TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 961

FLOW AROUND WINGS ACCOMPANIED BY SEPARATION OF VORTICES

By C. Schmieden

Luftfahrtforschung
Vol. 17, No. 2, February 20, 1940
Verlag von R. Oldenbourg, München und Berlin

Washington
December 1940

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 961

FLOW AROUND WINGS ACCOMPANIED BY SEPARATION OF VORTICES*

By C. Schmieden

SUMMARY

The flow around wings computed by the usual method leads in the case of a finite trailing edge to a stagnation point in the trailing edge due to the Kutta-Joukowsky condition of flow governing this region. As a result, the theoretical pressure distribution differs substantially from the experimental values in the vicinity of the trailing edge.

The present report describes an alternative method of calculation in which the rear stagnation point no longer appears. The stream leaves the trailing edge tangentially on the pressure side and a similar tangential separation occurs on the suction side of the profile at a point slightly in front of the trailing edge.

The result is a closer agreement in pressure distribution with reality.

INTRODUCTION

The flow around a given airfoil computed by the usual method differs in various aspects from real flow, even if the errors resulting from the substitution of ideal for real flow, i.e., the omission of the friction effects, are disregarded. The principal departure from the real flow occurs by the pressure distribution in proximity of the trailing edge because of the Kutta-Joukowsky condition of flow governing this region; by finite trailing edge this condition locates a stagnation point in the trailing edge and the streamline enveloping the profile leaves the profile in direction of the bisectrix of the trailing edge (fig. 1). But the real flow does not show this behavior.

*"Über Tragflügelströmungen mit Wirbelablösung." Luftfahrtforschung, vol. 17, no. 2, Feb. 1940, pp. 37-46.

The flow leaves the trailing edge tangentially on the pressure side and a similar tangential separation occurs on the suction side in the neighborhood of the trailing edge by formation of a vortex zone behind the wing. In order to construe an ideal flow that fits this actual flow pattern better than the Kutta-Joukowsky flow, the vortex zone is idealized by two vortex sheets of equal and opposite circulation, one leaving the trailing edge tangentially on the pressure side, the other leaving the trailing edge tangentially on the suction side (fig. 2). Sufficiently downstream from the profile the undisturbed parallel flow must re-establish itself; hence it follows that the two vortex layers must approach one another and possess equal and opposite circulation, because then only is the undisturbed parallel flow able to re-establish itself at infinity behind the profile. In the immediate vicinity of each of the vortex sheets the flow is parallel to a sudden increase in velocity proportional to the vortex strength on passing through the sheet. The flow velocity therefore must everywhere on the two vortex sheets have an equal and constant value, that is, be equal to the flow velocity because undisturbed parallel flow exists at infinity. Finally, since no fluid particle can penetrate from without in the zone between the two sheets, it follows from the continuity that the fluid between these sheets must be at rest, i.e., must form a dead-air region wherein the static pressure of infinite distance prevails. With this we have found the conditions which the looked-for flow must satisfy: On the two vortex sheets which at the same time form the boundaries of the dead-air region (free streamlines) a stream velocity constant and equal to the flow velocity must prevail. The course of the free streamlines is, moreover, to be so determined that, aside from this velocity condition, the width of the dead air diminishes to zero with increasing distance from the profile.

Now it is shown that out of the multitude of possible flows for a given profile at a given angle of attack, only one flow characterized by a definite break-away point in the suction side leads to the condition of constant speed along the trailing streamlines together with a dead-air region which disappears at infinity. The calculation is carried out first explicitly for the simple case of a flat plate and then extended to a form of modified Joukowsky profiles which are amenable to simpler treatment.

2. Flow around a Flat Plate

The flow approaches the plate of length l at an angle of attack α with velocity $v_\infty = 1$ from the left (fig. 3). The lower free streamline leaves the trailing edge at E, whereas the upper leaves at the temporarily unknown or arbitrarily assumed point C on the suction side. The first is consistently convex to the flow, its direction approaching the direction of flow asymptotically; the other is, first, concave, it reaches an inflection point at W, and it beomes convex like the first. The stagnation point is at A; the flow around leading edge B is with infinitely great velocity

With

$$w(z) = \varphi(x,y) + i\psi(x,y) \tag{1}$$

as the complex potential of this flow, and

$$\frac{dw}{dz} = v_x - i\,v_y = q\,e^{-i\theta} \tag{1a}$$

as complex velocity,the boundary conditions to be satisfied are as follows:

1) ψ = const., for the complex potential past the plate and on the free streamlines, the constant being posted at zero;

2) For the complex velocity the direction along the parts of the plate in the stream is given, i.e.,

$$\theta = -\alpha \quad \text{on AE and BC}$$

$$\theta = \pi - \alpha \quad \text{on AB}$$

whereas $q = v_\infty = 1$ on the free streamline.

With the function

$$\zeta = \ln\frac{dz}{dw} = \ln\frac{1}{q} + i\theta \tag{2}$$

the region of the z-plane filled by flowing fluid is transformed to the region of the ζ-plane shown in figure 4. The zero point of this plane is D since

$$\underline{R}(\zeta) = +\infty \text{ in } A$$

and

$$\underline{R}(\zeta) = -\infty \text{ in } B$$

the pieces of the plate furnish the horizontal boundaries of a parallel strip at distance $\pi - \alpha$ or $-\alpha$ to the real axis, and the free streamlines, because of $\underline{R}(\zeta) = 0$, the twice to be counted piece of the imaginary axis CWDE. All streamlines in the ζ-plane as reflections of the streamlines of the z-plane begin and end in point D. It is readily seen that the tangential direction in the inflection point of the upper free streamline must always $< (\pi - \alpha)$, and that in addition with increasing approach on this boundary the tip W of the breakwater in the ζ-plane consistently approaches the upper strip wall, so that the door between the two halves becomes steadily narrower. Therefore, the volume which the image flow sends through this door must also drop continuously. In the z-plane this indicates an unrestricted approach of W toward C and also of C to B; in the limiting case all three points meet in B, the transformation degenerates: the ζ-plane yields a half-strip, the z-plane the common Helmholtz flow with separation of two consistently convex free streamlines on the plate ends (fig. 5).

On the contrary, by decreasing the tangential slope in the inflection point we obtain a second essential limiting case of $\theta_w = 0$. Here the tip of the breakwater coincides with the image of the infinity, the inflection point in the z-plane has shifted to infinity; hence the upper free streamline is everywhere concave. Together with the lower, consistently convex streamline a dead-air region is formed, the width of which diminishes to zero (fig. 6).

Shortening the length of the breakwater still more leaves a dead-air region which closes at finity. These cases can be realized only when the flow velocity and the velocity on the free streamlines are no longer equal but in a certain ratio to one another. In the subsequent shortening of the breakwater it therefore must be shifted parallel to itself as a whole out of the imaginary axis, whereby the shortening of the breakwater is accompanied by a continuously smaller dead-air region. The breakwater itself shifts toward the right where, in the extreme case of vanishing length, it terminates in point

$$\zeta = \ln \sec \alpha - i \alpha$$

The dead-air region has then become zero; hence it leaves in the z-plane the well-known Kutta-Joukowsky flow around a flat plate with circulation (fig. 7).

