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By E. Cambilargiu

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THE TORSION OF BOX BEAMS WITH ONE SIDE LACKING*

By E. Cambilargiu

The torsion of box beams of rectangular section, the edges of which are strengthened by flanges, and of which one side is lacking, is analyzed by the energy method. The torsional stresses are generally taken up by the bending of the two parallel walls, the rigidity of which is augmented by the third wall. The result was checked experimentally on duralumin and plywood boxes. The torsion recorded was 10 to 30 percent less than that given by the calculation, owing to self-stiffening.

I. INTRODUCTION

Box beams (rectangular) lacking one side, with or without bulkheads, find frequent use in airplane design as, for instance, on the wing near fuel tanks, or bomb racks, or the landing gear, or even in the fuselage in the vicinity of a wide door, or of the load compartment for droppable loads (bombs, provisions), or incidental to armement installation. Shell constructions of rectangular, rounded-off section, as customary on wings and fuselages near openings extending to an inside wall, or in any case of considerable width, can also be approximately treated as such.

Section II explains the unsuitability of Bredt's method. The analysis is made according to Minelli's procedure.

Section III describes an experimental procedure for the exact derivation of the values of shear modulus G. The experimental solution of G and Young's modulus E of the employed material is followed subsequently by a torsion test and a torsion analysis of thin-walled prismatic beams of rectangular section with one side lacking. The materials are duralumin and plywood. The analytical data are discussed and compared with the test data.

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*"Berechnung der Verdrehung kastenförmiger Träger, denen eine Wand fehlt." Luftfahrtforschung, vol. 16, no. 8, August 20, 1939, pp. 403-411. ("Il calcolo torsionale delle travi a cassone mancanti di una parete.")

II. THEORETICAL ANALYSIS OF TORSION

Bredt's theory of torsion of hollow cylinders and thin-walled prisms states that the torsional stiffness of a body of this type is zero when a part lying between two generatrices is lacking or, more simply expressed, when the body along a generatrice is cut up. It is sufficient to check Bredt's formula

$$B = \frac{4\ G\ S^2}{\int \frac{dc}{s}}$$

If $s = 0$, for no matter how small the region, then $B = 0$.

Experience, on the other hand, shows that it is possible to maintain a certain, not negligible, torsional stiffness for hollow, thin-walled prisms, which are partitioned and open. (The partitioning corresponds to that considered in Bredt's theory.) It only stipulates the prisms to be built in at one end or both - this case is technically little probable - in such a way as to preclude axial warping of the end section.

Torsional stiffness can be markedly increased by flanges running along the edges of the hollow, open prism. In consequence, the torsional stress involved does not correspond to the classical St. Venant-Bredt stress, but rather to one under which the particular hollow body, which the external force tries to twist, and actually twists, not merely reacts with shearing stresses but also with normal stresses. In other words, rather than a true <u>hollow body</u>, it represents a <u>system</u> of beams joined along the edges, each of which is stressed separately in shear and bending.

With a type of fixity not prohibiting axial warping of the end section, the hollow body could have no torsional stiffness differing from zero. If the upright walls were joined at the point of fixity with a cylindrical hinge with vertical axis, and the horizontal wall with a cylindrical hinge with vertical axis, thus permitting the built-in section to warp at will, the torsional stiffness would be zero. Hence, it is assumed that the restraint is actually as previously indicated, so that the problem becomes that of torsion of an open, thin-walled prism with flanges and partitions, as illustrated in figure 1.

The torque is transmitted by means of two vertical forces P of equal magnitude and opposite direction and applied at the vertical walls in the end section. What are the elementary form changes of which the state of the total form change consists?

The two vertical walls are strained by antisymmetrical bending moments of equal and opposite magnitude. Let y_1 be the bending ordinate, and y_2 the shear ordinate of a vertical wall.

The tendency of the left or right wall to deflect upward or downward is counteracted by the liaison of the vertical walls with the horizontal wall along the edges. Actually the horizontal wall prevents the lower edge of the vertical left wall from becoming shorter and that of the right-hand wall from becoming longer. The result is a countereffect on the upright wall facing the horizontal wall along the edge, which produces an axial strain in these walls. It is therefore necessary to take into account a total axial displacement $\xi(x)$ of the sections on the vertical walls, naturally in the opposite direction; that is, toward the negative x axis for the left wall, and toward the positive x axis for the right wall.

The horizontal wall itself receives axial reactions from the vertical walls, of equal magnitude and opposite direction, against which it can react only with the bending y_3 and, if necessary, with the shear y_4. But it is not strained as a whole; i.e., its center line retains its original length.

The state of deformation is therefore reduced to the five parameters y_1, y_2, ξ, y_3, y_4. Their positive directions are those shown in figure 1. As seen, the x axis for each wall was assumed with point of origin in the outside free end. The positive direction points from free end toward the restraint. The strain condition is explored by means of the energy method, which is based on the principle of virtual energy (reference 1). (The notation used in the present report is the same as Minelli's (reference 1).)

Let L denote the strain energy, and U the sum of the scalar products of external forces and displacements of their applied points, then form the difference L - U. The principle states that between all strain conditions reconciled with the type of support, the state for which $L - U$ becomes a minimum is the true state. Hence, expressing $L - U$ in relation to the form changes and defining these to the satisfaction of the support conditions and the minimum condition for $L - U$ gives the true state of strain. For the case in point, it is:

$$U = P\left\{y_1(0) + y_2(0)\right\} = -\int_0^l P(y_1' + y_2')\,dx$$

Now L is to be expressed.

Let J_v and J_o, respectively, denote the moment of inertia of one vertical or the horizontal wall, Ω the cross-sectional area of a vertical wall, s_1 the thickness of a vertical wall of height h, and s_2 of the horizontal wall of height b.

The flanges contained in Ω are assumed equal, and they are also counted in, in J_v. If, instead, the lower flanges are ascribed to the horizontal wall, J_v would have to be given a value which would correspond to the vertical wall without the lower flange, whereby the neutral axis of the wall would be displaced upward. The stress τ along the wall web is assumed uniformly distributed, and hence $\tau = -G\,y_2'$ in the vertical walls as well as in the horizontal wall. Then

$$\left.\begin{aligned} L = \int_0^l \Big\{ & 2\,\frac{1}{2}\,E\,J_v\,y_1''^2 + \frac{EJ_o}{2}\,y_3''^2 + 2\,\frac{1}{2}\,E\,\Omega\,\xi'^2 \\ & + 2\,\frac{1}{2}\,G\,s_1\,h\,y_2'^2 + \frac{G s_2 b}{2}\,y_4'^2 \Big\}\,dx \end{aligned}\right\} \qquad (0)$$

and $L - U$ may be expressed with

$$\left.\begin{aligned} L - U = \int_0^l \Big\{ & E\,J_v\,y_1''^2 + \frac{EJ_o}{2}\,y_3''^2 + E\Omega\xi'^2 + G s_1\,h y_2'^2 \\ & + \frac{G s_2 b}{2}\,y_4'^2 + 2P(y_1' + y_2') \Big\}\,dx \end{aligned}\right\} \qquad (1)$$

