Structures Research Section

TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 861

COMPRESSION STRUTS

WITH NONPROGRESSIVELY VARIABLE MOMENT OF INERTIA

By B. Radomski

Luftfahrtforschung
Vol. 14, No. 9, September 20, 1937
Verlag von R. Oldenbourg, München und Berlin

Washington
May 1938

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 861

COMPRESSION STRUTS WITH NONPROGRESSIVELY VARIABLE MOMENT OF INERTIA*

By B. Radomski

SUMMARY

The buckling failure conditions for a bar with nonprogressively variable moment of inertia J_n, although constant over length l_n, are developed.

For two cases: 1) bar consisting of two lengths l_1 and l_2 with J_1 and J_2; 2) bar consisting of three lengths l_1, l_2 and again l_1, with J_1, J_2 and again J_1 (symmetrical with respect to center), graphs are plotted for different ratios $J_1 : J_2$ over $l_1 : l_2$, showing a mean moment of inertia J_m, with the aid of which the buckling strength P_k of the bar with sudden variations of the moment of inertia can be represented in the Euler form.

I. INTRODUCTION

The following investigation is only valid in the Euler range, that is, only when

$$\sigma_k = \frac{P_k}{f_{min}} \leq \sigma_G \quad \text{(fig. 1)}$$

P_k is buckling load at failure, f_{min}, minimum section. In addition, it is assumed that the moments of inertia at the points of sudden variations are effective in their full magnitude immediately before and after the jump, whereas in reality there is a brief compensating zone for welded and a longer zone for riveted bars.

For the rest, the assumptions are that the bar was

*"Knickstäbe mit sprungweise veränderlichem Trägheitsmoment." Luftfahrtforschung, vol. 14, no. 9, September 20, 1937, pp. 438-443.

previously exactly straight and perfectly pin-jointed, that the effect of the cross force and the change in buckling length due to axial load and flexure is negligibly small and that the differential equation of the elastic line $E J y'' = - M$ is applicable.

II. GENERAL DERIVATION

Consider an originally straight bar, consisting of n joined parts stiff in bending with n different - but along the lengths l_n constant - moments of inertia. The axial load P has an eccentricity e of finite magnitude, small in relation to the cross-sectional dimensions, that has no effect on the result of the investigation and is, for the sake of simplicity, constant (fig. 2).

Within a length l_r the differential equation of the elastic line

$$E J_r y_r'' = -P(e + y_r),$$

is applicable, which through $v_r = e + y_r$ reduces to:

$$E J_r v_r'' = -P v_r \tag{1}$$

The general solution of this equation is as known:

$$P v_r = A_r \cos \varphi_r + B_r \sin \varphi_r \tag{2}$$

with A_r and B_r as integration constants, while

$$\varphi_r = \frac{x_r}{k_r} \ (3) \quad \text{and} \quad k_r = \sqrt{\frac{E J_r}{P}} \tag{4}$$

For each of the n lengths an equation corresponding to equation (2) can be established, with 2 n temporarily unknown integration constants, for whose determination the following conditions are available:

1) 1 time $y_1 = 0$ for $x_1 = 0$

2) n - 1 time $y_{r-1} = y_r$ for $x_{r-1} = l_{r-1}$ and $x_r = 0$

3) n - 1 time $y'_{r-1} = y_r'$ for $x_{r-1} = l_{r-1}$ and $x_r = 0$

4) 1 time $y_n = 0$ for $x_n = l_n$

or altogether $2n$ - conditions from which the values A and B may be obtained. The differentiation of equation (2) gives:

$$k_r\ P\ v_r' = -A_r \sin \varphi_r + B_r \cos \varphi_r \qquad (5)$$

and the differentiation of equation (5) with considera- tion of equation (4) gives:

$$k_r^2\ P\ v_r'' = -A_r \cos \varphi_r - B_r \sin \varphi_r \qquad (6)$$

i.e., $E\ J_r\ v_r'' = -P\ v_r$.

Then the determinative equations read with

$$\alpha_r = \frac{l_r}{k_r} \qquad (7)$$

as follows:

1. $A_1 = P\ e$

2. $A_{r-1} \cos \alpha_{r-1} + B_{r-1} \sin \alpha_{r-1} - A_r = 0$

3. $-A_{r-1} \sin \alpha_{r-1} + B_{r-1} \cos \alpha_{r-1} - \frac{k_{r-1}}{k_r} B_r = 0$

4. $A_n \cos \alpha_n + B_n \sin \alpha_n = P\ e$

The equation scheme is as follows:

The constants A_r and B_r are obtained as quotients of two determinants and may be written in equation (2). The bar then fails under a buckling load if the right-hand side of equation (2) becomes great beyond all limits. Since $\cos \varphi_r$ and $\sin \varphi_r$ are consistently ≤ 1, A_r, and B_r must become very great in order that $P\ v_r$ goes beyond all limits.

