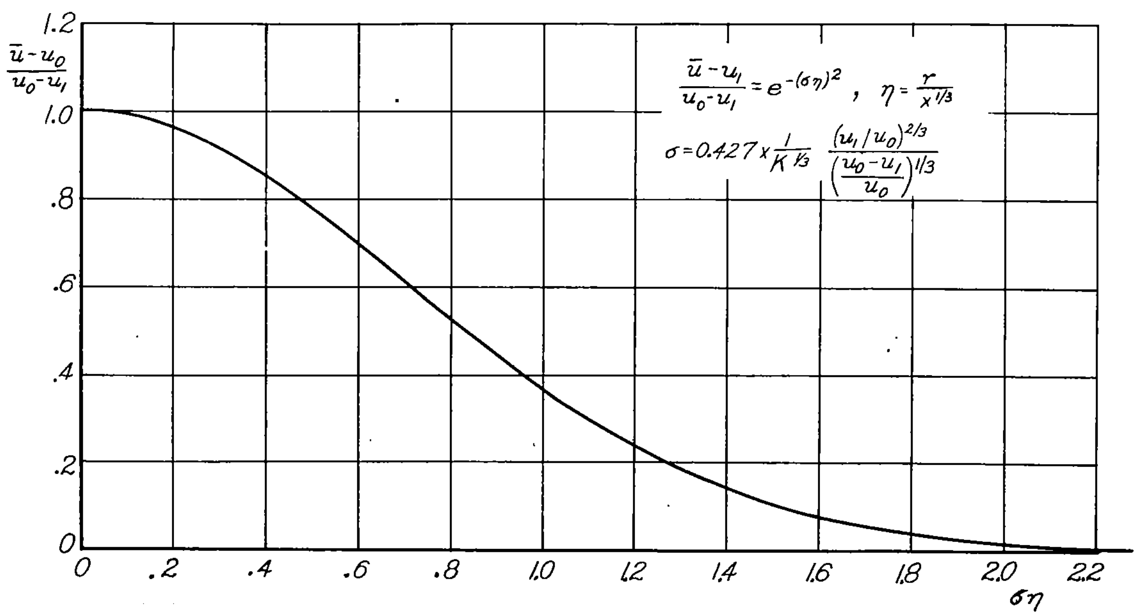

Figure 8.- Asymptotic distribution function for outside air in motion $(u_1 \neq 0)$.

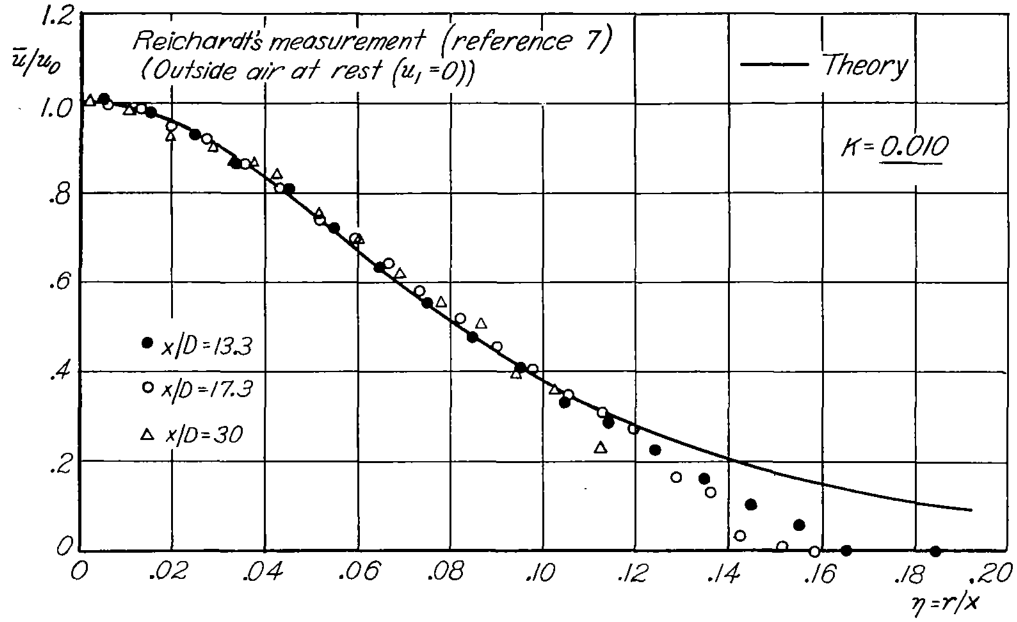

Figure 9.- Comparison of theoretical and experimental distribution function.

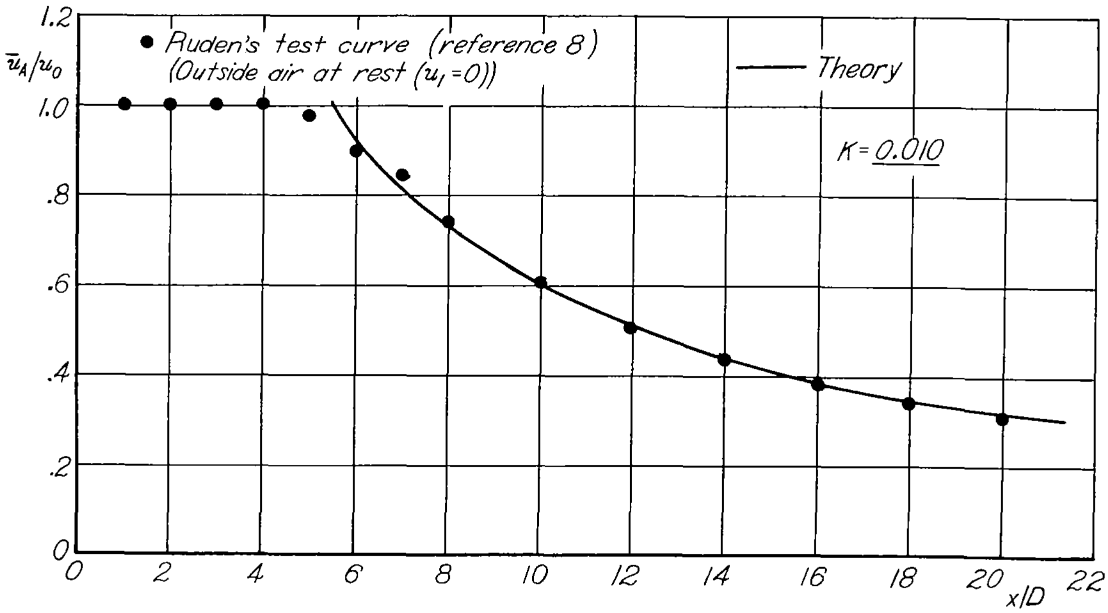

Figure 10.- Comparison of theoretical and experimental velocity decrease along jet axis.

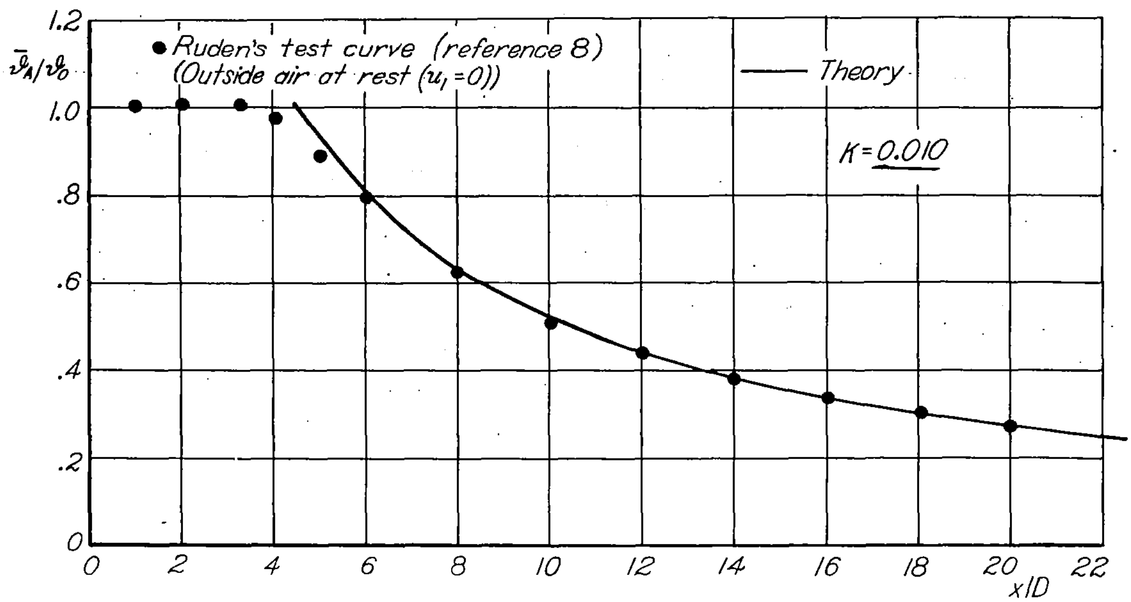

Figure 11.- Comparison of theoretical and experimental temperature drop along jet axis.

