TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 971

THE ELLIPTIC WING BASED ON THE POTENTIAL THEORY

By Klaus Krienes

Zeitschrift für angewandte Mathematik und Mechanik
Vol 20, No. 2, April 1940

Washington
March 1941

NOTICE

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 971

THE ELLIPTIC WING BASED ON THE POTENTIAL THEORY*

By Klaus Krienes

SUMMARY

The present report deals with the elliptical wing in straight and angular flow on the basis of the potential theory. Conformably to the theory of first approximation upon which the calculation rests, the known requirements regarding the shape of the surface and its angle of attack must be met. A further condition is that the slope of the surface toward the streamlines must be a continuously differentiable function of the points of the surface. If this is not the case, in a given example (for instance, by aileron deflection or wing dihedral - the latter being of importance in sideslips), the discontinuities must be replaced by suitable rounding off. In general, the calculation of a given elliptic surface requires a series of infinitely many potential functions, the coefficients of which are afforded from linear infinite systems of equations. The expansion is stopped with a certain term, depending upon the degree of accuracy desired. Its effect on the integral quantities, lift and lift moment, is practically negligible. An immediate prediction of the induced drag is ruled out, since it would involve all the coefficients of the infinite number of potential functions. Otherwise, the lift distribution at the wing tips does not approach zero or the downwash becomes infinite, which is due to the fact that the load distribution of the lifting line is developed here by spherical functions (equation (80)) which do not approach zero at the wing tips as do the trigonometric functions employed elsewhere. On the wing in sideslip, which can be summarily replaced by a lifting line, the so-called parasite drag (reference 2)

$$W_F = - \int \int (p_u - p_{ob}) \frac{w}{V} \, d\,x \, d\,y$$

---

*"Die elliptische Tragfläche auf potentialtheoretischer Grundlage." Z.f.a.M.M., vol. 20, no. 2, April 1940, pp. 65-88.

would have to be defined first and the suction force on the leading edge subtracted therefrom, where, however, extrapolations are recommended because of the finite number of computed coefficients. Even the resulting pressure distribution is only conditionally valid by few expansion terms, especially near the wing tips.

It may be mentioned that the computed potential and downwash functions change on transition to $\kappa^2 \longrightarrow 0$ into Kinner's functions for the circular wing.

A large portion of the computations were made on the calculating machine, the accuracy of the slide rule being insufficient in the calculation of the elliptic integral for higher n.

INTRODUCTION

This article is intended as a contribution to the theory of the lifting surface. The aerodynamics of the elliptic wing in straight and oblique flow are explored on the basis of the potential theory. The foundation of the calculation is the linearized theory of the acceleration potential (references 1 and 2) in which all small quantities of higher order are disregarded. This affords the following simplifications:

1. The z coordinate of every wing point is neglected, i.e., the variation of the potential function corresponding to the pressure pump on the surface is situated on the base ellipsoid (fig. 1).

2. The streamlines, along which the convective integration of the acceleration is effected, are straight lines parallel to the direction of the stream.

In the case of the elliptic boundary, solutions of Laplace's differential equation $\Delta\psi = 0$, as products of Lamé's function, are known.

<u>The acceleration potential and Lamé's functions.</u>- Suppose that the elliptic wing is in a stationary parallel flow with the velocity V. The fluid is homogeneous, incompressible, frictionless, and not subjected to gravity and non-vortical outside the lifting surface and the shedding vortex band. Then:

$$\frac{DW}{dt} = -\frac{1}{\rho}\ \text{grad}\ p \tag{1}$$

The pressure p is expressed by the potential function ψ

$$p - p_\infty = -\rho\ V^2\ \psi \tag{2}$$

where p_∞ is the static pressure at infinity. Function ψ satisfies Laplace's differential equation

$$\Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \tag{3}$$

and may be visualized as being the result of a superposition of sources and sinks of intensity $\sigma(x_F, y_F, z_F)$ on every point of the surface.

$$\psi(x,y,z) = \iint \sigma(x_F,y_F,z_F)\ \frac{\partial}{\partial n}\left(\frac{1}{R}\right) dF \tag{4}$$

where n denotes the direction of the normals of the surface in (x_F,y_F,z_F) and

$$R = \sqrt{(x - x_F)^2 + (y - y_F)^2 + (z - z_F)^2}$$

the distance of the starting point (x,y,z) from (x_F, y_F, z_F). The integral can be exchanged for one taken over the surface of the ground ellipse.

$$\psi(x,y,z) = \iint \sigma(x_F,y_F)\frac{\partial}{\partial z_F}\left(\frac{1}{R}\right) d\,x_F\ d\,y_F \tag{4a}$$

The mathematical treatment of the present case becomes possible by the introduction of ellipsoidal-hyperboloidal coordinates (reference 3). The semi-axes of the base ellipsoid being c in direction y and $c\sqrt{1 - \kappa^2}$ in the direction of x (so that $2\kappa c$ becomes the distance of the aerodynamic centers on axis y) the new system of coordinates is:

$$\frac{x^2}{\rho^2 - \kappa^2} + \frac{y^2}{\rho^2} + \frac{z^2}{\rho^2 - 1} = c^2$$

$$\frac{x^2}{\mu^2 - \kappa^2} + \frac{y^2}{\mu^2} - \frac{z^2}{1 - \mu^2} = c^2 \qquad \infty \geq \rho \geq 1 \geq \mu \geq \kappa \geq \nu \geq -\kappa \tag{5}$$

$$-\frac{x^2}{\kappa^2 - \nu^2} + \frac{y^2}{\nu^2} - \frac{z^2}{1 - \nu^2} = c^2$$

The solution of the equations leads to:

$$\left.\begin{aligned} x &= \frac{c}{\sqrt{1-\kappa^2}}\sqrt{\rho^2-\kappa^2}\sqrt{\mu^2-\kappa^2}\;\frac{\sqrt{\kappa^2-\nu^2}}{\kappa}, \qquad y = c\rho\mu\frac{\nu}{\kappa} \\ z &= \frac{c}{\sqrt{1-\kappa^2}}\sqrt{\rho^2-1}\,\sqrt{1-\mu^2}\,\sqrt{1-\nu^2} \end{aligned}\right\} \tag{6}$$

The surfaces ρ, μ, and ν = const., respectively, are confocal surfaces of the second degree; $\rho = 1$ yields the elliptic base surface, $\mu = 1$ the plane $z = 0$ outside of the elliptic disk. The surface element of the base ellipsoid is

$$dx\,dy = c^2\frac{\mu^2-\nu^2}{\sqrt{\mu^2-\kappa^2}\sqrt{\kappa^2-\nu^2}}\,d\mu\,d\nu \tag{7}$$

besides which

$$\frac{c}{\sqrt{1-\kappa^2}}\sqrt{1-\mu^2}\,\sqrt{1-\nu^2} = \sqrt{c^2-y^2-\frac{\kappa^2}{1-\kappa^2}} \tag{8}$$

is valid for $\rho = 1$.

The introduction of new coordinates u, v, w

$$\wp(u) = \rho^2 - \frac{1}{3}(1+\kappa^2), \qquad \wp(v) = \mu^2 - \frac{1}{3}(1+\kappa^2),$$
$$\wp(w) = \nu^2 - \frac{1}{3}(1+\kappa^2) \tag{9}$$

for ρ, μ, ν by means of Weierstrass' $\wp$ function (reference 4), appearing in equation (10)

$$\frac{d\wp(u)}{du} = 2\sqrt{(\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3)} \tag{10}$$

where the quantities

$$e_1 = \frac{1}{3}(2-\kappa^2); \quad e_2 = \frac{1}{3}(2\kappa^2-1); \quad e_3 = -\frac{1}{3}(1+\kappa^2);$$
$$e_1 + e_2 + e_3 = 0 \tag{11}$$

i.e.,

$$\sqrt{e_1 - e_2} = \sqrt{1 - \kappa^2}; \quad \sqrt{e_2 - e_3} = \kappa; \quad \sqrt{e_1 - e_3} = 1$$

are posted, gives

$$u = \int_{\infty}^{\gamma(u)} \frac{d\, p}{2\sqrt{p - e_1}\sqrt{p - e_2}\sqrt{p - e_3}} = \int_{\infty}^{\rho} \frac{d\, \rho}{\sqrt{\rho^2 - 1}\sqrt{\rho^2 - \kappa^2}} \tag{12}$$

and Laplace's equation reads

$$[\gamma(v) - \gamma(w)]\frac{\partial^2\psi}{\partial u^2} + [\gamma(w) - \gamma(u)]\frac{\partial^2\psi}{\partial v^2} + [\gamma(u) - \gamma(v)]\frac{\partial^2\psi}{\partial w^2} = 0 \tag{13}$$

Posting the solution in the form

$$\psi(u,v,w) = E(u)E(v)E(w) \tag{14}$$

gives, for each of the three functions E, Laplace's differential equation

$$\frac{d^2E(u)}{d\, u^2} = [A + B\ \gamma(u)]\ E(u) \tag{15}$$

with the separation constants A and B. After posting

$$B = n(n + 1) \quad \text{and} \quad v_m = -\frac{A}{1 + \kappa^2} + \frac{B}{3}$$

and again introducing μ, ν, ρ by means of equation (9), Lamé's differential equation (reference 3, vol. I, p. 359) reads

$$\left.\begin{aligned}(\mu^2 - \kappa^2)(\mu^2 - 1)\frac{d^2E_n{}^m(\mu)}{d\,\mu^2} + \mu(2\mu^2 - \kappa^2 - 1)\frac{dE_n{}^m(\mu)}{d\mu}\\ + [(1 + \kappa^2)v_m - n(n+1)\mu^2]E_n{}^m(\mu) = 0\end{aligned}\right\} \tag{15a}$$

For $B = n(n + 1)$ there are precisely two $2n + 1$ values v_m, for which $E_n{}^m(\mu)$ has the form (reference 3, p.360)

$$E_n{}^m(\mu) = \sqrt{1-\mu^2}^{\,\epsilon_1}\sqrt{\mu^2-\kappa^2}^{\,\epsilon_2}\ \mu^{\epsilon_3}(a_0\mu^{n-\epsilon_1-\epsilon_2-\epsilon_3} + a_2\mu^{n-\epsilon_1-\epsilon_2-\epsilon_3-2} + \ldots);$$

$$\epsilon_\alpha = \begin{cases}0\\1\end{cases} \tag{16}$$

$O_n^m(\mu) = a_0\mu^{n-\epsilon_1-\epsilon_2-\epsilon_3} + \ldots$ is an even polynomial in μ of degree $n - \epsilon_1 - \epsilon_2 - \epsilon_3$. The values v_m are dissimilar solutions of an algebraic equation, resulting from the condition that $O_n^m(\mu)$ is a polynomial. With a view to calculations later on, $E_n^m(\nu)$ is defined as follows:

$$E_n^m(\nu) = \sqrt{1 - \nu^{2\epsilon_1}} \left[\frac{\sqrt{\kappa^2 - \nu^2}}{\kappa}\right]^{\epsilon_2} \left[\frac{\nu}{\kappa}\right]^{\epsilon_3} O_n^m(\nu)$$

$$= \Delta\varphi^{\epsilon_1}(-\cos\varphi)^{\epsilon_2}(\sin\varphi)^{\epsilon_3} O_n^m(\varphi) \tag{16a}$$

by putting

$$\frac{\nu}{\kappa} = \sin\varphi; \quad \frac{\sqrt{\kappa^2 - \nu^2}}{\kappa} = -\cos\varphi; \quad \Delta\varphi = \sqrt{1 - \kappa^2 \sin^2\varphi} \tag{17}$$

The solution

$$\psi'{}_n^m(\mu, \nu, \rho) = E_n^m(\mu) E_n^m(\nu) E_n^m(\rho) \tag{18}$$

achieved with equation (13) can also be represented in different form (reference 5). For, on denoting the zero places of the polynomial $O_n^m(\mu)$ with ρ_s^2, it is readily seen from equations (5) and (6) that

$$E_n^m(\mu) E_n^m(\nu) E_n^m(\rho)$$

$$= \frac{\text{const}}{c^n} \left\{1 \begin{matrix} x \\ y \\ z \end{matrix} \begin{matrix} xy \\ yz \\ zx \end{matrix} \; xyz\right\} \prod_{\rho_s^2} \left(\frac{x^2}{\rho_s^2 - \kappa^2} + \frac{y^2}{\rho_s^2} + \frac{z^2}{\rho_s^2 - 1} - c^2\right) \tag{19}$$

One of the factors contained in the parentheses is selected and the product $\prod_{\rho_s^2}$ is formed over all zero places beonging to $O_n^m(\mu)$. The zero places ρ_s^2 can also be determined direct by applying Laplace's operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ to the right side of equation (19) and making the result equal zero. The system of equations for ρ_s^2 is then as follows:

$$\frac{3^{\epsilon_2}}{\rho_s^2 - \kappa^2} + \frac{3^{\epsilon_3}}{\rho_s^2} + \frac{3^{\epsilon_1}}{\rho_s^2 - 1} + \sum_{q \neq s} \frac{4}{\rho_s^2 - \rho_q^2} = 0 \qquad \begin{matrix} s = 1, 2, \ldots \frac{m}{2} \\ m = n - \epsilon_1 - \epsilon_2 - \epsilon_3 \end{matrix} \tag{20}$$

The potential functions in the respective form of (18) and (19) are so-called "inner" solutions of the potential equation, not suitable for our purposes, since we require potential functions which ordinarily disappear at infinity. These are secured by taking the Lamé function of the second type in the variable ρ. This is the solution of Lamé's equation (15a), which, for $\rho \longrightarrow \infty$ as $\frac{\text{const}}{\rho^{n+1}} \longrightarrow 0$. It has the form

$$F_n^m(\rho) = E_n^m(\rho) \int_\rho^\infty \frac{d\rho}{[E_n^m(\rho)]^2 \sqrt{\rho^2 - 1} \sqrt{\rho^2 - \kappa^2}} \tag{21}$$