Now

Equation System

A_1	B_1	A_2	B_2	A_3	B_{r-1}	A_r	B_r	A_{r+1}	B_{n-1}	A_n	B_n	
1												$=P e$
$\cos \alpha_1$	$\sin \alpha_1$	-1										
$-\sin \alpha_1$	$\cos \alpha_1$		$-\frac{k_1}{k_2}$									
		$\cos \alpha_2$	$\sin \alpha_2$	-1								
		$-\sin \alpha_2$	$\cos \alpha_2$		$-\frac{k_{r-2}}{k_{r-1}}$							
				$\cos \alpha_3$	$\sin \alpha_{r-1}$	-1						
				$-\sin \alpha_3$	$\cos \alpha_{r-1}$		$-\frac{k_{r-1}}{k_r}$					
						$\cos \alpha_r$	$\sin \alpha_r$	-1				
						$-\sin \alpha_r$	$\cos \alpha_r$		$-\frac{k_{n-2}}{k_{n-1}}$			
								$\cos \alpha_{r+1}$	$\sin \alpha_{n-1}$	-1		
								$-\sin \alpha_{r+1}$	$\cos \alpha_{n-1}$		$-\frac{k_{n-1}}{k_n}$	
										$\cos \alpha_n$	$\sin \alpha_n$	$=P e$

$$A_r = \frac{Z}{N} \tag{8}$$

Since Z as the sum of the products of k_r, $\sin \alpha_r$, $\cos \alpha_r$, P, and e cannot go beyond all limits for finite values of P, e, J_r, and l_r, there remains as criterion for the buckling of the bar

$$N \longrightarrow 0 \tag{9}$$

No buckling occurs for values $N > 0$. The bar buckles at $N = 0$. Here N is the determinant of the previously cited equation system and is of the $(2\ n)$th degree for a bar of n different lengths.

Examples: For the cases $n = 2$ to $n = 5$ the solutions of the determinants afford the following buckling conditions:

1. $n = 2$

$$k_1 \cot \alpha_2 + \cot \alpha_1 k_2 \geq 0 \tag{10}$$

which, simplified, gives with $\nu_r' = 1 - \alpha_r \cot \alpha_r$

$$\frac{l_1}{l} \nu_2' + \frac{l_2}{l} \nu_1' \leq 1 \tag{11}$$

2. $n = 3$

$$k_1 \cot \alpha_2 \cot \alpha_3 + \cot \alpha_1 k_2 \cot \alpha_3 + \cot \alpha_1 \cot \alpha_2 k_3 - \frac{k_1 k_3}{k_2} \geq 0 \tag{12}$$

For the symmetrical case $l_1 = l_3$ and $J_1 = J_3$ the expression reduces to

$$2 \cot \alpha_1 \cot \alpha_2 + \cot^2 \alpha_1 \frac{k_2}{k_1} - \frac{k_1}{k_2} \geq 0 \tag{13}$$

3. $n = 4$

$$+ k_1 \cot \alpha_2 \cot \alpha_3 \cot \alpha_4 - \cot \alpha_1 \frac{k_2 k_4}{k_3}$$

$$+ \cot \alpha_1 k_2 \cot \alpha_3 \cot \alpha_4 - \cot \alpha_2 \frac{k_1 k_4}{k_3}$$

$$+ \cot \alpha_1 \cot \alpha_2 k_3 \cot \alpha_4 - \cot \alpha_3 \frac{k_1 k_4}{k_2}$$

$$+ \cot \alpha_1 \cot \alpha_2 \cot \alpha_3 k_4 - \cot \alpha_4 \frac{k_1 k_3}{k_2} \geq 0 \qquad (14)$$

4. $n = 5$

$$+ k_1 \cot\alpha_2 \cot\alpha_3 \cot\alpha_4 \cot\alpha_5 - \cot\alpha_1 \cot\alpha_2 \frac{k_3 k_5}{k_4}$$

$$+ \cot\alpha_1 k_2 \cot\alpha_3 \cot\alpha_4 \cot\alpha_5 - \cot\alpha_1 \cot\alpha_3 \frac{k_2 k_5}{k_4}$$

$$+ \cot\alpha_1 \cot\alpha_2 k_3 \cot\alpha_4 \cot\alpha_5 - \cot\alpha_1 \cot\alpha_4 \frac{k_2 k_5}{k_3}$$

$$+ \cot\alpha_1 \cot\alpha_2 \cot\alpha_3 k_4 \cot\alpha_5 - \cot\alpha_1 \cot\alpha_5 \frac{k_2 k_4}{k_3}$$

$$+ \cot\alpha_1 \cot\alpha_2 \cot\alpha_3 \cot\alpha_4 k_5 - \cot\alpha_2 \cot\alpha_3 \frac{k_1 k_5}{k_4}$$

$$- \cot\alpha_2 \cot\alpha_4 \frac{k_1 k_5}{k_3}$$

$$- \cot\alpha_2 \cot\alpha_5 \frac{k_1 k_4}{k_3}$$

$$- \cot\alpha_3 \cot\alpha_4 \frac{k_1 k_5}{k_2}$$

$$- \cot\alpha_3 \cot\alpha_5 \frac{k_1 \cdot k_4}{k_2}$$

$$- \cot\alpha_4 \cot\alpha_5 \frac{k_1 \; k_3}{k_2}$$

$$+ \frac{k_1 \; k_3 \; k_5}{k_2 \; k_4} \geq 0 \qquad (15)$$

The mathematical solution of equations (14) and (15) is already quite elaborate and unusually sensitive, because the result appears as small difference of great numbers. As n increases the terms become consistently more extensive and so become void as far as practical application is concerned.

As it is impossible to express explicitly the buckling load P contained in α and k from the preceding expressions, a method covering the usual cases of $n = 2$ and $n = 3$ by means of a substitute inertia moment is given.