The flows in question therefore afford a steady transition from the unsteady Helmholtz flow to the steady circulation flow.* Every one of these flows is definitely defined by the break-away point on the suction side, where, of course, the break-away point of the flow which presents the transition from infinite to finite dead-air region is not, a priori, available. The force acting on the plate also varies continuously with the break-away point: On the Helmholtz flow it is at right angle to the plate; on flows with infinite dead-air region it gradually turns in the vertical, and for flows with finite dead-air region inclusive of the two limiting cases it is always pure lift. Of particular interest, of course, is the flow with dead-air region the width of which diminishes to zero, since it is the first of the system for which, sufficiently aft of the plate, undisturbed parallel flow is re-established. Besides, it satisfies the other requirements

---

*Aside from this system of possible discontinuous flows there is still another system of such flows which, in wholly similar manner, leads from the Helmholtz to the circulation-free stable plate flow (pure deflection flow). This class of unstable flows, especially for the circular contour, have been discussed in an earlier report entitled "Critical Solutions of the Theory of Unstable Flows," Ing. Arch., vol. 3, 1932. Since there is, besides the deflection flow and the pure circulation flow with smooth efflux in the trailing edge, the - infinite - system of circulation flow with flow around the trailing edge which, of itself, represents a steady transition from pure deflection flow to pure circulation flow, it may be assumed that for each of these circulation flows there are also a number of unsteady flows which lead from the Helmholtz to this circulation flow. The proof, which would have to be carried out as in the two cases treated here, is omitted, but the conclusion is given as follows:

For every contour there is a twofold infinite system of unsteady flows, for which the position of the break-away points of the two free streamlines on the contour plays the part of a parameter. The positions of the break-away points are forwardly limited by the break-away points of the usual Helmholtz flow with everywhere convex free streamlines, in case this flow exists by the given contour.

formulated in the introduction, namely, separation of a vortex layer on the trailing edge tangentially on the pressure side and break-away of another vortex layer of equal and opposite circulation on the suction side in the vicinity of the trailing edge. This flow is therefore dealt with in particular.

In order to formulate the results obtained in the ζ-plane analytically and, above all, quantitatively, the region of the ζ-plane is so transformed on the upper half of the unit circle of the τ-plane (fig. 8) that the horizontal boundaries of the strip, that is, the images of the parts of the plate in the stream fall on the semicircle, and the image of the free streamlines on the piece of the real axis between -1 and +1. Furthermore, the transformation is so regulated that the zero point of the ζ-plane, i.e., the image of the infinity of the z-plane for dead-air region extending to infinity - falls in the zero point of the τ-plane. Point W is to fall with respect to $-\tau_0$ between -1 and 0; points A and B, to $e^{i\vartheta_S}$ and $e^{i\vartheta_0}$. Carried out, this transformation gives

$$e^{\zeta} = \frac{dz}{dw} = e^{-i\alpha} \frac{1 + 2\kappa^2\tau + \tau^2 + i \tan\frac{\alpha}{2}(1 - \tau^2)}{1 - 2\kappa^2\tau + \tau^2 - i \tan\frac{\alpha}{2}(1 - \tau^2)} ;$$

$$0 < \kappa^2 < 1 \qquad (3)$$

This equation satisfies all conditions, to wit:

1) For real τ the numerator is conjugate complex to the denominator, hence $\left|\frac{dz}{dw}\right| = 1$;

2) For $\tau = 0$, that is, at infinity of plane z:

$$\frac{dz}{dw} = e^{-i\alpha} \frac{1 + i \tan\frac{\alpha}{2}}{1 - i \tan\frac{\alpha}{2}} = 1;$$

3) For $\tau = e^{i\vartheta}$:

$$\frac{dz}{dw} = e^{-i\alpha} \frac{\kappa^2 + \cos\vartheta + \sin\vartheta \tan\frac{\alpha}{2}}{\kappa^2 + \cos\vartheta - \sin\vartheta \tan\frac{\alpha}{2}}$$

the direction of $\frac{dz}{dw}$ is either $-\alpha$ or $\pi - \alpha$;

4) $\frac{dz}{dw}$ disappears once on the upper half of the unit circle, and also becomes once infinite. For ϑ_o and ϑ_S

$$\kappa^2 + \cos\vartheta_o + \sin\vartheta_o \tan\frac{\alpha}{2} = 0$$

$$\kappa^2 + \cos\vartheta_S + \sin\vartheta_S \tan\frac{\alpha}{2} = 0$$

whence

$$\tan\frac{\vartheta_o}{2} = \frac{1}{(1 - \kappa^2)\cos\frac{\alpha}{2}}\left\{\sin\frac{\alpha}{2} + \sqrt{1 - \kappa^4\cos^2\frac{\alpha}{2}}\right\}$$

$$\tan\frac{\vartheta_S}{2} = \frac{1}{(1 - \kappa^2)\cos\frac{\alpha}{2}}\left\{- \sin\frac{\alpha}{2} + \sqrt{1 - \kappa^4\cos^2\frac{\alpha}{2}}\right\}$$

The root carries the + sign. The other two roots are on the lower half of the unit circle. It is readily proved that

$$\vartheta_o - \vartheta_S = \alpha$$

5) Inflection points occur on the free streamline when $\frac{d}{d\tau}\left(\frac{dz}{dw}\right)$ disappears. Between -1 and $+1$ this derivation disappears only once for

$$\tau_o = -\frac{1}{\kappa^2}(1 - \sqrt{1 - \kappa^4})$$

with the special values $\tau_o = -1$ for $\kappa^2 = 1$ and $\tau_o = 0$ for $\kappa^2 = 0$

6) Lastly,

$$\frac{dz}{dw} = e^{-i\alpha}$$

for $\tau = \pm 1$ and $\kappa^2 \neq 1$

Hence, the free streamlines leave the plate tangentially, and equation (3) meets all of the required conditions.

To complete the determination of the flow, the complex potential as a function of τ is necessary. This is most readily achieved by the method of singularities. Since the infinite distance of the z-plane is a boundary point in the τ-plane the doublet creating the parallel flow of the z-plane becomes in the τ-plane a quadruplet in the origin whose axes directions agree with the coordinate directions, and the vortex at infinity of the z-plane - the counter vortex of the plate circulation - becomes a doublet with horizontal axis which may be visualized as being the result of convergence of two vortices of opposite equal circulation with fixed vortex moment (circulation × vortex distance). These singularities themselves give the real axis as streamline, and to make the unit circle streamline also, it is reflected on this circle, whence, after adding a nonessential constant for w:

$$w = \frac{C}{2}\left(\tau + \frac{1}{\tau}\right)^2 - 2\lambda C\left(\tau + \frac{1}{\tau}\right) \tag{4}$$

The constant defines the scale of the z-plane; λ is so defined that it makes the stagnation point of the upper half of the unit circle in $e^{i\vartheta_S}$ coincide.

Computing $\frac{dw}{d\tau}$ at:

$$\frac{dw}{d\tau} = - C\,\frac{1 - \tau^2}{\tau^3}\,(1 + \tau^2 - 2\lambda\tau)$$

so that

$$\lambda = \frac{1 + \tau_S{}^2}{2\tau_S} = \cos\vartheta_S$$

finally gives

$$\frac{dw}{d\tau} = - C\,\frac{(1 - \tau^2)(1 + \tau^2 - 2\tau\cos\vartheta_S)}{\tau^3} \tag{5}$$

With equations (3) and (5), the problem is largely solved because

$$\frac{dz}{d\tau} = \frac{dz}{dw} \cdot \frac{dw}{d\tau}$$

hence $\frac{dz}{d\tau}$ itself is known and, after integration, the transformal function $z(\tau)$ as well. After this the stream pattern in the z-plane can be readily plotted by transferring the easily constructed streamline system of the τ-plane.

Although this integration is elementary, the calculation is carried out for two specific cases only, so as to avoid tedious paper work.

