TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

---

No. 842

---



# UNIVERSAL LOGARITHMIC LAW OF VELOCITY DISTRIBUTION AS APPLIED TO THE INVESTIGATION OF BOUNDARY LAYER AND DRAG OF STREAMLINE BODIES AT LARGE REYNOLDS NUMBER

By G. Gurjienko

Central Aero-Hydrodynamical Institute, Moscow, 1936



Washington
November 1937

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 842

# UNIVERSAL LOGARITHMIC LAW OF VELOCITY DISTRIBUTION AS APPLIED TO THE INVESTIGATION OF BOUNDARY LAYER AND DRAG OF STREAMLINE BODIES AT LARGE REYNOLDS NUMBER*

By G. Gurjienko

## INTRODUCTION

Previous investigation devoted to the study of the turbulent boundary layer and drag of wings and figures of revolution were based essentially on the assumption of the power law of velocity distribution in the boundary layer, the exponent $n$ in the velocity distribution formula

$$u = u_\delta \left(\frac{y}{\delta}\right)^n \tag{1}$$

(where $u$ is the velocity in the boundary layer, $u_\delta$ = the velocity at the outer limit of the boundary layer, $y$ = the ordinate taken normal to the wall, and $\delta$ = the thickness of the boundary layer) being taken equal to 1/7. Using this method the problem of the cylindrical wing was solved by Miller (reference 1) and that of the airship body by Millikan (reference 2).

However, as various tunnel tests have shown (see Nikuradse, reference 2), the form of curve (1) varies with the Reynolds Number in such a manner that to a first approximation the exponent successively assumes the values 1/8, 1/9, 1/10, etc., as the Reynolds Number increases.

A certain forward step in investigating the effect of a change in $n$ with Reynolds Number on the boundary layer and the drag of figures of revolution was taken in 1932 by K. K. Fediaevsky (reference 14) who, making use of

---

*Report No. 257, of the Central Aero-Hydrodynamical Institute, Moscow, 1936.

the test data of Nikuradse for smooth tubes, found an approximate relation between the exponent $n$ and the Reynolds Number formed from the velocity at infinity and the length of the body $(Re = VL/\nu)$.

The above-mentioned method possessed, however, two unavoidable theoretical defects, namely: 1) the computation, on the basis of Nikuradse's tests, of the relation $n = f(Re)$ was very approximate and 2) the variation of $n$ along the body at constant $Re$ was neglected. The first defect may only be removed by conducting numerous tedious wind-tunnel tests with figures of revolution, similar to the tests of Nikuradse, and these would hardly give satisfactory results on account of the smallness of the Reynolds Numbers attainable in our wind tunnels. With regard to taking account of the variation of $n$ along the length of the body due to the gradual increase in the local Reynolds Number along the body, we should be confronted with very great mathematical difficulties in the solution of the boundary-layer equation in attempting to find and to substitute in it some relation between $n$ and the

Reynolds Number $R_\delta = \dfrac{\delta u_\delta}{\nu}$. (2)

With the above considerations in mind, we proceeded to work out a method for applying to the solution of the problem of the turbulent boundary layer and drag of streamline bodies, the universal logarithmic law of velocity distribution theoretically developed by Prandtl (reference 5) and Kármán (reference 6) in 1929-1930. As will appear below, the universal logarithmic law - by the very principle on which it is based - avoids both the difficulties mentioned above and offers, if not a final solution, at least one that much more closely approaches actual conditions.

The universal logarithmic formula of Kármán was used, as giving better agreement with experiment and as being derived from more rigorous assumptions than the corresponding formula of Nikuradse-Prandtl. As will be shown below, however, our equations are entirely applicable also to the case where the Nikuradse-Prandtl velocity distribution formula is taken as a starting point since the difference will evidence itself only in the change of certain constant coefficients.

In the present paper we shall consider a figure of revolution, so that the formulas applicable to the more simple cases as, for example, a wing or a flat plate will follow from our equations as corollaries.

For checking the results of our theory, we made use of the data derived from the excellent tests of Freeman on a 1/40-scale model of the airship "Akron" conducted in the large N.A.C.A. wind tunnel.

At the time when our work was already completed in its essentials, we obtained a preliminary report on the work of Moore (reference 7), in which the problem of a figure of revolution is solved by application of the Nikuradse-Prandtl law. Our equations, however, appear to be more general in the sense that they are applicable to various bodies and appear moreover to be exact solutions of the integral relation of Kármán which lies at the basis of the boundary-layer theory, since we are the first - so far as we know - to take full account of the curvature of the concentric elements of the boundary layer, i.e., their various distances from the axis of the body of revolution and to show numerically that the variation is small and does not appreciably affect the final result - thus rigorously proving that the curvature may be neglected. In addition we give, in our opinion, a more rational procedure for the numerical integration of the equations obtained than does Moore. The greatest difference, however, between our work and the similar work of other authors lies in bringing to light the large deviations of the results of application of the Kármán theory in its original form from the results of experiments. In the second part of this paper we have attempted to explain these discrepancies by subjecting the logarithmic theory to a general criticism. We have thus succeeded in deriving theoretically a sufficiently justifiable method of rendering the logarithmic theory more precise, after which it was found to be in excellent agreement with experiment.

In the first part we shall derive the fundamental equation for a body of revolution according to the Kármán theory in its original form, and in the second part we shall give all the comparisons of the results of tests with the modified theory.

## PART I

### 1. UNIVERSAL LOGARITHMIC FORMULAS

Before proceeding with the problem itself, we shall briefly review the essentials of the Kármán theory.

Kármán starts out from the following two fundamental assumptions: 1) Both the turbulent mixing characteristics (the mean mixing length) and the turbulent friction at a given point are completely determined by the characteristic magnitudes of the mean flow at the given point - in particular, by the derivatives of the mean flow; 2) The turbulent flow pulsation patterns are similar, i.e., differ only in the time and distance scales and do not depend on the viscosity.

As the tests of Stanton and the later ones of Fritch have shown, the velocity-distribution patterns at pipe sections having different wall roughness - i.e., having a different shear stress at the walls - appear to be entirely identical provided the difference between the maximum velocity at the channel axis and the velocity at some other point is expressed in terms of $\sqrt{\frac{\tau_0}{\rho}}$ where $\tau_0$ is the shearing stress or skin friction per unit area at the wall and $\rho$ is the mass density:

$$\frac{u_{max} - u}{\sqrt{\frac{\tau_0}{\rho}}} = f\left(\frac{z}{r}\right) \qquad (3)$$

In the above $z$ is the distance from the pipe axis at right angles to the mean flow direction and $r$ is the pipe radius. Thus the expression at the left is a function only of the coordinates and does not depend on the viscosity. This, naturally, is true only for regions not too near the wall.

The above suggests the existence of actual mechanical similitude between turbulent flows having different dynamical boundary conditions (shear at the wall).

