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STATICS OF CIRCULAR-RING STIFFENERS FOR MONOCOQUE FUSELAGES

By W. Stieda

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STATICS OF CIRCULAR-RING STIFFENERS FOR MONOCOQUE FUSELAGES*

By W. Stieda

For circular-ring stiffeners in monocoque fuselages the bending moments, axial forces, and shear forces under the action of applied external forces or a moment are accurately computed by known methods. Circular-ring stiffeners with variable moments of inertia are likewise considered. In comparison with the step-by-step and partially graphical procedure, the one here described is a more accurate and at the same time a simpler method.

I. GENERAL

For airplane pressure cabins, a circular-shaped fuselage cross section is generally chosen as most economical in weight. As will be shown in what follows, the internal pressure - which is the important factor in the design of stiffeners - gives rise, in the case of the circular cross section, to axial stresses only, which are taken up mainly by the adjacent skin. For a noncircular cross section, on the contrary, the stiffeners under internal pressure must also be designed for bending moments which lead to considerably increased weight.

In the present report, circular-ring stiffeners with constant and variable moments of inertia under the application of external forces are computed, the support given by the adjacent cylindrical fuselage skin being taken into account. For various cases of loading, the bending moments, axial forces, and shear forces for the ring are determined. The methods of Pohl and Wise (references 1 and 2) are used as basis for the computation. Both papers consider only circular rings with constant moments of inertia.

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*"Zur Statik von Kreisringspanten in Flugzeugdruckkabinen." Luftfahrtforschung, vol. 18, no. 6, June 30, 1941, pp. 214-222.

Pohl employs the elastic center, according to the method of Müller-Breslau for the determination of the static redundants, so that his procedure for variable moment of inertia is not directly applicable due to the displacement of the elastic center from the ring center. Wise dispenses with the simplified determination of the two or three static redundants by means of the elastic center but determines instead all nine or five displacement values δ_{ik}, respectively, according to Müller-Breslau. Since this method is laborious, he does not present the entire computational procedure. He also determines the axial and shear stresses for the three load cases: radial force, tangential force, and moment, for which Pohl determined only the bending moments. Wise, similarly, does not consider the case of variable moment of inertia in the ring stiffener.

In the present report, the two or three static redundants are determined from Castigliano's law of minimum work of deformation. Cases of arbitrarily variable moment of inertia can be treated without too much computation on the basis of the obtained derivative of the work with respect to the static redundants. A few examples are computed.

The equations for bending moment, axial and shear forces for constant moment of inertia, and for several cases of variable moments of inertia are computed for $\varphi = 15^\circ$ 30°, 45°, etc., and graphically represented, the method of representation of Pohl being used. In addition to the three principal load cases given by Pohl and Wise, namely, radial force, tangential force, and moment applied to the circular ring with tangential support, there is also considered the case of sine-shaped load distribution which is required for taking into account the air wake forces on the fuselage.

As in the papers of Pohl and Wise, the eccentricity between the supporting cylinder skin and center of gravity of the ring section is neglected since, with the usual construction of the fuselage stiffener, only a small error is involved.

II. INTERNAL PRESSURE FOR CIRCULAR CROSS SECTION

Consider a cylindrical fuselage under internal pressure. To determine the forces and moments, we make a cut along the top of the ring and introduce the three static redundants X_a, X_b, and X_c, of which the shear force X_b becomes zero on account of symmetry (figs. 1 and 2). The internal pressure p then produces at the statically determinate principal system, the following bending moments, upbending moments always being considered positive:

$$dP = p\,ds;\quad ds = r\,d\alpha$$

$$dM_{o\varphi} = r \sin(\varphi-\alpha)\, p\, r\, d\alpha$$

$$M_{o\varphi} = +\int_o^{\varphi} r \sin(\varphi-\alpha)\, p\, r\, d\alpha$$

$$= +\,p\,r^2 - p\,r^2\cos\varphi \quad (1)$$

Further, we have for $X_a = -1$ (fig. 2):

$$M_a = +\,(r - r\cos\varphi) \quad (2)$$

and for $X_c = -1$:

$$M_c = +\,1 \quad (3)$$

We now determine, as in all the other load cases, the static redundants, according to the Castigliano principle of minimum work of deformation.

$$M = M_0 - X_a\cdot M_a - X_c\cdot M_c$$
$$= p\,r^2 - p\,r^2\cos\varphi - X_a\cdot(r - r\cos\varphi) - X_c;$$
$$\frac{\partial M}{\partial X_a} = -(r - r\cos\varphi);\quad \frac{\partial M}{\partial X_c} = -1;$$
$$\frac{\partial A}{\partial X_a} = 0 = \int \frac{M}{EJ}\cdot\frac{\partial M}{\partial X_a}\cdot ds;$$

$$\frac{\partial A}{\partial X_a} = 0 = -\int_0^{2\pi}(p\,r^2 - p\,r^2\cos\varphi - X_a(r - r\cos\varphi) - X_c)\cdot(r - r\cos\varphi)\,r\,d\varphi$$
$$= \Big[2\,p\,r^2\sin\varphi - \frac{p\,r^2}{4}\sin 2\varphi - \frac{3}{2}\,p\,r^2\varphi$$
$$- 2\,X_a\cdot r\sin\varphi + \frac{3}{2}\,X_a\cdot r\cdot\varphi + X_a\cdot\frac{r}{4}\sin 2\varphi$$
$$- X_c\cdot\sin\varphi + X_c\cdot\varphi\Big]_0^{2\pi} \quad \ldots\ldots\ldots\ldots (4)$$
$$= -3\,r^2\,p + 3\,r\,X_a + 2\,X_c; \quad \ldots\ldots\ldots\ldots (5)$$

$$\frac{\partial A}{\partial X_c} = 0 = \int \frac{M}{EJ}\cdot\frac{\partial M}{\partial X_c}\cdot ds$$
$$= -\int_0^{2\pi}(p\,r^2 - p\,r^2\cos\varphi - X_a\cdot r + X_a\cdot r\cos\varphi - X_c)\,r\,d\varphi$$
$$= \Big[p\,r^2\cdot\varphi - p\,r^2\sin\varphi - X_a\cdot r\cdot\varphi + X_a\cdot r\sin\varphi - X_c\cdot\varphi\Big]_0^{2\pi}$$
$$\ldots\ldots (6)$$
$$= +2\,r^2\,p - 2\,r\,X_a - 2\,X_c \quad \ldots\ldots\ldots\ldots (7)$$

From equations (5) and (7), we obtain:

$$X_a = p\cdot r;\quad X_c = 0;$$
$$M = p\,r^2 - p\,r^2\cos\varphi - p\,r\,(r - r\cos\varphi) = 0!$$

For uniform internal loading, therefore, of a cross section, no bending moments occur in the ring stiffener - only axial forces N = p r = const, which remain essentially in the cylinder skin. The initial assumption that the internal pressure is transmitted from the cylinder skin to the ring stiffener thus holds only to a small extent.

We next consider the case that the ring stiffener has variable moment of inertia, the ring as previously being loaded by internal pressure (fig. 3).

$$J_1 = n\,J_2$$

$$J_1 = J_c;\quad \frac{J_c}{J_2} = n;\quad \frac{J_c}{J_1} = 1$$

Integrating equations (4) and (6) between the limits 0 to π and n times the interval from π to 2π:

$$\frac{\partial A}{\partial X_a} = 0 = \Big[-\frac{3}{2}\,p\,r^2\pi + \frac{3}{2}\,X_a\,r\pi + X_c\cdot\pi\Big] + n\Big[-\frac{3}{2}\,p\,r^2\pi + \frac{3}{2}\,X_a\cdot r\pi + X_c\cdot\pi\Big];$$

$$\frac{\partial A}{\partial x_c} = 0 = [p\,r^2\pi - X_a\cdot r\cdot\pi - X_c\cdot\pi] + n\,[p\,r^2\pi - X_a\cdot r\pi - X_c\cdot\pi].$$

Adding the equations:

$$0 = -\frac{3}{2}\,p\,\pi\,r^2\,(n+1) + \frac{3}{2}\,X_a\cdot r\pi\,(n+1) + X_c\cdot\pi\,(n+1) + p\,r^2\pi\,(n+1) - X_a\cdot r\pi\,(n+1) - X_c\cdot\pi\,(n+1);$$

$$0 = \Big(-\frac{n}{2} - \frac{1}{2}\Big)(p\cdot r - X_a);\quad X_a = p\cdot r;\quad X_c = 0.$$

For the case also of variable moment of inertia of the circular ring, no bending moments arise - only axial forces. This proof can readily be extended to any discontinuously variable moment of inertia.