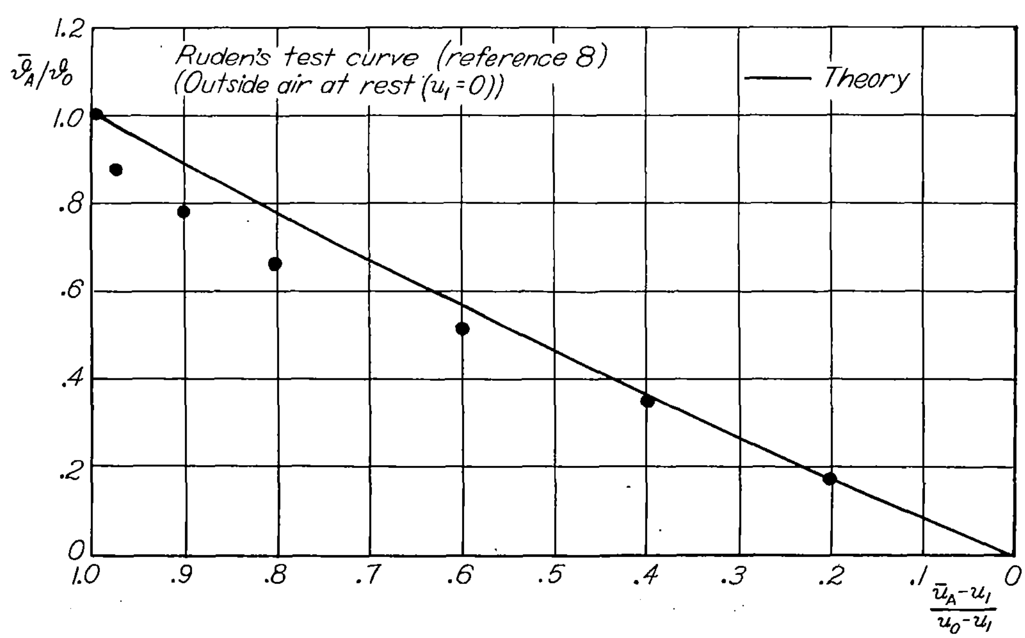

Figure 12.- Comparison of theoretical and experimental relationship between temperature drop and velocity decrease.

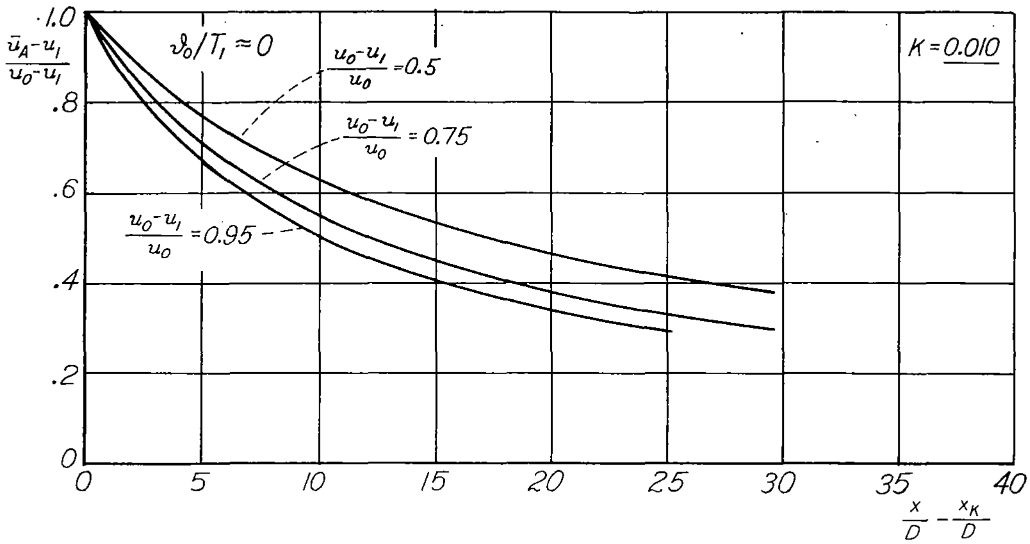

Figure 13.- Velocity decrease along jet axis for small density differences.

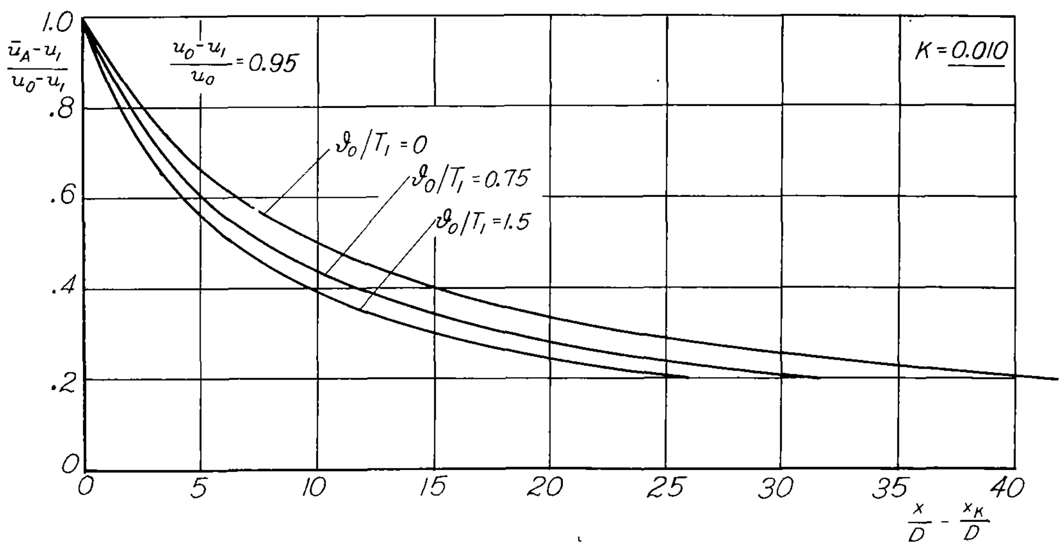

Figure 14.- Velocity decrease along jet axis for greater density differences.

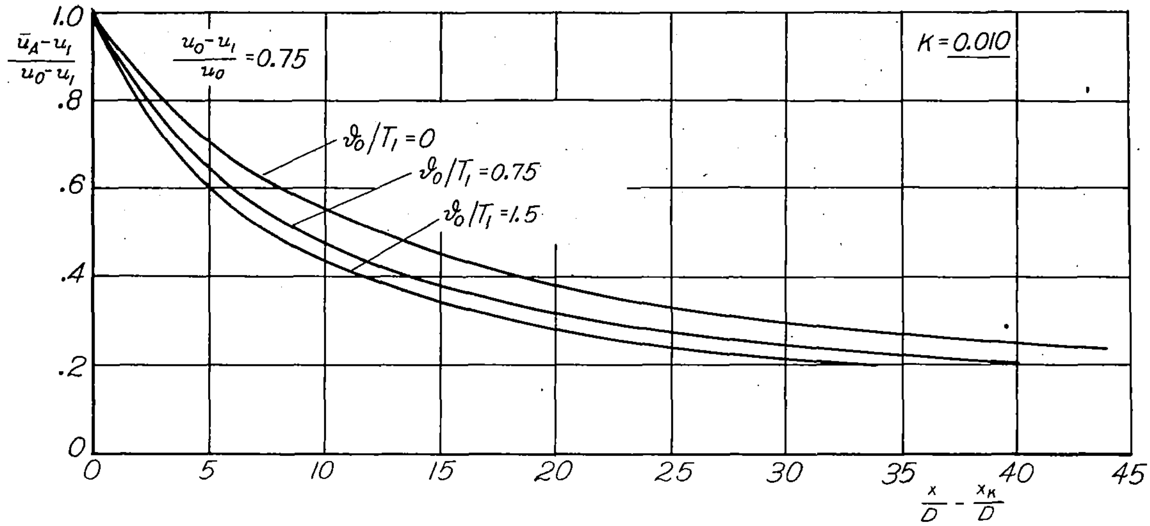

Figure 15.- Velocity decrease along jet axis for greater density differences.