If the form of the function $f(z/r)$ in expression

(3) is found, the problem is easily solved with regard to the velocity distribution in the pipe, the maximum velocity $u_{max}$, and the pipe resistance.

The most important feature of formula (3) is that it is true for any sufficiently large Reynolds Number; i.e., that the form of the function $f(z/r)$ does not depend on the Reynolds Number. For small Reynolds Numbers, where the effect of the viscosity shows up more strongly, formula (3) has less validity.

The problem of finding the function $f(z/r)$ was solved by Kármán (reference 4) by applying the above assumptions of mechanical similitude of the flow pulsation patterns and the absence of the effect of viscosity to the fundamental hydrodynamic equations (reduced to the Helmholtz form on the assumption of the existence of a flow function for the actual velocities). He obtained a new formula for the mixing length $l$:

$$l = K \frac{u'}{u''}$$

where $u'$ and $u''$ are derivatives of the velocity with respect to the coordinate at right angles to the mean flow and $K$ is a universal constant. Eliminating $l$ by the known expression

$$\tau \sim - \rho \, l^2 \, u'^2 \tag{4}$$

(which, as Kármán has shown, also appears as the result of the assumption of similitude of the pulsation field) where $\tau$ is the shearing stress at any point of the flow, and assuming $\tau$ to vary linearly with the distance $z$ from the axis of the pipe; that is, $\tau = \tau_0 \frac{z}{r}$ [1]), Kármán obtains the fundamental differential equation

$$K^2 \frac{u'^4}{u''^2} = \frac{\tau_0}{\rho} \frac{z}{r} \tag{5}$$

Upon integrating, with the corresponding choice of the constants of integration (at $z = 0$ $u = u_{max}$ and at $z = r$ $u' = \infty$), there is obtained:

---

1) This expression is a result of the requirement of dynamic equilibrium of the concentric walls of the fluid cylinders for the established mean flow.

$$\frac{u_{max} - u}{\sqrt{\frac{\tau_o}{\rho}}} = - \frac{1}{K}\left[\ln\left(1 - \sqrt{\frac{z}{r}}\right) + \sqrt{\frac{z}{r}}\right] \qquad (6)$$

i.e., the form of the function $f(z/r)$ is completely determined.

Introducing, in place of the distance $z$ from the axis of the pipe, the distance $y$ from the wall $(z = r - y)$ from (6), we find for the ratio $u/u_{max}$

$$\frac{u}{u_{max}} = 1 + \frac{1}{K\ u_{max}} \sqrt{\frac{\tau_o}{\rho}}\left[\ln\left(1 - \sqrt{1 - \frac{y}{r}}\right) + \sqrt{1 - \frac{y}{r}}\right] \qquad (6')$$

As may be seen, the ratio $u/u_{max}$ is not subject to the power law. It is interesting to note that as the Reynolds Number increases - that is, with decreasing value of $\sqrt{\tau_o/\rho}$, curve (6') changes its form, the curves obtained closely approaching the power law with consecutively decreasing exponents - 1/7, 1/8, 1/9, etc.

The dependence of the velocity profile on the Reynolds Number is thus already contained in formula (6'). It is not difficult to show that equation (6') has a discontinuity at $y = r$; that is, on the axis of the pipe - a result which is due to the fact that derivatives of $u$ with respect to $y$ higher than the second were neglected. The effect of this is only slight since the break in the smoothness of the curve (6) at $y = r$ is very small. At $y = 0$, that is, at the wall itself, the function (6) gives infinitely large velocities $u$ of logarithmic order. This is also explainable theoretically since the entire theory ceases to be true near the wall as a result of the strong increase in the effect of the viscosity. The latter results in the formation of a thin laminar layer at the wall, the existence of such a layer having been verified by numerous tests. The thickness of this layer and the velocity just outside it can only depend on the friction at the wall, the viscosity, and the fluid density. On the basis of dimensional considerations this thickness may be expressed as

$$\delta_{lam} = B \frac{\nu}{\sqrt{\frac{\tau_o}{\rho}}} \qquad (7)$$

and the velocity $u_{\delta_{lam}}$ just outside the layer as

$$u_{\delta_{lam}} = A \sqrt{\frac{\tau_0}{\rho}} \tag{8}$$

By the use of the above expressions Kármán was able to derive the law for the resistance of a round pipe. Assuming law (6) to be valid from the outer boundary of the laminar layer to the center of the pipe, that is, from $y = \delta_{lam}$ to $y = r$, the value of the velocity at the pipe axis may be found from expression (6), written for $y = \delta_{lam}$:

$$u_{max} = u_{\delta_{lam}} - \frac{1}{K}\sqrt{\frac{\tau_0}{\rho}}\left[\ln\left(1 - \sqrt{1 - \frac{\delta_{lam}}{r}}\right) + \sqrt{1 - \frac{\delta_{lam}}{r}}\right]$$

Since $\delta_{lam}$ is very small by comparison with $r$, this expression may be simplified by extracting the approximate root

$$\sqrt{1 - \frac{\delta_{lam}}{r}} \approx 1 - \frac{\delta_{lam}}{2r}$$

There is then obtained

$$u_{max} = u_{\delta_{lam}} + \frac{1}{K}\sqrt{\frac{\tau_0}{\rho}}\left[\ln\frac{2r}{\delta_{lam}} - 1\right] \tag{9}$$

Substituting expressions (7) and (8), we obtain:

$$u_{max} = \frac{1}{K}\sqrt{\frac{\tau_0}{\rho}}\left(\ln\frac{\sqrt{\frac{\tau_0}{\rho}}\, r}{\nu} + c\right) \tag{10}$$

where $C$ includes the constants A, B, etc.

Substituting in expression (9) the Reynolds Number of the pipe $R = \frac{u_{max}\, r}{\nu}$ and the nondimensional frictional coefficient $C_f = \frac{\tau_0}{\frac{1}{2}\rho\, u^2_{max}}$, we obtain:

$$\frac{K\sqrt{2}}{\sqrt{C_f}} = \ln\left(R\sqrt{C_f}\right) + C' \tag{11}$$

where $C' = C + \ln \sqrt{2}$.

This dependence of the resistance coefficient on the Reynolds Number agrees excellently with tests over a wide range of Reynolds Numbers.

In 1930, Prandtl obtained a universal logarithmic law of velocity distribution somewhat different from the law discussed above, of Kármán. Prandtl considered the infinite flow about a fixed wall and assumed the mixing length to be proportional to the distance $y$ from the wall - i.e., in place of expression (4), he had

$$l = K y \tag{4'}$$

This gave, after integrating the expression $\tau = \rho l^2 u^2$,

$$\frac{u}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{1}{K} \ln y + c$$

Nikuradse, on the basis of his tests, obtained the formula

$$\frac{u - u_{max}}{\sqrt{\frac{\tau_0}{\rho}}} = - \frac{1}{K} \ln \frac{r}{y} \tag{6''}$$

which may easily be obtained from the Prandtl formula by applying it to a pipe, excluding the constant $c$, and using the condition $u = u_{max}$ at $y = r$. This is the formula also used by Moore.