The result is a functional, depending on five functions which must make it a minimum. The equation of continuity between two walls along the edge eliminates one. Owing to the equality of strain along the edge, we have for the two vertical walls:

$$\xi + \frac{h}{2} y_1{}' = \frac{b}{2} y_3{}'$$

from which follows:

$$y_3{}' = \left(\frac{2}{b}\xi + \frac{h}{b} y_1{}'\right) \tag{2}$$

The substitution of this expression for y_3 in equation (1) reduces the new functional to the four functions y_1, y_2, y_4, ξ, giving

$$\left.\begin{aligned} L - U = \int_0^l \Big\{ & E\, J_v\, y_1''^2 + \frac{EJ_o}{2}\left(\frac{2}{b}\xi' + \frac{h}{b} y_1''\right)^2 + E\Omega\xi'^2 \\ & + G\, s_1\, h\, y_2'^2 + \frac{G s_2 b}{2} y_4'^2 + 2P(y_1' + y_2')\Big\} dx \end{aligned}\right\} \tag{3}$$

In the search of the functions y_1, y_2, y_4, ξ, which make this functional a minimum, the following limiting conditions which reflect the geometric constraint at the point fixity, should be observed:

$$y_1(l) = y_2(l) = y_4(l) = 0;\ y_1{}'(l) = 0;\ \xi(l) = 0 \tag{4}$$

A variation $\epsilon\eta$ is applied separately to each of the four functions. Considering the corresponding functional as function $\Phi(\epsilon)$, we can write:

$$\left(\frac{\partial\Phi(\epsilon)}{\partial\epsilon}\right)_{\epsilon=0} = 0 \tag{5}$$

η must naturally comply with the established limiting conditions which y_1, y_2, y_4, ξ themselves satisfy. Equation (5) is the well-known equation of the calculus of variations.

The variation is first applied to y_4.

$$\left.\begin{aligned}\Phi(\epsilon) \int_0^l \Big\{ & E\, J_v\, y_1''^2 + \frac{EJ_o}{2}\left(\frac{2}{b}\,\xi' + \frac{h}{b}\,y_1''\right)^2 + E\,\Omega\,\xi'^2 \\ & + G\, s_1\, h\, y_2'^2 + \frac{G s_2 b}{2}\,(y_4' + \epsilon\eta_4')^2 + 2P(y_1' + y_2')\Big\}\, dx\end{aligned}\right\} \quad (5a)$$

Applying equation (5) gives at once:

$$\left(\frac{\partial\Phi(\epsilon)}{\partial\epsilon}\right)_{\epsilon=0} = \int_0^l G\, s_2\, b\, y_4\, \eta_4'\, dx = 0 \qquad (5b)$$

or

$$\int_0^l y_4'\, \eta_4'\, dx = 0 \qquad (6)$$

Partial integration gives:

$$\left.\begin{aligned}\int_0^l y_4'\,\eta_4'\,dx = \Big|y_4'\eta_4\Big|_0^l - \int_0^l y_4''\,\eta_4\,dx = - y_4'(0)\eta_4(0) \\ - \int_0^l y_4''\,\eta_4\,dx = 0\end{aligned}\right\} \quad (6a)$$

This equation is complied with then and then only for any form of the function η_4 when

$$y_4'' = 0;\; y_4'(0) = 0 \qquad (7)$$

Adding the known geometric condition $y_4(l) = 0$ gives y_4 for which

$$y_4(x) = 0 \qquad (8)$$

Returning to the functional while posing $y_4 = 0$, and applying the variation to y_2 gives

$$\left.\begin{aligned}\Phi(\epsilon) = \int_0^l \Big\{ & EJ_v y_1''^2 + \frac{EJ_o}{2}\left(\frac{2}{b}\,\xi' + \frac{h}{b}\,y_1''\right)^2 + E\,\Omega\,\xi'^2 \\ & + G s_1 h (y_2' + \epsilon\eta_2')^2 + 2P(y_1' + y_2' + \epsilon\eta_2')\Big\}\, dx\end{aligned}\right\} \quad (8a)$$

Equation (5) gives a relation which, divided by 2, has the form

$$\int_0^l (G\ s_1\ h\ y_2'\ \eta_2' + P\ \eta_2')\ dx = 0 \qquad (9)$$

or

$$\int_0^l \eta_2' (G\ s_1\ h\ y_2' + P)\ dx = 0 \qquad (9a)$$

Equation (9a) is satisfied for any function η_2 if

$$G\ s_1\ h_2 y_2' + P = 0 \qquad (10)$$

The integration of equation (10) with regard to equation (4) for y_2, that is, $y_2(l) = 0$, gives:

$$y_2 = \frac{P}{G\ s_1\ h} (l - x) \qquad (11)$$

Applying the variation to ξ gives:

$$\left.\begin{aligned}\Phi(\epsilon) = \int_0^l \Big\{ & E\ J_v\ y_1''^2 + \frac{EJ_o}{2} \left[\frac{2}{b} (\xi' + \epsilon\eta') + \frac{h}{b}\ y_1''\right]^2 \\ & + E\Omega(\xi' + \epsilon\eta')^2 + G s_1 h y_2'^2 + 2P(y_1' + y_2')\Big\}\ dx\end{aligned}\right\} \qquad (11a)$$

The application of equation (5) leaves:

$$\int_0^l \left\{E\ J_o \left(\frac{2}{b}\ \xi' + \frac{h}{b}\ y_1''\right) \frac{2}{b}\ \eta' + 2\ E\ \Omega\ \xi'\ \eta'\right\} dx = 0 \qquad (12)$$

which, multiplied by $b^2/2$, and the common factor η' placed in brackets, affords

$$\int_0^l \eta' \left\{(2\ E\ J_o + E\ \Omega\ b^2)\ \xi' + h\ E\ J_o\ y_1''\right\} dx = 0 \qquad (13)$$

This equation is satisfied for any function η if

$$\xi' = \frac{-h E J_o}{2 E J_o + E \Omega b^2} y_1'' \tag{14}$$

Then the variation is applied to y_1

$$\left.\begin{aligned}\Phi(\epsilon) = \int_0^l \Big\{ & E J_v (y_1'' + \epsilon \eta_1'')^2 \\ & + \frac{EJ_o}{2}\left[\frac{2}{b}\xi' + \frac{h}{b}(y_1''+\epsilon\eta_1'')\right]^2 + E\Omega\xi'^2 \\ & + G s_1 h y_2'^2 + 2P(y_1'+\epsilon\eta_1'+y_2')\Big\} dx\end{aligned}\right\} \tag{15}$$