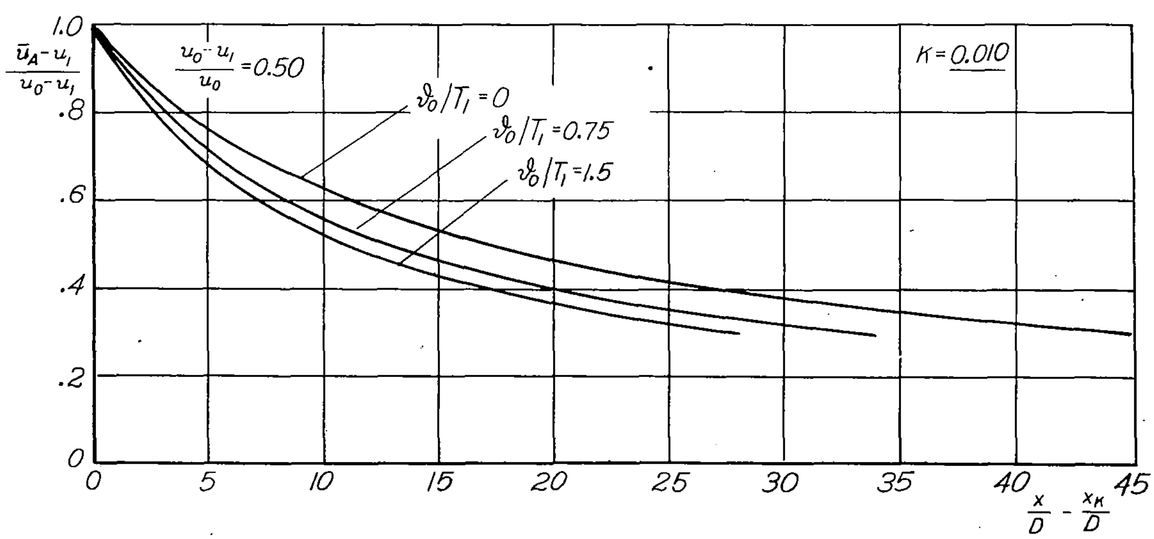

Figure 16.- Velocity decrease along jet axis for greater density differences.

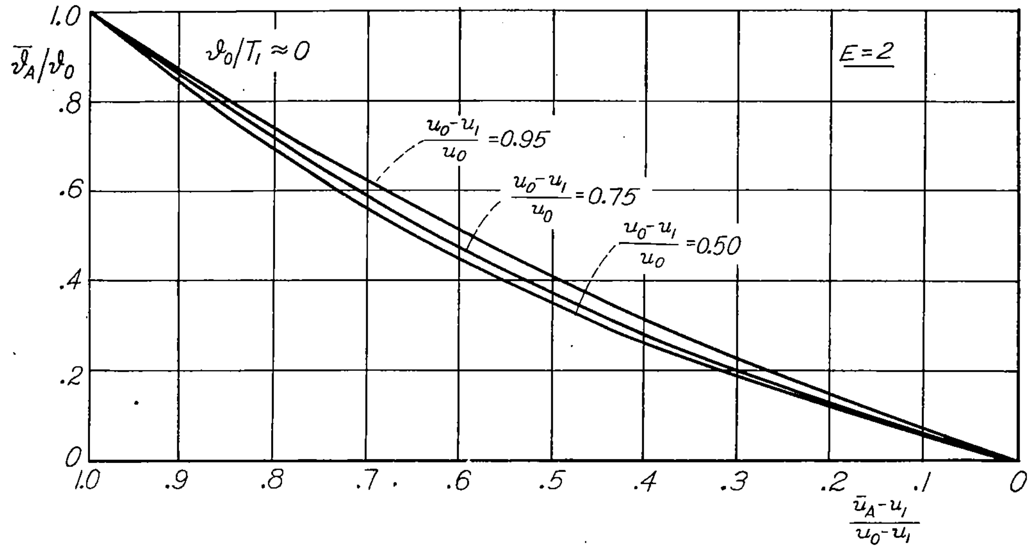

Figure 17.- Temperature on the jet axis plotted against the velocity on the jet axis.

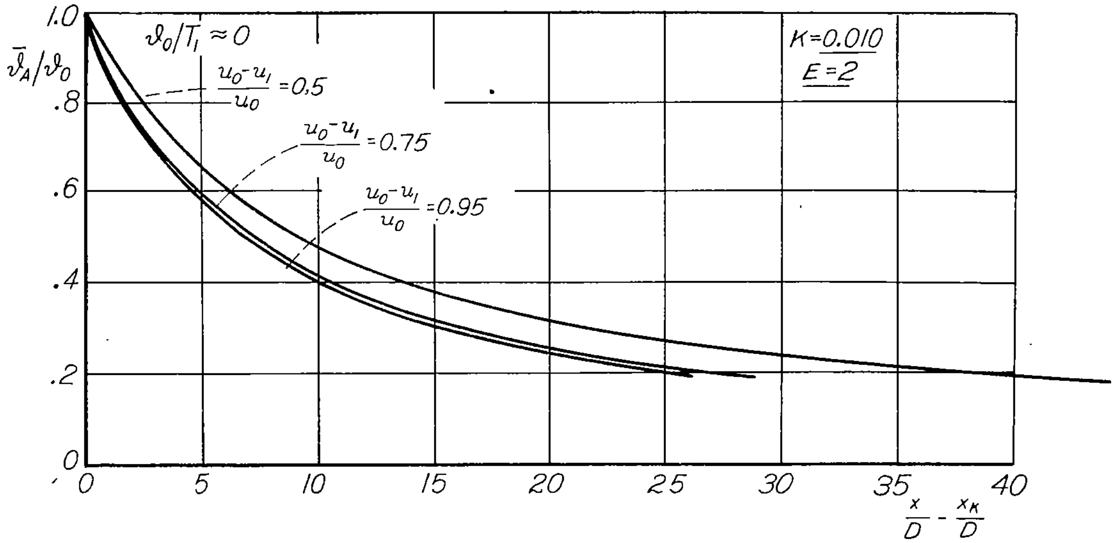

Figure 18.- Temperature drop along jet axis for small density differences.

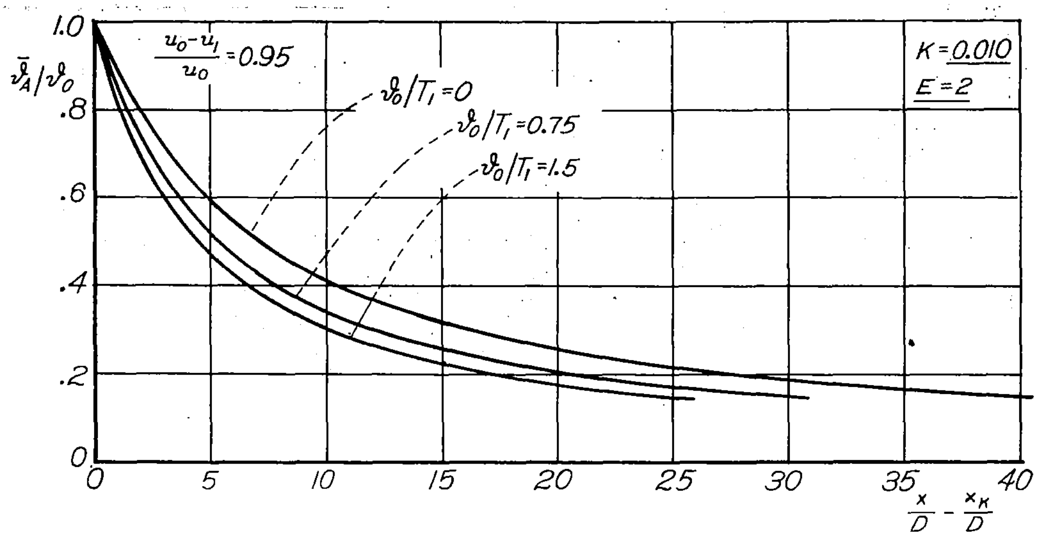

Figure 19.- Temperature drop along jet axis for greater density differences.