1. The case $\kappa = 1$ gives the Helmholtz flow (fig. 5). It yields

$$\tau_o = -1; \quad \vartheta_o = \pi; \quad \vartheta_S = \pi - \alpha$$

and

$$\left. \begin{aligned} \frac{dz}{dw} &= e^{-2i\alpha} \frac{\tau + e^{i\alpha}}{\tau + e^{-i\alpha}}; \\ \frac{dw}{d\tau} &= -C \frac{1 - \tau^2}{\tau^3}(1 + \tau^2 + 2\tau \cos \alpha) \end{aligned} \right\} \qquad (6)$$

For the direction of the velocity in the starting points of the free streamlines, it gives

$$\tau = +1 \qquad \frac{dz}{dw} = e^{-i\alpha}$$

$$\tau = -1 \qquad \frac{dz}{dw} = -e^{-i\alpha} = e^{i(\pi-\alpha)}$$

When the direction of the tangent of the upper free streamline at the break-away point is $\pi - \alpha$, as now, instead of $-\alpha$, as before, it is due to the fact that now $\kappa^2 = 1$ is to be posted in equation (3) first and then $\tau = -1$ plotted. In this case equation (3) simplifies to equation (6) because the singularities at ϑ_o and $-\vartheta_o$ coincide in $\vartheta_o = \pi$, and the corresponding linear factors in equation (3) are shortened accordingly.

Because of

$$1 + \tau^2 + 2\tau \cos \alpha = (\tau + e^{i\alpha})(\tau + e^{-i\alpha})$$

we have

$$\frac{dz}{d\tau} = -C\, e^{-2i\alpha}\, \frac{(1 - \tau^2)(\tau + e^{i\alpha})^2}{\tau^3}$$

and, with the coordinate origin in the trailing edge:

$$z = -C\, e^{-i\alpha} \left\{ \cos\alpha - \frac{1}{2}\left(\tau^2 e^{-i\alpha} + \frac{e^{i\alpha}}{\tau^2}\right) + 4 - 2\left(\tau + \frac{1}{\tau}\right) - 2\, i \sin\alpha \ln\tau \right\} \tag{7}$$

The length of the plate l and the distance of the stagnation point from the trailing edge l_S for $\tau = -1$ and for $\tau = e^{i(\pi - \alpha)}$ is:

$$l = 2\, C(4 + \pi \sin\alpha);$$

$$l_S = 2\, C \left\{ \sin\alpha \sin 2\alpha + 2(1 + \cos\alpha) + (\pi - \alpha)\sin\alpha \right\}$$

or

$$\frac{l_S}{l} = \frac{\sin\alpha \sin 2\alpha + 2(1 + \cos\alpha) + (\pi - \alpha)\sin\alpha}{4 + \pi\sin\alpha}$$

$$= \frac{1 + \cos\alpha}{2}\left(1 + \frac{\alpha^2}{2} + \ldots\right) \tag{8}$$

Since the stagnation point of the pure circulation flow is at distance $l\,\dfrac{1 + \cos\alpha}{2}$ from the trailing edge, the stagnation point of the Helmholtz flow is slightly shifted forward. The force on the plate, which here is naturally at right angle to the plate, is, according to equation (34):

$$P = \frac{\pi \rho\, l\, v_\infty^2 \sin\alpha}{4 + \pi\sin\alpha} \tag{9}$$

or slightly less than one-fourth of the lift of the pure circulation flow.

2. The case of $\kappa = 0$ (fig. 6)

Here $\tau_0 = 0$; $\tan \vartheta_0 = -\cot \frac{\alpha}{2}$; $\tan \vartheta_S = \cot \frac{\alpha}{2}$

or

$$\vartheta_0 = \frac{\pi + \alpha}{2} \qquad \vartheta_S = \frac{\pi - \alpha}{2}$$

The singularities are symmetrical with the imaginary axis. Besides, we have:

$$\left.\begin{aligned} \frac{dz}{dw} &= e^{-2i\alpha} \frac{\tau^2 + e^{i\alpha}}{\tau^2 + e^{-i\alpha}}; \\ \frac{dw}{d\tau} &= -C \frac{1 - \tau^2}{\tau^3} \left(1 + \tau^2 - 2\tau \sin \frac{\alpha}{2}\right) \end{aligned}\right\} \tag{10}$$

because

$$\lambda = \cos \vartheta_S = \sin \frac{\alpha}{2}$$

The inflection point of the upper free streamline now situated at infinity, results in a continuously concave upper free streamline which unrestrictedly approaches the consistently convex lower free streamline. For great distances the two dead-air boundaries act as $y \sim -\ln x$, that is, their distance from axis x becomes logarithmically infinite.

We compute $z(\tau)$. Because

$$1 + \tau^2 - 2\tau \sin \frac{\alpha}{2} = \left(\tau + ie^{i\frac{\alpha}{2}}\right)\left(\tau - ie^{-i\frac{\alpha}{2}}\right)$$

we have:

$$\frac{dz}{d\tau} = \frac{dz}{dw} \frac{dw}{d\tau}$$

$$= -C\, e^{-2i\alpha} \frac{1 - \tau^2}{\tau^3} \frac{\tau - i e^{i\frac{\alpha}{2}}}{\tau + i e^{-i\frac{\alpha}{2}}} \left(\tau + i e^{i\frac{\alpha}{2}}\right)^2$$

The division in partial fractures gives:

$$\frac{dz}{d\tau} e^{i\alpha} = -C\left\{\frac{e^{i\alpha}}{\tau^3} - \tau e^{-i\alpha} - 2 \sin\frac{\alpha}{2}\left(\frac{e^{i\alpha}}{\tau^2} - e^{-i\alpha}\right) - \frac{2 i \sin\alpha\, e^{i\alpha}}{\tau} - \frac{4 \sin^2\alpha}{\tau + i e^{-i\frac{\alpha}{2}}}\right\} \tag{11}$$

and, after integration:

$$z e^{i\alpha} = C\left\{4 \sin^2\alpha \ln\left(\tau + i e^{-i\frac{\alpha}{2}}\right) + 2 i \sin\alpha\, e^{i\alpha} \ln\tau + \frac{1}{2}\left(\frac{e^{i\alpha}}{\tau^2} + \tau^2 e^{-i\alpha}\right) - 2 \sin\frac{\alpha}{2}\left(\frac{e^{i\alpha}}{\tau} + \tau e^{-i\alpha}\right)\right\} + D \tag{12}$$

The plate length is obtained by insertion of the limits of $\tau = 1$ to $\tau = i e^{i\frac{\alpha}{2}}$ at

$$l = 2C\left\{1 - 2 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + \frac{\pi + \alpha}{4} \sin 2\alpha - \sin^2\alpha \ln\left[2\left(1 - \sin\frac{\alpha}{2}\right)\right]\right\} \tag{13}$$

For the distance d of the break-away point of the upper free streamline from the trailing edge, it is:

$$d = 2C\left\{\frac{\pi}{2} \sin 2\alpha - 4 \sin\frac{\alpha}{2} \cos\alpha + \sin^2\alpha \ln\frac{1 + \sin\frac{\alpha}{2}}{1 - \sin\frac{\alpha}{2}}\right\} \tag{14}$$

and for the distance of the stagnation point from the leading edge:

$$l - l_S = 2C\left\{\alpha \sin\alpha \cos\alpha + 4 \sin^4\frac{\alpha}{2} - 2 \sin^2\alpha \ln\cos\frac{\alpha}{2}\right\} \tag{15}$$

Lastly, equation (12) discloses that, for small real τ, i.e., on the free boundaries far downstream from the plate:

$$x \sim \frac{C}{2\ \tau^2} - \frac{2\ C \sin \frac{\alpha}{2}}{\tau}\ ; \quad y \sim 2\ C \sin \alpha \ln |\tau|$$

or

$$y \sim - C \sin \alpha \ln x \tag{16}$$

wherewith the previous assumption of asymptotic course of the free streamlines is proved. The two free streamlines approach one another unrestrictedly, according to equation (12).