III. MOMENT ON A TANGENTIALLY SUPPORTED CIRCULAR RING

The shear flow in the skin for an applied moment is (fig. 4):

$$t = \frac{M_d}{2\pi r^2} = \text{const} \qquad (8)$$

To determine the three static redundants, we again make a top cut on the ring. The moment M_o due to M_d and the shear flow t, corresponding to figure 4, are determined as follows: A load element $t\,r\,d\alpha$ produces at the point defined by the angle φ a moment:

$$dM_{0\varphi} = t(r - r\cos(\varphi-\alpha))\,r\,d\alpha \text{ (positive)}$$

$$M_{0\varphi} = tr^2\int_0^{\varphi}(1-\cos(\varphi-\alpha))\,d\alpha$$

$$= tr^2(\varphi - \sin\varphi) \text{ in the 1st and 2nd quadrants} \quad (9)$$

In the third and fourth quadrants, we substitute in place of φ the value $(2\pi - \varphi)$ and therefore obtain a down-bending - hence negative moment:

$$M_{0\varphi} = -tr^2(2\pi - \varphi - \sin(2\pi-\varphi))$$

$$= tr^2(\varphi - \sin\varphi - 2\pi) \quad \ldots\ldots (10)$$

The moments of the three static unknowns are (fig. 5):

$M_a = +(r - r\cos\varphi)$;

$M_b = +r\sin\varphi$; $M_c = +1$

We thus obtain:

$$\frac{\partial M}{\partial X_a} = -(r - r\cos\varphi);\quad \frac{\partial M}{\partial X_b} = -r\sin\varphi;\quad \frac{\partial M}{\partial X_c} = -1;$$

$$\frac{\partial A}{\partial X_a} = 0 = \int \frac{M}{EJ}\cdot\frac{\partial M}{\partial X_a}\cdot r\,d\varphi$$

$$= -\int_0^{2\pi}(t r^2\varphi - t r^2\sin\varphi - X_a\cdot r + X_a\cdot r\cos\varphi - X_b\,r\sin\varphi - X_c)(r - r\cos\varphi)\,r\,d\varphi$$

$$-\int_{\pi}^{2\pi}(-t r^2 2\pi)(r - r\cos\varphi)\,r\,d\varphi$$

$$= \left[\frac{tr^2}{2}\left(\varphi^2 - \varphi\sin\varphi + \frac{1}{2}\sin^2\varphi\right)\right.$$

$$\left. - X_a\cdot r\cdot\frac{3}{2}\varphi + X_a\cdot r\left(2\sin\varphi - \frac{1}{4}\sin 2\varphi\right)\right.$$

$$\left. + X_b\cdot r\left(\cos\varphi + \frac{1}{2}\sin 2\varphi\right) - X_c(\varphi + \sin\varphi)\right]_0^{2\pi}$$

$$+\left[t r^2 2\pi(\sin\varphi - \varphi)\right]_{\pi}^{2\pi} \quad \ldots\ldots (11)$$

$$\frac{\partial A}{\partial X_a} = 0 = -3X_a\cdot r - 2X_c; \quad \ldots\ldots (12)$$

$$\frac{\partial A}{\partial X_b} = 0 = -\int_0^{2\pi}(t r^2\varphi - t r^2\sin\varphi - X_a\cdot r + X_a\cdot r\cos\varphi - X_b\,r\sin\varphi - X_c)\,r\sin\varphi\,r\,d\varphi$$

$$-\int_{\pi}^{2\pi}(-t r^2 2\pi)\,r\sin\varphi\,r\,d\varphi$$

$$= \left[t r^2\left(\sin\varphi - \varphi\cos\varphi + \frac{1}{4}\sin 2\varphi - \frac{\varphi}{2}\right)\right.$$

$$\left. + X_a\cdot r\left(\cos\varphi + \frac{1}{2}\sin^2\varphi\right) + X_b\cdot r\left(\frac{1}{4}\sin 2\varphi - \frac{\varphi}{2}\right)\right.$$

$$\left. + X_c\cdot\cos\varphi\right]_0^{2\pi} + \left[t r^2 2\pi\cos\varphi\right]_{\pi}^{2\pi} \quad \ldots\ldots (13)$$

$$= +t\pi r^2 - X_b\pi r \quad \ldots\ldots (14)$$

$$X_b = +tr = +\frac{Md}{2\pi r};$$

$$\frac{\partial A}{\partial X_c} = 0 = -\int_0^{2\pi}(t r^2\varphi - t r^2\sin\varphi - X_a\cdot r + X_a\cdot r\cos\varphi - X_b\cdot r\sin\varphi - X_c)\,r\,d\varphi$$

$$-\int_{\pi}^{2\pi}(-t r^2 2\pi)\,r\,d\varphi$$

$$= \left[t r^2\frac{\varphi^2}{2} + t r^2\cos\varphi - X_a\cdot r\varphi\right.$$

$$\left. + X_a\cdot r\sin\varphi + X_b\cdot r\cos\varphi - X_c\cdot\varphi\right]_0^{2\pi}$$

$$+\left[-t r^2 2\pi\varphi\right]_{\pi}^{2\pi} \quad \ldots\ldots (15)$$

$$= -2X_a\cdot r - 2X_c \quad \ldots\ldots (16)$$

From equations (12) and (16) there follows $X_a = 0$, $X_c = 0$. In the first and second quadrants there is obtained:

$$M = \frac{M_d}{2\pi}(\varphi - 2\sin\varphi) \qquad (17)$$

and in the third and fourth quadrants:

$$M = \frac{M_d}{2\pi}(\varphi - 2\sin\varphi - 2\pi) \qquad (18)$$

The obtained momentum curve is at the same time the line of influence for the bending moment due to a displaced load moment.

We shall also determine the axial and shear forces in the circular ring stiffener due to the moment M_d (fig. 6).

$$N = N_o - N_b X_b$$

$d N_o = t\, r \cos(\varphi-\alpha)\, d\alpha$ (compression)

$$N_o = -\int_o^{\varphi} t\, r \cos(\varphi-\alpha)\, d\alpha$$

$$= -\, t\, r \sin\varphi \qquad (19)$$

With $N_b = +1 \sin\varphi$, the total axial force becomes

$$N = -\, t\, r \sin\varphi - t\, r \sin\varphi$$

$$= -\, 2t\, r \sin\varphi$$

$$= -\frac{M_d}{2\pi}(\sin\varphi) \qquad (20)$$

The force is thus compressive in quadrants I and II and tensile in quadrants III and IV.

Similarly we determine the shear force:

$$Q = Q_o - X_b N_b$$

$$d Q_o = t\, r \sin(\varphi-\alpha)\, d\alpha$$

$$Q_o = \int_0^{\varphi} t\, r \sin(\varphi-\alpha)\, d\alpha = t\, r - t\, r \cos\varphi \quad \ldots (21)$$

With $Q_b = 1 \cos\varphi$, we thus have:

$$Q = t\, r - t\, r\cos\varphi - t\, r\cos\varphi = t\, r\,(1-2\cos\varphi)$$

$$= \frac{M\, d}{2\pi\, r}\cdot(1-2\cos\varphi) \quad \ldots (22)$$

Substituting various values of φ in equations (17), (18), (20), and (22), the distribution of M, N, and Q over the circular ring is obtained. (See graph 1.)