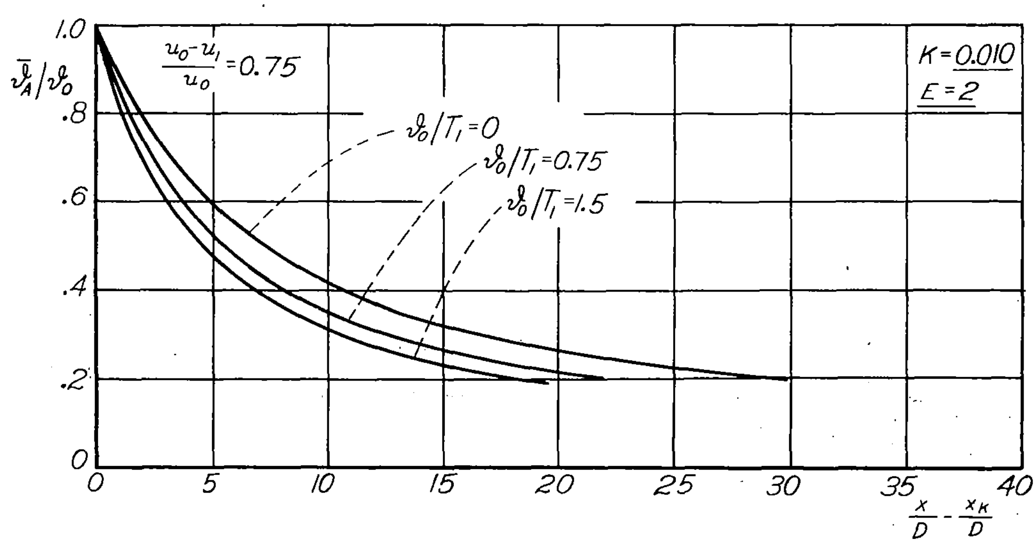

Figure 20.- Temperature drop along jet axis for greater density differences.

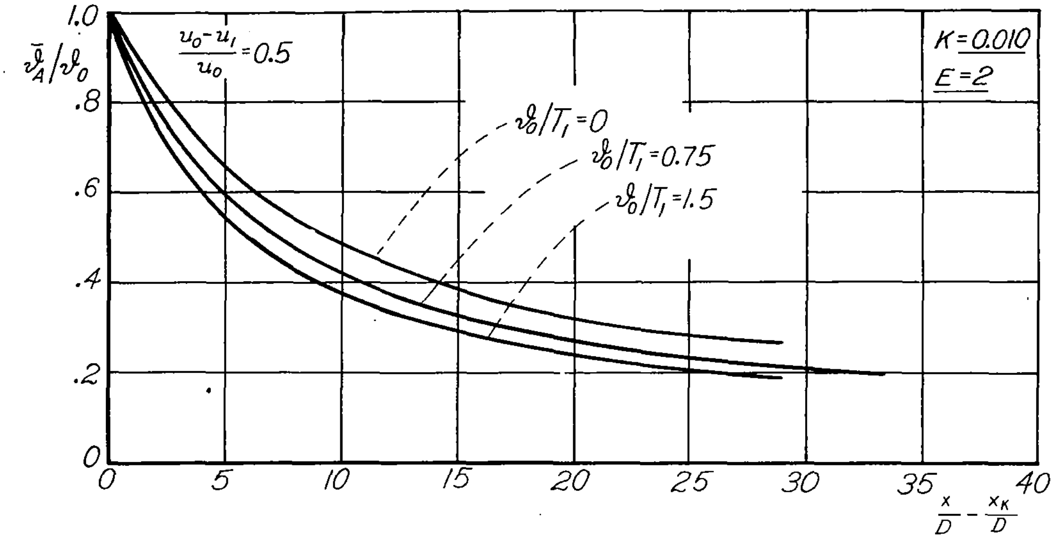

Figure 21.- Temperature drop along jet axis for greater density differences.

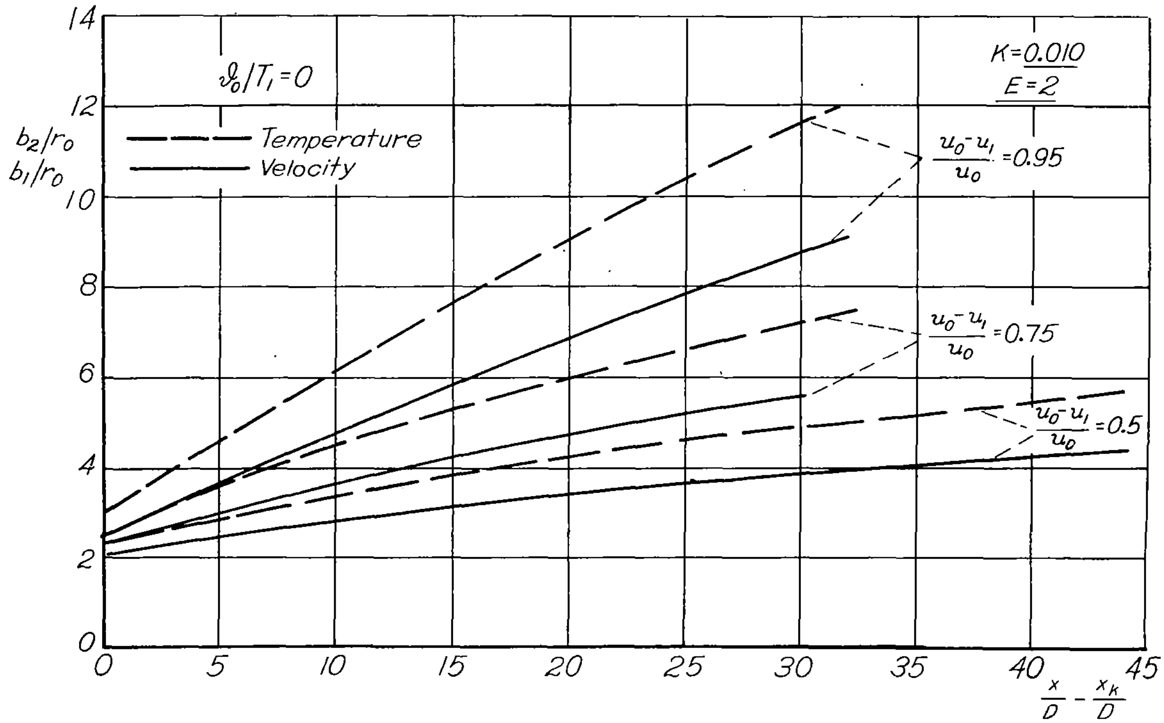

Figure 22.- Breadth of mixing along jet axis for small density differences.

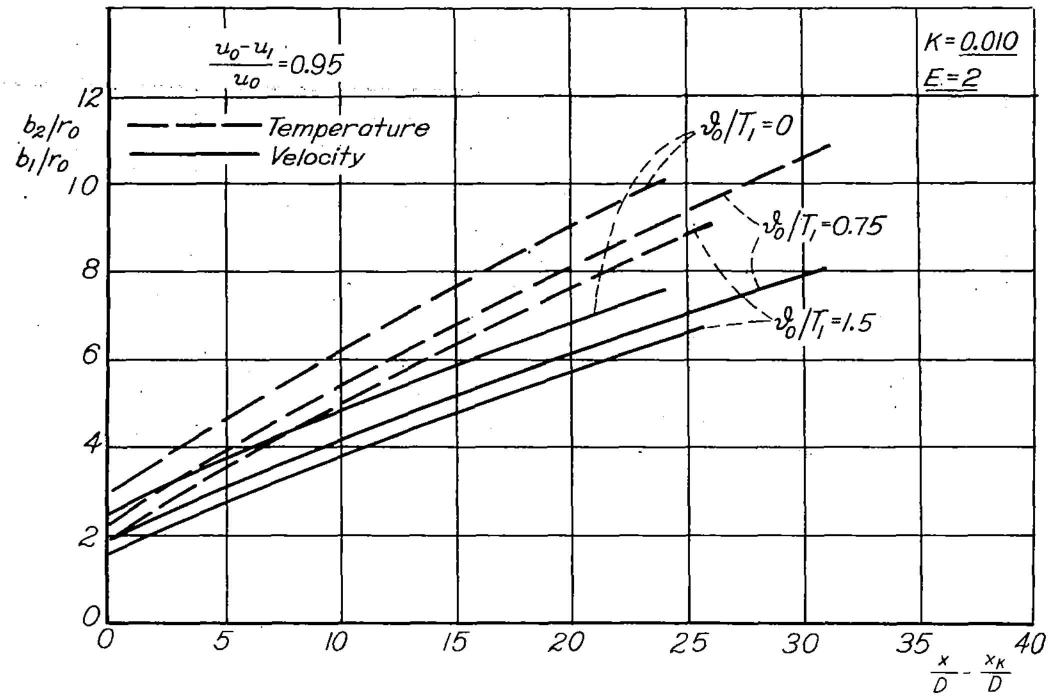

Figure 23.- Breadth of mixing along jet axis for greater density differences.