Evaluation of equation (34) gives the force on the plate at

$$P_x = 0;\ P_y = 2\ \pi\ \rho\ C\ v_\infty^2 \sin \alpha \tag{17}$$

The lift is therefore a pure lift, as it should be. For the practical range of small angles of attack the use of series expansions with respect to the angle of attack is recommended in place of formulas (13), (14), (15), and (17). We have:

$$l = 2\ C \left\{ 1 + \frac{\alpha}{2} (\pi - 2) + \alpha^2 \left(\frac{3}{4} - \ln 2 \right) + \ldots \right\}$$

and hence for small α

$$\frac{d}{l} = \alpha\ (\pi - 2) \left[1 - \frac{\alpha}{2}(\pi - 2) \right]$$

$$\frac{l - l_s}{l} = \alpha^2 \left[1 - \frac{\alpha}{2}(\pi - 2) \right]$$

and finally

$$A = \frac{\pi\ \rho\ l\ v_\infty^2 \sin \alpha}{1 + \frac{\alpha}{2}(\pi - 2) + \alpha^2 \left(\frac{3}{4} - \ln 2 \right)} = A_z f(\alpha)$$

$$\text{with}\quad A_z = \pi\ \rho\ l\ v_\infty^2 \sin \alpha \tag{18}$$

The lift is, in consequence, smaller than that of the circulation flow A_z because $f(\alpha) < 1$. For small

α the lift as a function of α is no longer a straight line, but a parabola which is tangent to the lift line of a Kutta-Joukowsky flow in the origin. A compilation of numerical values for $\frac{A}{A_Z}$, $\frac{d}{l}$, $\frac{l_S}{l}$, and of $\frac{l_{SZ}}{l}$ for stagnation point distance of the circulation flow, likewise counted from the trailing edge, is given in table I for a number of angles of attack.

TABLE I

α	A/A_Z	d/l	l_S/l	l_{SZ}/l
3°	0.9708	0.0580	0.9973	0.9973
6°	.9430	.1127	.9897	.9891
9°	.9166	.1643	.9775	.9755
12°	.8912	.2130	.9612	.9568
90°	.4915	.8663	.1679	.0000

For the adjoining series of the flows with finite wake, the singularities in the zero point of plane τ must be divided: the quadruplet becomes a pair of doublets, one of which lies within the upper half of the unit circle, the other, reflected, in the lower half. In accord with it the doublet splits into two vortices reflective to the real axis of equal and opposite circulation. The location of the singularities and the orientation of the doublet axis are to be so determined that the contour consisting of plate and dead-air region closes in the z-plane. The velocity on the free streamlines follows from the position of the break-away point of the upper free streamline, which is assumedly predetermined in the admissible region of the suction side. The resultant complex potential is much more complicated than for the flows studied so far, since the images of the singularities with respect to unit circle are themselves now located at infinity. So, since they do not contribute anything new, these flows are no longer discussed.

Figure 9 illustrates the pressure distribution on the flat plate at $\alpha = 12^\circ$ with the width of dead-air region diminishing to zero, and for comparison, in broken lines, the corresponding pressure distribution for pure circulation flow; figure 10 shows the curve of the free streamlines for the same angle α.

3. General Profile Flows

In this chapter our theorem is extended so as to afford, for any profile form, a corresponding flow as computed previously for the flat plate. No general proofs of existence are adduced; the fundamental facts of a practical calculation method are simply developed, since for physical reasons it seems certain that the series of dead-air flows which affords the transition from Helmholtz to pure circulation flow equally exists on every profile, as on the flat plate.

Aside from that, the derivation of the formulas is limited to the case of dead-air region the width of which diminishes to zero, for which the complex velocity of the corresponding plate flow reads, according to equation (10):

$$\zeta = \ln \frac{dz}{dw} = \ln \frac{\tau + i\, e^{i\frac{\alpha}{2}}}{\tau - i\, e^{-i\frac{\alpha}{2}}} + \ln \frac{\tau - i\, e^{i\frac{\alpha}{2}}}{\tau + i\, e^{-i\frac{\alpha}{2}}} - 2\, i\, \alpha \qquad (19)$$

The first term contains the conjugate singularities of the stagnation point in conjugate points of the unit circle, the second term the singularities of the sharp leading edge. The latter singularity must not occur for profiles of finite thickness with continuous camber. Its elimination is achieved by shifting points B and $\overline{B}$ (figs. 8 and 11) outward from the unit circle, but naturally so as to become conjugate again after the shifting. In other words, we replace, say $i\, e^{i\, \alpha/2}$ in (19) by $r\, e^{i\beta}$ with $r > 1$. By doing this, the second term in (19) becomes regular in and on the unit circle, so that it can be expanded in powers of τ, the radius of convergence of which equals $r > 1$. The coefficients of this series are purely imaginary; hence the series may be written

$$i \sum_{0}^{\infty} n\, a_n\, \tau^n \qquad a_n \text{ real}$$

Here the imaginary and now unessential constant $-2\,i\alpha$ may be assumed included in the absolute term of this series.

Thus, it is plain that by formula

$$\zeta = \ln \frac{\tau + i\,e^{i\frac{\alpha}{2}}}{\tau - i\,e^{-i\frac{\alpha}{2}}} + i\sum_{1}^{\infty} n\,a_n\,\tau^n \qquad a_{n\,real} \tag{20}$$

curved contours are equally included. Moreover, bearing in mind that the constants do not necessarily have to agree with those obtained by the development of the logarithm for satisfying the requirements to be made on (20) suggests that (20) itself is already the most general formula for arbitrary profiles. This is true, in fact, as will be demonstrated with several additional conditions for the constant a_n.

To begin with, it is clear that ζ is purely imaginary for $-1 \leqq \tau \leqq +1$, i.e., on the free streamlines; thus it gives the constant velocity equal to 1. But this is not enough to create a dead-air region of width converging to zero. The upper free streamline must also be continuously concave and the lower, either continuously convex - as on the flat plate - or first concave then continuously convex, as on profiles with finite trailing edge. The necessary occurrence of such cases at small angles of attack is readily seen from figure 2. Lastly, at infinity the curvature of both free streamlines of the same order must disappear as in the corresponding case of the flat plate, in order to avoid intersections of the free streamlines. The corresponding conditions for the constant a_n are developed by computing the curvature of the free streamlines as function of τ. Since the velocity on the free streamlines equals 1 and the stream function is constant, it is

$$\frac{\partial \varphi}{\partial S} = 1 \longrightarrow \varphi = S + C \qquad w = S + C + i\psi_0$$

In addition, $\zeta = i\theta$ and, since the curvature of a curve is given by

$$k = \frac{d\theta}{dS}$$

it follows that:

$$k = \frac{d\theta}{d\tau}\frac{d\tau}{dw}\frac{dw}{dS} = \frac{d\theta}{d\tau}:\frac{dw}{d\tau}$$