IV. RADIAL FORCE ON TANGENTIALLY SUPPORTED CIRCULAR RING

When a cylindrical tube is loaded by a force acting along the axis of symmetry, the shear flow, as is known, is not constant but varies with the static moment (fig. 7)

$$t = \frac{P}{2}\frac{2}{J}\int dF\, y = \frac{P}{J}\int \delta\, y\, ds \qquad (23)$$

In polar coordinates with the polar moment of inertia

$$J = \pi\, r^3\, \delta$$

we have

$$t = \frac{P}{\pi r^3 \delta}\int \delta\, r \cos\alpha\, r\, d\alpha$$

$$= \frac{P}{\pi r}\sin\alpha \qquad (24)$$

We again consider a cut made on the ring above. A load element $t\, r\, d\alpha$ produces at the position defined by the angle φ a moment:

$$d M_{0\eta} = -\, t\, r\, d\alpha\,(r - r\cos(\varphi-\alpha))$$

$$= -\frac{P}{\pi r}\sin\alpha\cdot r^2(1-\cos(\varphi-\alpha))\, d\alpha;$$

$$M_{0\eta} = -\frac{P}{\pi r}\int_0^{\eta}(\sin\alpha - \sin\alpha\cos(\varphi-\alpha))\, d\alpha$$

$$= -\frac{P}{\pi r}\left(1-\cos\varphi-\frac{\varphi}{2}\cdot\sin\varphi\right)$$

valid in range 0 to π (25)

On account of symmetry $X_b = 0$, so that we determine X_a and X_c

$$M = -\frac{P r}{\pi}\left(1-\cos\varphi-\frac{\varphi}{2}\sin\varphi\right) - X_a\cdot r(1-\cos\varphi) - X_c;$$

$$\frac{\partial M}{\partial X_a} = -(r - r\cos\varphi);\quad \frac{\partial M}{\partial X_c} = -1;$$

$$\frac{\partial A}{\partial X_a} = 0 = -2\int_0^{\pi}\left(-\frac{P\cdot r}{\pi}\left(1-\cos\varphi-\frac{\varphi}{2}\sin\varphi\right)\right.$$

$$\left. - X_a\cdot(r-r\cos\varphi) - X_c\right)(r-r\cos\varphi)\, r\, d\varphi$$

$$= \left[-\frac{P r}{\pi}\left(\varphi - \sin\varphi + \frac{\varphi}{2}\cos\varphi - \frac{1}{2}\sin\varphi\right.\right.$$

$$\left. - \sin\varphi + \frac{1}{4}\sin 2\varphi + \frac{\varphi}{2} + \frac{\varphi}{4}\cdot\sin^2\varphi\right.$$

$$\left. - \frac{\varphi}{8} + \frac{1}{16}\sin 2\varphi\right) - X_a\cdot r\left(\varphi - 2\sin\varphi + \frac{1}{4}\sin 2\varphi + \frac{\varphi}{2}\right)$$

$$\left. - X_c(\varphi - \sin\varphi)\right]_0^{\pi} \quad \ldots (26)$$

$$= -\frac{7}{8}P\cdot r - X_a\cdot\frac{3}{2}\cdot\pi r - X_c\cdot\pi; \quad \ldots (27)$$

$$\frac{\partial A}{\partial X_c} = 0 = -2\int_0^{\pi}\left(-\frac{P r}{\pi}\left(1-\cos\varphi-\frac{\varphi}{2}\sin\varphi\right)\right.$$

$$\left. - X_a\cdot(r-r\cos\varphi) - X_c\right) r\, d\varphi$$

$$= \left[-\frac{P r}{\pi}\left(\varphi - \sin\varphi + \frac{\varphi}{2}\cos\varphi - \frac{1}{2}\sin\varphi\right)\right.$$

$$\left. - X_a\cdot r(\varphi - \sin\varphi) - X_c\cdot\varphi\right]_0^{\pi} \quad \ldots (28)$$

$$= -\frac{P r}{2} - X_a\cdot\pi\cdot r - X_c\cdot\pi \quad \ldots (29)$$

From equations (27) and (29), there is obtained:

$$X_a = -\frac{3P}{4\pi} \; ; \; X_c = +\frac{Pr}{4\pi}$$

and therefore,

$$M = -\frac{Pr}{\pi}\left(1-\cos\varphi-\frac{\varphi}{2}\sin\varphi\right)+\frac{3P}{4\pi}(r-r\cos\varphi)-\frac{Pr}{4\pi}$$
$$= -\frac{Pr}{4\pi}(2-\cos\varphi-2\varphi\sin\varphi) \quad \ldots\ldots\ldots (30)$$

The obtained moment curve is simultaneously the line of influence for the bending moment due to a displaced radial concentrated force P = 1.

There will now be determined the axial and shear forces in the circular ring for this case:

$$dN_{0\eta} = t\,r\cos(\varphi-\alpha)\,d\alpha$$
$$= +\frac{P}{\pi}\cdot\sin\alpha\cos(\varphi-\alpha)\,d\alpha \quad (\text{tension})$$

$$N_{0\eta} = \int_0^{\eta} + \frac{P}{\pi}(\sin\alpha\cdot\cos(\varphi-\alpha))\,d\alpha$$
$$= +\frac{P\cdot\varphi}{2\pi}\sin\varphi \quad \ldots\ldots\ldots (31)$$

With $N_a = -1\cos\varphi$; $N_c = 0$, we have:

$$N = +\frac{P\cdot\varphi}{2\pi}\cdot\sin\varphi-\frac{3P}{4\pi}\cdot\cos\varphi$$
$$= +\frac{P}{2\pi}\left(\varphi\cdot\sin\varphi-\frac{3}{2}\cos\varphi\right) \quad \ldots\ldots (32)$$

The shear force is similarly determined:

$$Q_{0\varphi} = \int_0^{\varphi}\frac{P}{\pi}\sin\alpha\sin(\varphi-\alpha)\,d\alpha$$
$$= \frac{P}{2\pi}(\sin\varphi-\varphi\cos\varphi) \quad \ldots\ldots\ldots (33)$$
$$Q_a = 1\cdot\sin\varphi;\; Q_c = 0;$$
$$Q = \frac{P}{2\pi}(\sin\varphi-\varphi\cos\varphi)-\frac{3P}{4\pi}\sin\varphi$$
$$= -\frac{P}{2\pi}\left(\frac{1}{2}\sin\varphi+\varphi\cos\varphi\right) \quad \ldots\ldots (34)$$

By substituting various values of φ in equations (30), (32), and (34), there are obtained the bending moments, axial forces, and shear forces. The curves for N and Q also represent the influence lines due to a displaced radial force P (graph 2).

V. TANGENTIAL FORCE P ON A TANGENTIALLY SUPPORTED CIRCULAR RING

On applying a tangential force on the tangentially supported circular ring, the supporting skin forces may be determined by adding two radial oppositely directed forces. Two shear flows (fig. 8) are obtained (see sections III and IV):

$$t_1 = \frac{P\,r}{2\pi r^2} = \frac{P}{2\pi r}$$

and

$$t_2 = \frac{P}{\pi r}\sin\varphi$$

In quadrants I and II these shear flows add up, while in quadrants III and IV they subtract. Again a cut is made on the top of the ring and the three static redundants are determined. The action of P extends only to quadrant II.