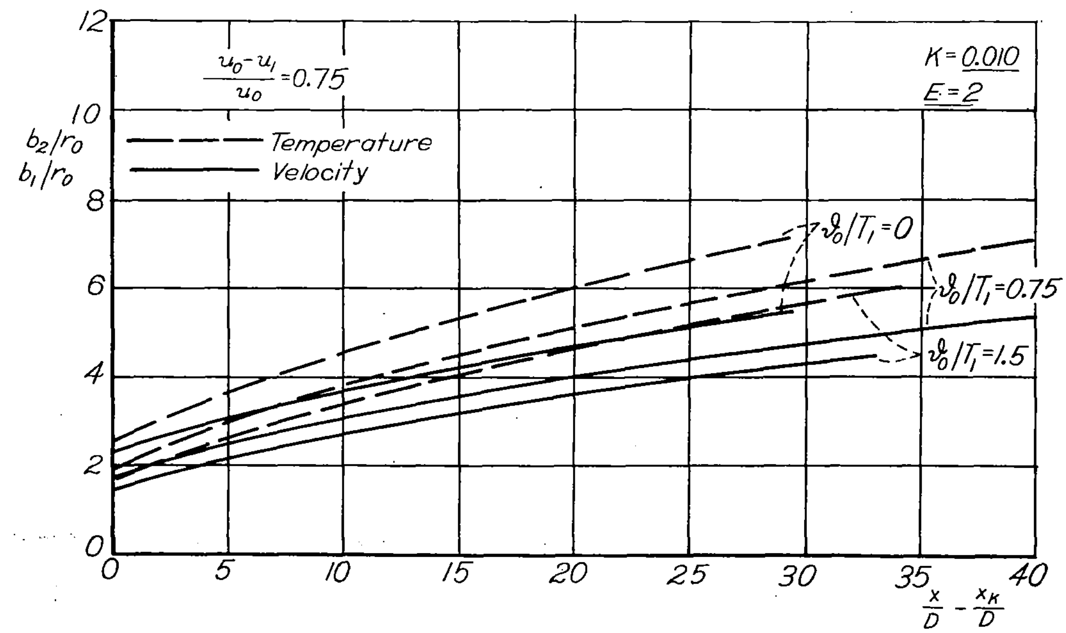

Figure 24.- Breadth of mixing along jet axis for greater density differences.

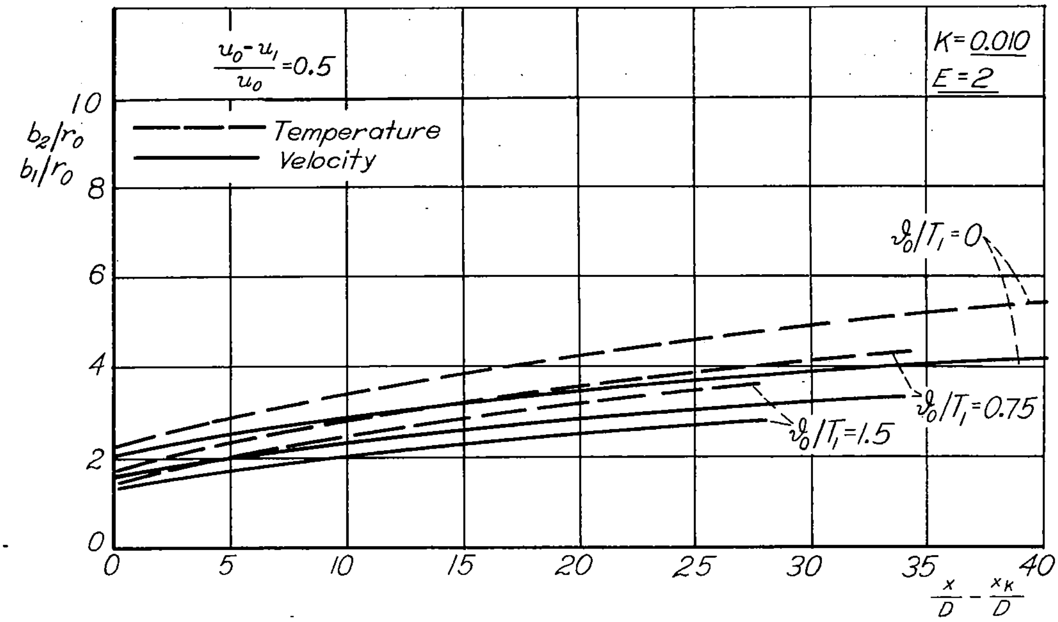

Figure 25.- Breadth of mixing along jet axis for greater density differences.

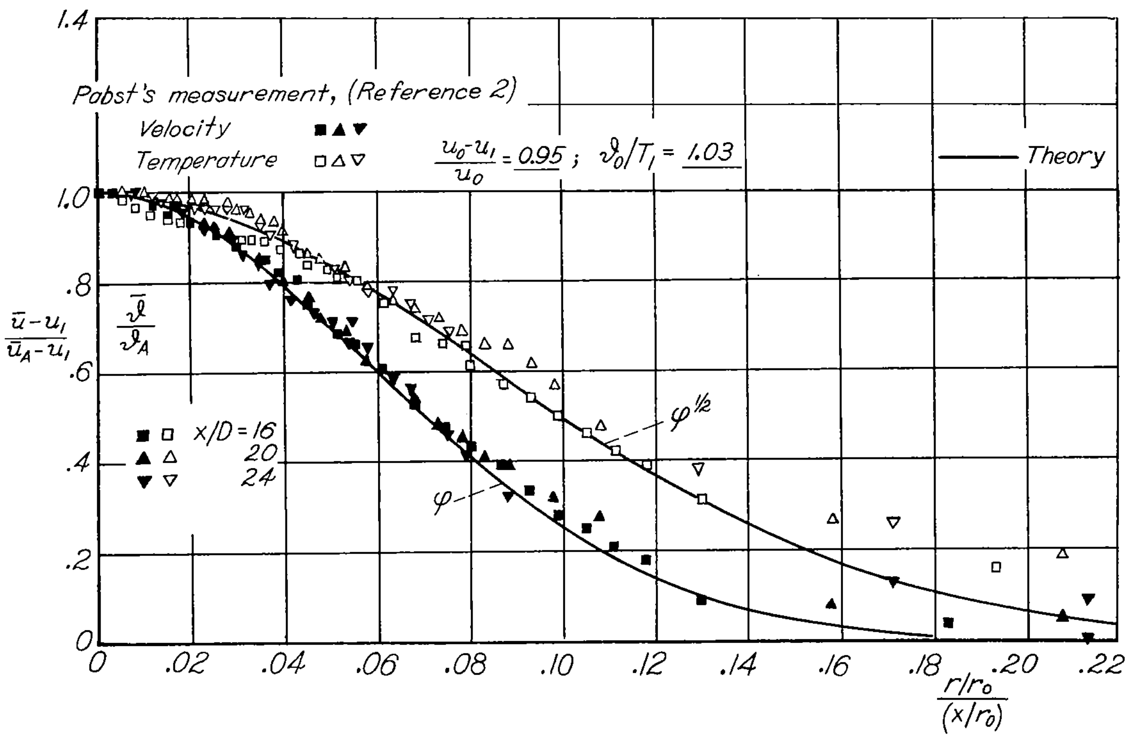

Figure 26.- Velocity and temperature distribution over the breadth of the mixing region.