The application of equation (5) gives:

$$\int_0^l \left\{2EJ_v y_1''\eta_1'' + EJ_o\left(\frac{2}{b}\xi' + \frac{h}{b}y_1''\right)\frac{h}{b}\eta_1'' + 2P\eta_1'\right\} dx = 0 \tag{15a}$$

or, rearranged,

$$\int_0^l \left\{\left(2EJ_v + \frac{h^2}{b^2}EJ_o\right)y_1''\eta_1'' + \frac{2h}{b^2}EJ_o\xi'\eta_1'' + 2P\eta_1'\right\} dx = 0 \tag{15b}$$

Partially integrated, the first expression of the integral reads:

$$\int_0^l y_1''\eta_1'' dx = \left|y_1''\eta_1'\right|_0^l - \int_0^l y_1'''\eta_1' dx =$$

$$= -y_1''(0)\eta_1'(0) - \left|y_1'''\eta_1\right|_0^l + \int_0^l y_1^{IV}\eta_1 dx =$$

$$= -y_1''(0)\eta_1'(0) + y_1'''(0)\eta_1(0) + \int_0^l y_1^{IV}\eta_1 dx \tag{16}$$

and the second expression:

$$\int_0^l \xi'\eta_1''dx = \left|\xi'\eta_1'\right|_0^l - \int_0^l \xi''\eta_1'dx = -\xi'(0)\eta_1'(0) - \left|\xi''\eta_1\right|_0^l$$

$$+\int_0^l \xi'''\eta_1 dx = -\xi'(0)\eta_1'(0) + \xi''(0)\eta_1(0) + \int_0^l \xi'''\eta_1\,dx \quad (17)$$

while the third can, as we know, be written as $-2P\,\eta_1(0)$, so that in conjunction with equations (16) and (17), equation (15a)

$$\left.\begin{aligned}
&-\left(2EJ_v + \frac{h^2}{b^2}EJ_o\right)y_1''(0)\,\eta_1'(0) + (2EJ_v \\
&\quad + \frac{h^2}{b^2}EJ_o)y_1'''(0)\,\eta_1(0) + \int_0^l \left(2EJ_v + \frac{h^2}{b^2}EJ_o\right)y_1^{IV}\eta\,dx \\
&\quad - \frac{2h}{b^2}EJ_o\,\xi'(0)\,\eta_1'(0) + \frac{2h}{b^2}EJ_o\,\xi''(0)\,\eta_1(0) \\
&\quad + \int_0^l \frac{2h}{b^2}EJ_o\,\xi'''\,\eta_1\,dx - 2P\,\eta_1(0) = 0
\end{aligned}\right\} \quad (18)$$

The integrals are arranged into a single integral, the expressions with respect to $\eta(0)$ and $\eta'(0)$ being divided into two groups:

$$\left.\begin{aligned}
&-\eta_1'(0)\left[\frac{2h}{b^2}EJ_o\,\xi'(0) + \left(2EJ_v + \frac{h^2}{b^2}EJ_o\right)y_1''(0)\right] \\
&+ \eta_1(0)\left[\frac{2h}{b^2}EJ_o\,\xi''(0)\right. \\
&\left.+ \left(2EJ_v + \frac{h^2}{b^2}EJ_o\right)y_1'''(0) - 2P\right] \\
&+ \int_0^l \eta_1\left[\frac{2h}{b^2}EJ_o\,\xi''' + \left(2EJ_v + \frac{h^2}{b^2}EJ_o\right)y_1^{IV}\right]dx = 0
\end{aligned}\right\} \quad (18a)$$

Equation (18a) is satisfied only when the following conditions are complied with:

$$\frac{2h}{b^2} EJ_o \xi''' + \left(2EJ_v + \frac{h^2}{b^2} EJ_o\right) y_1^{IV} = 0 \tag{19}$$

$$\frac{2h}{b^2} EJ_o \xi'' (0) + \left(2EJ_v + \frac{h^2}{b^2} EJ_o\right) y_1'''(0) - 2P = 0 \tag{20}$$

$$\frac{2h}{b^2} EJ_o \xi' (0) + \left(2EJ_v + \frac{h^2}{b^2} EJ_o\right) y_1''(0) = 0 \tag{20a}$$

The result is a differential equation, namely (19), and two limiting conditions (20) and (20a).

Integration of equation (19), with due regard to equation (20), gives:

$$\frac{2h}{b^2} EJ_o \xi'' + \left(2EJ_v + \frac{h^2}{b^2} EJ_o\right) y_1''' - 2P = 0 \tag{21}$$

and of equation (21) with due regard to equation (20a):

$$\frac{2h}{b^2} EJ_o \xi' + \left(2EJ_v + \frac{h^2}{b^2} EJ_o\right) y_1'' - 2Px = 0 \tag{21a}$$

Aside from equation (21a), we again write equation (14) obtained from the minimum condition with respect to ξ.

$$\xi' = \frac{-hEJ_o}{2EJ_o + E\Omega b^2} y_1'' \tag{14}$$

which, after elimination of ξ' from equations (21a) and (14), and minor changes, leaves:

$$y_1'' = \frac{Px}{EJ_v + \cfrac{h^2}{2\left(\cfrac{2}{E\Omega} + \cfrac{b^2}{EJ_o}\right)}} \tag{22}$$

Integration, with allowance for y_1 (equation (4)), gives:

$$y_1 = \frac{P}{6} \frac{x^3 - 3l^2x + 2l^3}{EJ_v + \cfrac{h^2}{2\left(\cfrac{2}{E\Omega} + \cfrac{b^2}{EJ_o}\right)}} \tag{23}$$

Integration of equation (14) with due regard to equation (4) for ξ and y_1 gives:

$$\xi = \frac{-\,h\,E\,J_o}{2EJ_o + E\Omega b^2}\,y_1' \tag{14a}$$

which, since y_1 is known from equation (23), gives:

$$\xi = \frac{E\,J_o\,h}{2EJ_v(2EJ_o + b^2E\Omega) + h^2EJ_o\,E\Omega}\,P(l^2 - x^2) \tag{24}$$

There remains then the solution of y_3. Equations (2) and (14a) afford a relation between y_3' and y_1' which, with allowance for $y_3(l) = y_1(l) = 0$ ultimately gives:

$$y_3 = \frac{b\,h\,E\,\Omega}{2EJ_o + b^2E\Omega}\,y_1 \tag{25}$$

The strain condition is therefore completely defined through $y_1(x)$ from equation (23), $y_2(x)$ from equation (11), $y_3(x)$ from equation (25), $\xi(x)$ from equation (24), and because $y_4(x) = 0$.

The construction of equation (23) discloses that $y_1(x)$, that is, the bending line of a vertical wall, agrees with the bending line of a built-in cantilever beam under load P at the free end, with the inertia moment:

$$J_v + \frac{h^2}{2\left(\frac{2}{\Omega} + \frac{b^2}{J_o}\right)} = J_v\left(1 + \frac{h^2}{2J_v\left(\frac{2}{\Omega} + \frac{b^2}{J_o}\right)}\right) \tag{23a}$$

while the natural inertia moment of the wall section is J_v only. Formula (23a) definitely expresses the effect of the presence of the horizontal wall on the bending stiffness of the vertical walls.