It is interesting to note that if expression (6") is applied to the laminar layer at the wall as was done above with formula (6), we obtain the same expression (10) but with the constant $C$ differently combined out of the unknown coefficients $A$, $B$, etc. Thus the same resistance law is obtained. The constants $C'$ and $K$ may be determined only experimentally. Up to the present these constants have been determined by Kármán and Nikuradse entirely by experiments on the dependence of the resistance of pipes on the Reynolds Number $R$, that is, by comparing formula (11) with tests. In doing this it was assumed that the universal constant $K$ thus determined is identical with the constant $K$ that enters into the velocity formulas (6) or (6"). In the second part of our work we shall give special consideration to this problem and show that actually this is far from being true. For the present, we shall pass on to the derivation of the equation for the solution of the boundary-layer problem and the drag of a figure of revolution, using as a basis the velocity and drag-distribution formulas of Kármán in the general form in which he derived them.

## 2. DERIVATION OF THE FUNDAMENTAL EQUATION

In the previous section we considered the laws of flow and the resistance of a circular pipe as derived by Kármán from the theory of mechanical similitude. The solution of Kármán refers to the so-called "internal problem." Our object in this work is to obtain a solution of the "external problem," i.e., the case where in place of a flow enclosed within a cylindrical pipe we have the flow of an infinite mass of fluid about a body. This body in our case will be a figure of revolution - for example, the hull of an airship.[2] We shall not distinguish the case of motion of a body in an undisturbed medium (airship flight) from that of a turbulent stream flowing up to a body fixed in position (aerodynamic tunnel) although these cases are not equivalent from the point of view of drag. The drag, as we know, depends chiefly on the character of the boundary layer which in turn is determined exclusively by the Reynolds Number $VL/\nu$ computed from the velocity of the motion (or the velocity of the mean flow at infinity) and some linear dimension of the body. The Reynolds Number affects both the absolute thickness of the boundary layer and the position of transition of the layer from the laminar to the turbulent condition. Since both of these factors also depend on the degree of turbulence of the flow in the tunnel, this degree of turbulence may be taken into account, in the final computation, by a corresponding increase in the Reynolds Number. It is true that since up to the present we possess no method for the quantitative determination of the degree of turbulence, we cannot as yet perform such a numerical operation. For this reason, strictly speaking, we may only study the motion of a body in an undisturbed medium. This is the case of greatest practical interest.

---

[2]In this work we shall consider only bodies with smooth surface.

We shall apply the conclusions of the previous section in their entirety to the external problem. It is therefore necessary to consider the justifiability of this procedure.

The turbulent boundary layer about a body situated in a flow is that region in which the action of the turbulent friction destroys the velocity distribution given by the potential flow. We have the same condition also in a circular pipe. In this case these frictional forces give the velocity profile of expression (6'). Since the latter was derived on the assumption that the effect of the turbulent friction shows up throughout the pipe cross section, it follows that in this case the entire region within the pipe is a "rolled up" boundary layer whose thickness is equal to the pipe radius and whose outer limit has become a single straight line, namely, the axis of the pipe. The velocity at this axis is, according to formula (10), a function of the frictional intensity at the wall, i.e., a function of the state of the boundary layer. In the case of the external problem this velocity just outside the boundary layer will appear as the boundary condition determined by the potential flow outside the boundary layer. This is where the difference lies between the internal and external problems. The velocity just outside the boundary layer we shall in what follows denote by $u_\delta$.

Strictly speaking, we are not quite justified in passing directly from the internal flow along the rectilinear wall of the pipe to the external flow along the curved boundary of a wing or figure of revolution. It has been established by tests that the flow, for example, in diverging and converging channels is not subject to the logarithmic law. It would be more rigorous if the transition to the external problem were made from the latter two cases. Since, however, we have up to the present, no solution of the problem of turbulent flow in diverging and converging channels, we are required to make use of the generally adopted method of transition to the external problem from straight pipes.

Thus, denoting by $\delta$ the thickness of the boundary layer, we shall have for the external case as the velocity distribution in the boundary layer

$$\frac{u - u_\delta}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{1}{\kappa}\left[\ln\left(1 - \sqrt{1 - \frac{y}{\delta}}\right) + \sqrt{1 - \frac{y}{\delta}}\right] \qquad (12)$$

where $y$ is taken normal to the surface of the body.

Formula (11) giving the dependence of the frictional coefficient on the Reynolds Number, becomes:

$$\frac{K\sqrt{2}}{\sqrt{C_f}} = \ln\left(R_\delta \sqrt{C_f}\right) + C' \tag{13}$$

where the coefficient $C_f$ is the ratio of the local frictional intensity to the dynamic velocity just outside the boundary layer[3)]

$$C_f = \frac{\tau_o}{\frac{1}{2}\,\rho\, u_\delta^2} \tag{14}$$

and the Reynolds Number is computed from the velocity just outside the boundary layer and the thickness of the layer:

$$R_\delta = \frac{u_\delta\,\delta}{\nu} \tag{15}$$

As may be seen, in the case of the external problem we bring into consideration, in place of the known radius of the pipe the unknown thickness of the boundary layer which, as is shown by both theory and experiment, varies along the hull. Thus, for a complete solution of the problem, we must have still another condition that connects the variable thickness $\delta$ of the boundary layer with the other variables, of which $u_\delta$ is considered as given by the assumption of a potential flow just outside the boundary layer.

An additional relation of this kind appears to be the well-known integral relation of Kármán which we shall now consider. This relation, derived for two-dimensional, steady flow assumes, for a figure of revolution, the following form:

$$\frac{\partial}{\partial s}\int_0^\delta 2\pi r' \rho u^2\, dy - u_\delta \frac{\partial}{\partial s}\int_0^\delta 2\pi r' \rho u\, dy = -\frac{\partial p}{\partial s}\int_0^\delta 2\pi r'\, dy - 2\pi r\, \tau_o \tag{16}$$

3)The local frictional intensity as we shall see farther on, will vary along the hull of the body.

where $s$ is the length along the hull of the figure of revolution, measured from the nose.

$r'$, radius of a concentric element of the surface. (See fig. 1.)

$\rho$, density of the fluid.

$y$, distance from the surface measured along the normal.

$\delta$, thickness of the boundary layer.

$u$ and $u_\delta$, mean velocities within and just outside the layer.

$p$, static pressure at boundary layer.

$r$, radius of circular section of figure of revolution.