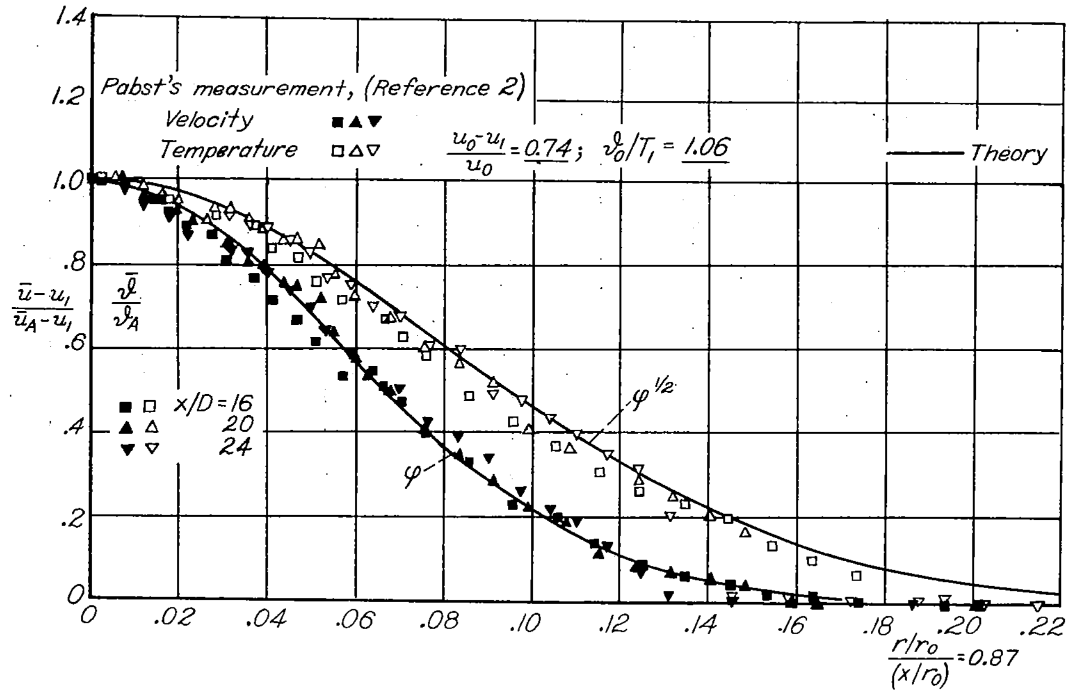

Figure 27.- Velocity and temperature distribution over the breadth of the mixing region.

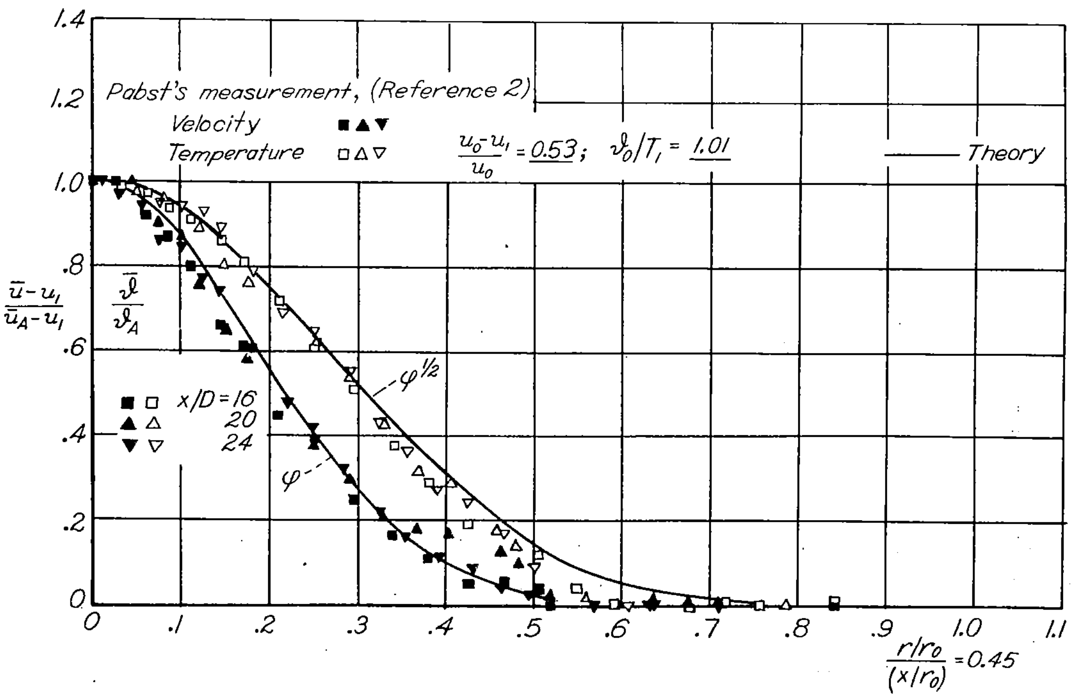

Figure 28.- Velocity and temperature distribution over the breadth of the mixing region.

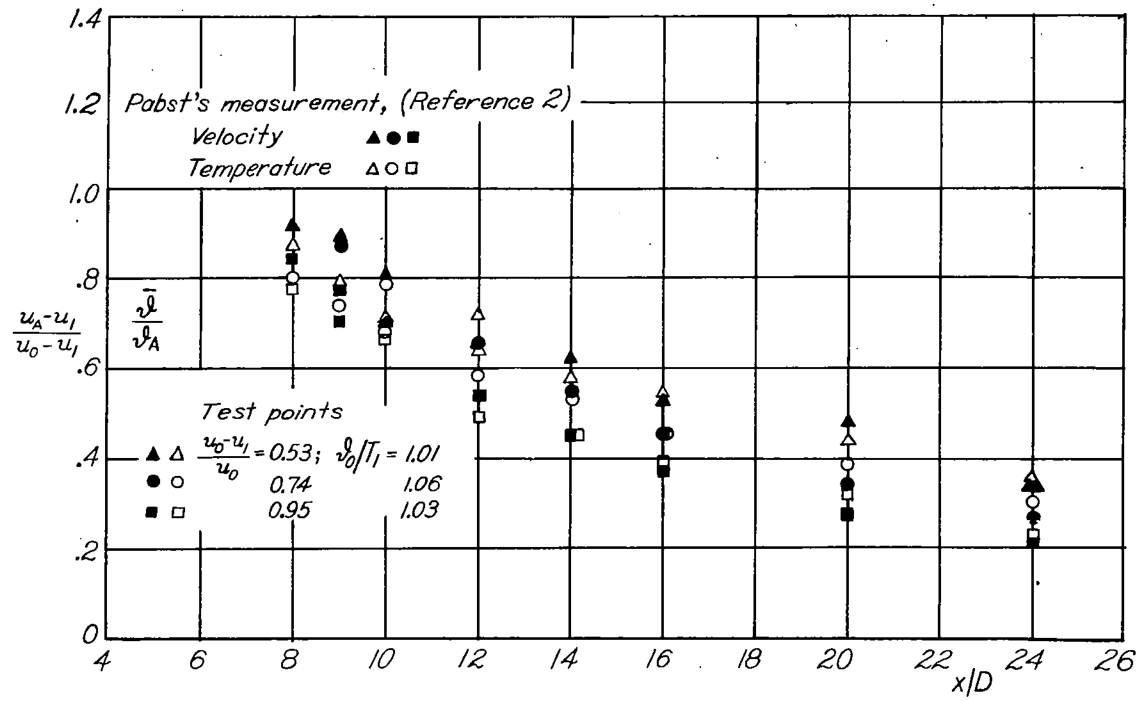

Figure 29.- Decrease of velocity and temperature along the jet axis.

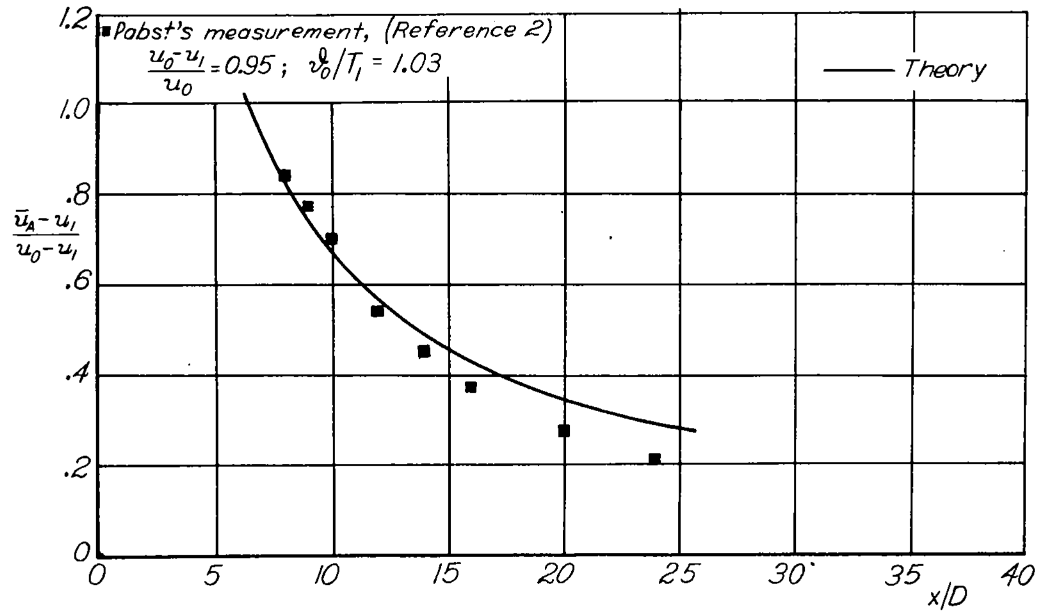

Figure 30.- Decrease of velocity along the jet axis.