The integral can always be reduced to elliptic integrals of the first and second categories only. The aspect of the outer solution of the potential equation is then as follows:

$$\begin{aligned} \psi_n^m(\rho,\mu,\nu) &= E_n^m(\mu)E_n^m(\nu)F_n^m(\rho) \\ &= E_n^m(\mu)E_n^m(\nu)E_n^m(\rho) \int_\rho^\infty \frac{d\rho}{[E_n^m(\rho)]^2 \sqrt{\rho^2 - 1} \sqrt{\rho^2 - \kappa^2}} \end{aligned} \tag{22}$$

and equation (19) yields

$$\psi_n^m(x,y,z) = \frac{\text{const}}{c^n} \left\{ \begin{matrix} & x & xy & \\ 1 & y & yz & xyz \\ & z & zx & \end{matrix} \right\} \prod_{\rho_s^2} \left(\frac{x^2}{\rho_s^2 - \kappa^2} + \frac{y^2}{\rho_s^2} + \frac{z^2}{\rho_s^2 - 1} - c^2 \right) \int_\rho^\infty \frac{d\rho}{[E_n^m(\rho)]^2 \sqrt{\rho^2 - 1} \sqrt{\rho^2 - \kappa^2}} \tag{23}$$

From the representation of the potential function by (4a) as source-sink superposition on the elliptic disk, it is apparent that the potential functions in the plane $z = 0$ in the outside zone of the disk must be zero. Hence ψ_n^m containing factor z must be taken according to (23) i.e., only such Lamé functions as are of the form (cf. (6) and (16)):

$$E_n^m(\mu) = \sqrt{1 - \mu^2}\, M_n^m(\mu) \tag{24}$$

Lamé's functions have orthogonality characteristics similar to those of the spherical functions (reference 3, vol. I, pp. 369 and 379):

$$\int_{-\kappa}^{+\kappa} E_n{}^m(\nu)\, E_n{}^t(\nu) \frac{d\nu}{\sqrt{1-\nu^2}\,\sqrt{\kappa^2-\nu^2}} = \begin{cases} 0, & \text{if } m \neq t \\ J_n{}^m, & \text{if } m = t \end{cases} \tag{25}$$

$$\int_{\mu=\kappa}^{1}\int_{\nu=-\kappa}^{+\kappa} E_n{}^m(\mu)E_n{}^m(\nu)E_s{}^t(\mu)E_s{}^t(\nu)\frac{\mu^2-\nu^2}{\sqrt{1-\mu^2}\sqrt{1-\nu^2}\sqrt{\mu^2-\kappa^2}\sqrt{\kappa^2-\nu^2}}\,d\mu\, d\nu$$

$$= \begin{cases} 0, & \text{if } n \neq s \\ & \text{or } m \neq t \\ I_n{}^m, & \text{if } n = s \\ & \text{and } m = t \end{cases} \tag{26}$$

The connection with equation (4a) is established by having recourse to the following representation of $\frac{1}{R}$ $\left(R = \sqrt{(x-x_F)^2 + (y-y_F)^2 + (z-z_F)^2}\right)$ (reference 3, v. II, p. 172):

$$\frac{1}{R} = \frac{\pi}{2c}\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} E_n{}^m(\mu)E_n{}^m(\nu)F_n{}^m(\rho)E_n{}^m(\mu_F)E_n{}^m(\nu_F)E_n{}^m(\sigma_F) \qquad \rho > \sigma_F \tag{27}$$

whence

$$\frac{\partial}{\partial z_F}\left(\frac{1}{R}\right)_{z_F=0}$$

$$= \frac{\pi\sqrt{1-\kappa^2}}{2c^2}\sum_{n=1}^{\infty}\sum_{m} E_n{}^m(\mu)E_n{}^m(\nu)F_n{}^m(\rho)E_n{}^m(\mu_F)E_n{}^m(\nu_F)\frac{M_n{}^m(1)}{\sqrt{1-\mu_F{}^2}\sqrt{1-\nu_F{}^2}}$$

leaving the summation over $E_n{}^m(\mu)$ in the form (24) to be effected. Then by assuming that the source-sink distribution on the elliptic disk approaches zero on the edge with the root from the edge distance, i.e., with $\sqrt{1-\mu_F{}^2}$, $\sigma(x_F,y_F)$ can be developed conformably to products of Lamé's function of the type (24):

$$\sigma(x_F,y_F) = \sum_{s=1}^{\infty}\sum_{t} A_s{}^t\, E_s{}^t(\mu_F)E_s{}^t(\nu_F) \tag{28}$$

after which the formulation of the integral (4a) gives, based on the orthogonality of Lamé's products (cf. (26)),

$$\psi(\mu,\nu,\rho) = \frac{\pi}{2}\sqrt{1-\kappa^2} \sum_{n=1}^{\infty} \sum_{m} A_n^m I_n^m M_n^m(1) E_n^m(\mu) E_n^m(\nu) F_n^m(\rho) \tag{29}$$

Equation (29) proves the potential function (4a) to be a certain linear combination of the functions ψ_n^m, which are analyzed next.

<u>Representation of the potential function</u> ψ_n^m <u>as a definite integral</u>.- If

$$X = \sum_{\alpha=1}^{3} \frac{1}{(e_\alpha - e_\beta)(e_\alpha - e_\gamma)} \sqrt{\gamma(t)-e_\alpha}\,\sqrt{\gamma(u)-e_\alpha}\,\sqrt{\gamma(v)-e_\alpha}\,\sqrt{\gamma(w)-e_\alpha} \tag{30}$$

(e_α defined by equation (11)) or if (6) and (9) are taken into account

$$X = \frac{\sqrt{\gamma(t)-e_2}}{\sqrt{e_2-e_1}\sqrt{e_2-e_3}}\,\frac{x}{c} + \frac{\sqrt{\gamma(t)-e_3}}{\sqrt{e_3-e_1}\sqrt{e_3-e_2}}\,\frac{y}{c} + \frac{\sqrt{\gamma(t)-e_1}}{\sqrt{e_1-e_2}\sqrt{e_1-e_3}}\,\frac{z}{c} \tag{31}$$

it can be shown that

$$\psi_n^m(x,y,z) = \frac{1}{2\pi i}\int_{\pi_1}^{\pi_2} Q_n(X)\; E_n^m(t)\,dt \tag{32}$$

$$E_n^m(t) = \sqrt{\gamma(t)-e_1}\left(\frac{\sqrt{\gamma(t)-e_2}}{i\sqrt{e_2-e_3}}\right)^{\epsilon_2}\left(\frac{-\sqrt{\gamma(t)-e_3}}{\sqrt{e_2-e_3}}\right)^{\epsilon_3}$$

$$\times \text{ polynomial } [\gamma(t)] \quad \epsilon_\alpha = \begin{cases} 0 \\ 1 \end{cases} \tag{33}$$

if a loop about the points of the complex t plane corresponding to $X = \pm 1$ is taken as integration path from π_1 toward π_2. $Q_n(X)$ satisfies Legendre's differential equation

$$(1-X^2)\frac{d^2 Q_n(X)}{dX^2} - 2X\frac{d\,Q_n(X)}{dX} + n(n+1)\,Q_n(X) = 0 \tag{34}$$

and $E_n^m(t)$ complies with Lamé's differential equation (15). Then

$$\frac{\partial^2 Q_n(X)}{\partial u^2} - \frac{\partial^2 Q_n(X)}{\partial t^2} = n(n+1)[\gamma(u) - \gamma(t)]\, Q_n(X) \qquad (35)$$

for any two variables each of t, u, v, w; i.e.,

$$E_n{}^m(t)\left[\frac{\partial^2 Q_n(X)}{\partial u^2} - \left\{A_m + n(n+1)\gamma(u)\right\} Q_n(X)\right]$$
$$= E_n{}^m(t)\,\frac{\partial^2 Q_n(X)}{\partial t^2} - Q_n(X)\,\frac{d^2 E_n{}^m(t)}{dt^2} \qquad (36)$$

and

$$\frac{\partial^2}{\partial u^2}\,\psi_n{}^m(u,v,w) - [A_m + n(n+1)\gamma(u)]\,\psi_n{}^m(u,v,w)$$
$$= \frac{1}{2\pi i}\left[E_n{}^m(t)\,\frac{\partial Q_n(X)}{\partial t} - Q_n(X)\,\frac{d\,E_n{}^m(t)}{d\,t}\right]_{\pi_1}^{\pi_2} \qquad (37)$$

The same holds true if u is replaced by v or w. The integration path is next so chosen that

$$\left[E_n{}^m(t)\,\frac{\partial Q_n(X)}{\partial t} - Q_n(X)\,\frac{d\,E_n{}^m(t)}{dt}\right]_{\pi_1}^{\pi_2} = 0$$

i.e., $\psi_n{}^m$ in every one of the three variables u, v, w, satisfies Lamé's equation (15) and is accordingly a third representation of the potential function defined by (22). The points $X = \pm 1$ in plane t are given by

$$t = v + w + u \quad (X = -1); \qquad t = v + w - u \quad (X = +1) \qquad (38)$$

On the elliptic disk $(\rho = 1)$, $\gamma(u) = e_1$; i.e.,

$$u = \int_{\infty}^{1} \frac{d\,\rho}{\sqrt{\rho^2 - 1}\,\sqrt{\rho^2 - \kappa^2}} = -\omega_1 \qquad (39)$$

The potential function $\psi_1{}^1$ is cited as an example. The sole Lamé function of the first type and first degree equipped with the factor $\sqrt{\rho^2 - 1}$ is:

$$E_1{}^1(\rho) = \sqrt{\rho^2 - 1}$$

According to (22) and (23), respectively, the potential function then reads, respectively:

$$\psi_1{}^1(\rho,\mu,\nu)=\sqrt{\rho^2-1}\sqrt{1-\mu^2}\sqrt{1-\nu^2}\int_\rho^\infty \frac{d\rho}{(\rho^2-1)\sqrt{\rho^2-1}\sqrt{\rho^2-\kappa^2}}$$

and

$$\psi_1{}^1(x,y,z) = \frac{\sqrt{1-\kappa^2}}{c} z \int_\rho^\infty \frac{d\rho}{(\rho^2-1)\sqrt{\rho^2-1}\sqrt{\rho^2-\kappa^2}}.$$

Lift and lift moments.- The lift is given by

$$A = \int_F \int_{ell} (p_u - p_{ob})dxdy = \rho V^2 \iint (\psi_{ob} - \psi_u)dxdy \tag{40}$$

whereby

$$\left.\begin{aligned} \psi_n{}^m\Big|_{\rho=1} &= E_n{}^m(\mu)E_n{}^m(\nu)F_n{}^m(1) = \frac{\pm E_n{}^m(\mu)E_n{}^m(\nu)}{M_n{}^m(1)\sqrt{1-\kappa^2}} \\ &= \pm\frac{1}{c}\sqrt{c^2 - y^2 - \frac{x^2}{1-\kappa^2}}\;\frac{M_n{}^m(\mu)M_n{}^m(\nu)}{M_n{}^m(1)} \end{aligned}\right\} \tag{41}$$

Based on the orthogonality of Lamé's functions, only $\psi_1{}^1$ contributes to the total lift

$$A = \frac{8}{3}\,\frac{\rho}{2}\,V^2\,F_{ell} \tag{42}$$

$\psi_2{}^1$ furnishes the pitching moment about the y axis

$$M = \frac{8}{15}\,c\,\sqrt{1-\kappa^2}\,\frac{\rho}{2}\,V^2\,F_{ell} \tag{43}$$

$\psi_2{}^{-1}$, the rolling moment about the x axis

$$L = \frac{8}{15}\,c\,\frac{\rho}{2}\,V^2\,F_{ell} \tag{44}$$

(The negative index refers to the odd functions in y.)