The moment due to t_1 in quadrants I and II is

$$M_{0\varphi} = -r^2 t_1(\varphi-\sin\varphi)$$
$$= -\frac{Pr}{2\pi}(\varphi-\sin\varphi) \quad (35)$$

In quadrants III and IV, by substituting $(2\pi-\varphi)$ for φ:

$$M_{0\varphi} = +\frac{Pr}{2\pi}(2\pi-\varphi+\sin\varphi) \quad (36)$$

The moment due to t_2 in quadrants I and II is

$$M_{0\varphi} = -\frac{Pr}{\pi}\left(1-\cos\varphi-\frac{\varphi}{2}\sin\varphi\right) \quad (37)$$

In quadrants III and IV by substituting $(2\pi-\varphi)$ for φ:

$$M_{0\varphi} = -\frac{Pr}{\pi}\left(1-\cos\varphi+\frac{2\pi-\varphi}{2}\sin\varphi\right) \quad (38)$$

In quadrant II we have, in addition, a moment due to P:

$$M_{\varphi\varphi} = + P r (1 - \sin \varphi) \qquad (39)$$

From the total moment we again obtain $\partial M/\partial X_a$, $\partial M/\partial X_b$, and $\partial M/\partial X_c$, and through integration:

$$\frac{\partial A}{\partial X_a} = 0 = \left[\frac{Pr}{2\pi}\left(\frac{\varphi^2}{2} - \varphi \sin\varphi + \frac{1}{2}\sin^2\varphi\right)\right]_0^{\pi}$$
$$+ \left[\frac{Pr}{\pi}\left(\frac{11}{8}\varphi - \frac{5}{2}\sin\varphi + \frac{\varphi}{2}\cos\varphi + \frac{5}{16}\sin 2\varphi + \frac{\varphi}{4}\sin^2\varphi\right)\right]_0^{\pi}$$
$$- \left[\frac{Pr}{2\pi}\left(2\pi\varphi - \frac{\varphi^2}{2} - 2\pi\sin\varphi + \varphi\sin\varphi - \frac{1}{2}\sin^2\varphi\right)\right]_{\pi}^{2\pi}$$
$$+ \left[\frac{Pr}{\pi}\left(\frac{11}{8}\varphi - \frac{5}{2}\sin\varphi + \frac{\varphi}{2}\cos\varphi + \frac{5}{16}\sin 2\varphi\right.\right.$$
$$\left.\left. - \frac{\pi}{2}\sin^2\varphi + \frac{\varphi}{4}\sin^2\varphi - \pi\cdot\cos\varphi\right)\right]_{\pi}^{2\pi}$$
$$- \left[Pr\left(\varphi + \cos\varphi - \sin\varphi + \frac{1}{2}\sin^2\varphi\right)\right]_{\frac{\pi}{2}}^{\pi}$$
$$+ \left[x_a \cdot r\left(\frac{3}{2}\varphi - 2\sin\varphi + \frac{1}{2}\sin 2\varphi\right)\right.$$
$$\left. + x_b \cdot r\left(-\cos\varphi - \frac{1}{2}\sin^2\varphi\right) + X_c \cdot (\varphi - \sin\varphi)\right]_0^{2\pi} \qquad (40)$$

$$0 = \frac{9}{4}Pr - \frac{1}{2}Pr\pi + X_a \cdot 3\pi r + X_c \cdot 2\pi \quad \ldots \quad (41)$$

$$\frac{\partial A}{\partial X_b} = 0 = \left[\frac{Pr}{2\pi}\left(-\varphi\cos\varphi + \sin\varphi + \frac{1}{4}\sin 2\varphi - \frac{\varphi}{2}\right)\right]_0^{\pi}$$
$$+ \left[\frac{Pr}{\pi}\left(-\cos\varphi - \frac{1}{2}\sin^2\varphi - \frac{\varphi^2}{4}\sin^2\varphi\right.\right.$$
$$\left.\left. - \frac{\varphi^2}{8}\cos 2\varphi + \frac{\varphi}{8}\sin 2\varphi + \frac{1}{16}\cos 2\varphi\right)\right]_0^{\pi}$$
$$- \left[\frac{Pr}{2\pi}\left(-2\pi\cos\varphi + \varphi\cos\varphi - \sin\varphi - \frac{1}{4}\sin 2\varphi + \frac{\varphi}{2}\right)\right]_{\pi}^{2\pi}$$
$$+ \left[\frac{Pr}{\pi}\left(-\cos\varphi - \frac{1}{2}\sin^2\varphi + \frac{\pi}{2}\varphi - \frac{\pi}{4}\sin 2\varphi - \frac{\varphi^2}{4}\sin^2\varphi\right.\right.$$
$$\left.\left. - \frac{\varphi^2}{8}\cos 2\varphi + \frac{\varphi}{8}\sin 2\varphi + \frac{1}{16}\cos 2\varphi\right)\right]_{\pi}^{2\pi}$$
$$- \left[Pr\left(-\cos\varphi + \frac{1}{4}\sin 2\varphi - \frac{\varphi}{2}\right)\right]_{\frac{\pi}{2}}^{\pi}$$
$$+ \left[X_a r\left(-\cos\varphi - \frac{1}{2}\sin^2\varphi\right)\right.$$
$$\left. + X_b r\left(\frac{\varphi}{2} - \frac{1}{4}\sin 2\varphi\right) - X_c \cdot \cos\varphi\right]_0^{2\pi} \quad \ldots \quad (42)$$

$$0 = -\frac{Pr}{2} + \frac{Pr\pi}{4} + X_b \cdot r\pi \quad \ldots \quad (43)$$

$$X_b = -\frac{P}{4\pi}(\pi - 2)$$

$$\frac{\partial A}{\partial X_c} = 0 = \left[\frac{Pr}{2\pi}\left(\frac{\varphi^2}{2} + \cos\varphi\right)\right]_0^{\pi} + \left[\frac{Pr}{\pi}\left(\varphi - \frac{3}{2}\sin\varphi + \frac{\varphi}{2}\cos\varphi\right)\right]_0^{\pi}$$
$$- \left[\frac{Pr}{2\pi}\left(2\pi\varphi - \frac{\varphi^2}{2} - \cos\varphi\right)\right]_{\pi}^{2\pi}$$
$$+ \left[\frac{Pr}{\pi}\left(\varphi - \frac{3}{2}\sin\varphi - \pi\cos\varphi + \frac{\varphi}{2}\cos\varphi\right)\right]_{\pi}^{2\pi}$$
$$- \left[Pr(\varphi + \cos\varphi)\right]_{\frac{\pi}{2}}^{\pi} + \left[X_a r(\varphi - \sin\varphi) - X_b r\cos\varphi + X_c\varphi\right]_0^{2\pi}$$
$$\ldots \quad (44)$$

$$0 = 2Pr - \frac{1}{2}Pr\pi + X_a \cdot 2\pi r + X_c \cdot 2\pi \quad \ldots \quad (45)$$

From equations (41) and (45), we obtain:

$$X_a = -\frac{P}{4\pi}; \quad X_c = -\frac{Pr}{4\pi}(3 - \pi)$$

The final moments in quadrants I to IV, therefore, are:

$$M_{I} = -\frac{Pr}{2\pi}(\varphi - \sin\varphi) - \frac{Pr}{\pi}\left(1 - \cos\varphi - \frac{\varphi}{2}\sin\varphi\right)$$
$$+ \frac{P}{4\pi}(r - r\cos\varphi) + \frac{P}{4\pi}(\pi - 2)\, r\sin\varphi + \frac{Pr}{4\pi}(3 - \pi)$$
$$= -\frac{Pr}{4\pi}[(2\varphi + \pi)(1 - \sin\varphi) - 3\cos\varphi] \quad \ldots \quad (46)$$

$$M_{II} = +\frac{Pr}{4\pi}[(3\pi - 2\varphi)(1 - \sin\varphi) + 3\cos\varphi] \quad \ldots \quad (47)$$

$$M_{III\text{-}IV} = +\frac{Pr}{4\pi}[(3\pi - 2\varphi)(1 - \sin\varphi) - 3\cos\varphi] \quad \ldots \quad (48)$$