For the flat plate, equations (10) and (12) give

$$k = \frac{2\ C\ \tau^4 \sin\alpha}{(1 - \tau^2)\left(1 + \tau^2 - 2\tau \sin\frac{\alpha}{2}\right)^2\left(1 + \tau^2 + 2\tau \sin\frac{\alpha}{2}\right)} \tag{21}$$

whereas our formula (20) for any contour yields

$$k = \frac{C\ \tau^3}{(1 - \tau^2)\left(1 + \tau^2 - 2\tau \sin\frac{\alpha}{2}\right)}$$

$$\left\{\frac{2 \cos\frac{\alpha}{2}}{1 + \tau^2 - 2\tau \sin\frac{\alpha}{2}} - \sum_1^{\infty} n\ a_n\ \tau^{n-1}\right\} \tag{22}$$

The known characteristics of the free streamlines of the plate flow are read directly from equation (21): for positive τ - lower free streamline - k is positive, i.e., the free streamline is convex. The curvature becomes infinite at the break-away point and disappears at infinity of the z-plane ($\tau = 0$) of the fourth order in τ. For negative τ - the upper free streamline - k is positive; hence the free streamline concave. For the rest the same holds true as for $\tau > 0$. From this action of the plate flow there follows immediately a condition for a_1 for the general profile forms: to insure a qualitatively equal flow curve at infinity as on the flat plate, the braces $\{\}$ in (22) must disappear for $\tau = 0$, i.e.:

$$\alpha_1 = 2 \cos\frac{\alpha}{2} \tag{23}$$

This condition must be absolutely satisfied if neither a dead-air region of infinite width nor a physically impossible overlap of free streamlines is to occur. This condition is termed, for short, the closing condition.

It is further required that the a_n be so constituted that the brace does not disappear at all between -1 and 0 and only once at the most between 0 and +1. This signifies

$$-1 < \tau < 0 \quad \frac{2 \cos \frac{\alpha}{2}}{1 + \tau^2 - 2\tau \sin \frac{\alpha}{2}} < \sum_1^\infty n \quad n\, a_n\, \tau^{n-1}$$

specifically:

$$\frac{\cos \frac{\alpha}{2}}{1 + \sin \frac{\alpha}{2}} < \sum_1^\infty n \; (-1)^{n-1} n\, a_n$$

This condition is, because of the magnitude of a_1 practically always satisfied. The two possible cases for the lower free streamline depend upon the action of the braces for $\tau = 1$. If

$$\frac{\cos \frac{\alpha}{2}}{1 - \sin \frac{\alpha}{2}} > \sum_1^\infty n \quad n\, a_n$$

then the lower free streamline is consistently convex. If the less-than sign (<) applies, the free streamline is first concave, then convex. With the equal sign (=), the consistently convex free streamline with finite curvature starts at the break-away point. Here it is presumed that the braces for $0 < \tau < 1$ disappear once at the most, as, of course, will be the case with logical profile forms. However, it is stressed that all cases where a_n fails to satisfy these conditions, either yield a dead-air region of infinite width or else result in overlapping in the plane of flow.

Like the curvature of the free streamlines as a function of ϑ, the contour curvature itself can be represented as a function of $\vartheta (\tau = e^{i\vartheta})$. Taking (1) and (2) into account, it follows that

$$k = \frac{d\theta}{dS} = \frac{d\theta}{d\vartheta}\frac{d\vartheta}{d\varphi}\frac{d\varphi}{dS} = \frac{d\theta}{d\vartheta}\frac{q}{\frac{dw}{d\vartheta}} \tag{24}$$

Likewise the direction of the velocity θ and hence the tangential direction of the contour as imaginary part of ζ are known in relation to ϑ. So, in order to compute the corresponding flow for a known profile with given orientation of flow direction, we first establish the contour curvature with respect to the tangent slope. Herewith the left side of (23) in relation to θ is known and with it also in relation to ϑ and a_n when θ is computed on the basis of (20). The right-hand side of (23) is also given with respect to ϑ and the a_n's whence at least the a_n's can be computed from this equation. Approximately this is always possible, when stipulating the compliance of this equation only for a number of discreet, suitably chosen values of ϑ. It affords a profile that is in agreement with the given one on a corresponding number of points in the curvature and the related tangential slope, that is, a precisely as close approximation as represented by the substitution of a curve by a basket curve, that is, a sequence of pieces of successively following curvature circles. In practice this calculation is likely to be extremely cumbersome even by small density of the partial points, especially since the whole determination of the constants would have to be repeated by a change in angle of attack, because this angle enters over the tangential slope of the suction side on the trailing edge in the determination of the constants.

The next chapter deals with a special class of profiles produced by a mechanism similar to that of the Joukowsky profiles and therefore amenable to simpler treatment.

4. Adjacent Profiles

They represent a special class of thin profiles which lend themselves to simpler treatment. Reverting to the arguments which afforded (20) we shift points B and $\overline{B}$ slightly outside of the circle in the $ire^{i\beta/2}$ and $-ire^{-i\beta/2}$ with $r > 1$. This rounds off the sharp leading edge and - depending upon the amount of displacement - affords a more or less thick profile whose mean line is the corresponding flat plate. However, the thus obtained forumla produces as yet no practical profiles because of the concurrently to be satisfied closing condition. So we extend it by locating further singularities in the τ-plane in point $iRe^{i\gamma}$ and $-iRe^{-i\gamma}$, whereby

$$\lim_{r \to 1} R \to \infty$$

so that in the extreme case the flat plate flow occurs again. Accordingly, we put (equation (19)):

$$\frac{dz}{dw} = -e^{-i(\alpha+\beta+2\gamma)} \frac{\tau + ie^{i\frac{\alpha}{2}}}{\tau - ie^{-i\frac{\alpha}{2}}} \frac{\tau - ire^{i\frac{\beta}{2}}}{\tau + ire^{-i\frac{\beta}{2}}} \frac{\tau - iRe^{i\gamma}}{\tau + iRe^{-i\gamma}} \tag{25}$$

This formula gives $\frac{dz}{dw} = 1$ for $\tau = 0$ and $\left|\frac{dz}{dw}\right| = 1$ on the free streamlines. The closing condition (23) here reads

$$\cos\frac{\alpha}{2} = \frac{\cos\frac{\beta}{2}}{r} + \frac{\cos\gamma}{R} \tag{26}$$

After putting

$$ire^{i\frac{\beta}{2}} = ie^{i\frac{\alpha}{2}} + \rho e^{i\delta} \tag{27}$$

a profile is termed "thin" when the profile parameters ρ and $\frac{1}{R}$ are so small that its squares and products are negligible.

Then equation (25) can be written as

$$\frac{dz}{dw} = e^{-i(\alpha+\beta)} \frac{\tau + ie^{i\frac{\alpha}{2}}}{\tau - ie^{-i\frac{\alpha}{2}}} \frac{\tau - ie^{i\frac{\alpha}{2}}}{\tau + ie^{-i\frac{\alpha}{2}}}$$

$$\left\{1 - 2i\rho \frac{\tau \sin\delta + \cos(\delta - \alpha/2)}{\left(\tau - ie^{i\frac{\alpha}{2}}\right)\left(\tau + ie^{-i\frac{\alpha}{2}}\right)} + \frac{2i\tau\cos\gamma}{R}\right\} \tag{28}$$

According to equations (26) and (28), only the value of $\frac{\cos\gamma}{R}$ is of importance on these profiles, so that without restriction of generalization we can put $\gamma = 0$ or $\gamma = \pi$.