The elliptic wing in straight flow.- Assume the elliptic wing in a stream in direction of the positive x axis with velocity V. Now equation (1) enables the calculation of the velocities induced by the pressure potential ψ in space and especially on the lifting surface. The z component of equation (1) reads in the stationary case

$$(V + u)\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} = V^2\frac{\partial \psi}{\partial z} \tag{45}$$

Small quantities of higher order are disregarded, i.e.,

$$\frac{\partial w}{\partial x} = V\frac{\partial \psi}{\partial z} \tag{46}$$

The z component w of the velocity vector $\underline{w}$ is hereafter called "downwash" for short. The downwash on the elliptic surface is obtained by integration of equation of equation (46) for $z = 0$ and $y \leq c$ over x:

$$\frac{w}{V} = \int_{-\infty}^{x} \frac{\partial \psi}{\partial z}\, d\, x \tag{47}$$

The calculation of the integral is readily secured by having recourse to the representation (32) of the potential function $\psi_n{}^m$. After formulating $\frac{\partial \psi_n{}^m}{\partial z}$ by differentiation below the integral sign, equation (47) gives

$$\left.\begin{aligned} \frac{w_n{}^m}{V} = \frac{1}{2\pi i}\int_{-\infty}^{x}\int_{\square} \frac{\partial}{\partial z} Q_n\Bigg(&\frac{\sqrt{\gamma(t) - e_2}}{\sqrt{e_2 - e_1}\,\sqrt{e_2 - e_3}}\,\frac{x}{c} \\ &+ \frac{\sqrt{\gamma(t) - e_3}}{\sqrt{e_3 - e_1}\,\sqrt{e_3 - e_2}}\,\frac{y}{c} + \frac{\sqrt{\gamma(t) - e_1}}{\sqrt{e_1 - e_2}\,\sqrt{e_1 - e_3}}\,\frac{z}{c}\Bigg) E_n{}^m(t)\,dt\,dx \end{aligned}\right\} \tag{48}$$

Now

$$\frac{\partial Q_n(X)}{\partial z} = \frac{dQ_n(X)}{dX}\,\frac{\partial X}{\partial z} = \frac{dQ_n(X)}{dX}\,\frac{\sqrt{\gamma(t) - e_1}}{\sqrt{e_1 - e_2}\,\sqrt{e_1 - e_3}}\,\frac{1}{c}$$

$$\frac{\partial Q_n(X)}{\partial x} = \frac{dQ_n(X)}{dX}\,\frac{\partial X}{\partial x} = \frac{dQ_n(X)}{dX}\,\frac{\sqrt{\gamma(t) - e_2}}{\sqrt{e_2 - e_1}\,\sqrt{e_2 - e_3}}\,\frac{1}{c}$$

that is

$$\frac{\partial Q_n(X)}{\partial z} = i\,\kappa\,\frac{\partial Q_n(X)}{\partial x}\,\frac{\sqrt{\gamma(t) - e_1}}{\sqrt{\gamma(t) - e_2}} \qquad (49)$$

which, when inserted in equation (48) and followed by integration over x, affords - since $\lim Q_n(X) = 0$ for $X \to \infty$ -

$$\frac{w_n{}^m}{V} = \frac{1}{2\pi i}\,i\,\kappa \int_{\boxed{\leftarrow}} Q_n(X)\,\frac{\sqrt{\gamma(t) - e_1}}{\sqrt{\gamma(t) - e_2}}\,E_n{}^m(t)\,dt \qquad (50)$$

$Q_n(X)$ is given by Neumann's representation

$$Q_n(X) = \frac{1}{2}\int_{-1}^{+1} \frac{P_n(Y)}{X - Y}\,dY \qquad (51)$$

Similarly $X(t)$ is expressed with

$$Y(s) = \frac{\sqrt{\gamma(s) - e_2}}{\sqrt{e_2 - e_1}\sqrt{e_2 - e_3}}\,\frac{x}{c} + \frac{\sqrt{\gamma(s) - e_3}}{\sqrt{e_3 - e_1}\sqrt{e_3 - e_2}}\,\frac{y}{c} + \frac{\sqrt{\gamma(s) - e_1}}{\sqrt{e_1 - e_2}\sqrt{e_1 - e_3}}\,\frac{z}{c} \qquad (52)$$

whence (50) becomes

$$\frac{w_n{}^m}{V} = \frac{1}{2\pi i}\,i\,\kappa \int_{\boxed{\leftarrow}} \frac{1}{2}\int_s \frac{P_n(Y(s))}{X(t) - Y(s)}\,\frac{dY}{ds}\,\frac{\sqrt{\gamma(t) - e_1}}{\sqrt{\gamma(t) - e_2}}\,E_n{}^m(t)\,ds\,dt \qquad (53)$$

The integrand has poles at $t = s$, because $X(t) - Y(s) = 0$ and at $t = \omega_1 + i\omega_2$, where $\sqrt{\gamma(t) - e_2}$ has a simple zero place. The behavior of the denominator near the zero place is defined by Taylor expansion. It affords

$$\left.\begin{aligned}\sqrt{p(t) - e_2} &= [t - (\omega_1 + i\omega_2)]\left(\frac{d}{dt}\sqrt{p(t) - e_2}\right)_{t=\omega_1 + i\omega_2} + \ldots \\ &= -[t - (\omega_1 + i\omega_2)]\sqrt{e_2 - e_1}\sqrt{e_2 - e_3} + \ldots\end{aligned}\right\} \qquad (54)$$

taking into account equation (1), as well as

$$X(t) - Y(s) = (t - s)\left(\frac{\partial X}{\partial t}\right)_{t=s} + \ldots = (t - s)\,\frac{\partial Y}{\partial s} + \ldots \qquad (55)$$

according to equation (52). Then the integration of $s = v + w + u$ as far as $s = v + w - u$ corresponding to $Y = -1$ to $Y = +1$ (equation (38)) gives

$$\frac{1}{2}\int \frac{P_n(Y(s))}{X(\omega_1 + i\,\omega_2) - Y(s)}\,\frac{dY}{ds} = Q_n(X(\omega_1 + i\,\omega_2)) \tag{56}$$

By

$$\int P_n(Y(s))\,\frac{\sqrt{\Upsilon(s) - e_1}}{\sqrt{\Upsilon(s) - e_2}}\,E_n{}^m(s)\,d\,s \tag{57}$$

it is to be noted that the integrand for $\rho = 1$, i.e., $u = -\omega_1$ has the period $2\omega_1$, so that the integration path can be shifted until s proceeds from $i\,\omega_2 - \omega_1$ to $i\,\omega_2 + \omega_1$ (fig. 2), which corresponds to $\epsilon = -\frac{\pi}{2}$ to $+\frac{\pi}{2}$, when $\sqrt{\Upsilon(s) - e_3} = -\kappa \sin\epsilon$, that is, $ds = \frac{d\epsilon}{\Delta\epsilon}$ $(\Delta\epsilon = \sqrt{1 - \kappa^2}\,\sin^2\epsilon)$. Moreover, let

$$\frac{x}{c\sqrt{1 - \kappa^2}} = \xi, \quad \frac{y}{c} = \eta \tag{58}$$

so that, because of

$$X(\omega_1 + i\omega_2) = -\frac{\sqrt{e_2 - e_3}}{\sqrt{e_3 - e_1}\sqrt{e_3 - e_2}}\,\frac{y}{c} = \eta \quad (\Upsilon(\omega_1 + i\omega_2) = e_2) \tag{59}$$

and, according to (33):

$$E_n{}^m(\omega_1 + i\omega_2) = i\,E_n{}^m(\lambda)_{\lambda=\kappa} \tag{60}$$

the downwash function on the elliptic disk becomes

$$\frac{w_n{}^m}{V} = E_n{}^m(\kappa)\,Q_n(\eta) - \frac{1}{2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} P_n(\xi\cos\epsilon + \eta\sin\epsilon)\,M_n{}^m(\epsilon)\,\Delta\epsilon\,\frac{d\epsilon}{\cos\epsilon} \tag{61}$$

The coefficients obtained in the polynomial of ξ and η are complete elliptic integrals of the first and second categories. The calculation of the downwash function in the case of $n = 1$ is expressed as:

$$E_1^{\,1}(\lambda) = \sqrt{1 - \lambda^2}, \qquad M_1^{\,1}(\epsilon) = 1$$

$$P_1(Y) = Y, \qquad Q_1(X) = \frac{X}{2} \ln \frac{X+1}{X-1} - 1$$

From (61) follows

$$\frac{w_1^{\,1}(\xi,\eta)}{V} = \sqrt{1 - \kappa^2}\; Q_1(\eta) - \frac{1}{2} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (\xi \cos\epsilon + \eta \sin\epsilon) \frac{\Delta\epsilon}{\cos\epsilon}\, d\epsilon$$

that is,

$$\frac{w_1^{\,1}(\xi,\eta)}{V} = \sqrt{1 - \kappa^2}\; Q_1(\eta) - \xi E\left(\frac{\pi}{2}\right); \qquad E\left(\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \Delta\epsilon\, d\epsilon$$

The lifting surface.- The shape of the surface is given by

$$z = z(x,y)$$

The slope of the surface in x direction must agree with the direction of the flow at the same point, that is,

$$\frac{\partial z(x,y)}{\partial x} = \frac{w(x,y)}{V} \tag{62}$$

from which follows, for $z = z(x,y)$

$$z(x,y) = \frac{1}{V} \int^{x} w(x,y)\, dx \tag{63}$$

the lower integration limit being arbitrary; we equate it to zero and add to the value of the integral an arbitrary function in y:

$$z(x,y) = \frac{1}{V} \int_0^{x} w(x,y)\, dx + g(y) \tag{63a}$$

The lifting line, the induced drag, and the suction force.- We merely refer to the corresponding chapters of

Kinner's report (reference 2), where these problems are treated in detail. The results are readily applicable to the elliptic disk. The lifting line, by which the lifting surface is assumed replaced, is obtained by correlating the lift elements through integration parallel to the x axis:

$$a_n{}^m(\eta) = \int_{-x_R}^{+x_R} (p_u - p_{ob})\, d\, x = 2\,\rho\, V^2 \int_{-x_R}^{+x_R} \psi_n{}^m\Big|_{\rho=1} d\, x$$

It affords, for instance,

$$\left.\begin{aligned} a_1{}^1(\eta) &= 4\pi\, \frac{\rho}{2}\, V^2\, c\, \sqrt{1 - \kappa^2}\, \frac{P_0(\eta) - P_2(\eta)}{3} \\ a_2{}^1(\eta) &\equiv 0 \\ a_3{}^s(\eta) &= 4\pi\, \frac{\rho}{2}\, V^2\, c\, \sqrt{1 - \kappa^2}\, \frac{P_2(\eta) - P_4(\eta)}{3}\, (\kappa^2 - \rho_s{}^2) \\ a_4{}^s(\eta) &\equiv 0 \end{aligned}\right\} \quad (64)$$

<u>The potential function of the second type</u>.- The foregoing potential functions lend themselves in any way to linear combination and yield the corresponding linear combinations for lift, lift moments, and downwash w. The potential functions dealt with so far afford the aerodynamic quantities of a correspondingly curved wing by shock-free entry of flow, that is, at a certain angle of attack where no flow around the leading edge occurs. The arbitrary angle of attack is obtained by superposing a flat elliptic disk with its flow, where, as is known, the leading edge is suction edge, that is, the lift density approaches infinity. All the potential functions of the first type approach zero, however, on the disk edge with the root from the edge distance; hence the task of finding potential functions that have these qualities. They are achieved by applying on the potential functions of the first type at constant x, y, z, and κ^2 the following boundary transition (reference 2):

$$\Phi_n{}^m = \frac{1}{c^{n-1}}\, \frac{d}{dc}\, [c^n\, \psi_n{}^m(x,y,z,c)] \qquad (65)$$

These potential functions Φ_n^m possess the quality of becoming infinite on the whole border of the disk. Because of the condition of smooth efflux on the trailing edge, it later is necessary to combine the functions linearly.

<u>The downwash function of the second type</u>.- In conformity with equation (65), the downwash functions are obtained by applying the operator $\frac{1}{c^{n-1}}\frac{d}{dc}c^n$ to equation (61), while observing the interrelationship:

$$\frac{1}{c^{n-1}}\frac{d}{dc}\left[c^n P_n(\eta)\right] = n\, P_n(\eta) - \eta\,\frac{d\, P_n(\eta)}{d\,\eta} = -\frac{d\, P_{n-1}(\eta)}{d\,\eta}$$

$$\eta = \frac{y}{c}$$

The same applies to $P_n(Y)$ and $Q_n(\eta)$. Equation (61) then gives

$$\frac{w_n^{m\,II}}{V} = -E_n^m(\kappa)\frac{d\, Q_{n-1}(\eta)}{d\,\eta} + \frac{1}{2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{d\, P_{n-1}(Y)}{d\,Y}\, M_n^m(\epsilon)\,\Delta\epsilon\,\frac{d\,\epsilon}{\cos\,\epsilon} \tag{66}$$

In dealing with the second fundamental problem, that is, in the calculation of a prescribed wing, potential functions are used, the downwash functions of which are independent of x on the disk. According to (46), this implies, since $\frac{\partial w}{\partial x} = 0$, that

$$\left.\frac{\partial\psi}{\partial z}\right|_{\rho=1} \equiv 0 \tag{67}$$

With coefficient b_n^m still to be defined, we put

$$\Phi_n(x,y,z) = \sum_m b_n^m\,\Phi_n^m(x,y,z) \qquad \Phi_{-n}(x,y,z) = \sum_{-m} b_n^{-m}\Phi_n^{-m}(x,y,z) \tag{68}$$

Correspondingly it is:

$$w_n = \sum_m b_n^m\, w_n^{m\,II}, \qquad w_{-n} = \sum_{-m} b_n^{-m} w_n^{-m\,II} \tag{68a}$$

$w_n(x,y)$ is designated as downwash function of the potential function of the second type Φ_n. According to (68a) and (66), it is:

$$\frac{w_n(x,y)}{V} = -\sum_m b_n{}^m E_n{}^m(\kappa)\ \frac{dQ_{n-1}(\eta)}{d\eta}$$

$$+\frac{1}{2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{d\ P_{n-1}(Y)}{d\ Y} \sum_m b_n{}^m M_n{}^m(\epsilon)\ \Delta\epsilon\ \frac{d\ \epsilon}{\cos\ \epsilon} \qquad (69)$$

The coefficients $b_n{}^m$ are now so defined that w_n is a function of η only. The first term in (69) already depends on η only; hence the second term itself, which usually depends on ξ also, must be a function of η only. $\frac{d\ P_{n-1}(Y)}{d\ Y}$ is a polynomial in Y with terms of the form

$$Y^{n-2p} = [\xi \cos\epsilon + \eta \sin\epsilon]^{n-2p}$$

hence is a sum of terms of the form

$$(\cos\epsilon)^{n-2p-\alpha}\ (\sin\epsilon)^{\alpha}\ \xi^{n-2p-\alpha}\ \eta^{\alpha} \qquad (70)$$

For the following arguments it is assumed that $M_n{}^m(\epsilon)$ is even in ϵ; so that all terms with odd powers of $\sin\epsilon$ disappear in the integration from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$. But if α is even, $(\sin\epsilon)^{\alpha} = 1 +$ sum of cos terms. When this is written in (70), Y^{n-2p} consists of the following summands:

$$(\cos\ \epsilon)^{n-2q}\ \xi^{n-2p-\alpha}\ \eta^{\alpha} \qquad (70a)$$

and the condition that

$$n - 2p - \alpha \leq n - 2\ q \leq n - 2\ p \qquad (71)$$

Putting

$$\sigma = \frac{n}{2} \text{ and } = \frac{n+1}{2}, \text{ respectively } (\sigma \text{ is an integer}) \qquad (72)$$

and stipulating that

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} (\cos\epsilon)^{n-2q} \sum_m b_n{}^m M_n{}^m(\epsilon)\ \Delta\epsilon\ \frac{d\ \epsilon}{\cos\ \epsilon} = 0; \quad q = 1,2,\ldots(\sigma-1) \qquad (73)$$

causes all terms containing the powers of $\cos \epsilon$, i.e., powers of ξ, to disappear. The sole nondisappearing summand of Y^{n-2p} is obtained when $n - 2q = 0$, that is, when first n is even and, according to (71) $n - 2p - \alpha = 0$; this then reads, according to (70a), η^{n-2p} and depends no longer on ϵ, so that it can be put before the integral. Combining the summands before the integral, which now has the same value for every p, there is obtained conformably to $\frac{d\ P_{n-1}(Y)}{d\ Y}$ in (69) for the integral

$$\frac{d\ P_{n-1}(\eta)}{d\ \eta} \frac{1}{2} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sum_m b_n^m\ M_n^m\ (\epsilon)\ \Delta\epsilon \frac{d\ \epsilon}{\cos\ \epsilon}$$

(The integral disappears for odd n.)