As in the previous cases, we also determine the axial and the shear forces:

$$N_{I} = +\frac{P}{2\pi}\left[\left(\varphi + \frac{\pi}{2}\right)\sin\varphi - \frac{1}{2}\cos\varphi\right] \quad \ldots \quad (49)$$

$$N_{II} = +\frac{P}{2\pi}\left[\left(\varphi - \frac{3}{2}\pi\right)\sin\varphi - \frac{1}{2}\cos\varphi\right] \quad \ldots \quad (50)$$

$$N_{III\text{-}IV} = +\frac{P}{2\pi}\left[\left(\varphi - \frac{3}{2}\pi\right)\sin\varphi - \frac{1}{2}\cos\varphi\right] \quad \ldots \quad (51)$$

$$Q_{I} = \frac{P}{2\pi}\left[1 + \frac{1}{2}\sin\varphi - \left(\varphi + \frac{\pi}{2}\right)\cos\varphi\right] \quad \ldots \quad (52)$$

$$Q_{II} = \frac{P}{2\pi}\left[1 + \frac{1}{2}\sin\varphi - \left(\varphi - \frac{3\pi}{2}\right)\cos\varphi\right] \quad \ldots \quad (53)$$

$$Q_{III\text{-}IV} = \frac{P}{2\pi}\left[1 + \frac{1}{2}\sin\varphi + \left(\frac{3\pi}{2} - \varphi\right)\cos\varphi\right] \quad \ldots \quad (54)$$

By substituting values of φ in the equations, we obtain the curves for the bending moments, axial and shear forces plotted in graph 3. Again these are the influence lines due to a displaced tangential force P.

VI. SINE-SHAPED LOAD DISTRIBUTION

In order to take into account air forces on the fuselage, there will also be considered the sine-shaped load distribution (fig. 9).

Let the loading be:

$$p = -p_{max} \cdot \cos\alpha \quad \left(\text{von } \frac{\pi}{2} \text{ bis } \frac{3\pi}{2}\right) \quad \ldots \quad (55)$$

$p_v = p_{max} \cdot \cos\alpha \cdot \cos(\pi - \alpha)$; Vertical Component

$$P_v = 2\int_{\frac{\pi}{2}}^{\pi} p_{max} \cdot \cos\alpha \cdot \cos(\pi - \alpha)\, r\, d\alpha = r\, p_{max} \cdot \frac{\pi}{2} \qquad (56)$$

The shear flow corresponding to equation (24) is then:

$$t = \frac{p_{max}}{2} \sin \alpha \qquad (57)$$

On making a cut on top of the ring, there follows from t:

$$M_0 = -\frac{r^2 p_{max}}{2}\left(1 - \cos\varphi - \frac{\varphi}{2}\sin\varphi\right) \qquad (58)$$

From $p = p_{max} \cos\alpha$, there is obtained in quadrants II and III:

$$dP = p \cdot r\,d\alpha;$$
$$dM_{0\varphi} = p \cdot r\,d\alpha \cdot \sin(\varphi - \alpha) \cdot r = -p_{max} \cdot r^2 \cos\alpha \sin(\varphi - \alpha)\,d\alpha;$$
$$M_{0\varphi} = -\int_{\frac{\pi}{2}}^{\varphi} r^2 p_{max} \cos\alpha \sin(\varphi - \alpha)\,d\alpha$$
$$= -r^2 p_{max}\left(\frac{\varphi}{2}\sin\varphi - \frac{\pi}{4}\sin\varphi + \frac{1}{2}\cos\varphi\right) \qquad (59)$$

Again on account of symmetry, $X_b = 0$; hence

$$\frac{\partial A}{\partial X_a} = 0 = -\int_0^{\pi} -\frac{r^2 p_{max}}{2}\left(1 - \cos\varphi - \frac{\varphi}{2}\sin\varphi\right)(1 - \cos\varphi)\,r^2 d\varphi$$
$$-\int_{\frac{\pi}{2}}^{\pi} -r^2 p_{max}\left(\frac{\varphi}{2}\sin\varphi - \frac{\pi}{4}\sin\varphi + \frac{1}{2}\cos\varphi\right)(1 - \cos\varphi)\,r^2 d\varphi$$
$$-\int_0^{\pi}(X_a\, r(1 - \cos\varphi) + X_c)(1 - \cos\varphi)\,r^2 d\varphi$$
$$= -r^2 p_{max}\left[+\frac{\pi}{4}\cos\varphi - \frac{\varphi}{2}\cos\varphi + \frac{1}{2}\sin\varphi + \frac{1}{2}\sin\varphi + \frac{\pi}{8}\sin^2\varphi - \frac{\varphi}{4}\sin^2\varphi + \frac{\varphi}{8} - \frac{1}{16}\sin 2\varphi - \frac{1}{8}\sin 2\varphi - \frac{\varphi}{4}\right]_{\frac{\pi}{2}}^{\pi} + \text{Gl. (26)} \qquad (60)$$

and replacing P in equation (26) by $\frac{r^2 p_{max}\,\pi}{2}$

$$= -\frac{10}{16}\pi r^2 p_{max} + r^2 p_{max} - \frac{3}{2} X_a \cdot \pi r - X_c \cdot \pi \qquad (61)$$

$$\frac{\partial A}{\partial X_c} = 0 = -\int_0^{\pi} -\frac{r^2 p_{max}}{2}\left(1 - \cos\varphi - \frac{\varphi}{2}\sin\varphi\right) r\,d\varphi$$
$$-\int_{\frac{\pi}{2}}^{\pi} -r^2 p_{max}\left(\frac{\varphi}{2}\sin\varphi - \frac{\pi}{4}\sin\varphi + \frac{1}{2}\cos\varphi\right) r\,d\varphi$$
$$-\int_0^{\pi}[X_a \cdot r(1 - \cos\varphi) + X_c]\,r\,d\varphi$$
$$= -r^2 p_{max}\left[\frac{\pi}{4}\cos\varphi - \frac{\varphi}{2}\cos\varphi + \sin\varphi\right]_{\frac{\pi}{2}}^{\pi} + \text{Gl. (28)} \qquad (62)$$

(substituting for P in equation (28) $\frac{r^2 p_{max}\,\pi}{2}$

$$= -r^2 p_{max} \cdot \frac{\pi}{2} + r^2 p_{max} - X_a \cdot \pi r - X_c \pi \qquad (63)$$

From equations (61) and (63) there is obtained:

$$X_a = -p_{max} \cdot \frac{r}{4}; \quad X_c = +r^2 p_{max}\left(\frac{1}{\pi} - \frac{1}{4}\right).$$

There is thus obtained for quadrants I to IV:

$$M_I = -\frac{r^2 p_{max}}{2}\left(1 - \cos\varphi - \frac{\varphi}{2}\sin\varphi\right) + \frac{r^2 p_{max}}{4}(1 - \cos\varphi) - r^2 p_{max}\left(\frac{1}{\pi} - \frac{1}{4}\right)$$
$$= -r^2 p_{max}\left(\frac{1}{\pi} - \frac{1}{4}\cos\varphi - \frac{\varphi}{4}\sin\varphi\right) \qquad (64)$$

$$M_{II} = -r^2 p_{max}\left(\frac{1}{\pi} + \frac{1}{4}\cos\varphi + \frac{\varphi}{4}\sin\varphi - \frac{\pi}{4}\sin\varphi\right) \qquad (65)$$

We determine also the axial and the shear forces (fig. 10)

due to t:

$$N_{0\varphi} = +\frac{r\,p_{max}}{4} \cdot \varphi \sin\varphi;$$

due to p:

$$dP = p \cdot ds = p_{max} \cdot \cos\alpha \cdot r\,d\alpha$$
$$dN_{0\varphi} = p_{max} \cdot \cos\alpha \cdot r\,d\alpha \sin(\varphi - \alpha)$$
$$N_{0\varphi} = p_{max} \cdot r \int_{\frac{\pi}{2}}^{\varphi} \cos\alpha \sin(\varphi - \alpha)\,d\alpha$$
$$= p_{max} \cdot r\left[\frac{\varphi}{2}\sin\varphi - \frac{\pi}{4}\sin\varphi + \frac{1}{2}\cos\varphi\right] \qquad (66)$$