III. THE SUBSTITUTE MOMENT OF INERTIA

Case 1 (fig. 5).- The bar consists of $n = 2$ lengths l_1 and l_2 so that $l_1 + l_2 = l$, with the moments of inertia J_1 and J_2 (fig. 3). Force $P_{n=2}$ is the ultimate buckling load of the bar and complies with equation (10)

$$k_1 \cot \alpha_2 + k_2 \cot \alpha_1 = 0.$$

Then visualize a bar of the same length l but constant moment of inertia J_m over the entire length, whereby, respectively,

$$J_1 > J_m > J_2 \text{ and } J_1 < J_m < J_2 \qquad (16)$$

J_m is to be such as to comply with

$$P_{n=2} = \frac{E\ J_m\ \pi^2}{l^2} \tag{17}$$

With the simplified relation P in place of $P_{n=2}$, equation (17) gives:

$$\sqrt{\frac{P}{E}} = \frac{\pi}{l}\sqrt{J_m}$$

Then

$$k_1 = \sqrt{\frac{E\ J_1}{P}} = \frac{l}{\pi}\sqrt{\frac{J_1}{J_m}} \quad \text{and} \quad k_2 = \sqrt{\frac{E\ J_2}{P}} = \frac{l}{\pi}\sqrt{\frac{J_2}{J_m}} \tag{18}$$

$$\alpha_1 = \frac{l_1}{k_1} = \pi\ \frac{l_1}{l}\sqrt{\frac{J_m}{J_1}} \quad \text{and} \quad \alpha_2 = \frac{l_2}{k_2} = \pi\ \frac{l_2}{l}\sqrt{\frac{J_m}{J_2}} \tag{19}$$

Having recourse to equation (18) transforms equation (10) into:

$$\sqrt{\frac{J_1}{J_2}}\ \cot\ \alpha_2 = -\ \cot\ \alpha_1 \tag{20}$$

Herefrom the correlated α_2 and $l_1 : l_2$ can be so determined for a constant ratio $J_1 : J_2$ and a variable α_1 that equation (20) is fulfilled.

Because it is:

$$\frac{l_1}{l_2} = \frac{\alpha_1}{\alpha_2}\sqrt{\frac{J_1}{J_2}} = \epsilon$$

and results in

$$\frac{l_1}{l} = \frac{\epsilon}{1 + \epsilon} \quad \text{and} \quad \frac{l_2}{l} = \frac{1}{1 + \epsilon}$$

after which:

$$\sqrt{\frac{J_m}{J_1}} = \frac{\alpha_1}{\pi}\frac{l}{l_1} \tag{21}$$

$$\sqrt{\frac{J_m}{J_2}} = \frac{\alpha_2}{\pi}\frac{l}{l_2} \tag{22}$$

Then the values $\beta = \sqrt{\frac{J_m}{J_2}}$ can be plotted for the parameter $p = \sqrt{\frac{J_1}{J_2}}$ against the abscissa $l_1 : l$ and the value β interpolated for the individual case with definite p and $l_1 : l$. Assuming $J_1 > J_2$, β remains ≥ 1 and the buckling load follows at:

$$P = \frac{E\ J_2\ \pi^2}{l^2}\beta^2 \tag{23}$$

The boundary curve of these curve systems follows at $p = \frac{J_1}{J_2} \longrightarrow \infty$, wherein J_2 is finite and J_1 exceeds all boundaries. Then it affords

$$k_1 = \sqrt{\frac{E\ J_1}{P}} \longrightarrow \infty, \quad \alpha_1 = \frac{l_1}{k_1} \longrightarrow 0, \quad \text{and} \quad \alpha_1 \cot\ \alpha_1 \longrightarrow 1$$

From the transformed equation (10)

$$l_1\ \alpha_2 \cot\ \alpha_2 + l_2\ \alpha_1 \cot\ \alpha_1 = 0$$

follows the condition for the boundary curve:

$$\frac{l_1}{l_2}\alpha_2 \cot\ \alpha_2 + 1 = 0$$

$$\frac{l_1}{l_2} = -\frac{\tan\ \alpha_2}{\alpha_2} \quad \text{or} \quad \frac{l_1}{l} = +\frac{1}{\nu_2{}'}$$

It exists only for values

$$\frac{\pi}{2} < \alpha_2 < \pi$$

<u>Case 2.</u>- The bar consists of $n = 3$ parts l_1, l_2, and l_1, so that $l_1 + l_2 + l_1 = l$ and of the inertia moments J_1, J_2, and J_1 symmetrical to the center (fig. 4). The force $P_{n=3}$ is the buckling load at failure and complies with equation (13):

$$2 \cot \alpha_1 \cot \alpha_2 + \cot^2 \alpha_1 \frac{k_2}{k_1} - \frac{k_1}{k_2} = 0.$$

Again a mean inertia moment J_m is introduced, so that:

$$P_{n=3} = P = \frac{E J_m \pi^2}{l^2} \tag{24}$$

As a result, equations (18) and (19) are applicable again and equation (13) becomes:

$$2 \cot \alpha_2 = \sqrt{\frac{J_1}{J_2}} \tan \alpha_1 - \sqrt{\frac{J_2}{J_1}} \cot \alpha_1 \tag{25}$$

From this it is possible to so determine for a constant ratio $J_1 : J_2$ and a variable angle α_1 the correlated angle α_2 and the ratio $l_1 : l_2$, that equation (25) is complied with.