The shear strains on a vertical wall are equal to those of a beam of the same size, built in, cantilevered, and loaded in the same manner.

The moments in the vertical walls, positive in the sense of the moment due to P, are in any section

$$M_v = E\,J_v\,y_1'' = \frac{Px}{1 + \dfrac{h^2}{2EJ_v\left(\dfrac{2}{E\Omega} + \dfrac{b^2}{EJ_o}\right)}} \tag{26}$$

The moment is, as seen, given by the product of $P \times x$ the moment existing in a vertical wall which is no longer connected with the horizontal wall, and a correction factor < 1, expressing the reduction in stress in a vertical wall by virtue of the attachment with the horizontal wall.

Equation (24) gives the specific strain ξ':

$$\xi' = \frac{-2\,E\,J_o\,h}{2EJ_v\,(2EJ_o + b^2 E\Omega) + h^2 EJ_o\,E\Omega}\,Px \tag{27}$$

The total normal stresses (positive, if tensile) in the upper and lower edges of the left vertical wall is given by the formula:

$$\left.\begin{aligned} \sigma_{\substack{\text{upper}\\ \text{lower}}} &= -E\,\xi' \pm \frac{M}{J_v}\,\frac{h}{2} = -E\xi' \pm \frac{E\,J_v\,y_1''}{J_v}\,\frac{h}{2} \\ &= E\left(-\xi' \pm \frac{h}{2}\,y_1''\right) \end{aligned}\right\} \tag{27a}$$

ξ' follows as function of y_1'' from equation (14), hence gives:

$$\sigma_{\substack{\text{upper}\\ \text{lower}}} = \frac{Eh}{2}\left(\frac{2E\,J_o}{2EJ_o + E\Omega b^2} \pm 1\right) y_1'' \tag{28}$$

In the corresponding edges of the right wall, the same equal and opposite stresses as in equation (28) are obtained.

III. EXPERIMENTS

The experiments were made on boxes of duralumin and plywood in order to test the conclusions of the preceding theory on open boxes and to evaluate the practical approximation.

It is anticipated that, because of the effect of self-stiffening as a result of the great form changes, the analytical results will be too unfavorable compared with reality; but in practice the analyst and the designer prefer to err on the safe side.

The foregoing theory states that the open boxes have a low over-all torsional stiffness as compared to a closed box of the same dimensions. But it does not equal zero as Bredt's theory stipulates in his particular case.

The foregoing theory further manifests that the torsional stiffness of the open box originates in the flexural stiffness of the vertical walls, the deflections of which the horizontal wall opposes. This wall undergoes no shear, it merely bends. The vertical walls are subjected to very little shear, which probably has little effect on the deformations of the system. It may be said that the system reacts predominantly with normal stresses to the applied torque, whence the term "twisting" is employed reluctantly to the type of stress considered here.

a) Experiments with Closed Duralumin Box for the Experimental Determination of G

The twisting test of the box beam with one wall lacking was preceded by the experimental determination of G for the employed duralumin sheet. This value is to be used in the calculations for the box with one wall removed. The determination of G is effected by the twisting of a thin-walled beam of square section (plots 1 and 2). The reasons for the square section were the following:

In a rectangular, hollow, closed-off prism so supported as to permit warping, the angles of warping or dislocation of a vertical wall and those of a horizontal wall, are proportional $\left(\frac{h}{s_1} - \frac{b}{s_2}\right)$. Now, since $h = b$ and $s_1 = s_2$ on a square section of constant thickness, the angles of warping are equal to zero; i.e., there is no warping. If such a box is restrained so as to prevent warping, it will have no effect whatsoever, because no normal stresses due to bending can occur. Even possibly existing flanges have no reason for positive or negative tension. All this is beneficial for the construction of a test box on which flanges along the edges are necessary. Moreover, it is simpler to clamp the box and provide a robust flange that definitely prevents warping. For this reason the shear modulus G, derived from the twist of the discussed box must be exactly correct.

For the box beam of square section of side length h, and thickness s, it gives:

$$B = \frac{4\,G\,h^4}{4\,\frac{h}{s}} = G\,h^3\,s$$

The angle at the extreme end amounts to

$$\vartheta(0) = \frac{M_t l}{B} = \frac{M_t l}{G\,h^3\,s}$$

For $l = 149.5$ cm, $h = 15$ cm, $s = 0.06$ cm, it is:

$$\vartheta(0) = M_t\,\frac{149.5}{G\,15^3 \times 0.06}$$

hence for G:

$$G = \frac{149.5}{15^3 \times 0.06}\,\frac{M_t}{\vartheta(0)} = 0.74\,\frac{M_t}{\vartheta(0)}$$

Measuring the vertical displacements y_1 and y_2 at the ends of a horizontal bar of 1.6 m length, and applying a torque with two equal and opposite loads P at 100 cm distance, gives:

$$M_t = P \times 100 \text{ kg/cm}$$

$$\vartheta(0) = \frac{(y_1 + y_2)}{160} \text{ (in radians)}$$

$$G = 0.74\,\frac{P \times 100}{\frac{y_1 + y_2}{160}} = 11,840 \times \frac{P}{y_1 + y_2} \text{ kg/cm}^2 \qquad (29)$$

The recorded values are compiled in table I.

TABLE I

P kg	y_1 cm	y_2 cm	$\frac{y_1+y_2}{cm}$	$\frac{P}{y_1+y_2}$	G kg/cm^2
10	0.2	0.2	0.4	25	296,000
20	.45	.45	.9	22.2	263,000
30	.65	.65	1.3	23	272,000
40	.95	.95	1.9	21	249,000
50	1.25	1.25	2.5	20	237,000

The last two values for G correspond to a strain condition by incipient buckling; they are therefore apparent, not actual, G values. The test average is G = 270,000 kg/cm^2.

Figures 2 and 3 show the test rig, and figure 3 is the set-up with Huggenberger strain gages, which were also used in order to obtain G by a different method.

b) Load Tests

The dimensions of the open box are given in plot 3; figure 4 shows the test procedure.

The torque is applied at the free end of the box by means of a double lever where the applied forces are 100 cm apart. The box being 20 cm wide, the force (P) is in each case $\frac{100}{20} = 5$ times greater than the load exerted at the two ends of the lever during the test.

The measurements included:

1. The two vertical, oppositely directed displacements y_s and y_d at both ends of the horizontal bar of 160 cm length, the test point lying on the median plane of the bar attachment.

2. The horizontal displacement y_o of the lower horizontal wall. The ordinates y_s, y_d, and y_o were measured at the free end of the box.

y_o corresponds to the value $y_3(0)$ of the theory. y_s and y_d, reduced in ratio of the horizontal differences, correspond to the quantity $y_1(0) + y_2(0)$, or, exactly:

$$y_1(0) + y_2(0) = \frac{20}{160}\, y_s = \frac{20}{160}\, y_d$$

For this reason the expression $\frac{20}{160} \times \frac{y_s + y_d}{2}$ is used for $y_1(0) + y_2(0)$ in table II.