There is thus obtained:

$$N_I = +p_{max} \cdot \frac{r}{4}(\varphi \sin\varphi - \cos\varphi) \qquad (67)$$

$$N_{II} = +p_{max} \cdot \frac{r}{4}(-\varphi\sin\varphi + \pi\sin\varphi - 3\cos\varphi) \qquad (68)$$

Similarly, there are obtained the shear forces:

$$Q_{0\varphi(p)} = p_{max} \cdot r\left[\frac{1}{2}\cos^2\varphi\sin\varphi + \frac{\varphi}{2}\cos\varphi - \frac{\pi}{4}\cos\varphi + \frac{1}{2}\sin^3\varphi - \frac{1}{2}\sin\varphi\right] \qquad (69)$$

and hence:

$$Q_I = -p_{max} \cdot \frac{r}{4} \varphi \cos\varphi \quad \ldots\ldots\ldots\ldots \quad (70)$$

$$Q_{II} = p_{max} \cdot \frac{r}{4} (2\cos^2\varphi \sin\varphi + \varphi\cos\varphi - \pi\cos\varphi + 2\sin^3\varphi - 2\sin\varphi) \quad (71)$$

By substituting various values of φ in equations (64), (65), (67), (68) and (70), (71), there are obtained the curves for bending moment, axial and shear forces shown in graph 4.

As a check for the maximum bending moment at the lowest point of the ring M_u, we use the moment curve of figure 2 as an influence curve for the sine-shaped load and obtain:

$$M_{(\varphi)} = \frac{r}{4\pi}(2 - \cos\varphi - 2\varphi\sin\varphi) \text{ influence curve}$$

$$dM = p_{max} \cdot \cos\varphi \cdot ds \cdot M_{(\varphi)};$$

$$M_u = 2\int_{\frac{\pi}{2}}^{\pi} \frac{p_{max} \cdot r^2}{4\pi}(2\cos\varphi - \cos^2\varphi - 2\varphi\sin\varphi\cos\varphi)\, d\varphi$$

$$= -p_{max} \cdot r^2 \cdot 0{,}068.$$

The value for M_u corresponds to the value computed from equation (64) for $\varphi = \pi$.

As a further check, the value of $\int M\, ds$ must become zero, a fact which holds for this case as for the three previous cases.

VII. TREATMENT OF THE ABOVE LOAD CASES FOR VARIABLE MOMENT OF INERTIA OF THE RING

For the case considered under III, let the moment of inertia vary according to figure 11:

$$J_1 = J_c$$

$$J_2 = n\, J_1; \quad \frac{J_c}{J_2} = \frac{1}{n}$$

Making use of equations (11), (13), and (15), we integrate over the ranges 0 to $\pi/2$, $\pi/2$ to $3\pi/2$, and $3\pi/2$ to 2π, the middle range being multiplied by 1/n:

$$\frac{\partial A}{\partial X_a} = 0 = \left[-X_a r\left(\frac{3\pi}{2} - 4\right) - X_c(\pi + 2)\right] + \frac{1}{n}\left[-X_a r\left(\frac{3\pi}{2} + 4\right) - X_c(\pi - 2)\right]$$

$$\frac{\partial A}{\partial X_c} = 0 = [-X_a r(\pi - 2) - X_c \cdot \pi] + \frac{1}{n}[-X_a r(\pi + 2) - X_c \pi].$$

Hence, for variable moment of inertia:

$$X_a = 0; \quad X_c = 0$$

$$\frac{\partial A}{\partial X_b} = 0 = \left[2tr^2 - tr^2\frac{\pi}{2} - X_b r\frac{\pi}{2}\right] + \frac{1}{n}\left[-2tr^2 + \frac{3}{2}tr^2\pi - X_b r\frac{\pi}{2}\right]$$

The following values are obtained for X_b:

$\frac{1}{n} = \frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
$X_b = \frac{M}{2\pi}\frac{d}{r} \cdot 1{,}0$	0,758	0,637	0,563	0,515	0,405

and, therefore, the total moments

$M = \frac{M_d}{2\pi}(\varphi - \eta \sin\varphi) \ldots (-2\pi)$ additive for quadrants III and IV

The corresponding values for η are:

$\frac{1}{n} = \frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
$\eta = 2{,}0$	1,758	1,637	1,563	1,515	1,405

The moments are plotted on graph 5. The check $\int M\, ds$ gives zero.

Finally, for the case considered under IV, the moment of inertia is varied, according to figure 12.

$$J_1 = J_c; \quad J_2 = n\, J_1; \quad \frac{J_c}{J_2} = \frac{1}{n}$$

The integration is between 0 and $\pi/2$ and between $\pi/2$ and π, the latter range being multiplied by 1/n. From equations (26) and (28) we then have:

$$\frac{\partial A}{\partial X_a} = 0 = \left[-\frac{Pr}{\pi}\left(\frac{13\pi}{16} - \frac{5}{2}\right) - X_a r\left(\frac{3\pi}{4} - 2\right) - X_c\left(\frac{\pi}{2} - 1\right)\right] + \frac{1}{n}\cdot\left[-\frac{Pr}{\pi}\left(\frac{\pi}{16} + \frac{5}{2}\right) - X_a r\left(\frac{3\pi}{4} + 2\right) - X_c\left(\frac{\pi}{2} + 1\right)\right];$$

$$\frac{\partial A}{\partial X_c} = 0 = \left[-\frac{Pr}{\pi}\left(\frac{\pi}{2} - \frac{3}{2}\right) - X_a r\left(\frac{\pi}{1} - 1\right) - X_c\frac{\pi}{2}\right] + \frac{1}{n}\cdot\left[-\frac{Pr}{\pi}\left(\frac{3}{2}\right) - X_a r\left(\frac{\pi}{2} + 1\right) - X_c\frac{\pi}{2}\right].$$

There are obtained the unknowns and factors:

$\frac{1}{n} = \frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
$X_a = -\frac{P}{\pi} \cdot 0{,}75$	0,706	0,675	0,642	0,622	0,555
$X_c = +\frac{Pr}{\pi} \cdot 0{,}25$	0,207	0,1875	0,1695	0,1613	0,1372
$\eta = 0{,}5$	0,505	0,5125	0,5272	0,5393	0,5822
$\xi = 0{,}25$	0,298	0.325	0,358	0,378	0,445

and substituting the bending factors, the bending moments are:

$$M = - \frac{Pr}{\pi} (\eta - \xi \cos\varphi - 0.5\, \varphi \sin \varphi)$$

The moments are plotted on graph 6. The check $\int M\, ds$ gives zero.

Similarly the bending moments for variable moments of inertia of the circular ring may be computed for the cases treated under sections V and VI by the procedure set up. The moments of inertia also may be otherwise discontinuously varied without too great increase in computation. It was not possible within the scope of this paper to consider further cases.

VIII. SUMMARY

For circular ring stiffeners with constant and variable moments of inertia, the bending moments and axial and shear forces were obtained for the cases of an applied moment, a radial force, a tangential force, and a sine load distribution, taking account of the tangential support of the ring. In order to be able to treat further cases of variable moment of inertia that may occur in practice, the integral values $\partial A/\partial X$ are computed. The practical application of the results presented in this paper gives a considerable saving in time and labor as compared with the usual method of dividing the ring moment areas into smaller sections according to Müller-Breslau and, moreover, provides greater accuracy.