For it is:

$$\frac{l_1}{l_2} = \frac{\alpha_1}{\alpha_2} \sqrt{\frac{J_1}{J_2}} = \epsilon$$

and one obtains

$$\frac{l_1}{l} = \frac{\epsilon}{2\epsilon + 1} \quad \text{and} \quad \frac{l_2}{l} = \frac{1}{2\epsilon + 1}$$

and as before:

$$\sqrt{\frac{J_m}{J_1}} = \frac{\alpha_1}{\pi}\,\frac{l}{l_1} \quad \text{and} \quad \sqrt{\frac{J_m}{J_2}} = \frac{\alpha_2}{\pi}\,\frac{l}{l_2}$$

For the symmetrically stepped bar two cases

a) $J_1 > J_2$ and b) $J_2 > J_1$

must be distinguished.

a) (fig. 6) $J_1 > J_2$

$$p_a = \sqrt{\frac{J_1}{J_2}} \geq 1 \quad \text{and} \quad \beta_a = \sqrt{\frac{J_m}{J_2}} \geq 1$$

are chosen as parameters of the curves. The buckling load follows at

$$P = \frac{E\ J_2\ \pi^2}{l^2}\ \beta_a{}^2 \tag{26}$$

For the boundary curve with parameter

$$p_a = \sqrt{\frac{J_1}{J_2}} \dashrightarrow \infty$$

where J_2 is finite and J_1 ∞, we obtain

$$k_1 = \sqrt{\frac{E\ J_1}{P}} \longrightarrow \infty, \quad \alpha_1 = \frac{l_1}{k_1} \longrightarrow 0, \quad \text{and} \quad \alpha_1 \cot \alpha_1 \longrightarrow 1$$

Equation (13) may also be written as:

$$2 \cot \alpha_2 + \cot \alpha_1\ \frac{l_2\ \alpha_1}{\alpha_2\ l_1} - \frac{l_1}{\alpha_1}\,\frac{\alpha_2}{l_2}\,\frac{1}{\cot \alpha_1} = 0$$

$$\text{or } + 2 \cot \alpha_2\ \frac{l_1}{l_2}\ \alpha_2 + 1 - \frac{l_1{}^2}{l_2{}^2}\alpha_2{}^2 = 0$$

$$1 + \cot^2 \alpha_2 = \cot^2 \alpha_2 - 2 \cot \alpha_2 \frac{l_1}{l_2} \alpha_2 + \left(\frac{l_1}{l_2} \alpha_2\right)^2$$

$$+ \frac{1}{\sin^2 \alpha_2} = \left(\cot \alpha_2 - \frac{l_1}{l_2} \alpha_2\right)^2$$

$$+ \frac{1}{\sin \alpha_2} = \pm \left(\cot \alpha_2 - \frac{l_1}{l_2} \alpha_2\right)$$

$$\frac{1}{\alpha_2} \frac{1 \mp \cos \alpha_2}{\sin \alpha_2} = \mp \frac{l_1}{l_2}$$

Only positive values of $\frac{l_1}{l_2}$ and α_2 being possible, the negative signs disappear. And the equation of the boundary curve becomes

$$\frac{1 + \cos \alpha_2}{\alpha_2 \sin \alpha_2} = \frac{l_1}{l_2} \quad \text{or} \quad \frac{\cot \frac{\alpha_2}{2}}{\frac{\alpha_2}{2}} = \frac{2\, l_1}{l_2}$$

b) (fig. 7) $J_2 > J_1$

Here $p_b = \sqrt{\frac{J_2}{J_1}} \geq 1$ is the chosen parameter. Then

$\beta_b = \sqrt{\frac{J_m}{J_1}} \geq 1$ and the buckling load becomes

$$P = \frac{E\, J_1\, \pi^2}{l^2} \beta_b^2 \tag{27}$$

If the parameter

$$p_b = \sqrt{\frac{J_2}{J_1}} \longrightarrow \infty$$

wherein J_1 is finite and $J_2 \longrightarrow \infty$, the buckling load of the whole bar becomes

$$P_k = \frac{1}{4} \frac{E J_1 \pi^2}{l_1^2}$$

because the bar l_1 may for reasons of symmetry be considered as being rigidly clamped at l_2. For the boundary curve the condition is:

$$\frac{J_1}{4 l_1^2} = \frac{J_m}{l^2} \quad \text{or} \quad \beta_b = \sqrt{\frac{J_m}{J_1}} = \frac{l}{2 l_1}$$

The graphs ensuing therefrom afford the solution of the buckling loads P_k with the aid of the substitute moments of inertia, that is

$$J_m = J_2 \beta^2 \quad \text{for case 1}$$

$$J_m = J_2 \beta_a^2 \quad " \quad " \quad 2a$$

$$J_m = J_1 \beta_b^2 \quad " \quad " \quad 2b$$

IV. NOTE CONCERNING THE GRAPHS

Charts 5 to 7 give the correlated changes in buckling load for different ratios $l_1 : l$ and $l_2 : l$ and different ratios of moments of inertia. They show that reinforcing or weakening a short end length of a compression strut within range

$$(0 < l_1 : l < 0.1 \quad \text{or} \quad 0 < l_2 : l < 0.1)$$

causes only a minor change in its buckling load, whereas, as figures 6 and 7, for example, show, a cross sectional change of a short central length already produces a substantial change in the buckling load. They readily show the effect of local cross-sectional changes and make it thus possible, even for a given inertia moment and a stipulated buckling load P_k, to determine length and dimension of the other piece by variation of p and $l_1 : l$.