Posing $\beta = \alpha + \Delta\alpha$ and denoting the derivation of z with respect to τ for the flat plate by subscript o gives, from equation (11) and (28),

$$\frac{dz}{d\tau} = e^{-i\Delta\alpha} \frac{dz_o}{d\tau} - C\, e^{-i(\alpha+\beta)} \frac{1 - \tau^2}{\tau^3} \left(\tau + ie^{i\frac{\alpha}{2}}\right)^2$$

$$\left\{ \frac{2\, i\tau \cos\gamma}{R} \frac{\tau - ie^{i\frac{\alpha}{2}}}{\tau + ie^{-i\frac{\alpha}{2}}} - 2\, i\rho \frac{\tau \sin\delta + \cos\left(\delta - \frac{\alpha}{2}\right)}{\left(\tau + ie^{-i\frac{\alpha}{2}}\right)^2} \right\} \tag{29}$$

The first term again gives the flat plate but with an angle of attack raised by $\Delta\alpha$; the second and third give the departure of the profile contour from the straight line and are to be considered as small of the first order.

From it $z(\tau)$ can then be computed by decomposition of the partial fraction and elementary integration. But the obtained expressions are so complicated that they are not reproduced here. Our chief interest is centered on the location of the break-away point on the suction side in relation to the trailing edge. Posting

$$z = e^{-i\Delta\alpha} z_o + \Delta z$$

while the coordinate origin is placed in the trailing edge, the complex vector becomes the break-away point on the suction side

$$d\, e^{i(\pi-\beta)} + \Delta z_A$$

with

$$\Delta z_A\, e^{i\beta} = \frac{\cos\gamma}{R} \Delta z_R + \rho \cos\left(\delta - \frac{\alpha}{2}\right) \Delta z_{\rho_1} + \sin\delta\, \Delta z_{\rho_2} \tag{30}$$

where d is taken from (14). With scale factor equal to one

$$\Delta z_R = -\frac{16}{3} i + 2\alpha \frac{16 - 3\pi}{3}$$

$$\Delta z_{\rho_1} = 2\pi - 2\, i\alpha(4 - \pi)$$

$$\Delta z_{\rho_2} = 8\, i + *$$

is valid up to terms in (α^2).

Now the term $\Delta z_A e^{i\beta}$ merely measures the displacement of the break-away point from its place on the flat plate with angle α - this plate, however, having as profile mean line the angle β - in its place on the thin profile, the real part of this expression of the displacement in plate direction (positive values indicate displacement toward trailing edge) and the imaginary part the displacement at right angle to the plate. (Positive values signify displacement upward.)

Thus for small α Δz_{ρ_1} is essentially real, whereas Δz_R and Δz_{ρ_2} are esentially imaginary. The location of the singular points is then so chosen that the displacement of the break-away point in plate direction becomes either negative or, when positive, becomes as small as possible in order to give the piece of the suction side in the dead-air region - in itself arbitrarily plotted - an aerodynamically beneficial curve.

At the same time the displacement at right angle to the plate must become positive or the profile itself overlaps. The break-away point must therefore be located in the shaded zone of figure 12.

For limiting the permissible zone for the profile constants R, ρ, and δ, which, moreover, must satisfy the closing conditions, several specific cases are analyzed.

1. $\beta = \alpha$ gives

$$\delta = \frac{\pi + \alpha}{2}; \; r = 1 + \rho$$

and with the closing condition

$$\frac{\rho}{1 + \rho} \cos \frac{\alpha}{2} = \frac{\cos \gamma}{R}; \quad \gamma = 0 \quad \frac{1}{R} \sim \rho \cos \frac{\alpha}{2}$$

hence, according to (30) and (31):

$$\Delta z\, e^{i\beta} = \frac{8\, i}{3}\, \rho \cos\frac{\alpha}{2} + 2\, \alpha\, \rho \cos\frac{\alpha}{2}\, \frac{16 - 3\pi}{3}$$

The displacement in plate direction, while positive, is very small; displacement at right angle to plate becomes positive; hence it affords a permissible case.

2. $\delta = \pi$ gives

$$r \cos\frac{\beta}{2} = \cos\frac{\alpha}{2}; \quad \gamma = 0$$

$$\frac{1}{R} = \frac{r^2 - 1}{r} \cos\frac{\beta}{2}$$

The displacement in plate direction becomes negative, but as that at right angle to the plate is likewise negative, the profile overlaps, so this case is ruled out.

3. $\frac{1}{R} = 0$. The closing condition now reads $\cos\frac{\alpha}{2} = \frac{\cos\frac{\beta}{2}}{r}$ and yields as location of the singularity the circle about the point $\left(0, \frac{1}{2\cos\frac{\alpha}{2}}\right)$ with radius $\frac{1}{2\cos\frac{\alpha}{2}}$ (fig. 13). Since, in any case, $\frac{\cos\frac{\beta}{2}}{r} > \cos\frac{\alpha}{2}$ for $r = 1$ and $\beta < \alpha$, it follows that the closing condition calls for $\gamma = \pi$ in the vertically shaded zone. And, since there δ must be less than $\frac{\pi}{2}$, all three of the terms in equation (30) give for this zone positive values, so that the break-away point shifts upward and toward the trailing edge.

4. Lastly, we determine the curve in the τ-plane separating the region of the profiles with self-overlap from the remaining zone. On this curve

$$8\, \rho \sin\delta = \frac{16}{3R} \qquad (32)$$

is at the same time applicable, according to (30) and (31), and the closing condition (26) into which ρ and δ are inserted for r and β conformably to (27). Posting $\frac{\alpha}{2} = \nu$, we obtain with allowance for (32) the equation of the boundary curve

$$2 \cos \nu - 3 \rho \sin \delta = 2 \frac{\cos \nu + \rho \sin \delta}{1 + \rho^2 + 2 \rho \sin (\delta - \nu)}$$

or in rectangular coordinates

$$\xi = \rho \cos \delta; \quad \eta = \rho \sin \delta$$

after a simple transformation

$$\eta = \frac{2}{3} \cos \nu \left[1 - \frac{5}{3(\xi - \sin \nu)^2 + 3(\eta + \cos \nu)^2 + 2}\right] \quad (33)$$

For negative ξ increasing in amount (for positive ξ, the curve is inside of the unit circle) η increases monotonic and approaches asymptotically the straight line

$$\eta = \frac{2}{3} \cos \nu$$

But for small ξ

$$\eta \sim - \frac{2 \sin 2 \nu}{1 + 4 \sin^2 \nu} \xi$$

in first approximation.

Figure 13 illustrates the approximate course of this boundary curve.

Summed up, the result is the following: If the singularity corresponding to the sharp leading edge is situated in the region below the boundary curve, the results are profiles with self-overlap. Between the straight $\beta = \alpha$ and the boundary curve, there are given profiles with a shifted break-away point in the favorable sense, negative in plate direction, positive at right angle to it. To the right of this straight line the positive displacement at right angle to the plate keeps on increasing but, as the displacement in plate direction becomes positive also, the curve of the suction-side piece in the dead-air region becomes unfavorable.[1] Beneficial profile shapes are therefore to be expected only when the singularity is located in the wedge between the limiting curve and the straight line $\beta = \alpha$.

An example of the profile forms obtained in this zone is shown in figure 14. The parameters for this profile are:

$$\alpha = 9^\circ; \quad \beta = 18^\circ; \quad r = 1.1; \quad \frac{l}{R} = 0.0990; \quad \gamma = 0$$

The profile coordinates are obtained by numerical integration from equation (25). The dotted piece on the suction side is in the dead-air region and, being by itself completely arbitrary, was so plotted as to give the suction side an aerodynamically beneficial shape. The direction of flow is at an angle of 13°30' to the horizontal.