TABLE II

P kg	y_s(left) mm	y_d(right) mm	y_0 mm	$y_1(0)+y_2(0)$ mm
5	12	12.5	0.5	1.53
10	24	25	1.0	3.06
15	35	36.5	1.5	4.47
20	48.5	51	2.0	6.22
25	59	62	2.5	7.56
30	71	73	3.0	9.00

There is a distinct proportionality between torque and strain, according to table II. This brings us to the formulas of the preceding theory. Making $x = 0$ in the expressions for y_1, y_2, and y_3 gives:

$$y_1(0) = \frac{P\,l^3}{3EJ_v\left(1 + \dfrac{h^2}{2J_v\left(\dfrac{2}{\Omega} + \dfrac{b^2}{J_0}\right)}\right)} \tag{30}$$

$$y_2(0) = \frac{Pl}{G s_1 h} \tag{31}$$

$$y_3(0) = \frac{b\,h\,\Omega}{2J_0 + b^2\Omega}\, y_1(0) \tag{32}$$

Next we compute J_v, Ω, and J_0. The cross section of a flange section is 30 mm² = 0.3 cm² (fig. 5). The inertia moment of a vertical wall with two flange sections - the centroids of the flange sections being $12 - (2\times0.4) = 11.2$ cm spaced apart - is:

$$J_v = \frac{0.06\times12^3}{12} + \frac{0.3\times11.2^2}{2} = 8.64 + 18.80 = 27.44 \text{ cm}^4$$

The inertia moment of the horizontal wall is:

$$J_o = \frac{0.06 \times 20^3}{12} = 40 \text{ cm}^4$$

The section Ω of one of the two vertical walls, inclusive of both flange sections is:

$$\Omega = 0.06 \times 12 + 2 \times 0.3 = 1.32 \text{ cm}^2$$

$$EJ_v\left(1 + \frac{h^2}{2J_v\left(\frac{2}{\Omega} + \frac{b^2}{J_o}\right)}\right) = \alpha\, E\, J_v \tag{33}$$

whereby

$$\alpha = 1 + \frac{h^2}{2J_v\left(\frac{2}{\Omega} + \frac{b^2}{J_o}\right)} \tag{34}$$

α is a kind of enlargement factor of $E\,J_v$, which includes the supplemental horizontal wall.

$$\alpha = 1 + \frac{12^2}{2 \times 27.44\left(\frac{2}{1.32} + \frac{20^2}{40}\right)} = 1.228$$

$$\alpha\, E\, J_v = 1.228 \times 750{,}000 \times 27.44 = 26.3 \times 10^6 \text{ kg/cm}^2$$

The box length without the clamping flange is about 135 cm.

$$\left.\begin{aligned} y_1(0) + y_2(0) &= \frac{P\,l^3}{3\,\alpha\,E\,J_v} + \frac{P\,l}{G\,s_1\,h} \\ &= P\left(\frac{135^3}{3 \times 26.3 \times 10^6} + \frac{135}{270000 \times 0.06 \times 12}\right) \\ &= P\,(0.0312 + 0.0007) = 0.0319\,P \end{aligned}\right\} \tag{35}$$

Then

$$y_1(0) = 0.0312\,P$$

hence

$$\left.\begin{aligned} y_3(0) &= \frac{b\,h\,\Omega}{2J_o + b^2\Omega}\,y_1(0) = \frac{20 \times 12 \times 1.32}{2 \times 40 + 20^2 \times 1.32}\,y_1(0) \\ &= 0.521\,y_1(0) = 0.521 \times 0.0312\,P = 0.01625\,P \end{aligned}\right\} \tag{36}$$

With $P = 30$ kg, $y_1(0) + y_2(0) = 0.319 \times 30 = 0.957$ cm against the 0.9 cm test value. For $y_3(0) = 0.01625 \times 30 = 0.48$ cm against the experimental 0.3 cm. The accord between theory and test is satisfactory.

c) Tests on Closed Plywood Box - Determination of G

The box was a thin-walled beam of square section. The dimensions were those of the duralumin specimen (plots 1 and 2). The walls were of 1.5 mm birch plywood. The walls and the partitions were connected by 10 x 10 mm² strips. The two partitions at the end were of 2 mm birch plywood. The 15 x 15 mm² flange strips were of spruce (figs. 6 and 7).

The distance of the couple and of the test scale was $d = 112$ cm. The test arrangement is shown in figure 8. It is

$$B = \frac{4S^2}{\int_c \frac{dc}{s}} G = \frac{4h^4}{4\frac{h}{s}} G = Gh^3 s \tag{37}$$

$$\vartheta(0) = \frac{M_t l}{B} = \frac{Pdl}{Gh^3 s} = \frac{y_d + y_s}{d} \tag{38}$$

y_d and y_s are the respective readings from the right and left test scale.

According to equation (38), the experimental value of G is:

$$G = \frac{P d^2 l}{h^3 s(y_d + y_s)} = 6{,}770 \frac{P}{y_d + y_s} \tag{39}$$

TABLE III

Load P kg	y_d mm	y_s mm	$y_d + y_s$ mm	$\frac{P}{y_d + y_s}$	G kg/cm²
5	12.5	12	24.5	2.04	13,800
10	25	24	49	2.04	13,800
15	41	39	80	1.875	12,700
20	58	55	113	1.77	12,000
25	78	76	154	1.62	11,000

Since critical phenomena appear at $P = 15$ kg, the value $G = 13,800 \text{ kg/cm}^2$ is maintained.

Putting $m = 3$, would give:

$$E = \frac{2(m + 1)}{m} G = 2.67 \times 13,800 = 36,800 \text{ kg/cm}^2$$

d) Load Tests with Open Plywood Box

The dimensions are given in figure 9. The side walls and the partitions are of 1.5 mm birch plywood, the outside bulkhead walls of 2 mm birch plywood. The flange strips are of 15 x 15 mm² spruce. The test arrangement is shown in figure 10.