These graphs (figs. 5 to 7) have the additional characteristic that the value β equals the parameter p in the extreme condition $l_1 : l = 1$.

However, since it is impossible to include all probable cases in one graph - for $l_1 : l \longrightarrow 1$ and $p > 2.2$ the graphs give no values β - the following modified graphs will be found to be more practical.

Case 1 (fig. 8).- γ as function of $l_1 : l$ for parameter

$$q = \frac{J_2}{\frac{J_1 + J_2}{2}}$$

substitute moment of inertia:

$$J_m = \frac{J_1 + J_2}{2} \gamma^2$$

buckling load:

$$P_k = \frac{E \pi^2 \frac{J_1 + J_2}{2}}{l^2} \gamma^2$$

Case 2a (fig. 9).- γ_a as function of $2\, l_1 : l$ for the parameter

$$q_a = \frac{J_2}{\frac{J_1 + J_2}{2}}$$

substitute inertia moment:

$$J_m = \frac{J_1 + J_2}{2} {\gamma_a}^2$$

buckling load:

$$P_k = \frac{E\,\pi^2\,\frac{J_1 + J_2}{2}}{l^2}\,\gamma_a{}^2$$

<u>Case 2b (graph, fig. 10).</u>- γ_b as function of $l_2 : l$ for parameter

$$q_b = \frac{J_1}{\frac{J_1 + J_2}{2}}$$

substitute inertia moment:

$$J_m = \frac{J_1 + J_2}{2}\,\gamma_b{}^2$$

buckling load:

$$P_k = \frac{E\,\pi^2\,\frac{J_1 + J_2}{2}}{l^2}\,\gamma_b{}^2$$

Graph 9 loses its clearness when $2\,l_1 : l \longrightarrow 1$. A close approximation in the range of $0.9 < 2\,l_1 : l < 1$ is found in Föppl's solution (cf. "Lectures on Technical Mechanics," vol. III, 7th ed., p. 384).

There:

$$P_k = \frac{E\,\pi^2\,J_1}{\left(2l_1 + \frac{J_1}{J_2}\,l_2\right)^2}$$

V. VALIDITY IN TETMAJER'S RANGE

Of course, it is understood that the determination of the failing load through the substitute moment of inertia alone is sufficient only when the compression strut does not fail as a result of local overstressing. In all cases which fail to satisfy the previously cited condition

$$\sigma_k = \frac{P_k}{f_{min}} \leq \sigma_G$$

the solution of the buckling load P_k must be followed by further investigation, especially if the range of f_{min} lies in bay center. The same applies to all struts with nonprogressively variable moment of inertia that do not fall in category 1, 2a, or 2b. Concerning the actual ultimate buckling load $P_{k,Br}$, it can be stated only for the time being that

$$P_{k,T} < P_{k,Br} < P_{k,E}$$

Therein the upper limit $P_{k,E}$ is formed by that buckling load which satisfies the buckling condition $N = 0$ with preservation of E even in the plastic range. The lower limit $P_{k,T}$ is obtained as follows:

Determine:

$$\sigma_{k,E} = \frac{P_{k,E}}{f_{min}} \quad (> \sigma_G)$$

Then let $\sigma_{k,T}$ denote Tetmajer's ultimate buckling stress at the same point λ to which the stress $\sigma_{k,E}$ belongs.

Next, let:

$$P_{k,T} = P_{k,E} \frac{\sigma_{k,T}}{\sigma_{k,E}}$$

The actual buckling load $P_{k,Br}$ then will lie between the two limits, because, first, elastic buckling is ascribed to the whole strut, while a buckling modulus $T < E$ exists in the range f_{min}, and secondly, the whole strut is given a buckling modulus $T < E$ while T prevails only in the range of f_{min}.

As the buckling modulus T varies with the stress only the buckling condition $N \gtrless 0$ can be investigated for a predetermined buckling load, whereby the buckling modulus T, which varies in the individual lengths l_n, replaces E. The ultimate buckling load $P_{k,Br}$ must be ascertained by trial, i.e., by determining the buckling condition $N \gtrless 0$ for several different values $P(P_{k,T} < P < P_{k,E})$, followed by graphical interpolation.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

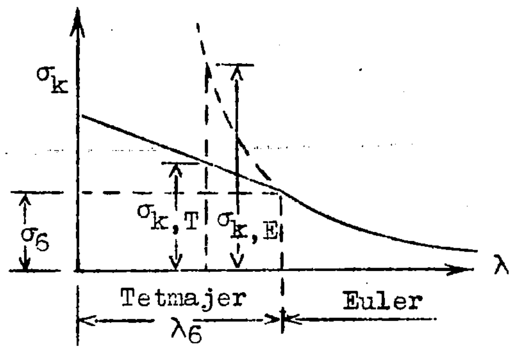


Figure 1.

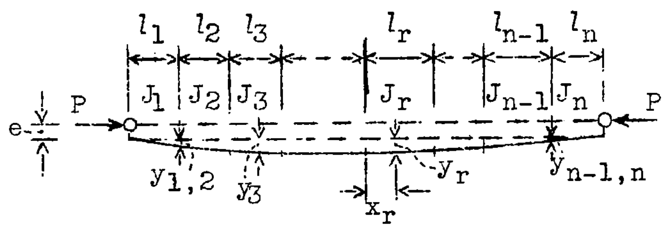


Figure 2.