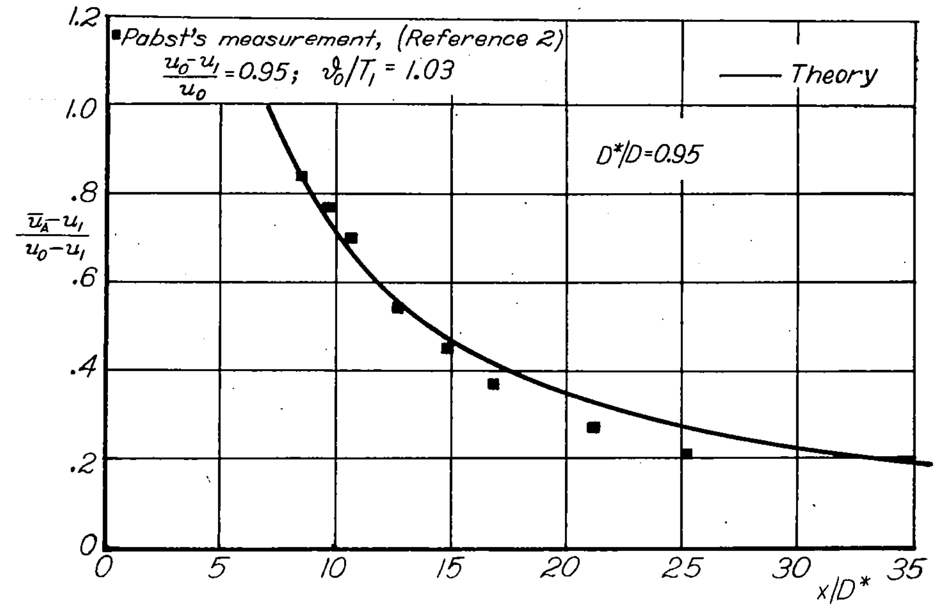

Figure 30a.- Comparison of theoretical and experimental velocity decrease along the jet axis with introduction of the effective nozzle diameter.

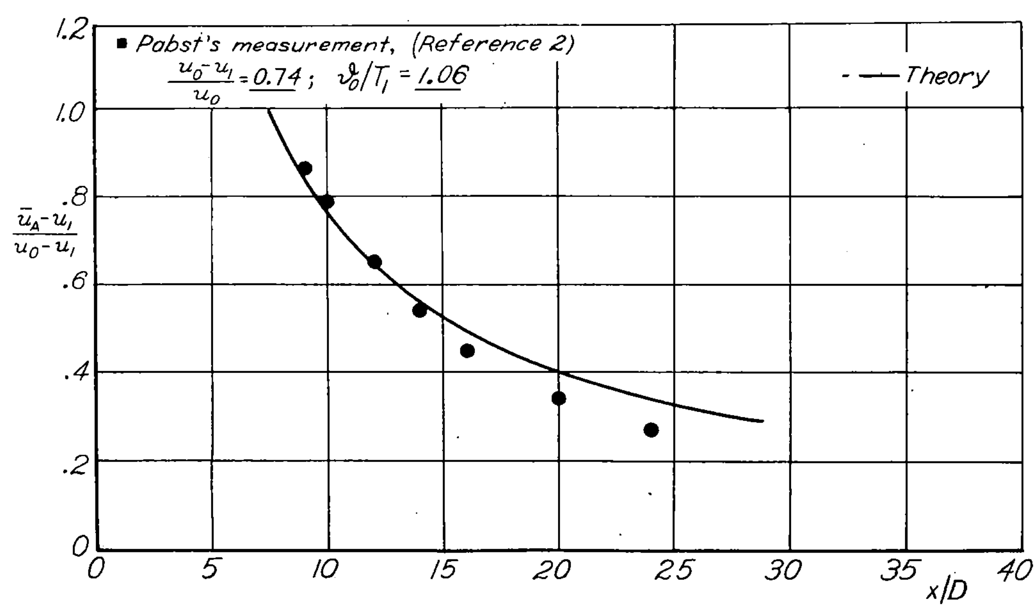

Figure 31.- Decrease of velocity along jet axis.

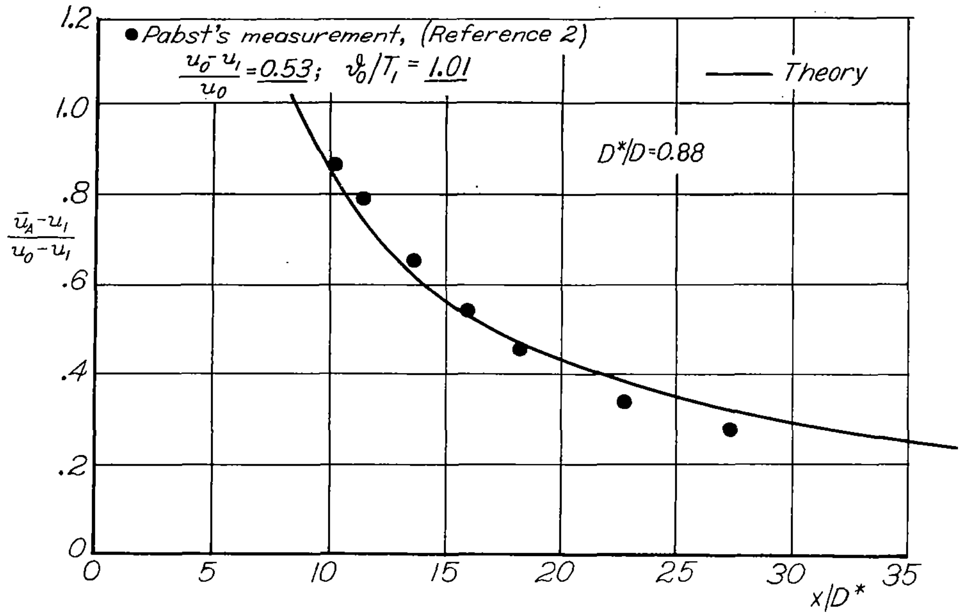

Figure 31a.- Comparison of theoretical and experimental decrease in velocity along the jet axis for effective nozzle diameter.

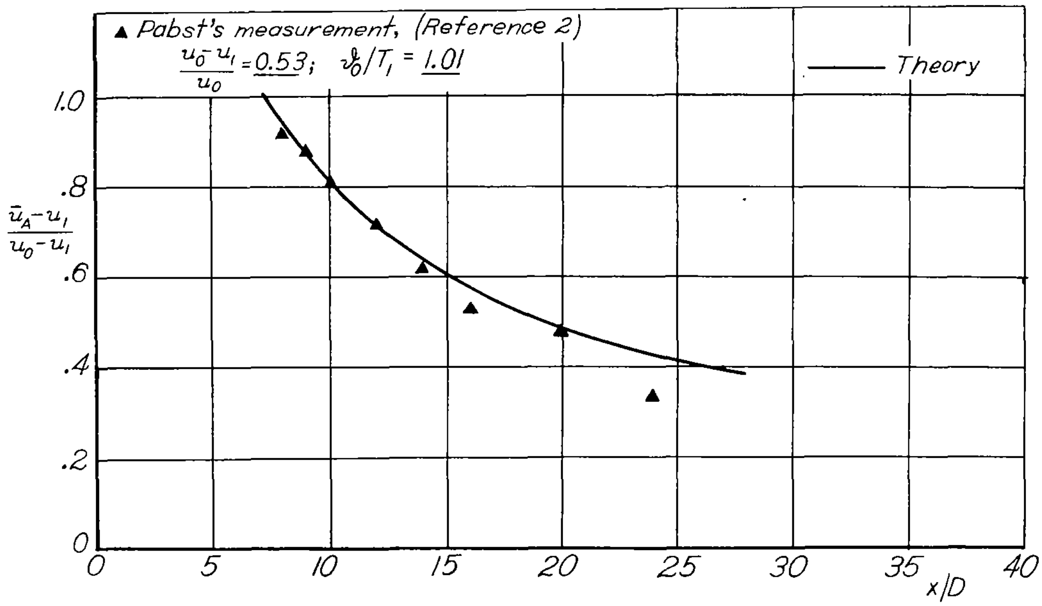

Figure 32.- Decrease of velocity along jet axis.