APPENDIX

5. Force Calculation

According to Bernoulli's equation,

$$p - p_0 = p^* = \frac{\rho}{2}\,(1 - q^2)$$

where p_0 is the static pressure at infinity; p^* disappears in the dead-air region. The component of the resultant force on the profile is read as

$$P_x = \oint p^* \cos(n_x) ds; \quad P_y = \oint p^* \cos(n_y) ds$$

or, because of

$$ds \cos(n_x) = -dy; \quad ds \cos(n_y) = dx$$

[1]If, for instance, $r = \frac{1}{\cos\alpha}$ and $\beta = 0$, the break-away point on the suction side is already to the right from the trailing edge.

$$P_x - iP_y = -\oint p^*(dy + idx) = -i\,\frac{\rho}{2}\oint (1 - q^2)(dx - idy)$$

Since

$$q = \frac{dw}{dz}\,e^{i\theta}$$

it follows that

$$P_x - iP_y = \overline{P} = -i\,\frac{\rho}{2}\oint\left(1 - \left(\frac{dw}{dz}\right)^2 e^{2i\theta}\right)\overline{dz}$$

or, because of

$$dz = \overline{dz}\,e^{2i\theta}$$

finally

$$\overline{P} = i\,\frac{\rho}{2}\oint\left(\left(\frac{dw}{dz}\right)^2 dz - \overline{dz}\right)$$

As the piece of the suction side in the dead-air region contributes nothing to the integral, we can write

$$\overline{P} = -i\,\frac{\rho}{2}\int_0^\pi\left(\frac{dw}{dz}\right)^2\frac{dz}{d\tau}\,d\tau + i\,\frac{\rho}{2}\int_0^\pi\overline{\frac{dz}{d\tau}}\,d\tau$$

Then, according to (20)

$$\frac{dz}{dw} = \frac{\tau + ie^{i\frac{\alpha}{2}}}{\tau - ie^{-i\frac{\alpha}{2}}}\,e^{i\Sigma\,a_n\tau^n}$$

hence on the unit circle with $\tau = e^{i\vartheta}$

$$\overline{\frac{dz}{dw}} = \frac{e^{-i\vartheta} - ie^{-i\frac{\alpha}{2}}}{e^{i\vartheta} + ie^{i\frac{\alpha}{2}}}\,e^{-i\Sigma a_n e^{-in\vartheta}}$$

$$\frac{dw}{dz} = \frac{e^{i\vartheta} - ie^{-i\frac{\alpha}{2}}}{e^{i\vartheta} + ie^{i\frac{\alpha}{2}}}\,e^{-i\Sigma a_n e^{in\vartheta}}$$

If both expressions are compared, it is seen that, valid on the unit circle,

$$\overline{\left(\frac{dz}{dw}\right)_z} = \left(\frac{dw}{dz}\right)_{\overline{z}}$$

Expressed in words: $\frac{dz}{dw}$ formed for z is conjugate complex to $\frac{dw}{dz}$ formed for $\overline{z}$.

Furthermore, since $\frac{dw}{d\tau}$ is real for real τ,

$$\overline{\left(\frac{dw}{d\tau}\right)_z} = \left(\frac{dw}{d\tau}\right)_{\overline{z}} \qquad \overline{(d\tau)}_z = (d\tau)_{\overline{z}}$$

so that we may write:

$$\int_0^\pi \overline{\frac{dz}{d\tau}}\, d\tau = \int_0^\pi \overline{\frac{dz}{dw}}\; \overline{\frac{dw}{d\tau}}\, d\tau = \int_0^\pi \left(\frac{dw}{dz}\frac{dw}{d\tau}\, d\tau\right)_{\overline{z}}$$

$$= \int_0^\pi \left[\left(\frac{dw}{dz}\right)^2 \frac{dz}{d\tau}\, d\tau\right]_{\overline{z}} = -\int_{-\pi}^0 \left(\frac{dw}{dz}\right)^2 \frac{dz}{d\tau}\, d\tau$$

which finally gives

$$\overline{P} = -i\frac{\rho}{2}\int_{-\pi}^{+\pi} \left(\frac{dw}{dz}\right)^2 \frac{dz}{d\tau}\, d\tau \equiv -i\frac{\rho}{2} \oint \left(\frac{dw}{dz}\right)^2 dz \qquad (34)$$

where the last integral in the positive sence of rotation is to be extended over the unit circle of plane τ.

The integrand for the flat plate is

$$\frac{dw}{dz}\frac{dw}{d\tau} = -C\, e^{2i\alpha} \frac{\tau - ie^{-i\frac{\alpha}{2}}}{\tau - ie^{i\frac{\alpha}{2}}} \frac{(\tau^2 + e^{-i\alpha})(1 - \tau^2)}{\tau^3}$$

The separation of the polar at $ie^{i\frac{\alpha}{2}}$ corresponding to the leading edge by a cut inward of small radius leaves the sole singularity located at $\tau = 0$ with the residuum

$$\operatorname{res}(0) = -2\, i\, C \sin\alpha$$

whence, according to (34):

$$\overline{P} = \pi \rho \operatorname{res}(0) = -2 i \pi \rho C \sin \alpha$$

The result is therefore a pure lift of magnitude

$$A = 2\pi \rho C \sin \alpha$$

In conclusion, we compute the suction force on the leading edge by evaluating the integral (34) over the small radius about the point $i e^{i \alpha/2}$. The result is:

$$S = - 2 \pi \rho C \sin^2 \alpha e^{-i\alpha}$$

As a check, we show that the lift diminished by the suction force is at right angle to the plate

$$\underline{A} - \underline{S} = 2\pi \rho C \sin \alpha (i + \sin \alpha e^{-i\alpha})$$

or, because of

$$i + \sin \alpha e^{-i\alpha} = i \cos \alpha e^{-i\alpha}$$

$$\underline{A} - \underline{S} = \pi \rho C \sin 2 \alpha e^{i\left(\frac{\pi}{2}-\alpha\right)}$$

For general profile forms, we find

$$\overline{P} = - \delta \rho i C(\sin \alpha - \alpha_2) \tag{35}$$

with allowance for (20) and (23), so that here also, as it should be, a pure lift results.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

FIGURE LEGENDS

Figure 1.- Flow past a profile conformable to the Kutta-Joukowsky condition of flow: the flow leaves in the direction of the angle bisectrix of the trailing edge; at other than zero edge angle, a stagnation point is located in the trailing edge.

Figure 2.- Flow around the profile of figure 1 with vortex separation; the flow leaves the trailing edge tangentially on the pressure side at a speed equal to the velocity of flow; a similar tangential separation occurs on the suction side near the trailing edge. Between the two limiting streamlines formed by vortex sheets of equal and opposite circulation is a dead-air region.

Figure 3.- Dead-air flow around a flat plate in the general case of a dead-air region, the width of which becomes infinite.

Figure 4.- Transformation of the plane of the stream on the plane of the logarithm of the complex velocity. The border corresponding to the plate is the heavy solid line, that of the corresponding free streamlines is the thin line.

Figure 5.- The common Helmholtz flow around a flat plate; the force is at right angle.

Figure 6.- Dead-air flow around the flat plate with dead-air width converging to zero; the force is at right angle to the direction of the stream.

Figure 7.- Pure circulation flow around the flat plate with smooth efflux from the trailing edge, the force is at right angle to the stream.

Figure 8.- Transformation of the plane of the stream on a semicircle; the edge representing the flat plate is shown as heavy line, that of the free streamlines as a thin line.