As shown in section II, it is:

$$y_1(0) = \frac{P\,l^3}{3EJ_v \; 1 + \dfrac{h^2}{2J_v\left(\dfrac{2}{\Omega} + \dfrac{b^2}{J_o}\right)}} \tag{40}$$

with $h = 15$ cm, $b = 25$ cm, and s (wall thickness) = 0.15 cm. Now the walls of the box have a $G = 13,800 \text{ kg/cm}^2$, and an $E = 36,800 \text{ kg/cm}^2$, as established by tests. But for the spruce strips, it is around $E = 100,000 \text{ kg/cm}^2$. So in the calculation of Ω (section of one wall including strips) and for J_v (their inertia moment) as is customary in reinforced concrete, the area of the spruce, i.e., that of the harder material, must be multiplied by

$$n = \frac{E_{spruce}}{E_{plywood}} = \frac{100000}{36000} = \sim 3$$

Hence (fig. 11):

$$\Omega = 0.15 \times 15 + 3 \times 2 \times 1.5^2 = 15.75 \text{ cm}^2 \tag{41}$$

$$J_v = 0.15 \times \frac{15^3}{12} + 3 \times 1.5^2 \frac{13.5^2}{2} = 656 \text{ cm}^4 \tag{42}$$

$$\frac{b^2}{J_o} = \frac{25^2}{0.15 \times \dfrac{25^3}{12}} = \frac{12}{0.15 \times 25} = 3.2 \tag{43}$$

$$\frac{2}{\Omega} + \frac{b^2}{J_o} = \frac{2}{15.75} + 3.2 = 0.127 + 3.2 = 3.327 \qquad (44)$$

$$\frac{h^2}{2J_v\left(\frac{2}{\Omega} + \frac{b^2}{J_v}\right)} = \frac{15^2}{2 \times 656 \times 3.327} = 0.0515 \qquad (45)$$

$$y_1(0) = \frac{P\ 135^3}{3 \times 36{,}800 \times 656 \times 1.0515} = 0.0336\ P \qquad (46)$$

$$y_2(0) = \frac{P\ 135}{13{,}800 \times 0.15 \times 15} = 0.00434\ P \qquad (47)$$

$$y_1(0) + y_2(0) = (0.0336 + 0.00434)\ P = 0.03794\ P \qquad (48)$$

$$\vartheta(0) = \frac{2 \times 0.038\ P}{25} = 0.00304\ P \qquad (49)$$

Take, for example, the 10 kg load, bear in mind that the distance between the couple is 120 cm, and that b = 25 cm. Then,

$$P = \frac{120}{25}\ 10 = 48\ \text{kg}$$

$$\vartheta(0)_{\text{computed}} = 0.00304 \times 48 = 0.146$$

instead of an observed test value of 7.8 cm, which corresponds to

$$\vartheta(0)_{\text{recorded}} = \frac{2 \times 7.8}{120} = 0.13$$

Here also the agreement between theory and test is satisfactory.

The loads, the proportional moments M_t, and the recorded strains along with the theoretical and experimental values of $\vartheta(0)$, are compiled in table IV.

TABLE IV

kg	$M_t = P \times 120$ kg/cm	y_d(right) mm	y_s(left) mm	$\vartheta(0)_{recorded}$ in radians	$\vartheta(0)_{computed}$ in radians
5	600	32	32	0.053	0.073
6	720	41	41	.067	.087
7	840	50	49	.082	.102
8	960	61	60	.101	.117
9	1080	69	67	.114	.131
10	1200	79	77	.130	.146
11	1320	88	86	.145	.160
0		6	5		

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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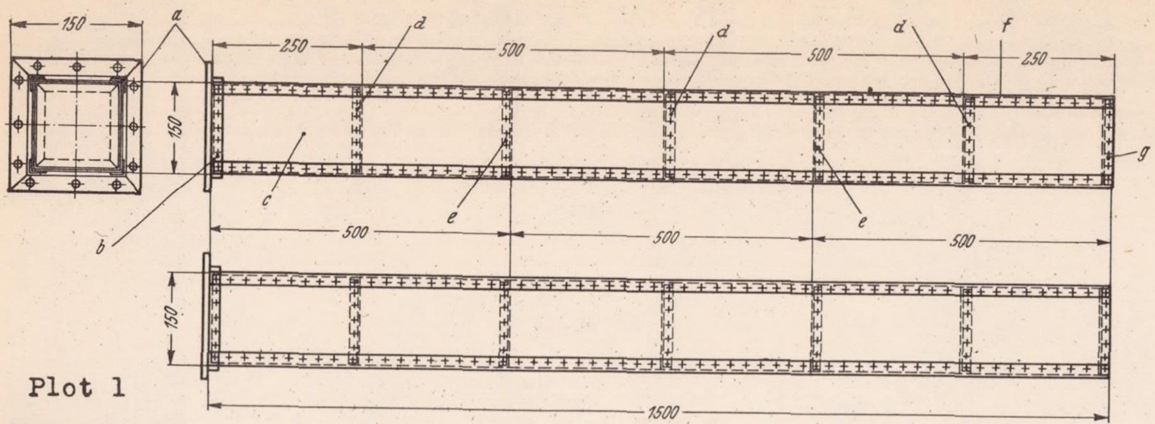
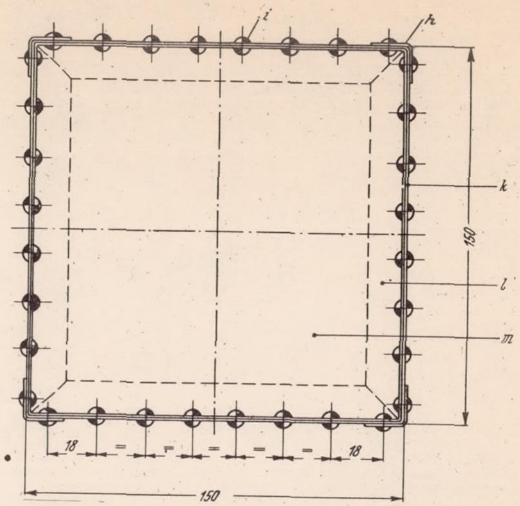
1. Minelli, C.: Nuovo calcolo energetico-variazionale di "travi a cassone" sottoposte a torsione. Ricerche di ingegneria, Nov.-Dec. 1937.