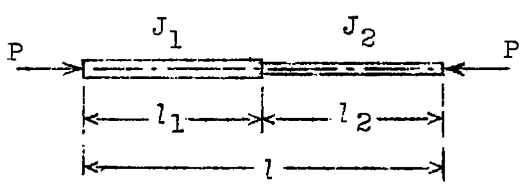


Figure 3.

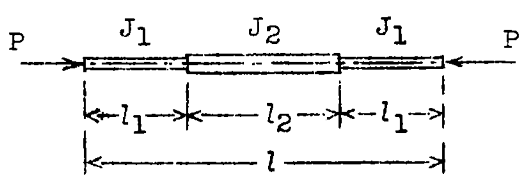
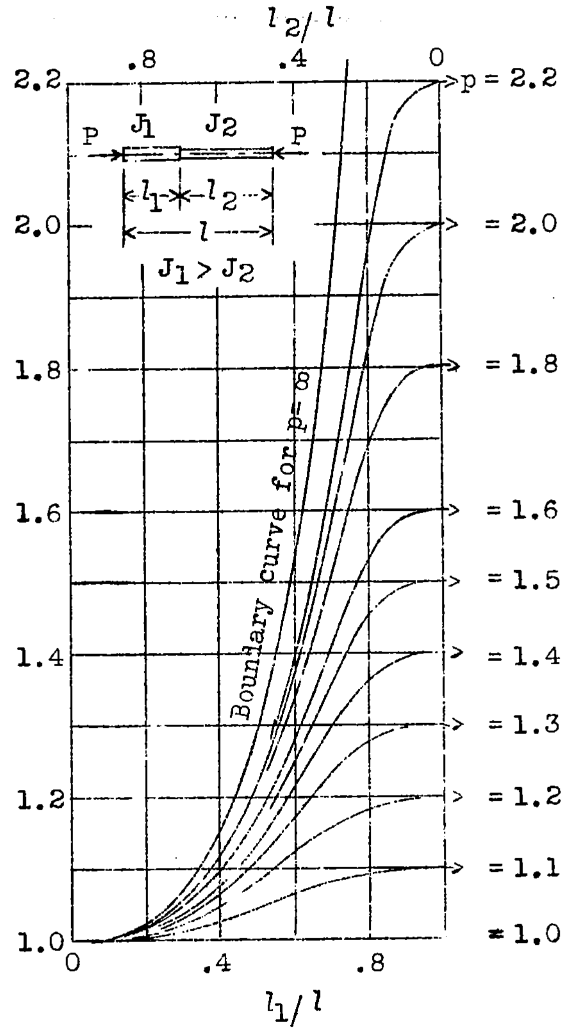


Figure 4.



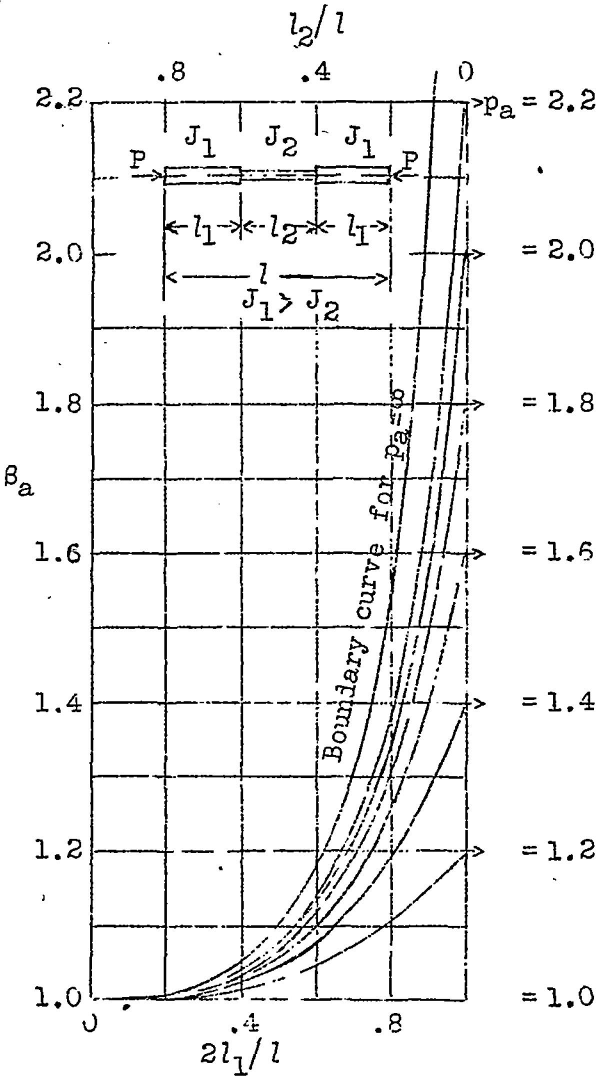
$\beta = \sqrt{\frac{J_m}{J_2}}$ as function of $l_1 : l$ or $l_2 : l$ for the parameter

$p = \sqrt{\frac{J_1}{J_2}}$

Failing load $P_k = \frac{E\, J_2\, \pi^2}{l^2}\ \beta^2$

Substitute moment of inertia: $J_m = J_2\ \beta^2$

Figure 5.- Buckling struts with non-progressively variable moment of inertia; substitute moment of inertia J_m.