$\tau_0$, local shearing stress (or skin friction per unit of area) and the variables have the following functional relations among each other:

$$r' = r'(s,y);\ \delta = \delta(s);\ u = u(s,y)$$

$$u_\delta = u_\delta(s);\ p = p(s);\ r = r(s)$$

Expression (16) thus gives the required additional relation expressing either $\tau_0$ as a function of $\delta$ and $s$, $\tau_0(\delta,s)$, or $\delta$ as a function of $\tau_0$ and $s$, $\delta(\tau_0,s)$ and the problem thus becomes determinate. In equation (16) the left-hand side represents the increase per second of the momentum of the fluid at an element of boundary layer of length $ds$ and the right-hand side gives the sum of the forces acting on the boundary layer. This relation will be proved below.

Figure 1 shows a portion of a section of a figure of revolution (generating line $ab$) with the boundary layer of thickness $\delta$. Through point A on the body, pass a conical surface normal to the surface of the body. Let the generating arc $ab$ receive an increment $ds = AA'$ and through point A' pass a conical surface A'B' normal to the surface of the body. Through the circle of intersection of surface A'B' with the outer surface of the boundary layer we pass the "surface of flow" a'b'. We thus obtain an annular element of a "tube of flow" bounded by the surface of flow and the body. To this element we may apply the "momentum law."

Let us denote the mass of fluid flowing per second through section AB of the boundary layer by Q and the momentum associated with this mass by J. These magnitudes are given by

$$Q = \int_0^\delta 2\pi r'\rho u dy$$

$$J = \int_0^\delta 2\pi r'\rho u^2 dy$$

In passing through the element ds, the mass Q and its momentum J increase by $\frac{\partial Q}{\partial s}$ ds and $\frac{\partial J}{\partial s}$ ds, respectively; that is, the mass and the momentum of the fluid flowing through section A'B' will be

$$Q + \frac{\partial Q}{\partial s} ds, \quad J + \frac{\partial J}{\partial s} ds$$

The increase in the mass $\frac{\partial Q}{\partial s}$ ds can only be obtained by the entrance of a mass of the potential flow fluid through portion $BB_1$ of the section of the tube of flow. Since the velocity of this inflow is $u_\delta$ the momentum of the fluid introduced through the entire section $AB_1$ will be

$$J + u_\delta \frac{\partial Q}{\partial s} ds$$

Our surface of flow was so drawn that the section A'B' of the boundary layer is at the same time a section of a tube of flow. Thus the increase in the momentum of the fluid through an element of the tube of flow will be

$$J + \frac{\partial J}{\partial s} ds - J - u_\delta \cdot \frac{\partial Q}{\partial s} ds = \frac{\partial}{\partial s}\left[\int_0^\delta 2\pi r'\rho u dy\right]$$

$$ds - u_\delta \frac{\partial}{\partial s}\left[\int_0^\delta 2\pi r'\rho u dy\right] ds \qquad (a)$$

This increase of momentum per second should, according to the momentum law, be equal to the sum of the forces acting on the element of the tube of flow in the direction of the momentum vector. If the static pressure at the left of the element is $p$ and that at the right $p + \frac{\partial p}{\partial s}\, ds$ the force, evidenced by the drop in pressure, will be

$$-\left[\left(p + \frac{\partial p}{\partial s}\, ds\right) \int_0^{\delta + \frac{\partial \delta}{\partial s} ds} 2\pi r' dy - p \int_0^{\delta + \frac{\partial \delta}{\partial s} ds} 2\pi r' dy\right] =$$

$$= -\left[\frac{\partial p}{\partial s} \int_0^{\delta + \frac{\partial \delta}{\partial s} ds} 2\pi r' dy\right] ds$$

The minus sign is chosen since a positive pressure gradient $\frac{\partial p}{\partial s}$, corresponds to a force in the opposite direction to the momentum vector.

Breaking up the integral thus obtained into two, we have

$$-\left[\frac{\partial p}{\partial s} \int_0^{\delta} 2\pi r' dy + \frac{\partial p}{\partial s} \int_{\delta}^{\delta + \frac{\partial \delta}{\partial s} ds} 2\pi r' dy\right] ds = -\left[\frac{\partial p}{\partial s} \int_0^{\delta} 2\pi r' dy\right] ds \qquad \text{(b)}$$

The second integral vanishes since it is infinitely small in comparison with the first. The resultant of the turbulent frictional forces on the element $ds$ will be

$$-\, 2\pi\, r\, \tau_0\, ds \qquad \text{(c)}$$

Equating expression (a) to the sum of expressions (b) and (c) and dividing through by $ds$, we obtain the integral relation (16) of Kármán.

We have assumed throughout that the static pressure does not vary along the thickness of the boundary layer. Actually there exists a certain, though very small, pressure gradient in the boundary layer (see, for example, reference 12), of which no quantitative account can as yet be taken. Attempts to determine the normal forces resulting from the pressure gradient normal to the surface by equating them to the integral of the centrifugal forces of the particles of fluid moving on a given curvilinear contour in the boundary layer gave negative results - the pressure gradient obtained as a result of the centrifugal forces making up, for example, only half the value that is actually observed.

Our problem now is to bring equation (16) into relation with expressions (13) and (14). We shall first transform expression (16). The radius $r'$ of the concentric element of the boundary layer, according to figure 1, is equal to

$$r' = r + y \cos \theta$$

where $\theta$ is the angle between the tangent to the generating line of the body of revolution and its axis. The quantity $y \cos \theta$ is generally neglected by comparison with $r$; that is, $r'$ is considered equal to $r$ and the effect of this simplification on the final result is not analyzed. We shall not introduce this simplification in order that we may have the possibility of determining the effect of neglecting $y \cos \theta$ on the final result. Substituting in expression (16) the value of $r'$, dividing by $2\pi \rho$ and transferring to the left-hand side the members containing $\cos \theta$, we obtain:

$$\frac{\partial}{\partial s}\int_0^\delta ru^2\,dy - u_\delta \frac{\partial}{\partial s}\int_0^\delta ru\,dy + \frac{\partial}{\partial s}\int_0^\delta \cos\theta\, u^2 y\,dy - u_\delta \frac{\partial}{\partial s}\int_0^\delta \cos\theta\, uy\,dy =$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial s}\int_0^\delta (r + y\cos\theta)\,dy - r\frac{\tau_0}{\rho}$$