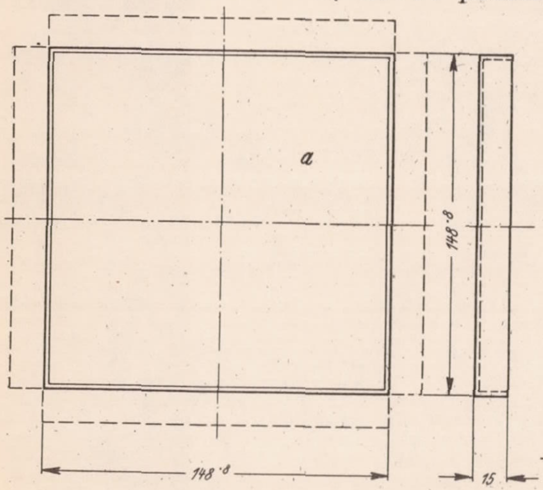


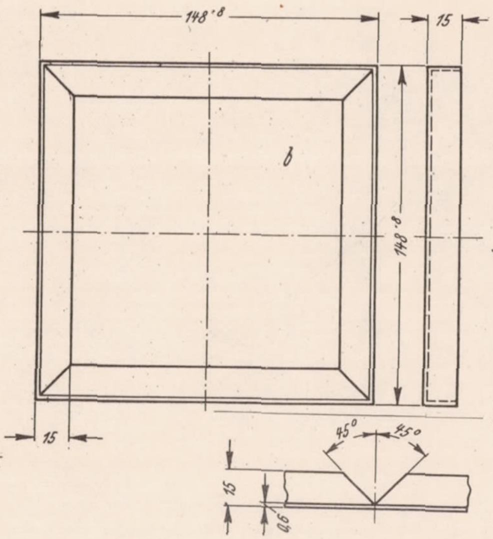
Plot 1

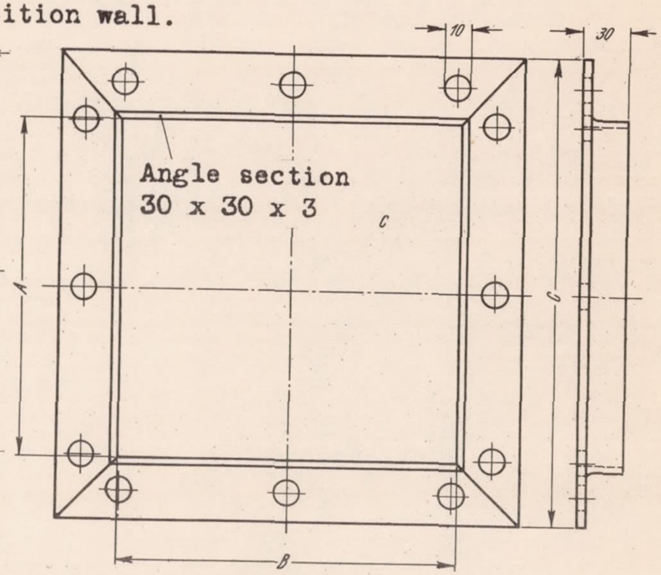
Duraluminum.
Pieces:1.
Scale 1:12.5 .
a, Attach flange.
b, Partition wall of Duraluminum.

g, Partition of 8/10 mm Duraluminum parts--scale 1:3 .
c, Duraluminum.
d, Duraluminum angles.
e, Duraluminum partition.
f, Rivet pitch, 18 mm.

h, Angle section.
i, Rivets.
k, 6/10 mm Duraluminum covering.
l, Angle section.
m, Partition wall.



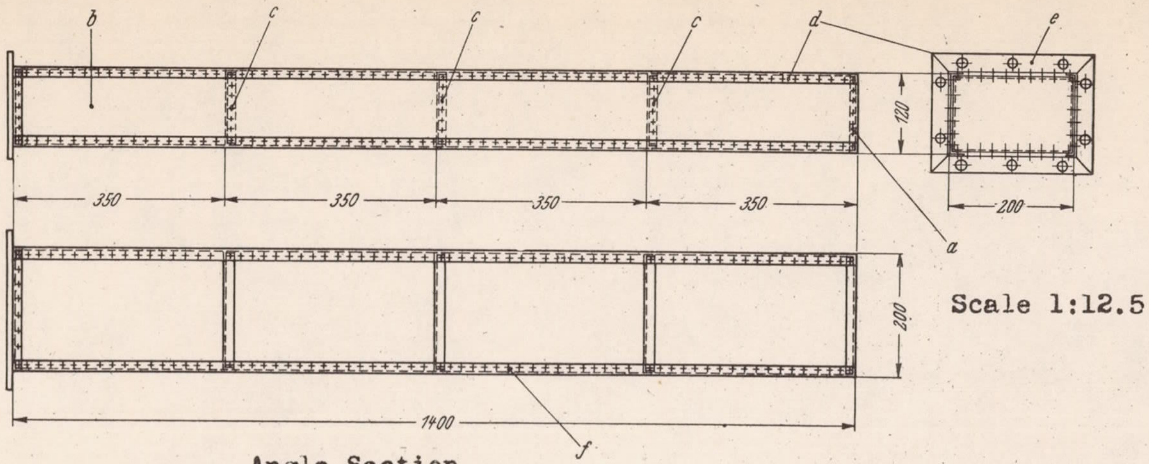




Plot 2

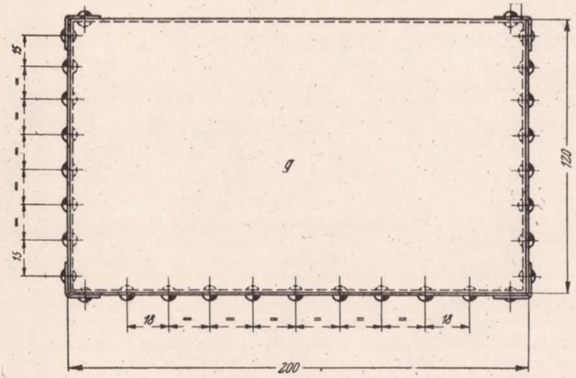
a, Partition wall of Duraluminum sheet.
2 pieces of 8/10 mm Duraluminum sheet.
2 pieces of 6/10 mm " " .

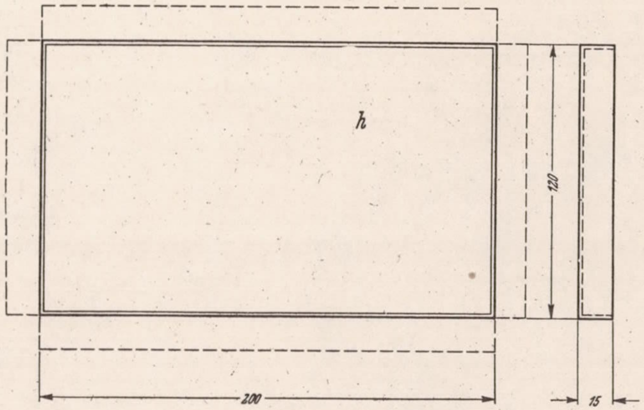
b, Duraluminum angle section.
3 pieces of 6/10 mm Duraluminum sheet.
c, Attachment flanges.



Scale 1:12.5

Angle Section
15 x 15 x 6/10





Plot 3
a, Partitions.
b, Duraluminum.
c, Partitions.
d, Angle sections.
e, Attach flange.
f, Rivet pitch.
g, Partition of duraluminum.
Scale 1:3.5.
h, Duraluminum.
2 pieces of 8/10 mm sheet.
3 pieces of 6/10 mm sheet.

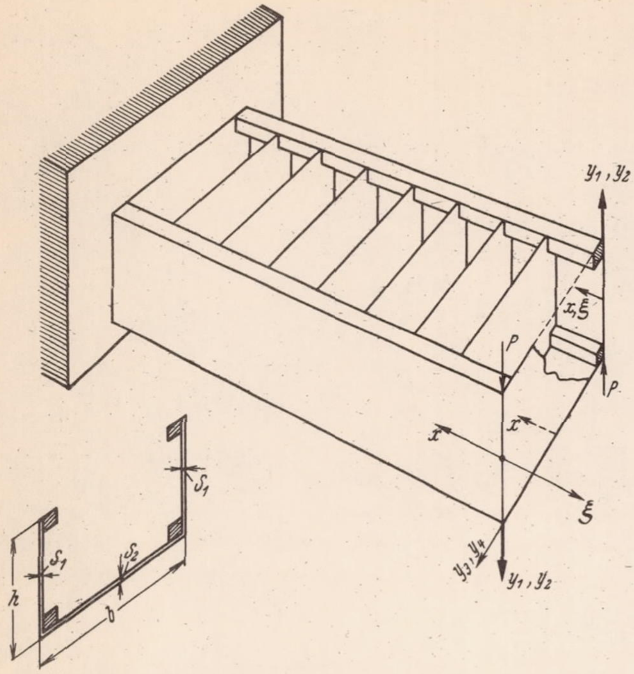


Figure 1.- Sketch plan of a box beam.