$M_n^{-m}(\epsilon)$ contains the factor $\sin \epsilon$; hence it is odd in ϵ. Considerations corresponding to the foregoing then give the condition

$$\left.\begin{array}{l} (\cos\ \epsilon)^{n-2q-1} \sin\ \epsilon \sum_{-m} b_n^{-m}\ M_n^{-m}(\epsilon)\ \Delta\epsilon \dfrac{d\ \epsilon}{\cos\ \epsilon} = 0, \\ q = 1,2,\ldots,(\tau - 1); \quad \tau = \dfrac{n}{2} \text{ and } = \dfrac{n-1}{2}, \\ \text{respectively} \end{array}\right\} \quad (74)$$

The value of the integral in (69) is other than zero only if n is odd. Summed up, it affords, by attention to $E^m_{2r}(\kappa) = E^{-m}_{2r+1}(\kappa) = 0$ (equation (16)):

$$\frac{1}{V} w_{2r+1}(\eta) = k_{2r+1} \frac{d\ Q_{2r}(\eta)}{d\ \eta}; \quad \frac{1}{V} w_{2r}(\eta) = i_{2r} \frac{d\ P_{2r-1}(\eta)}{d\ \eta}$$

$$\frac{1}{V} w_{-(2r+1)}(\eta) = j_{2r+1} \frac{d\ P_{2r}(\eta)}{d\ \eta}; \quad \frac{1}{V} w_{-2r}(\eta) = l_{2r} \frac{d\ Q_{2r-1}(\eta)}{d\ \eta}$$

whereby

$$\left.\begin{aligned}
k_{2r+1} &= -\sum_m b_{2r+1}^m \, E_{2r+1}^m \, (\kappa) \\
i_{2r} &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sum_m b_{2r}^m \, M_{2r}^m \, (\epsilon) \, \Delta\epsilon \, \frac{d\,\epsilon}{\cos\,\epsilon} \\
j_{2r+1} &= \frac{1}{2} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin\epsilon \sum_{-m} b_{2r+1}^{-m} \, M_{2r+1}^{-m}(\epsilon) \, \Delta\epsilon \, \frac{d\,\epsilon}{\cos\,\epsilon} \\
l_{2r} &= -\sum_{-m} b_{2r}^{-m} \, E_{2r}^{-m} \, (\kappa)
\end{aligned}\right\} \quad (75)$$

b_n^m and b_n^{-m} satisfy equations (73) and (74), respectively. They are $\sigma - 1$ homogeneous equations for the σ unknown b_n^m, and $\tau - 1$ equations for the τ unknown b_n^{-m}, respectively, which can be determined therefrom up to a common constant factor. The latter is so chosen that

$$k_{2r+1} = l_{2r} = -\sqrt{1 - \kappa^2}; \quad i_{2r} = j_{2r+1} = 1 \qquad (75a)$$

whence

$$\left.\begin{aligned}
&\frac{1}{V} w_{2r+1}(\eta) = -\sqrt{1-\kappa^2}\,\frac{d\;Q_{2r}(\eta)}{d\eta}; \quad \frac{1}{V} w_{2r}(\eta) = \frac{d\;P_{2r-1}(\eta)}{d\;\eta} \\
&\frac{1}{V} w_{-(2r+1)}(\eta) = \frac{d P_{2r}(\eta)}{d\;\eta}; \quad \frac{1}{V} w_{-2r}(\eta) = -\sqrt{1-\kappa^2}\,\frac{d\;Q_{2r-1}(\eta)}{d\;\eta}
\end{aligned}\right\} \quad (76)$$

Lift, lift moments, and the lifting lines of the potential function of the second type.- These quantities are obtained by the application of the operator $\frac{1}{c^{n-1}} \frac{d}{dc} c^n$ to the corresponding quantities of the potential factor of the first type (42, 43, 44, 64). It is pointed out that, during the differentiation, the areal content of the ellipse F_{ell} embodies the factor c^2. Then, bear in mind that (equations (68), (75), (75a))

Φ gives the lift $A = 8 \, \frac{\rho}{2} \, V^2 \, F_{ell}$ (77)

Φ_2 the pitching moment $M = \dfrac{1}{E\left(\kappa^2, \frac{\pi}{2}\right)} \dfrac{8}{3} c \sqrt{1-\kappa^2} \dfrac{\rho}{2} V^2 F_{ell}$ (78)

Φ_{-2} the rolling moment $L = \dfrac{8}{3} c \dfrac{\rho}{2} V^2 F_{ell}$ (79)

$$E\left(\kappa^2, \frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - \kappa^2 \sin^2 \epsilon}\, d\epsilon$$

and for the lifting lines

$$\left.\begin{aligned} a_1(\eta) &= 4\pi \frac{\rho}{2} V^2 c\sqrt{1 - \kappa^2}\, P_0(\eta) \\ a_2(\eta) &\equiv 0 \\ a_3(\eta) &= 4\pi \frac{\rho}{2} V^2 c\sqrt{1 - \kappa^2}\, P_2(\eta) \\ a_4(\eta) &\equiv 0 \end{aligned}\right\} \tag{80}$$

<u>The second fundamental problem of airfoil theory</u>.- This involves the calculation of the aerodynamic quantities of any given elliptic wing by means of the foregoing potential functions of the first and second types. The boundary condition to be met is given by

$$\frac{\partial z(x,y)}{\partial x} = \frac{w(x,y)}{V} = \int_{-\infty}^{x} \frac{\partial \psi}{\partial z}\, dx \tag{62a}$$

which, differentiated, leads to (46):

$$\frac{\partial w}{\partial x} = V \frac{\partial \psi}{\partial z} \tag{46}$$

Posting ψ as a linear combination of potential functions of the first and second types, leaves

$$\psi = \sum_{n=1} \sum_{m} a_n{}^m \psi_n{}^m + \sum_n C_n \Phi_n + \sum_n D_n \Phi_{-n} \tag{81}$$

leaves on the disk $(\rho = 1)$

$$\left.\frac{\partial \psi}{\partial z}\right|_{\rho=1} = \sum_n \sum_m a_n{}^m \frac{\partial \psi_n{}^m}{\partial z}, \quad \text{since } \left.\frac{\partial \Phi_n}{\partial z}\right|_{\rho=1} = \left.\frac{\partial \Phi_{-n}}{\partial z}\right|_{\rho=1} \equiv 0 \tag{81a}$$

where then, however,

$$\left.\frac{\partial\psi_n{}^m}{\partial z}\right|_{\rho=1} = \frac{\sqrt{1-\kappa^2}}{c}\,\frac{E_n{}^m(\mu)\,E_n{}^m(\nu)}{\sqrt{1-\mu^2}\,\sqrt{1-\nu^2}}\left[\frac{d\,F_n{}^m(\rho)}{d(\sqrt{\rho^2-1})}\right]_{\rho=1} \tag{82}$$

Now, if $\frac{\partial w}{\partial x}$ is developed according to the Lamé products

$$\sqrt{1-\mu^2}\,\sqrt{1-\nu^2}\,\frac{\partial w}{\partial x} = \frac{V}{c}\sum_n\sum_m g_n{}^m\,E_n{}^m(\mu)\,E_n{}^m(\nu);$$

$$E_n{}^m(\mu) = \sqrt{1-\mu^2}\,M_n{}^m(\mu)$$

then, because of equations (46), (81a), and (82), based upon the orthogonality of Lamé's product,

$$g_n{}^m = \sqrt{1-\kappa^2}\left[\frac{d\,F_n{}^m(\rho)}{d(\sqrt{\rho^2-1})}\right]_{\rho=1} a_n{}^m \tag{83}$$

This defines the coefficients $a_n{}^m$ of the potential function of the first type. On expanding $z = z(\xi,\eta)$ in a series of ξ and η, the $a_n{}^m$'s are determined by a comparison of coefficients in all terms affected with powers of ξ in the form of a screen method (reference 2).

From equation (46)

$$\frac{1}{V}\,w(\xi,\eta) - \sum_{n,m} a_n{}^m \int_{-\infty} \frac{\partial\psi_n{}^m}{\partial z}\,dx = \frac{1}{V}\,w(\eta) \tag{84}$$

is now only a function of η. This residuary condition is complied with through a suitable linear combination of the downward function of the second type, which depends solely on η. The condition reads

$$\sum_n C_n\,w_n(\eta) + \sum_n D_n\,w_{-n}(\eta) = w(\eta) \tag{85}$$

This equation is integrated from $\eta = 0$ to η, and then - multiplied by $P_{2\alpha-1}(\eta)$ and $P_{2\gamma}(\eta)$, respectively - integrated again from $\eta = -1$ to $+1$, and so yields through the intermediary of the inter-relations

$$\int_{-1}^{+1} P_{2r-1}(\eta)\ P_{2\alpha-1}(\eta)\ d\eta = \begin{cases} \dfrac{2}{2(2\alpha-1)+1} = \dfrac{2}{4\alpha-1} & \text{if } r = \alpha \\ 0 & \text{if } r \neq \alpha \end{cases}$$

$$\int_{-1}^{+1} Q_{2r}(\eta)\ P_{2\alpha-1}(\eta)\, d\eta = \frac{1}{\alpha(2\alpha-1) - r(2r+1)}$$

two infinite systems of equations for the coefficients C_n and D_n

$$\left.\begin{aligned} &\sum_{r=0} C_{2r+1} \frac{-\sqrt{1-\kappa^2}}{\alpha(2\alpha-1)-r(2r+1)} + C_{2\alpha}\frac{2}{4\alpha-1} = \frac{1}{V}\int_{-1}^{+1}\int_{0}^{\eta} w(\eta')\,d\eta'\,P_{2\alpha-1}(\eta)\,d\eta \\ &\qquad\qquad \alpha = 1,2,\ldots \\ &D_{2\gamma+1}\frac{1}{4\gamma+1} + \sum_{r=1} D_{2r}\frac{-\sqrt{1-\kappa^2}}{\gamma(2\gamma+1)-r(2r-1)} = \frac{1}{V}\int_{-1}^{+1}\int_{0}^{\eta} w(\eta')\,d\eta'\,P_{2\gamma}(\eta)\,d\eta \\ &\qquad\qquad \gamma = 1,2,\ldots \end{aligned}\right\} \quad (86)$$

<u>The outflow condition.</u>- Apart from the compliance of the equation systems (86), the coefficients C_n and D_n must also satisfy the outflow condition, that is, they must be so chosen that the lift density disappears on the trailing edge. The potential function of the first type satisfies this condition without that, since they disappear on the whole edge. The behavior of $\Phi_n{}^m$ on the boundary is known with the application of the differentiation process (65) to (41).

$$[\Phi_n{}^m]_{\substack{\rho=1\\ \mu\approx 1}} = \frac{c}{\sqrt{c^2 - y^2 - \dfrac{x^2}{1-\kappa^2}}}\, M_n{}^m(\nu) + ((\sqrt{1-\mu^2}))$$

(The term $((\sqrt{1-\mu^2}))$ disappears with the root from the edge distance.) We post, respectively,

$$\sum_m b_n{}^m\, M_n{}^m(\nu) = \underline{M}_n(\nu) \quad \text{and} \quad \sum_m b_n{}^m\, M_n{}^m(\varphi) = M_n(\varphi) \qquad (87)$$

whence the potential function of the second type becomes

$$[\Phi_n]_{\substack{\rho=1\\ \mu\approx 1}} = \sum_m b_n{}^m\, \Phi_n{}^m\Big|_{\rho=1} = \frac{c}{\sqrt{c^2 - y^2 - \dfrac{x^2}{1-\kappa^2}}}\, \underline{M}_n(\varphi) + ((\sqrt{1-\mu^2})) \qquad (88)$$

Now the functions $\underline{M}_n(\varphi)$ have orthogonality properties similar to the trigonometric functions, as may be proved by (73) and (74) if it is remembered that $\underline{M}_n(\varphi)$ can be written in the form

$$\underline{M}_n(\varphi) = (\sin\varphi)^{\epsilon_3}[a_0(\cos\varphi)^{n-\epsilon_3-1} + a_2(\cos\varphi)^{n-\epsilon_3-3} + \dots]$$

It is

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \underline{M}_\gamma(\varphi)\ \underline{M}_n(\varphi)\ \Delta\varphi\ d\varphi = 0, \quad \text{if} \quad n - \gamma = \pm 2, \pm 4, \dots \qquad (89)$$

The outflow condition reads, according to (88):

$$\sum_{r=1}^{\infty} C_{2r}\ \underline{M}_{2r}(\varphi) + \sum_{r=0}^{\infty} C_{2r+1}\ \underline{M}_{2r+1}(\varphi) \equiv 0 \qquad \frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2} \qquad (90a)$$

$$\sum_{r=1}^{\infty} D_{2r}\ \underline{M}_{-2r}(\varphi) + \sum_{r=1}^{\infty} D_{2r+1}\ \underline{M}_{-(2r+1)}(\varphi) \equiv 0 \quad \frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2} \qquad (90b)$$

which, after multiplication by $\underline{M}_{2\alpha-1}(\varphi)\,\Delta\varphi$ and $\underline{M}_{-2\gamma}(\varphi)\,\Delta\varphi$ respectively, and integration from $\varphi = \frac{\pi}{2}$ to $\frac{3\pi}{2}$, give

$$C_{2\alpha-1} I_{2\alpha-1,2\alpha-1} + \sum_{r=1} C_{2r}\ I_{2r,2\alpha-1} = 0 \qquad \alpha = 1,2,\dots \qquad (92a)$$

$$D_{2\gamma}\ I_{-2\gamma,-2\gamma} + \sum_{r=1} D_{2r+1}\ I_{-(2r+1),-2\gamma} = 0 \qquad \gamma = 1,2,\dots \qquad (92b)$$

whereby

$$I_{\gamma,\delta} = \int_{+\frac{\pi}{2}}^{+\frac{3\pi}{2}} \underline{M}_\gamma(\varphi)\ \underline{M}_\delta(\varphi)\ \Delta\varphi\ d\varphi \qquad (93)$$

The infinite equation systems (86) together with (92) then enable the determination of the coefficients of the potential functions C_n and D_n of the second type. These coefficients in conjunction with the previously computed coefficients a_n^m of the functions of the first type make the prediction of the aerodynamic quantities of the given wing possible, as will be illustrated on a model case.