Taking $r$ and $\cos \theta$ outside the integral signs as being independent of $y$ and performing all the possible differentiations and integrations, we have:

$$r \frac{\partial}{\partial s}\int_0^\delta u^2\,dy + \frac{\partial r}{\partial s}\int_0^\delta u^2\,dy - ru_\delta \frac{\partial}{\partial s}\int_0^\delta u dy - u_\delta \frac{\partial r}{\partial s}\int_0^\delta u dy +$$

$$+ \cos\theta \frac{\partial}{\partial s}\int_0^\delta u^2 y dy + \frac{\partial \cos\theta}{\partial s}\int_0^\delta u^2 y dy - u_\delta \cos\theta \frac{\partial}{\partial s}\int_0^\delta u y dy -$$

$$- u_\delta \frac{\partial \cos\theta}{s}\int_0^\delta u y dy = -\frac{1}{\rho}\frac{\partial p}{\partial s}\left(r\delta + \frac{1}{2}\delta^2\cos\theta\right) - r\frac{\tau_0}{\rho} \qquad (17)$$

To the region outside the boundary layer we may apply the Bernoulli theorem and thus express the pressure gradient entering into the expression on the right-hand side in terms of the velocity just outside the boundary layer. We have:

$$p + \frac{\rho\, u_\delta^2}{2} = \text{constant}$$

Differentiating with respect to $s$, we obtain:

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} = u_\delta \frac{\partial u_\delta}{\partial s}$$

Grouping the terms in expression (17) and substituting the value of $-\frac{1}{\rho_\delta}\frac{\partial p}{\partial s}$, we have:

$$r\left[\frac{\partial}{\partial s}\int_0^\delta u^2\,dy - u_\delta \frac{\partial}{\partial s}\int_0^\delta u dy\right] + \frac{\partial r}{\partial s}\left[\int_0^\delta u^2\,dy - u_\delta\int_0^\delta u dy\right] +$$

$$+\cos\theta\left[\frac{\partial}{\partial s}\int_0^\delta u^2 y dy - u_\delta \frac{d}{\partial s}\int_0^\delta u y dy\right] + \frac{\partial \cos\theta}{\partial s}\left[\int_0^\delta u^2 y dy - u_\delta\int_0^\delta u y dy\right] =$$

$$= u_\delta \frac{\partial u_\delta}{\partial s}\left(r\delta + \frac{1}{2}\delta^2 \cos\theta\right) - r\frac{\tau_0}{\rho} \qquad (18)$$

Making use of the velocity distribution law (12), we must now compute the integrals in expression (18).

We shall denote the expression $\sqrt{\frac{\tau_0}{\rho}}$ by $v_*$. This expression, which appears as the "dynamic scale" of velocities in equation (12) and has the dimensions of a velocity, we shall denote - following the suggestion of Professor L. G. Loitsansky - as the "dynamical velocity." We shall further introduce a new variable $x$:

$$x = 1 - \sqrt{1 - \frac{y}{\delta}} \tag{19}$$

noting that a variation in $y$ from $0$ to $\delta$ corresponds to a variation in $x$ from 0 to 1. We then obtain from expression (12) the velocity and its square:

$$\left.\begin{aligned} u &= u_\delta + \frac{v_*}{K}\,[\ln x + 1 - x] \\ u^2 &= u_\delta^2 + 2\,\frac{u_\delta\, v_*}{K}\,[\ln x - x + 1] + \frac{{v_*}^2}{K^2}\,(\ln x - x + 1)^2 \end{aligned}\right\} \tag{20}$$

We may now compute all the integrals, bearing in mind that in agreement with the change in variables (19):

$$y = \delta - \delta\,(1 - x)^2$$

$$dy = 2\delta\,(1 - x)\,dx$$

Without going through all the computations, we shall at once write down the final result:

$$\left.\begin{aligned} \int_0^\delta u\,dy &= \delta\left(u_\delta - \frac{5}{6}\,\frac{v_*}{K}\right) \\ \int_0^\delta u^2\,dy &= \delta\left(u_\delta^2 - \frac{10}{6}\,\frac{u_\delta v_*}{K} + \frac{14}{9}\,\frac{{v_*}^2}{K^2}\right) \\ \int_0^\delta u y\,dy &= \frac{1}{2}\,\delta^2\left(u_\delta - \frac{23}{60}\,\frac{v_*}{K}\right) \\ \int_0^\delta u^2 y\,dy &= \frac{1}{2}\,\delta^2\left(u_\delta - \frac{46}{60}\,\frac{u_\delta v_*}{K} + \frac{601}{1800}\,\frac{{v_*}^2}{K^2}\right) \end{aligned}\right\} \tag{21}$$

In the process of integration we meet with integrals with respect to $x$, and it is useful to write these down:

1) $$\int_0^1 \ln x dx = [x \ln x - x]_0^1 = -1$$

2) $$\int_0^1 x \ln x dx = \left[\frac{x^2 \ln x}{2} - \frac{x^2}{4}\right]_0^1 = -\frac{1}{4}$$

3) $$\int_0^1 x^2 \ln x dx = \left[\frac{x^3 \ln x}{3} - \frac{x^3}{9}\right]_0^1 = -\frac{1}{9}$$

4) $$\int_0^1 x^3 \ln x dx = \left[\frac{x^4 \ln x}{4} - \frac{x^4}{16}\right]_0^1 = -\frac{1}{16}$$

5) $$\int_0^1 \ln^2 x dx = [x \ln^2 x - 2x \ln x + 2x]_0^1 = 2$$

6) $$\int_0^1 x \ln^2 x dx = \left[\frac{x^2 \ln^2 x}{2} - \frac{x^2 \ln x}{2} + \frac{x^2}{4}\right]_0^1 = \frac{1}{4}$$

7) $$\int_0^1 x^2 \ln^2 x dx = \left[\frac{x^3 \ln^2 x}{3} - \frac{2x^3 \ln x}{9} + \frac{2x^3}{27}\right]_0^1 = \frac{2}{27}$$

8) $$\int_0^1 x^3 \ln^2 x dx = \left[\frac{x^4 \ln^2 x}{4} - \frac{x^4 \ln x}{8} + \frac{x^4}{32}\right]_0^1 = \frac{1}{32}$$

All the above integrals are readily integrated by parts noting that the limit of the expressions $x^n \ln^m x$, where $n$ and $m$ are positive integers, is zero as $x$ approaches zero, as may readily be proved by repeated application of the L'Hospital rule.