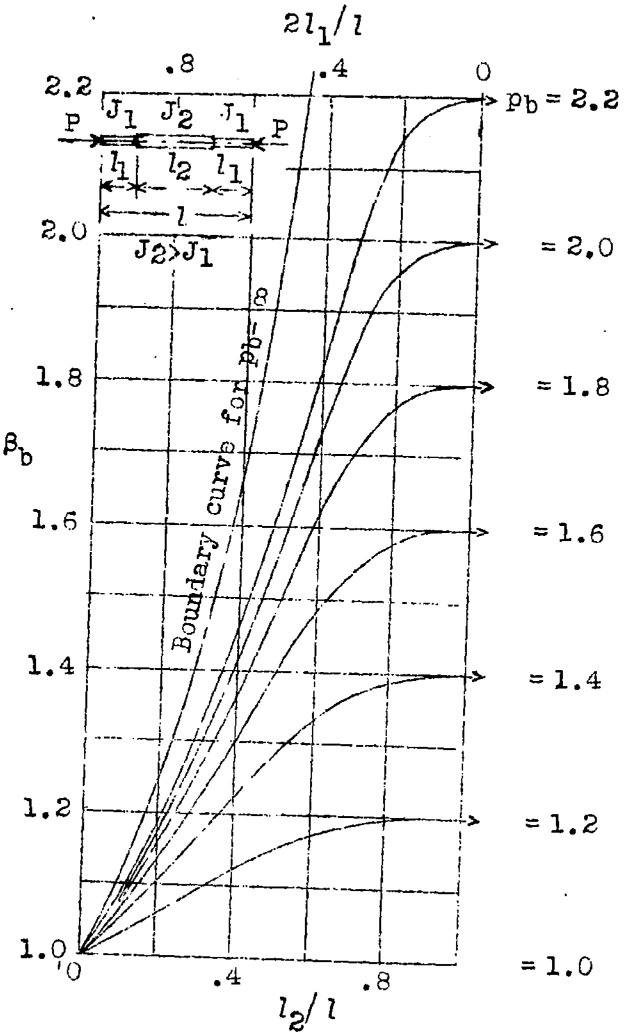
$\beta_a = \sqrt{\dfrac{J_m}{J_2}}$ as function of $2\,l_1 : l$ or $l_2 : l$ for the parameter

$$p_a = \sqrt{\frac{J_1}{J_2}}$$

Failing load $P_k = \dfrac{E\,J_2\,\pi^2}{l^2}\,\beta_a{}^2$

Substitute moment of inertia: $J_m = J_2\,\beta_a{}^2$

Figure 6.- Buckling struts with non-progressively variable moment of inertia; substitute moment of inertia J_m.



$\beta_b = \sqrt{\frac{J_m}{J_1}}$ as function of $l_2 : l$ or $2\,l_1 : l$ for the parameter

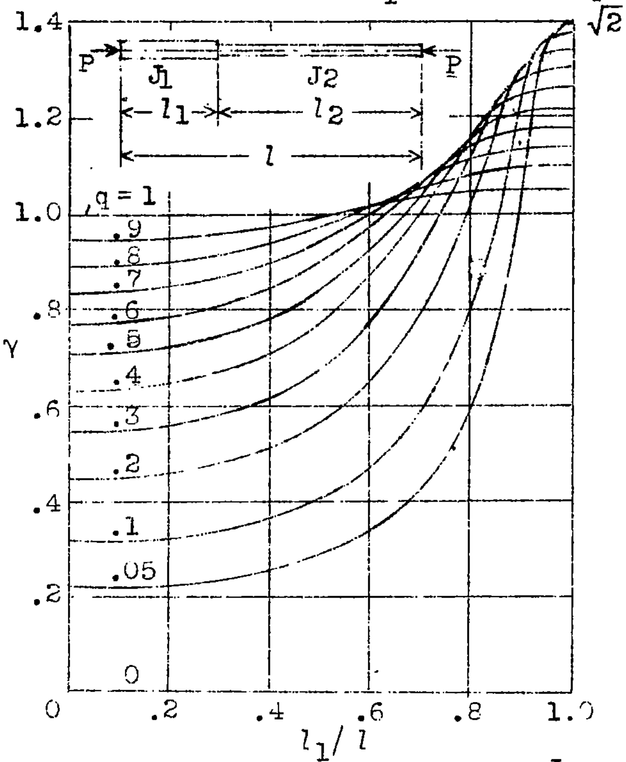
$$p_b = \sqrt{\frac{J_2}{J_1}}$$

Failing load $P_k = \frac{E\,J_1\,\pi^2}{l^2}\,\beta_b{}^2$

Substitute moment of inertia: $J_m = J_1\ \beta_b{}^2$

Figure 7.- Buckling struts with non-progressively variable moment of inertia; substitute moment of inertia J_m.

$\gamma = \sqrt{2} = 1.414 \cdots$ limit for $l_1/l \to 1$ and $q \to 0$.