Note.- The calculation is made on the assumption that the flow strikes the elliptic disk parallel to the minor principal axis. The length of the principal axis in y direction was c, in x direction $c\sqrt{1-\kappa^2}$. The calculation is readily applicable to the case of elliptic wing in flow parallel to the major principal axis when κ^2 is assumed negative, that is, supposing the aerodynamic center on the y axis. The conversion formulas for the elliptic integrals with negative κ^2 are:

$$\left.\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{d\,\epsilon}{\sqrt{1+(-\kappa^2)\sin^2\epsilon}} &= \frac{1}{\sqrt{1-\kappa^2}} \int_0^{\frac{\pi}{2}} \frac{d\,\epsilon}{\sqrt{1-\frac{-\kappa^2}{1-\kappa^2}\sin^2\epsilon}} \\ \int_0^{\frac{\pi}{2}} \sqrt{1+(-\kappa^2)\sin^2\epsilon}\,d\,\epsilon &= \sqrt{1-\kappa^2} \int_0^{\frac{\pi}{2}} \sqrt{1-\frac{-\kappa^2}{1-\kappa^2}\sin^2\epsilon}\,d\,\epsilon\end{aligned}\right\} \qquad (94)$$

The integrals are as far as the factor before the integral, the same as for the reciprocal axes ratio.

<u>The flat elliptic disk in straight flow</u>.- Let the angle of attack be α_0, that is, the elliptic area is given by

$$z = -x \tan \alpha_0 \approx -x\,\alpha_0$$

Then we have, according to (62),

$$\frac{w(x,y)}{V} = -\alpha_0; \quad \frac{dw}{dx} = 0$$

on the disk. According to (83) therefore, the coefficients a_n^m of the potential function of the first type are all zero; those of the second type must be computed conformably to (73, 74, 75a), as exemplified here for $n = 3$ and $n = -3$.

For an axis ratio of

$$\sqrt{1-\kappa^2} = \frac{1}{5} \qquad \kappa^2 = 0.96, \quad \text{that is,} \quad \Lambda = \frac{(2c)^2}{F_{ell}} = \frac{4}{\pi\sqrt{1-\kappa^2}} = 6.37$$

the complete elliptic integral is:

$$F = \int_0^{\frac{\pi}{2}} \frac{d\,\epsilon}{\sqrt{1-0.96\sin^2\epsilon}} = 3.01611;$$

$$E = \int_0^{\frac{\pi}{2}} \sqrt{1-0.96\sin^2\epsilon}\,d\epsilon = 1.05050$$

In the case of $n = 3$ and $\epsilon_1 = 1;\ \epsilon_2 = \epsilon_3 = 0$

$$E_3^s(\nu) = \sqrt{1-\nu^2}\,(\nu^2 - \rho_s^2) \qquad s = 1;2$$

$$M_3^s(\nu) = \nu^2 - \rho_s^2 \qquad M_3^s(\epsilon) = \kappa^2\sin^2\epsilon - \rho_s^2$$

and according to (20) it is:

$$\frac{1}{\rho_s^2-\kappa^2}+\frac{1}{\rho_s^2}+\frac{3}{\rho_s^2-1}=0, \quad \text{that is,} \quad \rho_1^2 = 0.19792; \quad \rho_2^2 = 0.97008$$

Conditions (73) and (75a) then read:

$$b_3^{(1)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(0.96\,\sin^2\epsilon - 0.19792)\Delta\epsilon\, d\epsilon + b_3^{(2)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(0.96\,\sin^2\epsilon - 0.97008)\,\Delta\epsilon\, d\epsilon = 0$$

$$-\sqrt{1-\kappa^2} = -b_3^{(1)}\sqrt{1-\kappa^2}(0.96-0.19792)-b_3^{(2)}\sqrt{1-\kappa^2}(0.96-0.97008)$$

whence $b_3^{(1)} = 1.317, \quad b_3^{(2)} = 0.3095$

i. e., $\underline{M}_3(\varphi) = 1.627\,\kappa^2\sin^2\varphi - 0.561$

For $n = 3$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, we find

$$E_3^{-1}(\nu) = \sqrt{1-\nu^2}\,\frac{\sqrt{\kappa^2-\nu^2}}{x}\,\frac{\nu}{\kappa}, \quad M_3^{-1}(\nu) = \frac{\sqrt{\kappa^2-\nu^2}}{\kappa}\,\frac{\nu}{\kappa},$$

$$M_3^{-1}(\epsilon) = \cos\epsilon\,\sin\epsilon$$

Equation (75a) reads:

$$1 = \frac{1}{2}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\sin\epsilon\; b_3^{-1}\cos\epsilon\,\sin\epsilon\,\Delta\epsilon\,\frac{d\epsilon}{\cos\epsilon} = b_3^{-1}\int_0^{\frac{\pi}{2}}\sin^2\epsilon\,\Delta\epsilon\, d\epsilon$$

that is,

$$b_3^{-1} = 2.654$$

Hence

$$\underline{M}_{-3}(\varphi) = -2.654\cos\varphi\,\sin\varphi, \qquad \sqrt{\kappa^2-\nu^2} = -\kappa\cos\varphi \quad (!)$$

The determination of the other necessary functions proceeds in similar manner. The integrals (93) can be computed by means of the functions $\underline{M}_n(\varphi)$, being either elliptic or reducible to elementary. The coefficients $I_{\gamma,\delta}$ of

(92) are herewith known. Calculation of the right side of (86) yields

$$-\alpha_0 \int_{-1}^{+1}\int_0^{\eta} d\,\eta'\; P_{2\alpha-1}(\eta)\,d\eta = -\alpha_0 \int_{-1}^{+1} \eta\; P_{2\alpha-1}(\eta)\,d\eta = \begin{cases} \frac{2}{3}\alpha_0 & \text{for } \alpha = 1 \\ 0 & \text{for } \alpha > 1 \end{cases}$$

For $\gamma = 1,2,\ldots$ the right side of (86) is always zero, whence no asymmetrical potential functions occur in y. In the effected calculation, the series (81) of the potential function was stopped with $n = 4$; hence equations (92) and (86) must be taken for $\alpha = 1;\ 2$. It gives

$$\left.\begin{aligned} 2.101\ C_1 + 1.5217\ C_2 + 0.527\ C_4 &= 0 \quad (\alpha = 1) \\ -\ 0.2410\ C_2 + 0.5132\ C_3 + 0.9248\ C_4 &= 0 \quad (\alpha = 2) \end{aligned}\right\} \quad (92a)$$

$$\left.\begin{aligned} -\frac{1}{5}\,C_1 + \frac{2}{3}\,C_2 + \frac{1}{10}\,C_3 \phantom{+ \frac{2}{7}\,C_4} &= -\frac{2}{3}\,\alpha_0 \quad (\alpha = 1) \\ -\frac{1}{30}\,C_1 \phantom{+ \frac{2}{3}\,C_2} - \frac{1}{15}\,C_3 + \frac{2}{7}\,C_4 &= 0 \quad (\alpha = 2) \end{aligned}\right\} \quad (86)$$

The direct solution gives

$$C_1 = 0.568\ \alpha_0, \quad C_2 = -0.7785\ \alpha_0, \quad C_3 = -0.342\ \alpha_0, \quad C_4 = -0.0135\,\alpha_0$$

and the lift, according to (77), at:

$$A = 0.568\ \alpha_0 \times 8 \times \frac{\rho}{2} V^2\ F_{ell} = 4.55\ \alpha_0\ \frac{\rho}{2}\ V^2\ F_{ell}$$

that is,

$$\frac{d\ c_a}{d\ \alpha_0} = 4.55$$

The moment about the y axis is, according to (78):

$$M = -0.7785\ \alpha_0 \times 0.9524 \times \frac{8}{3} c \sqrt{1 - \kappa^2}\ \frac{\rho}{2}\ V^2\ F_{ell}$$

$$M = -1.98\ \alpha_0\, c \sqrt{1 - \kappa^2}\ \frac{\rho}{2}\ V^2\ F_{ell}$$

the center of pressure is at $\dfrac{x}{c\sqrt{1 - \kappa^2}} = -0.435$, that is, at 28.3 percent of the maximum wing chord.

Incidental to the calculation of the induced drag, it is emphasized that the lift distribution of the lifting line does not disappear when allowing only for an infinite number of series terms in (81) at the wing tips (fig. 3), and hence must be included as a substitute lift distribution. In this case the elliptical is most suitable, giving for the induced drag the well-known formula

$$c_{wi} = c_a{}^2 \frac{F_{ell}}{\pi(2c)^2}$$

For the axes ratio

$$\sqrt{1 - \kappa^2} = \frac{1}{2}, \quad \kappa^2 = 0.75, \quad \Lambda = 2.55$$

the procedure is the same, the quantities $b_n{}^m$ being ascertained from (73, 74, 75a). This affords the functions $\underline{M}_n(\varphi)$ for the integrals $I_{\gamma,\delta}$, wherewith the coefficients of (92) are known. The solution gives

$$C_1 = 0.3741\ \alpha_o, \quad C_2 = -0.6347\ \alpha_o, \quad C_3 = -0.2347\ \alpha_o,$$
$$C_4 = -0.0138\ \alpha_o$$

the lift being

$$A = 2.99\ \alpha_o \frac{\rho}{2} V^2 F_{ell} \quad \text{and} \quad \frac{d\ c_a}{d\ \alpha_o} = 2.99$$

the pitching moment

$$M = -1.397\ \alpha_o\ c\sqrt{1 - \kappa^2} \frac{\rho}{2} V^2 F_{ell}$$

and the center of pressure at

$$\frac{x}{c\sqrt{1 - \kappa^2}} = -0.467 \quad \text{or 26.7 percent of maximum wing chord}$$

For $\sqrt{1 - \kappa^2} = 2 \quad \kappa^2 = -3 \quad \Lambda = 0.637$

it gives

$$C_1 = 0.124\ \alpha_o, \quad C_2 = -0.5245\ \alpha_o, \quad C_3 = -0.066\ \alpha_o$$
$$C_4 = -0.011\ \alpha_o$$

that is,

$$\frac{d\ c_a}{d\ \alpha_o} = 0.99$$

The center of pressure is situated at

$$\frac{x}{c\sqrt{1 - \kappa^2}} = -0.584 \qquad = 20.8 \text{ percent of the chord}$$

For comparison the values for the flat circular disk are repeated (reference 2):

$$\sqrt{1 - \kappa^2} = 1, \qquad \kappa^2 = 0, \qquad \Lambda = 1.272, \qquad \frac{d\, c_a}{d\, \alpha_o} = 1.82$$

center of pressure: $\frac{x}{c} = -0.515$, i.e., at 24.3 percent of chord.

The calculation method used here permits even a boundary transition to the lifting line $(\kappa' = \sqrt{1 - \kappa^2} = 0)$. It affords, when two series terms are taken into account,

$$\frac{d\, c_a}{d\, \alpha_o} = 2\pi \qquad \text{c.p. at 28.8 percent of chord.}$$

The latter result corresponds to an elliptic spanwise load distribution with a center of pressure at 1/4 chord in each airfoil section (c.g. of a homogeneous semiellipse). Development of all quantities appearing in the equation systems with respect to κ' affords for small κ':

$$\frac{d\, c_a}{d\, \alpha_o} = \frac{2\pi}{1 + \frac{63}{128}\pi\kappa' + \left(\ln\frac{4}{\kappa'} - \frac{7}{4}\right)\kappa'^2}$$

When $\frac{d\, c_a}{d\, \alpha_o}$ is calculated according to linear wing theory where, as is known,

$$\alpha_{eff} = \alpha_o + c_a \frac{1}{\pi\Lambda}, \qquad \frac{d\, c_a}{d\, \alpha_o} = \frac{2\pi\Lambda}{\Lambda + 2}$$

there appears a marked discrepancy at small Λ with respect to the values computed in accord with the theory of the lifting surface (fig. 4).