Before substituting the integrals (21) in expression (18) we shall eliminate from them $\delta$. Using the expressions for the friction coefficient $C_f$ and the Reynolds Number $R_\delta$ as given by (14) and (15), we obtain from (13):

$$\frac{K\sqrt{2}}{\sqrt{\frac{2\tau_o}{\rho u_\delta^2}}} = \ln\left(\frac{u_\delta \delta}{\nu}\sqrt{\frac{2\tau_o}{\rho u_\delta^2}}\right) + C'$$

or

$$\frac{K\ u_\delta}{v_*} = \ln\left(\frac{\delta\sqrt{2}}{\nu}\ v_*\right) + C'$$

and introducing the constant $C'$ under the logarithm sign, i.e., substituting $e^{C'_2} = C_2$, we obtain on solving the above equation for $\delta$:

$$\delta = \frac{\nu}{C_2\sqrt{2}v_*}\ e^{\frac{Ku_\delta}{v_*}} = \frac{n}{v_*}\ e^{\frac{Ku_\delta}{v_*}}$$

where we have set $\frac{\nu}{C_2\ \sqrt{2}} = n =$ constant

We shall introduce, in analogy to what was done by Kármán for the flat plate, a new variable $z$:

$$z = \frac{K\ u_\delta}{v_*}$$

whence

$$v_* = \frac{K\ u_\delta}{z}$$

and

$$\delta = \frac{n}{K}\ \frac{z\ e^z}{u_\delta} \tag{22}$$

Denoting the constant numerical values entering into the expression for the integrals (21) by letters:

$\frac{5}{6} = A$; $\frac{14}{9} = B$; $\frac{23}{60} = a$ $\frac{601}{1800} = b$; we obtain:

$$\int_0^\delta u^2\,dy = \frac{n}{K}\,\frac{ze^z}{u_\delta}\left(u_\delta^2 - \frac{2Au_\delta^2}{z} + \frac{Bu_\delta^2}{z^2}\right) = \frac{n}{K}\,u_\delta e^z\left(z - 2A + \frac{B}{z}\right)$$

$$\int_0^\delta u\,dy = \frac{n}{K}\,\frac{ze^z}{u_\delta}\left(u_\delta - \frac{Au_\delta}{z}\right) = \frac{n}{K}\,e^z\,(z - A)$$

$$\int_0^\delta uy\,dy = \frac{n^2}{2K^2}\,\frac{z^2e^{2z}}{u_\delta^2}\left(u_\delta - a\,\frac{u_\delta}{z}\right) = \frac{n^2}{2K^2}\,\frac{e^{2z}}{u_\delta}\,(z^2 - az)$$

$$\int_0^\delta u^2y\,dy = \frac{n^2}{2K^2}\,\frac{z^2e^{2z}}{u_\delta^2}\left(u_\delta^2 - \frac{2au_\delta^2}{z} + \frac{bu_\delta^2}{z^2}\right) = \frac{n^2}{2K^2}\,e^{2z}(z^2 - 2az + b)$$

The derivatives of these integrals with respect to s will be:

$$\frac{\partial}{\partial s}\int_0^\delta u\,dy = \frac{n}{K}\,\frac{dz}{ds}\,e^z\,(z - A + 1)$$

$$\frac{\partial}{\partial s}\int_0^\delta u^2\,dy = \frac{n}{K}\,\frac{du_\delta}{ds}\,e^z\left(z - 2A + \frac{B}{z}\right) + \frac{n}{K}\,\frac{dz}{ds}\,u_\delta e^z\left(z + \frac{B}{z} - \frac{B}{z^2} - 2A + 1\right)$$

$$\frac{\partial}{\partial s}\int_0^\delta uy\,dy = -\frac{n^2}{2K^2}\,\frac{1}{u_\delta^2}\,\frac{du_\delta}{ds}\,e^{2z}(z^2 - az) + \frac{n^2}{2K^2}\,\frac{dz}{ds}\,\frac{1}{u_\delta}\,e^{2z}(2z^2 - (2a - 2)z - a)$$

$$\frac{\partial}{\partial s}\int_0^\delta u^2y\,dy = \frac{n^2}{K}\,\frac{dz}{ds}\,e^{2z}\,[z^2 - (2a - 1)\,z + b - a]$$

On the right-hand side of these equations we may write the total for the partial derivatives since all the variables depend only on s.

Substituting all the values obtained into expression (18), dividing and collecting similar terms, we find:

$$\frac{n}{K} ru_\delta e^z\left(\frac{B}{z} - \frac{B}{z^2} - A\right) \frac{dz}{ds} + \frac{nr}{K} e^z \frac{du_\delta}{ds} \left(\frac{B}{z} - 2A\right) + \frac{dr}{ds} \frac{nu_\delta}{K} e^z \left(\frac{B}{z} - A\right) -$$

$$- \cos\theta \frac{n^2}{2K^2} e^{2z} \left[ (2az-2b+a) \frac{dz}{ds} + \frac{1}{u_\delta} \frac{du_\delta}{ds} az + \right.$$

$$\left. + \frac{1}{\cos\theta} \frac{d\cos\theta}{ds} (az-b) \right] = - r \frac{K^2 u_\delta^2}{z^2} \tag{23}$$

The terms containing $\cos\theta$ were grouped together so as to enable the effect of the term $y\cos\theta$ to be more advantageously studied.

Let us multiply the equation by the magnitude $\left[- \frac{Kz^2}{nu_\delta r}\right]$ and write the derivatives of $u_\delta$, $r$, and $\cos\theta$ in the form:

$$\frac{1}{u_\delta} \frac{du_\delta}{ds} = \frac{d \ln u_\delta}{ds}; \quad \frac{1}{r} \frac{dr}{ds} = \frac{d \ln r}{ds}; \quad \frac{1}{\cos\theta} \frac{d\cos\theta}{ds} = \frac{d \ln \cos\theta}{ds}$$

We then obtain:

$$e^z(Az^2-Bz+B) \frac{dz}{ds} + e^z(2Az^2-Bz) \frac{d \ln u_\delta}{ds} + e^z(Az^2-Bz) \frac{d \ln r}{ds} +$$

$$+ \frac{n \cos\theta\, z^2}{2K\, u_\delta\, r} e^{2z}\left[(2az-2b+a)\frac{dz}{ds} + az \frac{d \ln u_\delta}{ds} + (az-b)\frac{d \ln \cos\theta}{ds}\right] =$$

$$= \frac{K^3}{n} u_\delta \tag{24}$$

The above expression also serves as the fundamental differential equation for finding the relation between $z$ and $s$, and thus also $\delta = \delta(s)$ and $\tau_0 = \tau_0(s)$.

It is useful to introduce in the equation in place of $u_\delta$, its value obtained through the pressure distribution on the hull, which value may easily be found experimentally. This is the simplest method for finding the relation $u_\delta = u_\delta(s)$.

Following K. K. Fediaevsky (reference 14), let us set $\left(\frac{u_\delta}{V}\right)^2 = f$, where $V$ is the forward velocity of the body or the velocity of the flow at infinity.

From the Bernoulli equation $p + \frac{\rho u_\delta^2}{2} = p_o + \frac{\rho V^2}{2}$, where $p_o$ is the pressure at a great distance from the body, we obtain after dividing by $\frac{\rho V^2}{2}$

$$\left(\frac{u_\delta}{V}\right)^2 = f = 1 - \frac{p - p_o}{\frac{1}{2}\rho V^2} \tag{25}$$

The variation of $\frac{p - p_o}{\frac{1}{2}\rho V^2}$ along the hull is easily determined experimentally.