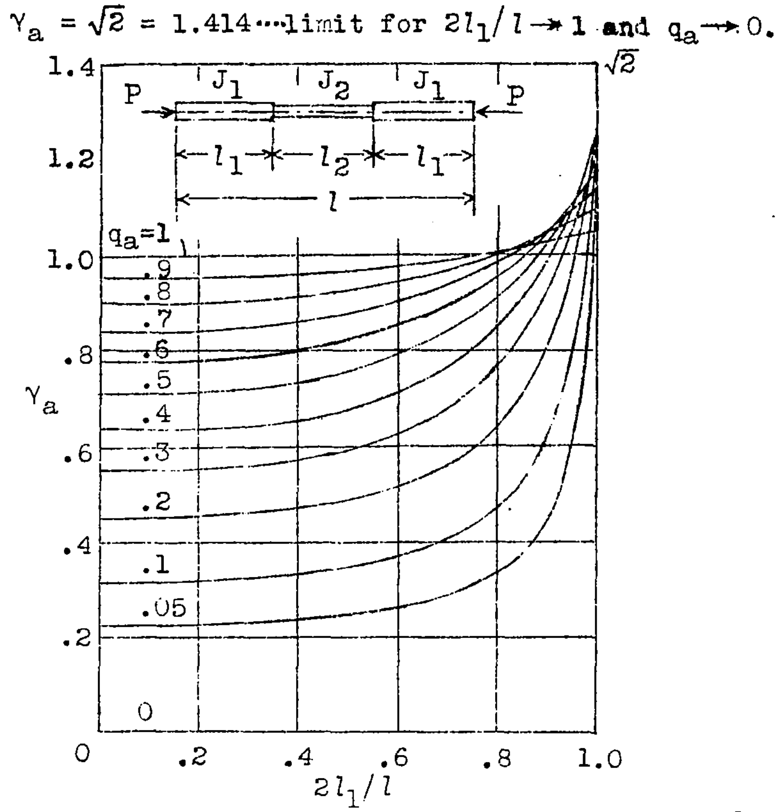


γ as function of $l_1 : l$ for parameter $q = \dfrac{J_2}{\dfrac{J_1 + J_2}{2}}$

Substitute moment of inertia: $J_m = \dfrac{J_1 + J_2}{2} \gamma^2$

Buckling load $P_k = \dfrac{E \pi^2 \dfrac{J_1 + J_2}{2}}{l^2} \gamma^2$

Figure 8.- Buckling struts with non-progressively variable moment of inertia.

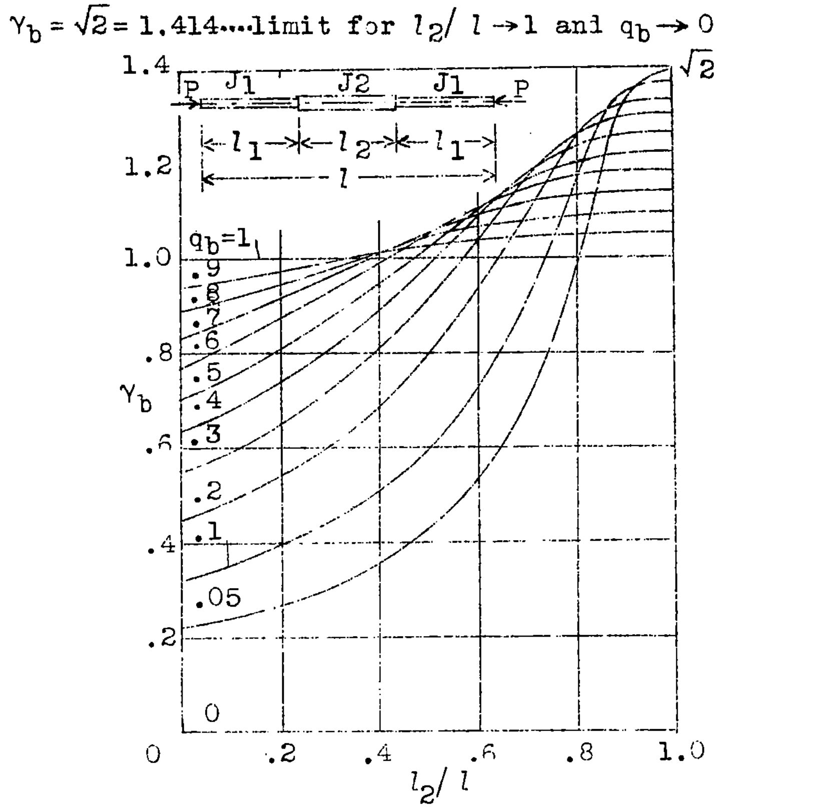


γ_a as function of $2l_1 : l$ for parameter $q_a = \dfrac{J_2}{\dfrac{J_1+J_2}{2}}$

Substitute moment of inertia: $J_m = \dfrac{J_1+J_2}{2}\gamma_a^2$

Buckling load $P_k = \dfrac{E\,\pi^2\,\dfrac{J_1+J_2}{2}}{l^2}\gamma_a^2$

Figure 9.- Buckling struts with non-progressively variable moment of inertia



γ_b as function of $l_2 : l$ for parameter $q_b = \dfrac{J_1}{\dfrac{J_1 + J_2}{2}}$

Substitute moment of inertia: $J_m = \dfrac{J_1 + J_2}{2} \gamma_b{}^2$

Buckling load $P_k = \dfrac{E \pi^2 \dfrac{J_1 + J_2}{2}}{l^2} \gamma_b{}^2$

Figure 10.- Buckling struts with non-progressively variable moment of inertia.