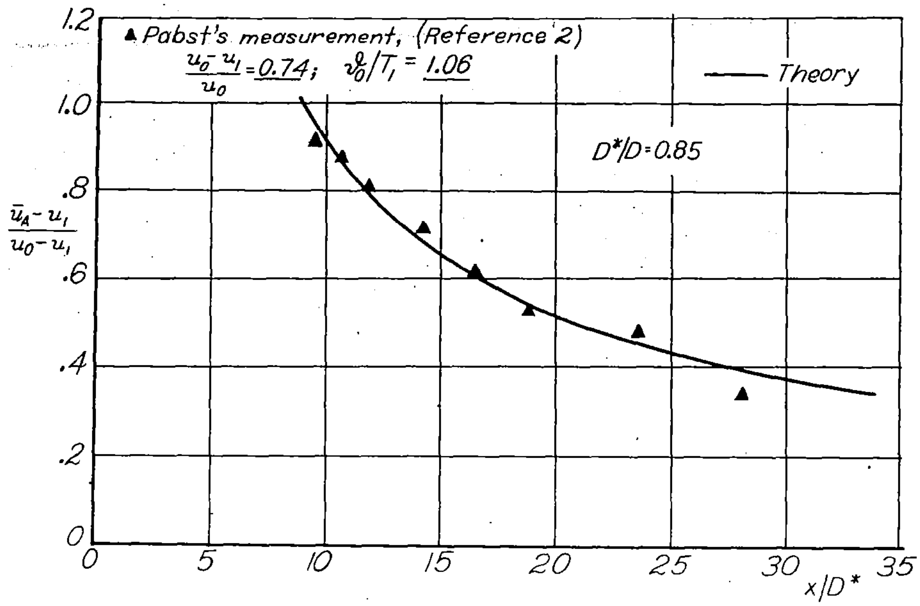

Figure 32a.- Comparison of theoretical and experimental velocity decrease along the jet axis for effective nozzle diameter.

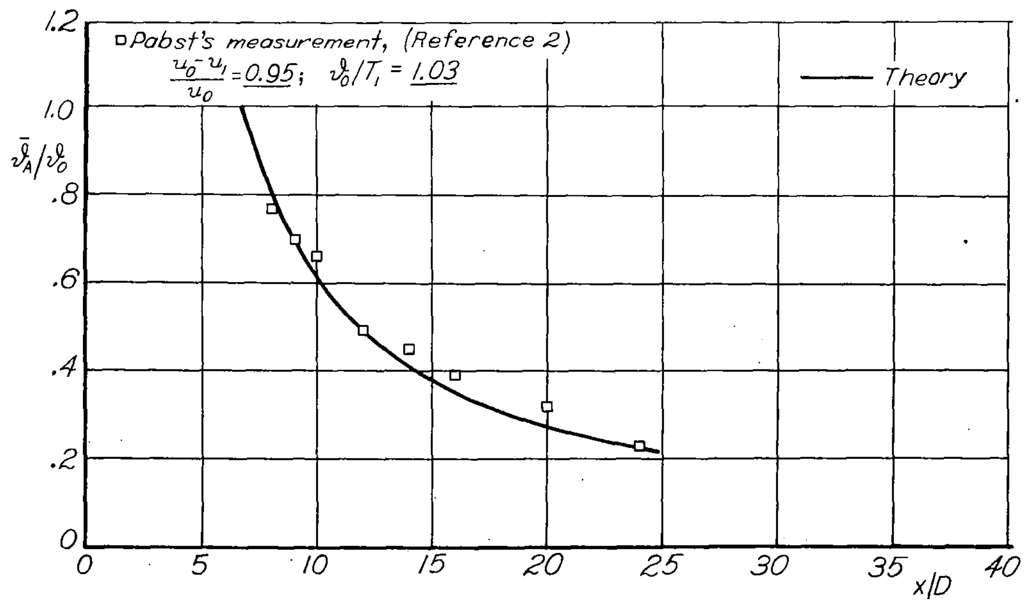

Figure 33.- Temperature drop along jet axis.

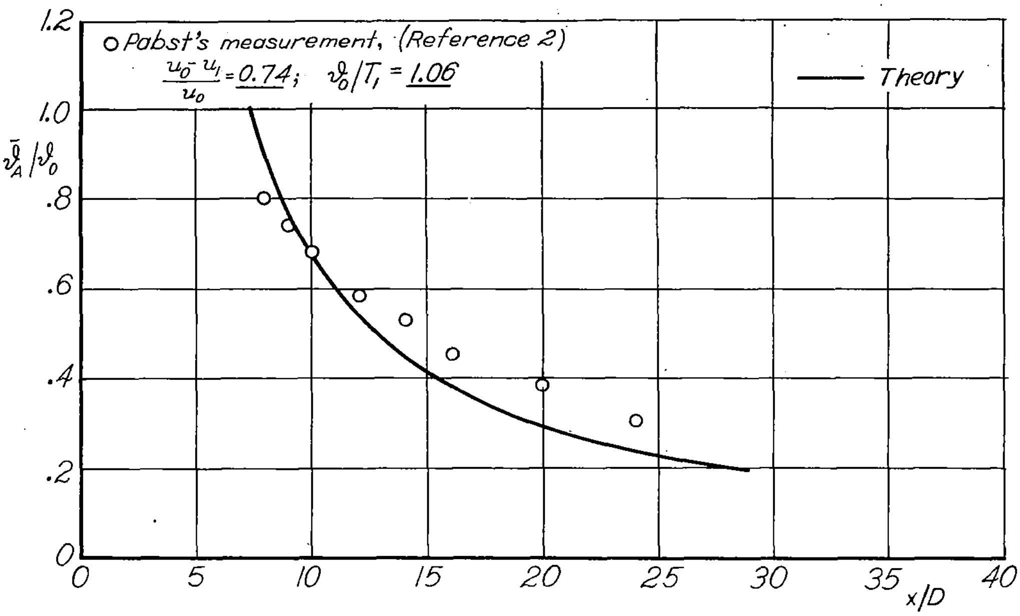

Figure 34.- Temperature drop along jet axis

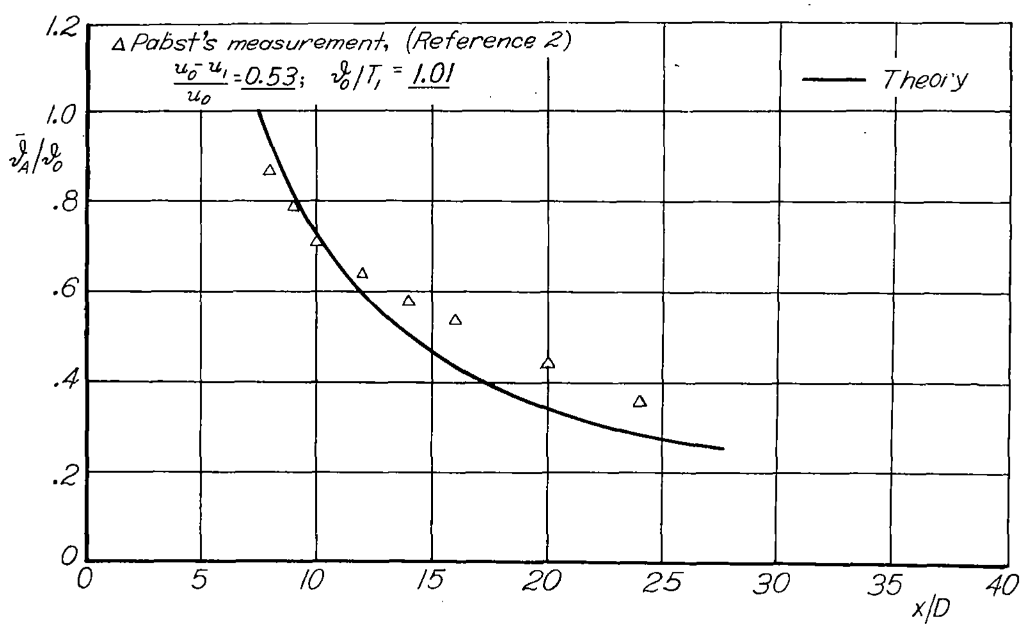

Figure 35.- Temperature drop along jet axis.