Making the change indicated above, we find:

$$u_\delta = V\sqrt{f}; \quad \ln u_\delta = \ln V + \ln \sqrt{f}; \quad d \ln u_\delta = d \ln \sqrt{f}$$

Substituting in equation (24) and introducing the Reynolds Number

$$Re = \frac{VL}{\nu} = \frac{VL}{n\ C_2\ \sqrt{2}}$$

whence

$$n = \frac{VL}{Re\ C_2\ \sqrt{2}}$$

we obtain:

$$e^z(Az^2 - Bz + B)\frac{dz}{ds} + e^z(2Az^2 - Bz)\frac{d \ln \sqrt{f}}{ds} + e^z(Az^2 - Bz)\frac{d \ln r}{ds} +$$

$$+ \frac{L}{2Re\ K\ C_2\ \sqrt{2}}\ \frac{\cos\theta\ z^2 e^{2z}}{r\sqrt{f}}\left[(2az - 2b + a)\frac{dz}{ds} + az\frac{d \ln \sqrt{f}}{ds} +\right.$$

$$\left. + (az - b)\frac{d \ln \cos\theta}{ds}\right] = \frac{Re\ K^3\ C_2\ \sqrt{2}}{L}\sqrt{f} \tag{26}$$

This differential equation of the first order with respect to $z$ has been set up for a figure of revolution. For the more simple case, i.e., that of the infinite cylindrical wing the equation is considerably simplified since $r$ then becomes infinite and the last two terms on the right-hand side vanish. Thus, for a wing, we obtain:

$$e^z(Az^2-Bz+B)\frac{dz}{ds}+e^z(2Az^2-Bz)\frac{d\ \ln\sqrt{f}}{ds}=\frac{Re\ K^3\ C_2\ \sqrt{2}}{L}\sqrt{f} \quad (27)$$

where $L$ is the wing chord.[4)]

For the even more simple case of the flat plate situated in a flow at zero angle of attack, the second term on the left-hand side of expression (26) likewise drops out since there is no pressure gradient along the plate $\left(\frac{d\ \ln\sqrt{f}}{ds}=0\right)$. The velocity just outside the boundary layer $u_\delta = V =$ constant, that is, $f = 1$.

Thus, for a plate, we obtain:

$$e^z(Az^2 - Bz + B)\frac{dz}{dx} = \frac{Re\ K^3\ C_2\ \sqrt{2}}{L} \quad (28)$$

where $x$ is the variable distance from the leading edge of the plate. Introducing in the above expression the Reynolds Number $R_x$ computed for any distance $x$ from the leading edge, i.e.:

$$R_x = \frac{Vx}{\nu} \qquad dx = \frac{\nu}{V}\,dR_x$$

we obtain:

$$e^z(Az^2 - Bz + B)\frac{dz}{dR_x} = \frac{\nu\ Re\ K^3\ C_2\ \sqrt{2}}{VL}$$

or

$$e^z(Az^2 - Bz + B)\,dz = K^3\ C_2\ \sqrt{2}d\ R_x$$

Integrating the above equation between the limits $R_o$, $R_x$, and $z_o$, $z$, we obtain:

$$R_x - R_o = \frac{1}{K^3\ C_2\ \sqrt{2}}\int_{z_o}^{z}(Az^2 - Bz + B)\ e^z\ dz$$

which gives, after substituting the numerical values of the coefficients $A$ and $B$ and taking the coefficient $A$ outside the integral sign:

$$R_x - R_o = \frac{5e^{-c'}}{6K^3\ \sqrt{2}}\int_{z_o}^{z}\left(z^2 - \frac{28}{15}\ z + \frac{28}{15}\right)e^z\ dz$$

---

4)It should be noted that, in contrast to the equation for the figure of revolution, the wing equation may be applied for any angles of attack, the latter affecting only the character of the function $\sqrt{f} = f(s)$.

i.e., we have obtained the equation for the flat plate in the form given by Kármán (reference 4).

## 3. INTEGRATION OF THE EQUATIONS OBTAINED

It is evident from even a superficial glance at the equations for the figure of revolution and for the wing, respectively, that they cannot be integrated by any known method. We have seen that integration is possible only for the flat plate. We shall therefore resort to a graphical integration using the method of successive approximations.

We set:

$$e^z(Az^2-Bz+B) = \varphi_1; \quad e^z(2Az^2-Bz) = \varphi_2; \quad e^z(Az^2-Bz) = \varphi_3$$

and modify somewhat the expression in brackets in the last terms on the left-hand side:

$$(2az-2b+a)\frac{dz}{ds} + az\frac{d\ln\sqrt{f}}{ds} + (az-b)\frac{d\ln\cos\theta}{ds}$$

whence, after grouping the terms, we obtain:

$$az\left(2\frac{dz}{ds} + \frac{d\ln\sqrt{f}}{ds} + \frac{d\ln\cos\theta}{ds}\right) - (2b-a)\frac{dz}{ds} - b\frac{d\ln\cos\theta}{ds}$$

or, writing the sum of the derivatives as the derivative of a sum:

$$az\frac{d}{ds}\left[2z+\ln(\sqrt{f}\cos\theta)\right] - \frac{d}{ds}\left[z(2b-a) + b\ln\cos\theta\right]$$

Substituting the above in our expression (26), we obtain:

$$\varphi_1\frac{dz}{ds} + \varphi_2\frac{d\ln\sqrt{f}}{ds} + \varphi_3\frac{d\ln r}{ds} + \frac{L}{2Re\ K\ C_2\sqrt{2}}\frac{\cos\theta}{\sqrt{f}\ r}$$

$$\left\{az^3\frac{d}{ds}\left[2z+\ln(\sqrt{f}\cos\theta)\right] - z^2e^{2z}\frac{d}{ds}\left[z(2b-a)+b\ln\cos\theta\right]\right\} =$$

$$= \frac{Re\ K^3\ C_2\sqrt{2}}{L}\sqrt{f}$$

Multiplying the equation by ds, integrating between the limits $s_o$ and $s$, and setting

$$k = \frac{R_e\ K^3\ C_2\ \sqrt{2}}{L} = \text{constant}$$

we obtain:

$$\int_{z_o}^{z} \varphi_1\, dz = k \int_{s_o}^{s} \sqrt{f}\, ds - \int_{\ln\sqrt{f_o}}^{\ln\sqrt{f}} \varphi_2\, d\ln\sqrt{f} - \int_{\ln r_o}^{\ln r} \varphi_3\, d\ln r$$

$$- \frac{aK^2}{2k} \int_{(2z_o+\ln(\sqrt{f_o}\cos\theta)}^{2z+\ln(\sqrt{f}\cos\theta_o)} \frac{\cos\theta}{r\sqrt{f}}\, z^2 e^{2z}\, d\,[2z+\ln(\sqrt{f}\cos\theta)] +$$

$$+ \frac{K^2}{2k} \int_{z_o(2b-a)+b\ln\cos\theta_o}^{z(2b-a)+b\ln\cos\theta} \frac{\cos\theta}{r\sqrt{f}}\, z^2 e^{2z} d\,[z(2b-a)+b\ln\cos\theta] \qquad (29)$$

where the letters with subscript o denote the initial values of the variables.