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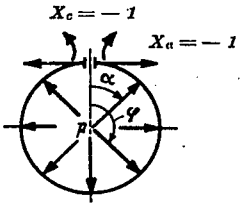


Figure 1.- Circular ring stiffener with internal pressure.

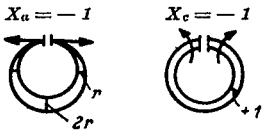


Figure 2.- Introduction of the static redundants.

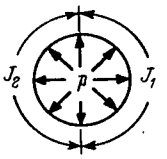


Figure 3.- Circular ring stiffener with variable moment of inertia.

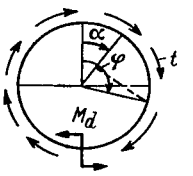


Figure 4.- Moment M_d on a tangentially supported circular ring.

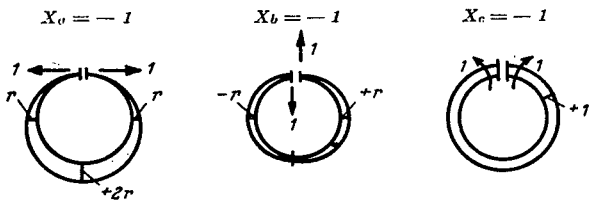


Figure 5.- Introduction of the static redundants.

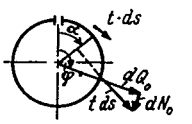


Figure 6.- Determination of the axial and shear forces.

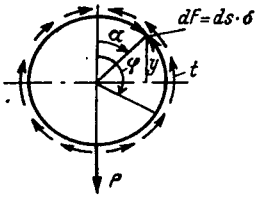


Figure 7.- Radial force P on a tangentially supported circular ring.

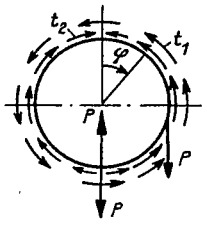


Figure 8.- Tangential force P on a tangentially supported circular ring.

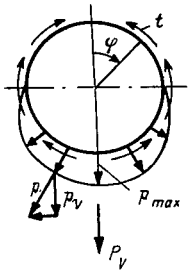


Figure 9.- Application of a sine load distribution.

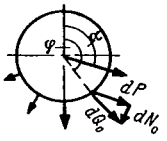


Figure 10.- Determination of the axial and shear forces.

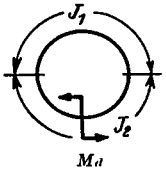


Figure 11.- Variable moment of inertia.

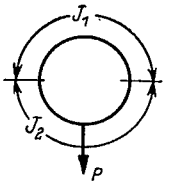
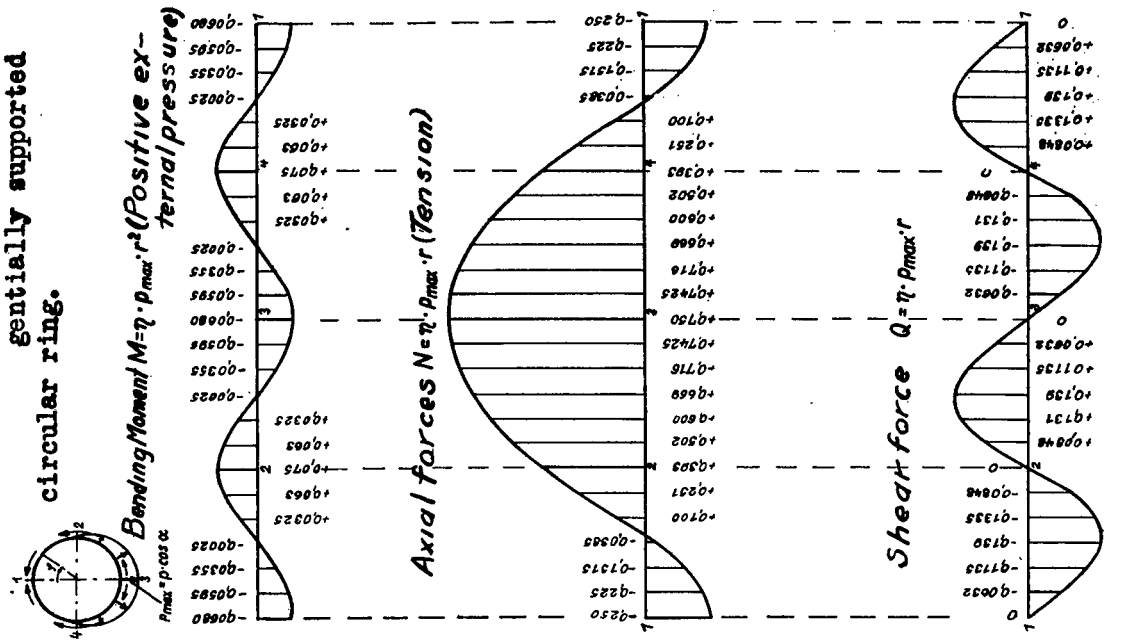


Figure 12.- Variable moment of inertia.

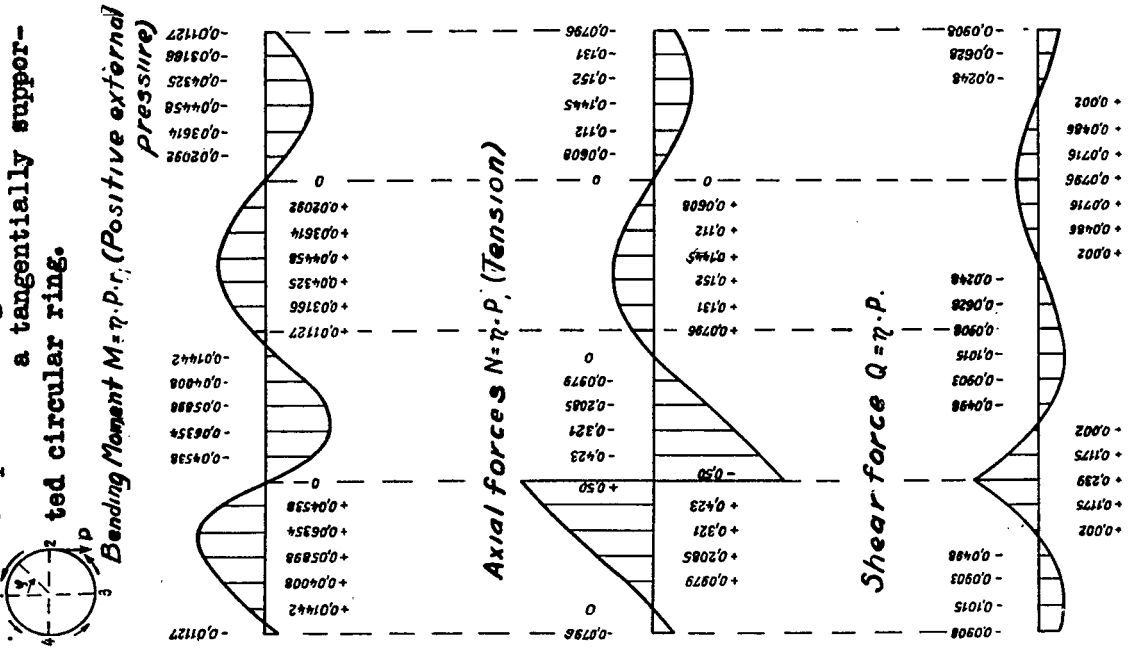
Graph 1.- Moment M on a tangentially supported circular ring.

Graph 2.- Radial force P on a tangentially supported circular ring.

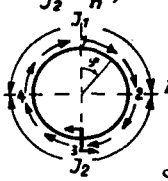
Graph 4.- Sine load on a tangentially supported circular ring.