On the other hand, the agreement with Weinig's results is good. (See reference 6.)

Figure 5 shows the center of pressure position plotted against aspect ratio.

The elliptic wing in yaw.- This problem can be treated under the same assumptions as the wing in straight flow. The so-called angle of yaw β is defined in figure 6.

Now the streamlines are straights in the plane $z = 0$, defined by

$$\left.\begin{aligned} y &= -(x - x_R)\tan\beta + y_R \\ &= -x\tan\beta + \text{const} \end{aligned}\right\} \qquad (95)$$

From equation (1)

$$(V\cos\beta + u)\frac{\partial w}{\partial x} + (-V\sin\beta + v)\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

under simplifying assumptions, there is found

$$V\cos\beta\frac{\partial w}{\partial x} - V\sin\beta\frac{\partial w}{\partial y} = V^2\frac{\partial\psi}{\partial z} \qquad (96)$$

But because of (95) it becomes

$$\frac{dw}{dx} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\tan\beta, \qquad \text{that is} \qquad \frac{dw}{dx}\cos\beta = \frac{\partial w}{\partial x}\cos\beta - \frac{\partial w}{\partial y}\sin\beta$$

Hence equation (96) can be written in the form

$$\frac{dw}{dx}\cos\beta = V\frac{\partial\psi}{\partial z} \qquad (y = -x\tan\beta + \text{const}) \qquad (96a)$$

The downwash follows from integration along a streamline again assumed as a straight line parallel to the direction of flow

$$\frac{w}{V} = \frac{1}{\cos\beta}\int_{-\infty}^{x}\frac{\partial\psi}{\partial z}\,dx \qquad (97)$$

The potential function ψ is again assumed as a linear combination of the potential function Ψ_n^m, Φ_n, Φ_{-n}, in the form (81), whence the same forumlas are obtained for lift and moments.

The downwash function is computed on the basis of an oblique-angle system of coordinates in the xy plane, given by the ellipse diameter parallel to the stream direction - ξ_β axis - and the related conjugate diameter - η_β axis. Posting

$$\tan \varphi_\beta = \sqrt{1 - \kappa^2} \tan \beta, \quad \text{that is,} \quad \sin \varphi_\beta = \frac{\sqrt{1 - \kappa^2} \sin \beta}{\Delta \beta};$$

$$\cos \varphi_\beta = \frac{\cos \beta}{\Delta \beta} \tag{98}$$

we have the quoted diameters given by

$$y = -\frac{\tan \varphi_\beta}{\sqrt{1 - \kappa^2}} x \quad \text{and} \quad y = -\frac{\tan\left(\varphi_\beta + \frac{\pi}{2}\right)}{\sqrt{1 - \kappa^2}} x, \text{ respectively} \tag{99}$$

On the disk, i.e., $\rho = 1$, it is according to equations (6) and (17):

$$\xi = \frac{x}{c\sqrt{1 - \kappa^2}} = -\frac{\sqrt{\mu^2 - \kappa^2}}{\sqrt{1 - \kappa^2}} \cos \varphi; \quad \eta = \frac{y}{c} = \mu \sin \varphi$$

Similarly, we post

$$\xi_\beta = -\frac{\sqrt{\mu^2 - \kappa^2}}{\sqrt{1 - \kappa^2}} \cos (\varphi - \varphi_\beta); \quad \eta_\beta = \mu \sin (\varphi - \varphi_\beta)$$

that is,

$$\xi_\beta = \frac{\cos \beta}{\Delta \beta} \xi - \frac{\sqrt{1 - \kappa^2} \sin \beta}{\Delta \beta} \eta, \quad \eta_\beta = \frac{\sqrt{1 - \kappa^2} \sin \beta}{\Delta \beta} \xi + \frac{\cos \beta}{\Delta \beta} \eta \tag{100}$$

In this instance the potential function $\psi_n{}^m$ conformable to (32) is utilized. During the respective differentiations and integrations along a streamline, the fact that $y = - x \tan \beta + \text{const}$ should be borne in mind.

$$\left.\begin{aligned} \psi_n{}^m(x,y,z) = \frac{1}{2\pi i} \int_{\boxed{}} Q_n \Bigg[& \frac{\sqrt{\Upsilon(t) - e_2}}{\sqrt{e_2 - e_1}\sqrt{e_2 - e_3}} \frac{x}{c} \\ & + \frac{\sqrt{\Upsilon(t) - e_3}}{\sqrt{e_3 - e_1}\sqrt{e_3 - e_2}} \frac{y}{c} + \frac{\sqrt{\Upsilon(t) - e_1}}{\sqrt{e_1 - e_2}\sqrt{e_1 - e_3}} \frac{z}{c} \Bigg] E_n{}^m(t)\, d\, t \end{aligned}\right\} \tag{32}$$

We find that

$$\frac{\partial\, Q_n(X)}{\partial z} = \frac{d Q_n(X)}{d\, X} \frac{1}{c} \frac{\sqrt{\Upsilon(t) - e_1}}{\sqrt{e_1 - e_2}\sqrt{e_1 - e_3}}$$

and

$$\frac{dQ_n(X)}{d\,x} = \frac{dQ_n(X)}{d\,X}\,\frac{1}{c}\left[\frac{\sqrt{\Upsilon(t) - e_2}}{\sqrt{e_2 - e_1}\sqrt{e_2 - e_3}} - \frac{\sqrt{\Upsilon(t) - e_3}}{\sqrt{e_3 - e_1}\sqrt{e_3 - e_2}}\tan\beta\right]$$

$\frac{\partial Q_n(X)}{\partial z}$ thus can be expressed again by $\frac{Q_n(X)}{d\,x}$; posting the result in (97) and integrating over x gives

$$\frac{w_n{}^m}{V} = \frac{1}{2\pi i}\,\frac{1}{\cos\beta}\int\limits_{\boxed{\leftarrow}} Q_n(X)\,\frac{\sqrt{\Upsilon(t) - e_1}}{\sqrt{e_1 - e_2}\sqrt{e_1 - e_3}}$$

$$\times\,\frac{1}{\frac{\sqrt{\Upsilon(t) - e_2}}{\sqrt{e_2 - e_1}\sqrt{e_2 - e_3}} - \frac{\sqrt{\Upsilon(t) - e_3}}{\sqrt{e_3 - e_1}\sqrt{e_3 - e_2}}\tan\beta}\,E_n{}^m(t)\,dt \qquad (101)$$

With equations (51) and (52) $Q_n(X)$ is now replaced again by

$$Q_n(X) = \frac{1}{2}\int_{v+w-\omega_1}^{v+w+\omega_1}\frac{P_n(\Upsilon(s))}{X - \Upsilon(s)}\,\frac{d\Upsilon}{ds}\,ds$$

The integrand has poles at $t = s$, where $X(t) - \Upsilon(s) = 0$ and at $t = t_\beta$, where

$$\frac{\sqrt{\Upsilon(t_\beta) - e_2}}{\sqrt{e_2 - e_1}\sqrt{e_2 - e_3}} - \frac{\sqrt{\Upsilon(t_\beta) - e_3}}{\sqrt{e_3 - e_1}\sqrt{e_3 - e_2}}\tan\beta = 0$$

i.e.,

$$\sqrt{\Upsilon(t_\beta) - e_2} = i\sqrt{e_2 - e_3}\sin\varphi_\beta;\quad \sqrt{\Upsilon(t_\beta) - e_3} = -\sqrt{e_2 - e_3}\cos\varphi_\beta$$

The denominator at this point is as

$$(t - t_\beta)\sqrt{\Upsilon(t_\beta) - e_1}\left[\frac{-\cos\varphi_\beta}{i\sqrt{e_1 - e_2}} - \frac{i\sin\varphi_\beta}{\sqrt{e_1 - e_3}}\tan\beta\right] + \ldots$$

The residuum at $t = t_\beta$ is therefore

$$2\pi i Q_n(X(t_\beta)) \frac{1}{i\cos\varphi_\beta + i\tan\varphi_\beta\sin\varphi_\beta} E_n^m(t_\beta)$$

$$= 2\pi i \frac{\cos\beta}{\Delta\beta} E_n^m(-\kappa\cos\varphi_\beta)\, Q_n(\eta_\beta) \tag{102}$$

with a view to the fact that, according to (100)

$$X(t_\beta) = \sin\varphi_\beta \frac{x}{c\sqrt{1-\kappa^2}} + \cos\varphi_\beta \frac{y}{c} = \eta_\beta$$

and according to equation (33)

$$E_n^m(t_\beta) = i\, E_n^m(\lambda)_{\lambda=-\kappa\cos\varphi_\beta}$$

For the residuum at $t = s$, the same holds true as in the case of straight flow. It affords:

$$2\pi i \int_{-\frac{\Pi}{2}}^{+\frac{\Pi}{2}} \frac{1}{2} P_n(\xi\cos\epsilon + \eta\sin\epsilon) \frac{i\Delta\epsilon}{\cos\epsilon - \sqrt{1-\kappa^2}\tan\beta\sin\epsilon} iM_n^m(\epsilon)\, d\epsilon \tag{103}$$

By replacing $\sqrt{1-\kappa^2}\tan\beta$ by $\tan\varphi_\beta$ and posting the coordinates ξ_β and η_β by means of equation (100), the values (102) and (103) from (101) lead to

$$\left.\begin{aligned} \frac{1}{V} w_n^m(\xi_\beta, \eta_\beta) &= \frac{1}{\Delta\beta} E_n^m(-\kappa\cos\varphi_\beta) Q_n(\eta_\beta) \\ &- \frac{1}{\Delta\beta}\frac{1}{2}\int_{-\frac{\Pi}{2}}^{+\frac{\Pi}{2}} P_n[\xi_\beta\cos(\epsilon+\varphi_\beta)+\eta_\beta\sin(\epsilon+\varphi_\beta)] \frac{M_n^m(\epsilon)}{\cos(\epsilon+\varphi_\beta)}\Delta\epsilon\, d\epsilon \end{aligned}\right\} \tag{104}$$

For the terms of P_n, containing powers of ξ_β, the denominator $\cos(\epsilon + \varphi_\beta)$ in the integral disappears, affording complete elliptic integrals of the first and second types. To compute the others, numerator and denominator are expanded with $\cos(\epsilon - \varphi_\beta)\Delta^2\beta$, which, with

$$\cos(\epsilon + \varphi_\beta)\cos(\epsilon - \varphi_\beta)\Delta^2\beta = \cos^2\beta - \Delta^2\beta\sin^2\epsilon$$

gives

$$\frac{1}{V} w_n^m (\xi_\beta, \eta_\beta) = \frac{1}{\Delta\beta} E_n^m(-\kappa \cos \varphi_\beta) Q_n(\eta_\beta)$$

$$-\frac{1}{2}\Delta\beta \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} P_n[\xi_\beta \cos(\epsilon+\varphi_\beta)+\eta_\beta \sin(\epsilon+\varphi_\beta)] \frac{\cos(\epsilon-\varphi_\beta) M_n^m(\epsilon)}{\cos^2\beta - \Delta^2\beta \sin^2 \epsilon} \Delta\epsilon \, d\epsilon \tag{104a}$$

The existent integrals are reducible to complete elliptic integrals of the first and second types and one integral of form

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 \epsilon}{\cos^2 \beta - \Delta^2 \beta \sin^2 \epsilon} \frac{d\epsilon}{\Delta\epsilon}$$

which, as a complete elliptic integral of the third type, is reducible to incomplete first and second types (reference 7).

$$\frac{(1-\kappa^2)\sin\beta\,\Delta\beta}{\cos\beta} \int_0^{\frac{\pi}{2}} \frac{\sin^2\epsilon}{\cos^2\beta - \Delta^2\beta \sin^2\epsilon} \frac{d\epsilon}{\Delta\epsilon}$$

$$= E(\beta) F\left(\frac{\pi}{2}\right) - E\left(\frac{\pi}{2}\right) F(\beta) - \Delta\beta \tan\beta\, F\left(\frac{\pi}{2}\right) \tag{105}$$

The calculation of the downwash function for the potential function ψ_1^1 is given as a model example.

$$E_1^1(\lambda) = \sqrt{1 - \lambda^2} \qquad M_1^1(\epsilon) = 1$$

$$P_1(Y) = Y \qquad Q_1(X) = \frac{X}{2} \ln \frac{X+1}{X-1} - 1$$

$$E_1^1(-\kappa \cos \varphi_\beta) = \sqrt{1 - \kappa^2 \cos^2 \varphi_\beta} = \frac{\sqrt{1+\kappa^2}}{\Delta\beta}$$

Then, according to (104a):

$$\frac{1}{V} w_1{}^1(\xi_\beta,\eta_\beta) = \frac{1}{\Delta\beta}\,\frac{\sqrt{1-\kappa^2}}{\Delta\,\beta}\,Q_1(\eta_\beta)$$

$$-\frac{1}{2}\Delta\,\beta\int_0^{\frac{\pi}{2}}[\xi_\beta\cos(\epsilon+\varphi_\beta)+\eta_\beta\sin(\epsilon+\varphi_\beta)]\,\frac{\cos\,(\epsilon-\varphi_\beta)\Delta\epsilon}{\cos^2\beta-\Delta^2\beta\sin^2\epsilon}\,d\,\epsilon$$

$$=\frac{\sqrt{1-\kappa^2}}{\Delta^2\beta}\,Q_1(\eta_\beta)$$

$$-\frac{1}{2}\xi_\beta\frac{1}{\Delta\beta}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\Delta\epsilon d\epsilon-\frac{1}{2}\eta_\beta\Delta\,\beta\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{\sin\varphi_\beta\cos\varphi_\beta+\sin\epsilon\,\cos\epsilon}{\cos^2\beta-\Delta^2\beta\,\sin^2\epsilon}\,\Delta\epsilon d\epsilon$$

The evaluation of the integral gives

$$\frac{1}{V} w_1{}^1(\xi_\beta,\eta_\beta) = \frac{\sqrt{1-\kappa^2}}{\Delta^2\beta}\,Q_1(\eta_\beta) - \frac{\xi_\beta}{\Delta\beta}\,E\left(\frac{\pi}{2}\right)$$

$$-\,\eta_\beta\,\frac{\sqrt{1-\kappa^2}}{\Delta^2\beta}\left[E(\beta)F\left(\frac{\pi}{2}\right)-E\left(\frac{\pi}{2}\right)F(\beta)\right]$$

<u>The downwash functions of the potential functions of the second type.</u>- According to (68)

$$\Phi_n = \sum_m b_n{}^m\,\Phi_n{}^m$$

whereby

$$\Phi_n{}^m = \frac{1}{c^{n-1}}\,\frac{d}{dc}\,[c^n\,\psi_n{}^m(x,y,z,c)]$$

according to (65). The downwash function $w_n(\xi_\beta,\eta_\beta)$ is computed from $w_n{}^m(\xi_\beta,\eta_\beta)$, that is,

$$\frac{1}{V} w_n = -\frac{1}{\Delta\beta}\sum_m b_n{}^m\,E_n{}^m(-\,\kappa\cos\varphi_\beta)\,\frac{dQ_{n-1}(\eta_\beta)}{d\,\eta_\beta}$$

$$+\frac{1}{2}\,\frac{1}{\Delta\beta}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{d\,P_{n-1}(Y)}{dY}\,\frac{M_n(\epsilon)}{\cos(\epsilon+\varphi_\beta)}\,\Delta\,\epsilon\,d\,\epsilon \qquad (106)$$

in the same manner.