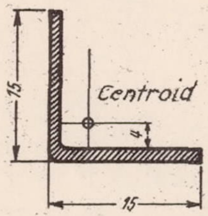


Figure 5.- Flange profile.

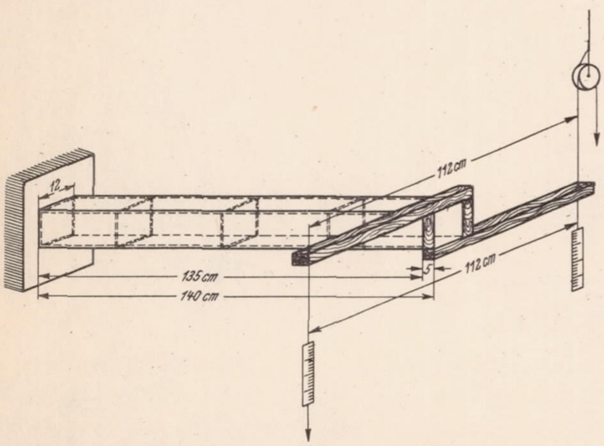


Figure 6.- Test rig with closed plywood box to define G.

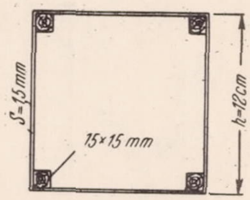


Figure 7.- Section through box.

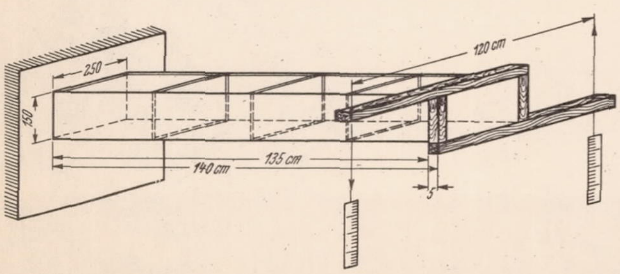


Figure 9.- Test rig with open plywood box.

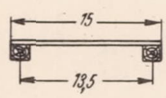


Figure 11.- Wall section.

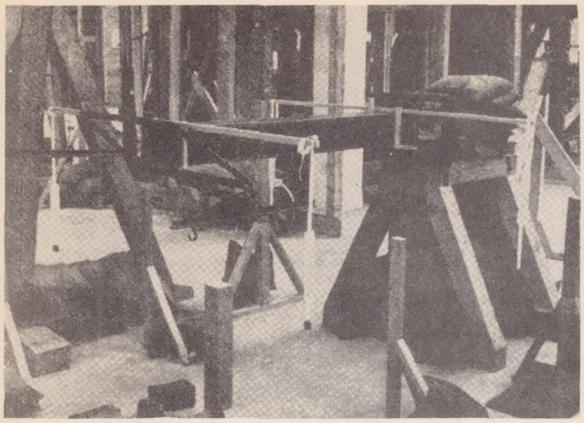


Figure 2.- Load test with closed duraluminum box to define G.

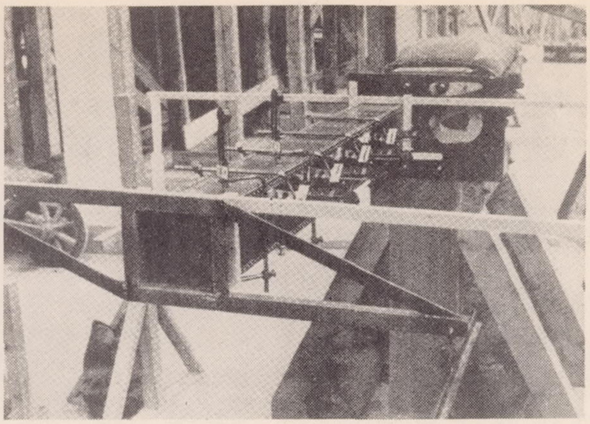


Figure 3.- Determination of G with Huggenberger tensiometers.

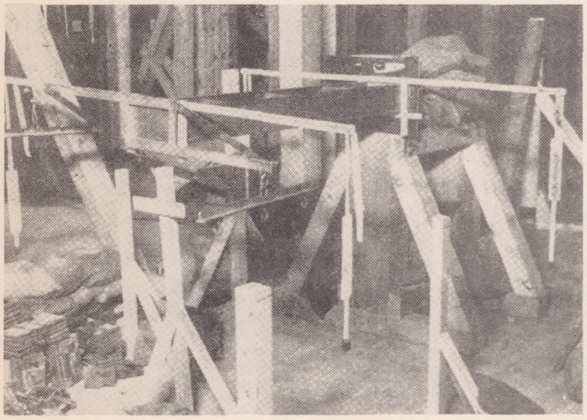


Figure 4.- Load test with open duraluminum box.

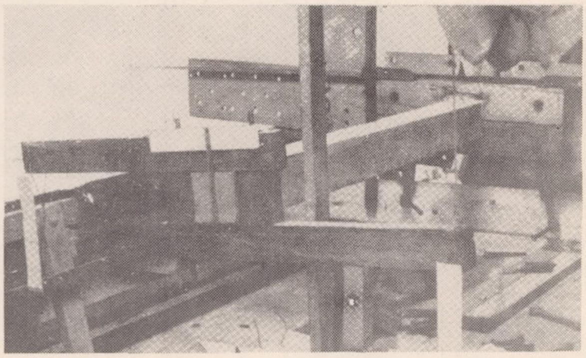


Figure 8.- Load test with closed plywood box to define G.

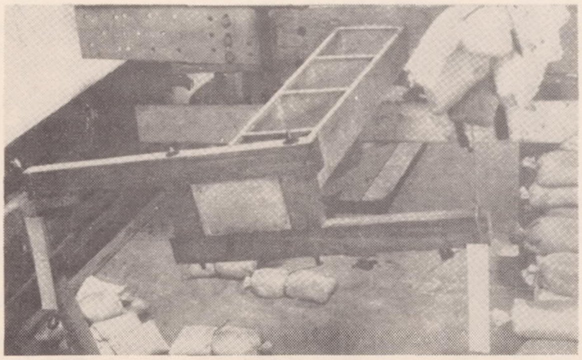


Figure 10.- Load test with open plywood box.