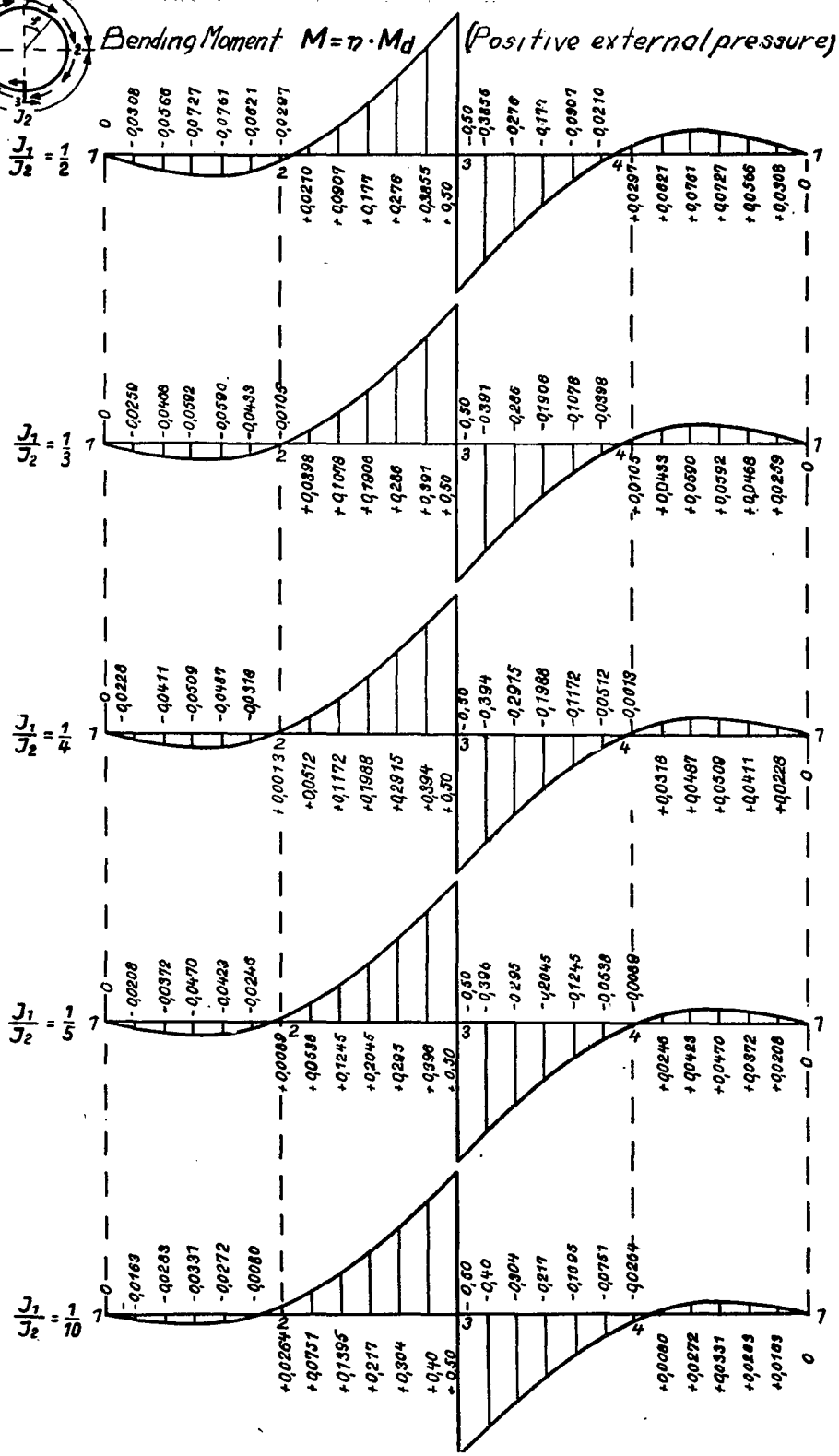
Graph 3.- Tangential force P on a tangentially supported circular ring.



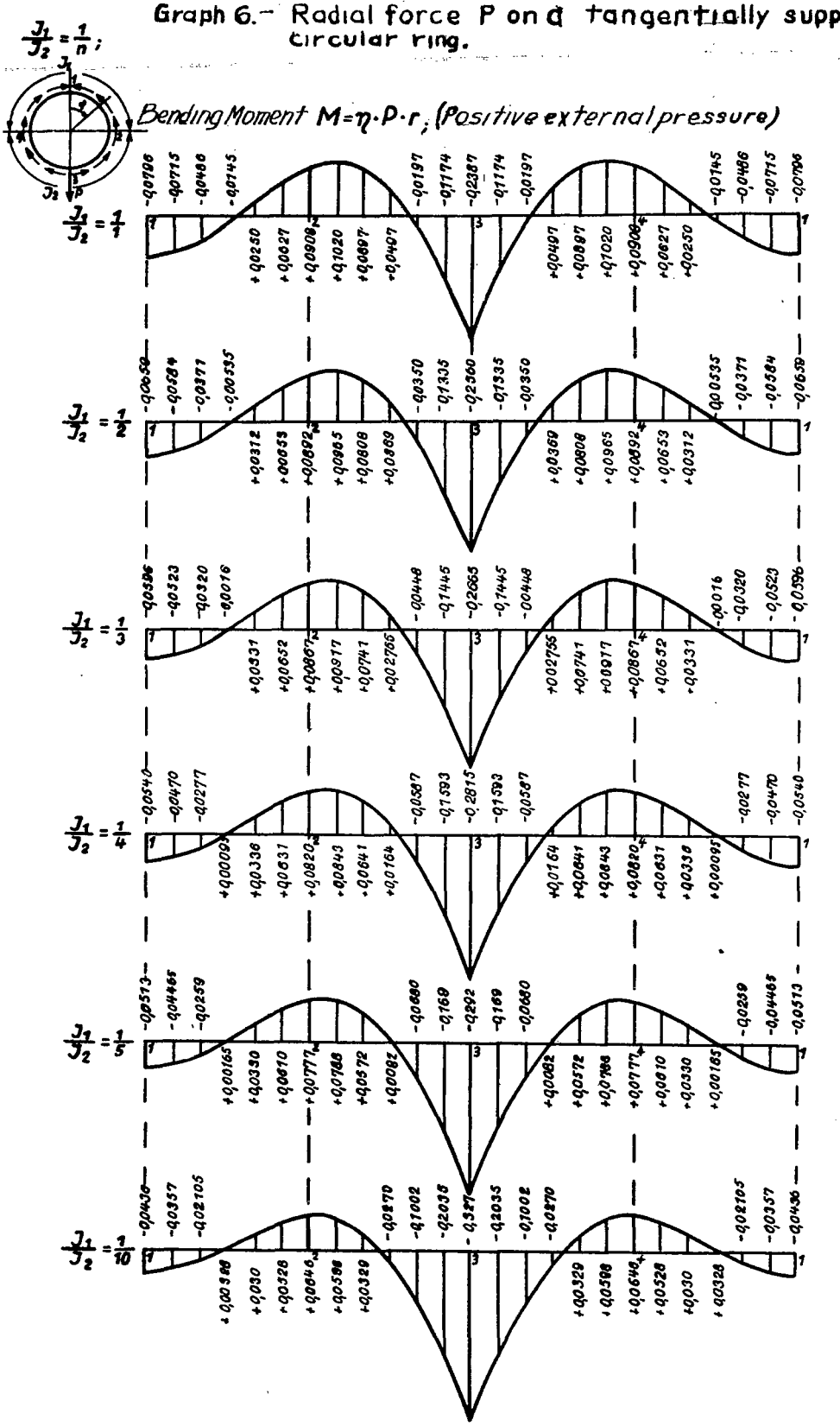
$\frac{J_1}{J_2} = \frac{1}{n}$; Graph 5.- Moment M_d on a tangentially supported circular ring.



Bending Moment $M = \eta \cdot M_d$ (Positive external pressure)



Graph 6.– Radial force P on a tangentially supported circular ring.



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