On the disk we have $\left.\frac{\partial \Phi_n}{\partial z}\right|_{\rho=1} = 0$, that is, on a streamline $(\eta_\beta = \text{const})$ the downwash function is constant; hence w_n is a function of η_β only, and we accordingly put $\xi_\beta = 0$ in $\frac{d}{dY} P_{n-1}(Y)$. To find the summand

$$\eta_\beta^{n-2p} [\sin (\epsilon + \varphi_\beta)]^{n-2p} \quad \text{from} \quad Y^{n-2p}$$

we put $\tau = \frac{n}{2}$ and $\frac{n-1}{2}$, respectively, whence

$$[\sin (\epsilon + \varphi_\beta)]^{n-2p} = [\sin(\epsilon + \varphi_\beta)]^{n-2\tau} [\sin(\epsilon + \varphi_\beta)]^{2\tau - 2p}$$

$$= [\sin(\epsilon + \varphi_\beta)]^{n-2\tau} (1 + \text{sum of cos terms})$$

Factor $\cos(\epsilon + \varphi_\beta)$ can be extracted from the sum, thus becoming shorter with respect to the denominator. The new sum, however, multiplied by $[\sin(\epsilon + \varphi_\beta)]^{n-2\tau}$, yields only terms which either satisfy odd in ϵ or else conditions (73) and (74), respectively, and accordingly disappear. A proportion other than zero is afforded only by $[\sin(\epsilon + \varphi_\beta)]^{n-2\tau} \times 1$, which is, however, no longer dependent on exponent p. In consequence, all η_β^{n-2p} before the integral can be combined conformably to

$\frac{d}{dY} P_{n-1}(Y)$ into $\frac{d}{d\eta_\beta} P_{n-1}(\eta_\beta)$, since the integral for every p is the same; i.e.,

$$\frac{w_n(\eta_\beta)}{V} = - \frac{1}{\Delta\beta} \sum_m b_n{}^m E_n{}^m(- \kappa \cos \varphi_\beta) \frac{dQ_{n-1}(\eta_\beta)}{d\eta_\beta}$$

$$+ \frac{dP_{n-1}(\eta_\beta)}{d\eta_\beta} \frac{1}{2} \Delta \beta \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} [\sin\epsilon \cos\varphi_\beta + \cos \epsilon \sin \varphi_\beta]^{n-2\tau}$$

$$\times \frac{\cos \epsilon \cos \varphi_\beta + \sin \epsilon \sin \varphi_\beta}{\cos^2 \beta - \Delta^2 \beta \sin^2 \epsilon} \underline{M}_n(\epsilon) \Delta \epsilon \, d\epsilon \tag{107}$$

Hence the downwash functions for the wing in yaw have the following form

$$\frac{1}{V} w_n(\eta_\beta) = k_n(\beta) \frac{dQ_{n-1}(\eta_\beta)}{d\,\eta_\beta} + i_n(\beta) \frac{dP_{n-1}(\eta_\beta)}{d\,\eta_\beta} \tag{107a}$$

$$-\frac{1}{V} w_{-n}(\eta_\beta) = l_n(\beta) \frac{dQ_{n-1}(\eta_\beta)}{d\,\eta_\beta} + j_n(\beta) \frac{dP_{n-1}(\eta_\beta)}{d\,\eta_\beta} \tag{107b}$$

<u>The second fundamental problem for the case of the wing in sideslip</u>.- The procedure is the same as in the case of the wing in horizontal flow, by putting

$$\sqrt{1-\mu^2}\sqrt{1-\nu^2}\cos\beta\,\frac{dw(x,y)}{dx} = \frac{V}{c}\sum_n\sum_m g_n{}^m\,E_n{}^m(\mu)\,E_n{}^m(\nu) \tag{108}$$

where

$$\frac{1}{V}\,w(x,y) = \cos\beta\,\frac{dz(x,y)}{dx} \tag{108a}$$

and $y = -x\tan\beta + \text{const}$; ψ to be given again by

$$\psi = \sum_n\sum_m a_n{}^m\,\psi_n{}^m + \sum_n C_n\,\Phi_n + \sum_n D_n\,\Phi_{-n} \tag{109}$$

Equation (96a) then gives

$$g_n{}^m = \sqrt{1-\kappa^2}\left[\frac{d\,F_n{}^m(\rho)}{d(\sqrt{\rho^2-1})}\right]_{\rho=1} a_n{}^m \tag{110}$$

and with it the coefficients of the potential functions of the first type. For the determination of the coefficients of the functions of the second type we take (97) in the following form:

$$\left.\begin{aligned}&\sum_n C_n\int_{-\infty}^{x}\frac{\partial\Phi_n}{\partial z}\,\frac{d\,x}{\cos\beta} + \sum_n D_n\int_{-\infty}^{x}\frac{\partial\Phi_{-n}}{\partial z}\,\frac{d\,x}{\cos\beta}\\ &\quad = \frac{1}{V}\,w(x,y) - \sum_n\sum_m a_n{}^m\int_{-\infty}^{x}\frac{\partial\psi_n{}^m}{\partial z}\,\frac{d\,x}{\cos\beta} = \frac{1}{V}\,w(\eta_\beta)\end{aligned}\right\} \tag{111}$$

The right-hand side is a function of η_β only because of the determination of the $a_n{}^m$ from equation (96a). The rest of the formula then reads:

$$\left.\begin{aligned}&\sum_n C_n\, k_n(\beta)\,\frac{dQ_{n-1}(\eta_\beta)}{d\,\eta_\beta}+\sum_n C_n\, i_n(\beta)\,\frac{dP_{n-1}(\eta_\beta)}{d\eta_\beta}\\&+\sum_n D_n l_n(\beta)\,\frac{dQ_{n-1}(\eta_\beta)}{d\,\eta_\beta}+\sum_n D_n j_n(\beta)\,\frac{dP_{n-1}(\eta_\beta)}{d\,\eta_\beta}=\frac{1}{V}\,w(\eta_\beta)\end{aligned}\right\}\quad(112)$$

This equation is integrated from $\eta_\beta = 0$ to η_β and then - multiplied by $P_{2\alpha-1}(\eta_\beta)$ and $P_{2\gamma}(\eta_\beta)$ - again from $\eta_\beta = -1$ to $+1$. This affords on the basis of the orthogonality characteristics of the spherical functions

$$\left.\begin{aligned}&\sum_{r=0}\left(C_{2r+1}k_{2r+1}(\beta)+D_{2r+1}l_{2r+1}(\beta)\right)\frac{1}{\alpha(2\alpha-1)-r(2r+1)}\\&+\left(C_{2\alpha}i_{2\alpha}(\beta)+D_{2\alpha}j_{2\alpha}(\beta)\right)\frac{2}{4\alpha-1}\\&=\frac{1}{V}\int_{-1}^{+1}\int_{0}^{\eta_\beta} w(\eta'_\beta)\,d\eta'_\beta\,P_{2\alpha-1}(\eta_\beta)\,d\eta_\beta;\qquad \alpha = 1,2,\ldots\end{aligned}\right\}\quad(113a)$$

$$\left.\begin{aligned}&\sum_{r=1}\left(C_{2r}\,k_{2r}\,(\beta)+D_{2r}\,l_{2r}\,(\beta)\right)\frac{1}{\gamma(2\gamma-1)-r(2r-1)}\\&+\left(C_{2\gamma+1}\,i_{2\gamma+1}\,(\beta)+D_{2\gamma+1}\,j_{2\gamma+1}(\beta)\right)\frac{1}{4\gamma+1}\\&=\frac{1}{V}\int_{-1}^{+1}\int_{0}^{\eta_\beta} w(\eta'_\beta)\,d\eta'_\beta\,P_{2\gamma}(\eta_\beta)\,d\eta_\beta;\qquad \gamma = 1,2,\ldots\end{aligned}\right\}\quad(113b)$$

The outflow condition.- The formulation of the outflow condition is prefaced by the following note. The downwash functions (107) on the wing in sideslip have the quality of becoming infinite by $\eta_\beta = \pm 1$ like $Q_{n-1}(\eta_\beta)$ and $\frac{dQ_{n-1}(\eta_\beta)}{d\,\eta_\beta}$, respectively. But $\eta_\beta = \pm 1$ are marginal points of the elliptic disk in which parallels to the flow direction touch the ellipse (fig. 6). Accordingly, it is to be assumed that the so-called "vortex tails" in these points leave the disk parallel to the direction of the stream. Hence it is postulated that no flow around the trailing edge occurs between the points $\eta_\beta = \pm 1$. In other words:

$$\sum_n C_n \underline{M}_n(\varphi) + \sum_n D_n \underline{M}_{-n}(\varphi) \equiv 0 \qquad \frac{\pi}{2} + \varphi_\beta \leq \varphi \leq \frac{3\pi}{2} + \varphi_\beta$$

This equation, multiplied by

$\underline{M}_{2\alpha-1}(\varphi)\ \Delta\varphi$ and $\underline{M}_{-2\gamma}(\varphi)\ \Delta\varphi$, respectively

and integrated from $\frac{\pi}{2} + \varphi_\beta$ to $\frac{3\pi}{2} + \psi_\beta$ gives the following systems of equations:

$$\left.\begin{aligned} & C_{2\alpha-1} I^{(\beta)}_{2\alpha-1,2\alpha-1} + \sum_{r=1} C_{2r} I^{(\beta)}_{2r,2\alpha-1} + \sum_{r=1} D_{2r} I^{(\beta)}_{-2r,2\alpha-1} = 0; \\ & \qquad\qquad\qquad\qquad \alpha = 1,2,\ldots \\ & D_{2\gamma} I^{(\beta)}_{-2\gamma,-2\gamma} + \sum_{r=1} C_{2r-1} I^{(\beta)}_{2r-1,-2\gamma} + \sum_{r=1} D_{2r+1} I^{(\beta)}_{-(2r+1),-2\gamma} = 0; \\ & \qquad\qquad\qquad\qquad \gamma = 1,2,\ldots \end{aligned}\right\} \qquad (114)$$

whereby

$$I^{(\beta)}_{\gamma,\delta} = \int_{\frac{\pi}{2}+\varphi_\beta}^{\frac{3\pi}{2}+\varphi_\beta} \underline{M}_\gamma(\varphi)\ \underline{M}_\delta(\varphi)\ \Delta\varphi\ d\varphi \qquad (114a)$$

These two equations ((113) and (114)) make it possible to determine the coefficients C_n and D_n. Together with the previously defined a_n^m's the aerodynamic quantities of an elliptic wing in yaw can be computed. As to the equation systems themselves, they form a coupling of the systems (82) and (92) for the case of straight flow, as is readily apparent from the similarity of the corresponding coefficients and which thus affords a first simple mathematical check.