Figure 9.- Pressure distribution of the flat plate at $\alpha = 12^\circ$, with separation (solid curve) and in pure circulation flow (dotted curve). For profiles with trailing edge other than zero the departure of both curves in vicinity of the trailing edge is much greater, since then the pressure difference in the trailing edge becomes equal to the dynamic pressure.

Figure 10.- Course of the free streamlines for the flat plate at $\alpha = 12^\circ$. The break-away on the upper free streamline occurs at 0.213 plate chord distance from the trailing edge. (The y-scale is three times greater than the x-scale for better representation.)

Figure 11.- Profiles of finite thickness obtained by shifting of the singularity related to the sharp leading edge of the flat plate in the outer space of the unit circle of the τ-plane.

Figure 12.- Permissible zone for location of break-away point of upper free streamline on transition from the flat plate to an adjacent profile.

Figure 13.- Permissible zone for the singularity in the τ-plane related to the sharp leading edge of the flat plate. The zero circle in the shaded zone corresponds to the profile plotted in figure 14.

Figure 14.- Shape of profile with parameters given in context; the dotted piece of the suction side is in the dead-air region.

Figure 15.- Calculation of force acting on the profile. The vector of the normals points toward the inside of the circle.

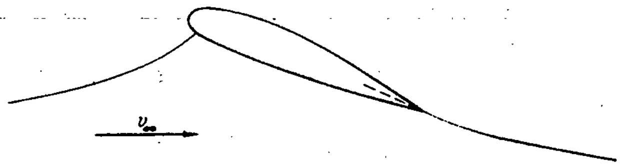

Figure 1.

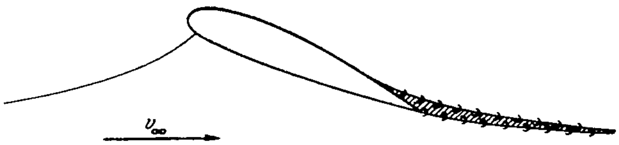

Figure 2.

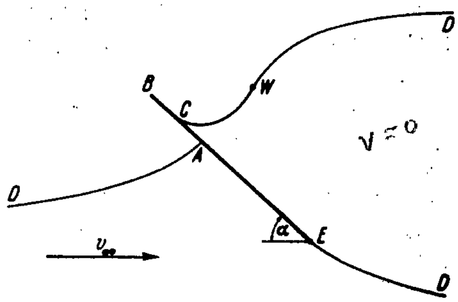

Figure 3.

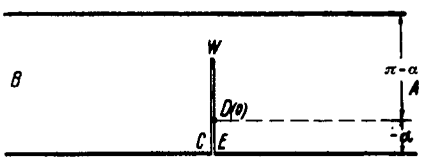

Figure 4.

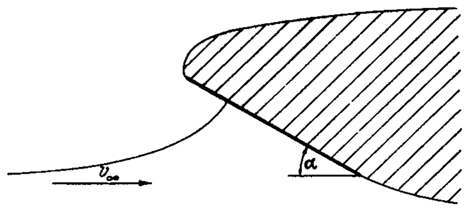

Figure 5. Flat plate Helmholtz flow

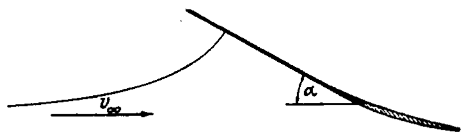

Figure 6.

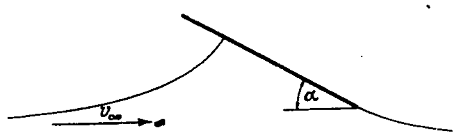

Figure 7.

Flat plate with Γ

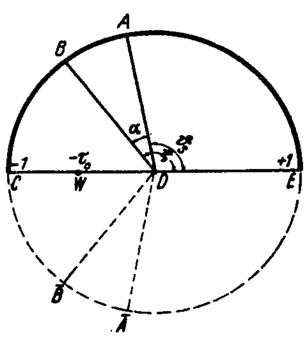

Figure 8.

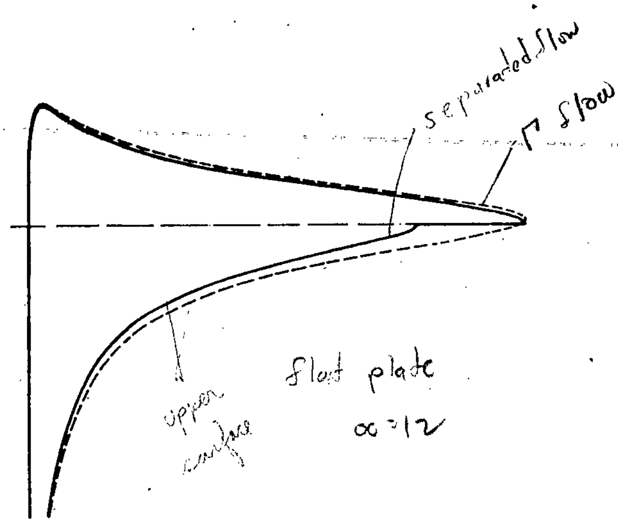

Figure 9.

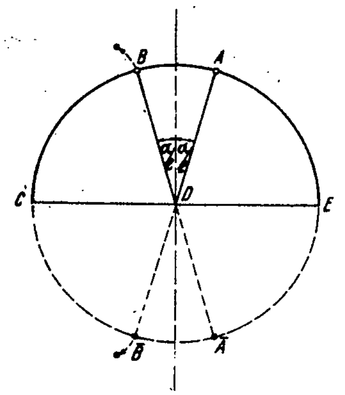

Figure 11.

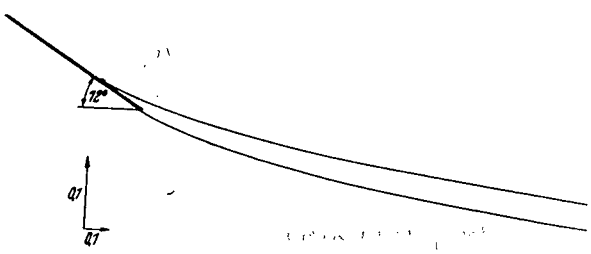

Figure 10.

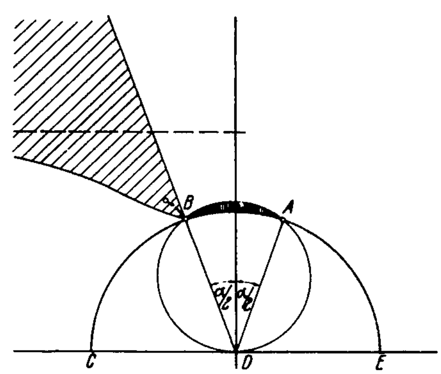

Figure 13.

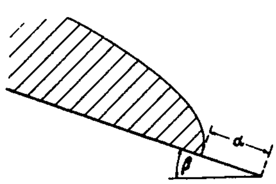

Figure 12.

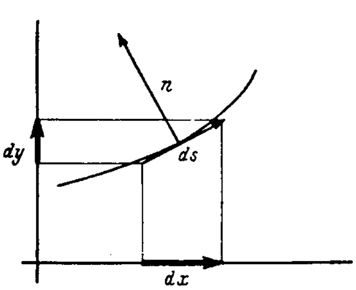

Figure 15.

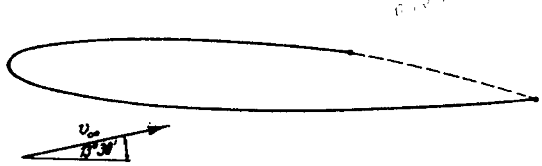

Figure 14.