The flat elliptic wing in yaw.- The calculations are carried out for the axes ratio

$$\sqrt{1 - \kappa^2} = \frac{1}{5}$$

and the angles of yaw

$$\beta = 15^\circ \quad \text{and} \quad \beta = 30^\circ$$

Again only the potential functions of the second type conformable to (108) and (110) are required. To simplify the voluminous paperwork only the functions up to the degree $n = 3$ are taken, i.e.,

$\Phi_1, \Phi_2, \Phi_3, \Phi_{-2}, \Phi_{-3}$ (Φ_{-1} does not exist)

The five unknown coefficients of these potential functions require five equations. Two are taken from the condition $\frac{1}{V} w(\eta_\beta) = -\cos\beta \tan\alpha_o$ on the disk, equation (113a) for $\alpha = 1$, and (113b) for $\gamma = 1$. For the other three the outflow condition (equation (114)) with $\alpha = 1.2$ and $\gamma = 1$) is used. The $\underline{M}_n(\varphi)$ are those previously defined in the case of straight flow. The integrals $I_{\gamma,\delta}^{(\beta)}$ obtain now only the limits $\frac{\pi}{2} + \varphi_\beta$ and $\frac{3\pi}{2} + \varphi_\beta$, while $I_{\gamma,\gamma}$ remain unchanged on account of the periodicity of the integrand. The values of the incomplete elliptic integrals of the first and second types necessary for the determination of the coefficients $k_n(\beta)$, $i_n(\beta)$, etc., were taken from Legendre's tables (reference 8)

For $\beta = 15^\circ$ it affords

$$C_1 = 0.5204\ \alpha_o$$

$$C_2 = -0.7194\ \alpha_o \qquad D_2 = 0.0144\ \alpha_o$$

$$C_3 = -0.3343\ \alpha_o \qquad D_3 = -0.0065\ \alpha_o$$

i.e.,

$$A = 4.16\ \alpha_o \frac{\rho}{2} V^2 F_{ell}$$

$$M = -1.83\ \alpha_o\ c \sqrt{1 - \kappa^2}\ \frac{\rho}{2} V^2 F_{ell}$$

$$L = 0.0384\ \alpha_o\ c \frac{\rho}{2} V^2 F_{ell}$$

For $\beta = 30^\circ$ it is:

$$C_1 = 0.407\ \alpha_o$$

$$C_2 = -0.562\ \alpha_o \qquad D_2 = 0.0277\ \alpha_o$$

$$C_3 = -0.265\ \alpha_o \qquad D_3 = -0.0124\ \alpha_o$$

i.e.,

$$A = 3.26\ \alpha_o \frac{\rho}{2} V^2 F_{ell}$$

$$M = -1.43\ \alpha_o\ c \sqrt{1 - \kappa^2}\ \frac{\rho}{2} V^2 F_{ell}$$

$$L = 0.074\ \alpha_o\ c \frac{\rho}{2} V^2 F_{ell}$$

The new additive moment L is positive according to the above calculations, which is synonymous with the fact that the leading wing half receives greater lift. The coordinates of the centers of pressure are:

$$\beta = 15^\circ: \frac{x}{c\sqrt{1 - \kappa^2}} = -0.440; \quad \frac{y}{c} = 0.00925 \quad (\varphi = 6^\circ 02')$$

$$\beta = 30^\circ: \frac{x}{c\sqrt{1 - \kappa^2}} = -0.439; \quad \frac{y}{c} = 0.0227 \quad (\varphi = 13^\circ 51')$$

For comparison we repeat the values obtained at $\beta = 0^\circ$ when the expansion is stopped with $n = 3$. The bracketed terms contain the change in percent with respect to the quantities computed with the four expansion terms, and from which inferences can be made regarding the convergence.

$$C_1 = 0.563\ \alpha_0 \quad (-0.9 \text{ percent})$$

$$C_2 = -0.776\ \alpha_0 \quad (-0.2 \text{ percent}) \quad D_2 = 0$$

$$C_3 = -0.365\ \alpha_0 \quad (+7.0 \text{ percent}) \quad D_3 = 0$$

i.e.,

$$A = 4.50\ \alpha_0 \frac{\rho}{2} V^2 F_{ell}$$

$$M = -1.98\ \alpha_0\ c\sqrt{1 - \kappa^2} \frac{\rho}{2} V^2 F_{ell}$$

Center of pressure: $\dfrac{x}{c\sqrt{1 - \kappa^2}} = -\ 0.438 \quad (+0.7 \text{ percent})$

The results are correlated in figures 7 and 8.

For great Λ an approximate formula for the rolling moment in relation to angle of yaw β and axes ratio $\kappa' = \sqrt{1 - \kappa^2}$ is again expedient:

$$\left.\frac{dc_L}{d\alpha_0} = \frac{15}{128} \pi^2 \kappa' \sin 2\beta \frac{\ln \frac{4}{\kappa'} - \frac{3}{2} - \dfrac{\ln \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right)}{\sin \beta}}{1 + \dfrac{\pi}{\cos \beta} \dfrac{243}{128} \kappa'} \right\} \quad (115)$$

The formula indicates that the moment on transition to the lifting line $(\kappa' = 0)$ disappears. But if the moment with the half wing chord instead of half the span is made nondimensional, thus voiding the factor κ' in (115) the moment coefficient becomes logarithmically infinite on limiting transition. In the extreme case the lift decreases with $\cos^2 \beta$, for the reason that the flow velocity in x direction is $V \cos \beta$.

Horner's results on wings of different plan forms in sideslip are in very close agreement with the values given here. The assumption that the rolling moment is, aside from the angle of yaw, largely dependent upon the aspect ratio rather than the chord distribution appears therefore justified.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

REFERENCES

1. Prandtl, L.: Beitrag zur Theorie der tragenden Fläche. Z.f.a.M.M., vol.16, no. 6, Dec. 1936, pp. 360-61.

2. Kinner, W.: Die kreisförmige Tragfläche auf potentialtheoretischer Grundlage. Ing.-Arch., vol. 8, 1937, pp. 47-80.

3. Heine, E.: Handbuch der Kugelfunktionen. vol. I, Berlin, 1878, p. 347 ff; vol. II, 1881.

4. Whittaker-Watson: Modern Analysis. Cambridge, 1920, p. 549 ff.

5. Hobson, E. W.: Spherical and Ellipsoidal Harmonics. Cambridge, 1931, p. 476.

6. Weinig, F.: Beitrag zur Theorie des Tragflügels endlicher, insbesondere kleiner Spannweite. Luftfahrtforschung, vol. 13, no. 12, Dec. 20, 1936, pp. 405-09.

7. Schlömilch, D.: Comp. d. höh. Analysis. vol. II, Braunschweig, 1865, pp. 336-39.

8. Legendre, A. M.: Tafeln der ellipt. Normalintegrale. Stuttgart, 1931.

9. Hoerner, S.: Kräfte und Momente schrägangeströmter Tragflügel. Luftfahrtforschung, vol. 16, no. 4, April 20, 1939, pp. 178-83.

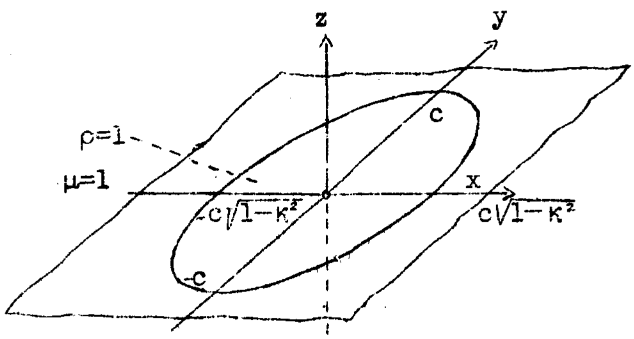


Fig.1 Base Ellipsoid

t plane

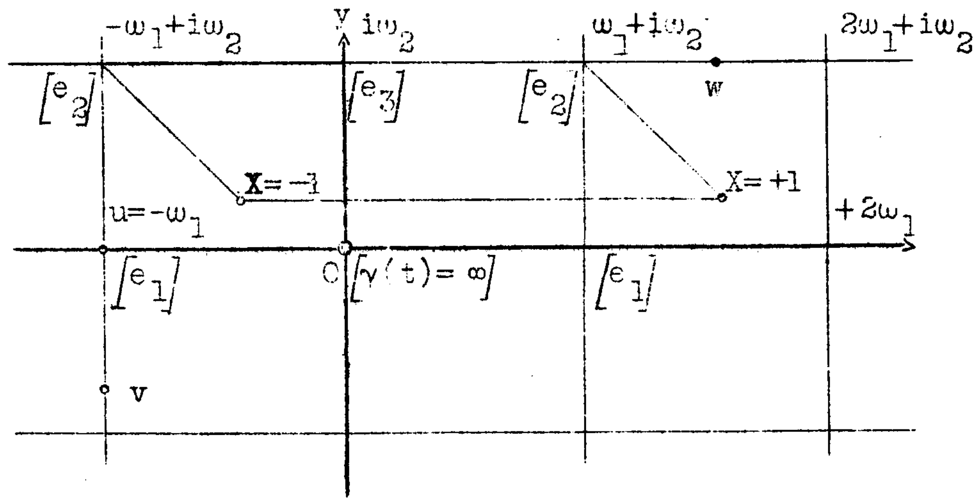
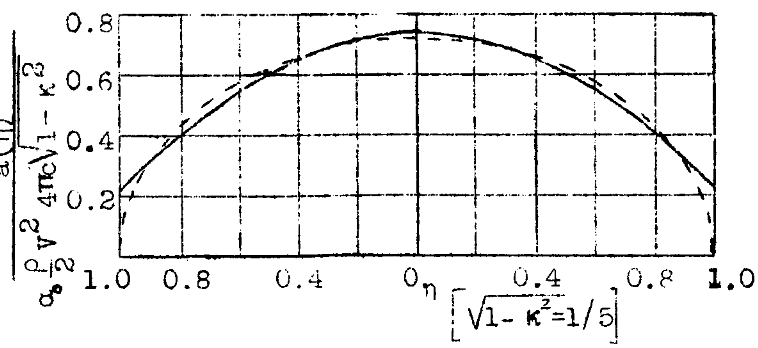


Fig. 2



Computed lift distribution of the lifting line. ———

Elliptic lift distribution with equal total lift. - - - - -

Fig. 3

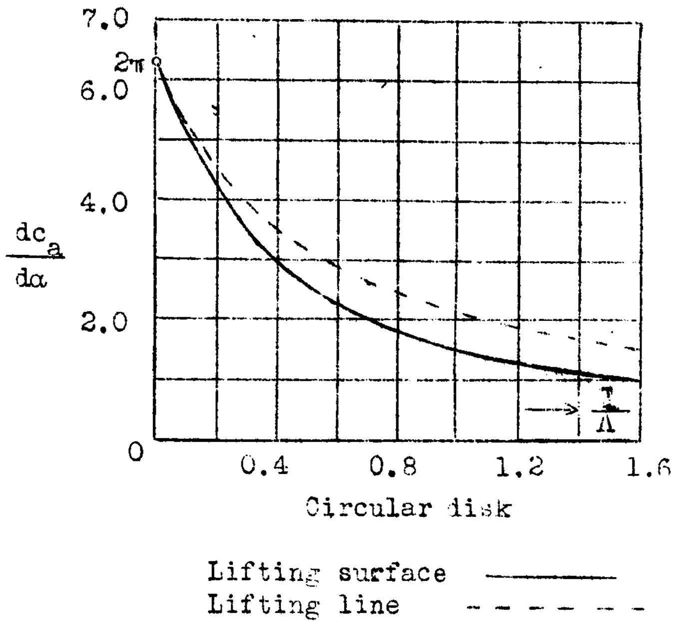


Figure 4.

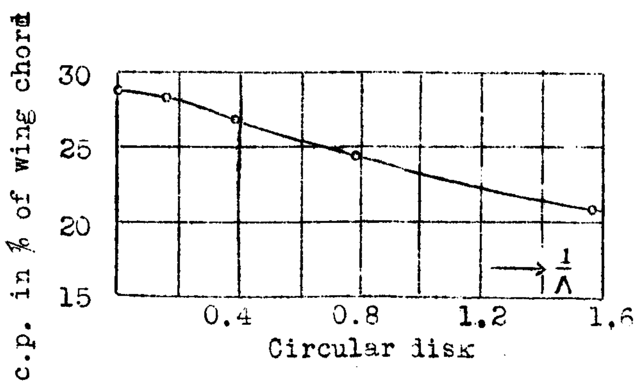


Figure 5.

45

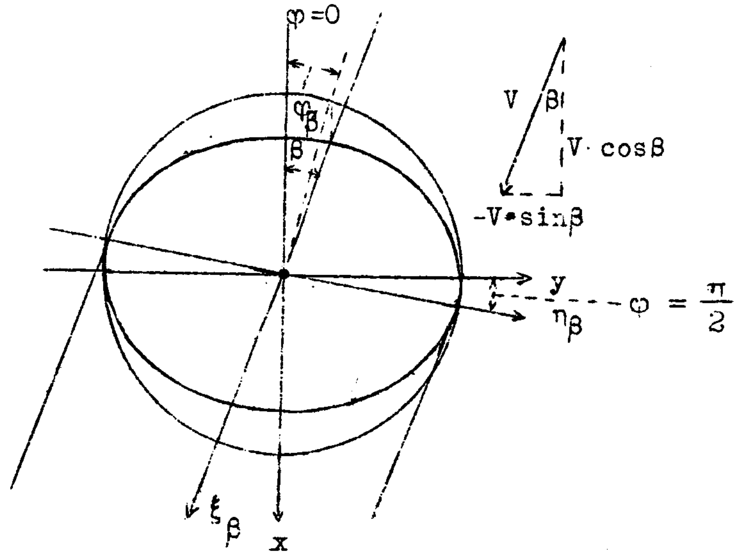


Figure 6.

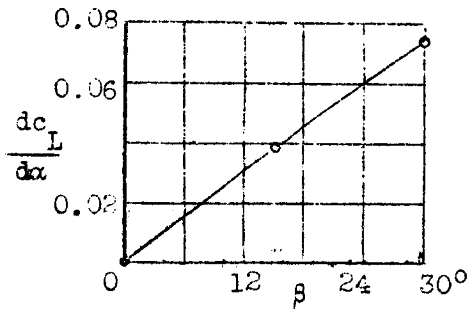


Figure 7 $\frac{dc_L}{d\alpha}$ plotted against angle of yaw β for aspect ratio Λ =6.37

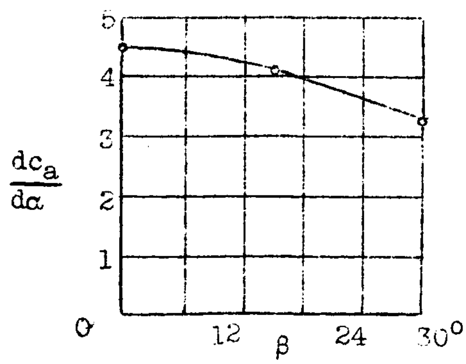


Figure 8. $\frac{dc_a}{d\alpha}$ plotted against angle of yaw β for aspect ratio Λ = 6.37.