The integration should rigorously start from the point of transition of the laminar into the turbulent layer, i.e., in accordance with the notation of K. K. Fediaevsky (reference 13) from $s_o = t$. The position of this point may be determined using the so-called critical Reynolds Number $R_{\delta_{kr}}$, which generally lies within the range:

$$1,600 \leqslant R_{\delta_{kr}} \leqslant 10,000$$

Using this critical Reynolds Number $R_{\delta_{kr}} = \frac{u_\delta\ \delta_{kr}}{\nu}$ and computing the laminar boundary layer, starting from the nose, by the laws determined by the two-term formula for the velocity distribution (reference 13), it is easy to determine $s_o = t$, which is the lower limit of integration of expression (29). If the hypothesis of conservation of momentum at the transition point is assumed, we should obtain:

$$\delta_{turb} \approx 1.4\ \delta_{lam}$$

In view of the fact, however, that $R_{\delta_{kr}}$ oscillates within a very wide range and depends entirely on the degree of turbulence of the flow, the above operation for finding $s_o$ can be carried out only with great difficulty.

On the other hand, as tests have shown, to neglect the laminar portion - i.e., to assume that the turbulent layer begins at the nose itself, has a negligible effect on the final result and this is particularly true for full-scale bodies, for which the relative length of the laminar portion is negligibly small.

Assuming therefore that $s_o = 0$, we shall have at the nose $\delta = 0$ and $u_{\delta_o} = V\sqrt{f_o} = 0$; whence from (22):

$$z_o\ e^{z_o - \frac{K}{n}\delta u_\delta} = 0$$

that is, $z_o = 0$. We likewise find:

$$\sqrt{f_o}=0;\ \ln\sqrt{f_o}=-\infty;\ r_o=0;\ \ln r_o=-\infty;\ 2z_o+\ln(\sqrt{f_o}\ \cos\theta_o)=-\infty$$

The initial value $\cos\theta_o$ depends on the shape of the nose of the hull. The tangent to the generating line of the figure of revolution is generally perpendicular to its axis, i.e., $\cos\theta_o = 0$. The integral on the left-hand side of equation (29) may be integrated. Integrating between the limits $z = 0$ to $z = z$, we obtain, denoting the integral by $\varphi_4$,

$$\varphi_4 = \int_0^z \varphi_1\, dz = \int_0^z (Az^2 - Bz + B)\, e^z\, dz =$$

$$= e^z[Az^2-(2A+B)z+2(A+B)]-2(A+B) \qquad (30)$$

Substituting the initial values and the integral obtained into our equation (29), we find

$$\varphi_4 = k\int_0^s \sqrt{f}ds - \int_{-\infty}^{\ln\sqrt{f}} \varphi_2 d \ln\sqrt{f} - \int_{-\infty}^{\ln r} \varphi_3 d \ln r -$$

$$- \frac{aK^2}{2k}\int_{-\infty}^{2z+\ln(\sqrt{f}\cos\theta)} \frac{\cos\theta}{r\sqrt{f}} z^2 e^{2z} d\left[2z+\ln(\sqrt{f}\cos\theta)\right] + \frac{K^2}{2k}\int_{-\infty}^{z(2b-a)+b\ln\cos\theta} \frac{\cos\theta}{r\sqrt{f}} z^2 e^{2z} d$$

$$\left[z(2b-a) + b\ln\cos\theta\right] \qquad (31)$$

The equation written in the above form is very suitable for performing a numerical integration by the method of successive approximations. The integration procedure we have proposed and successfully applied in practice will now be described.

First of all we consider the "zero approximation" $z = f(s)$. This zero approximation may be obtained by constructing $\delta = f(s)$ from the equation of K. K. Fediaevsky and solving the transcendental equation:

$$z\ e^z = \frac{KV}{n}\sqrt{f}\ \delta = \frac{K\ Re\ C_2\ \sqrt{2}}{L}\sqrt{f}\ \delta$$

Having the curve $z = f(s)$, the zero approximation, we plot the functions under the integral signs on the right-hand side against the corresponding expressions under the differential sign and perform the graphical integration up to the points corresponding to the chosen values of $s$ and thus compute the value of $\varphi_4$ from formula (31). With these values of $\varphi_4$, we find the next approximation for $z$ and repeat the entire operation - the process being continued until values of $z$ are obtained that differ very little from the preceding values. The latter values of $z$ give the integral curve. Generally an agreement to the second decimal place (which is entirely sufficient) is found after repeating the process three to four times.

The integration may also be performed not with respect to $s$ but with respect to $x$, for which purpose there should then be substituted in expression (31), $ds = dx/\cos\theta$. The above integrating procedure has the advantage in that graphical differentiation is entirely dispensed with (except for finding $\cos\theta = f(s)$ in the case where the shape of the generating line of the hull is not given analytically and where the possible errors have a very small effect on the final result due to the smallness of the last two terms in comparison with the preceding terms). This cannot be said of the method proposed by Moore who, neglecting the term $y \cos \theta$ in the expression $r' = r + y \cos \theta$ (this corresponds to the vanishing of the last two terms of equation (31)) succeeds in combining the integrals

$$\int_{-\infty}^{\ln \sqrt{f}} \varphi_2 \, d \ln \sqrt{f} \quad \text{and} \quad \int_{-\infty}^{\ln r} \varphi_3 \, d \ln r$$

into one by introducing a certain differential operator and making it necessary to perform graphical differentiation. In view of the fact that graphical differentiation is much more difficult than integration, we believe ourselves justified in saying that our method is superior to that of Moore.

The actual computation was simplified by: 1) determining the functions $\varphi_2$, $\varphi_3$, etc., not from tables but by making use of functional scales of such length that $z$ could be read off to an accuracy of three decimal places (these scales are given in the appendix); 2) performing the graphical integration with the aid of the integraph manufactured by the Coradi firm of Zurich, and which at once permitted the determination of the ordinates of the integral curve. The actual computation was thus reduced to a minimum.

The computations carried out using the complete equation (31) showed that the terms $\ln(\sqrt{f} \cos \theta)$ and $b \ln \cos \theta$ under the differential signs in the last two terms are very small by comparison with the terms $2z$ and $(2b - a)z$ so that they may be freely discarded.[5] This is particularly true at large Reynolds Numbers Re, in which case $z$ strongly increases. We may then combine the last two integrals into one:

---

[5] This is true except near the nose and the tail where $\ln \sqrt{f} \cos \theta$ approaches $-\infty$ which fact, however, only slightly affects the results.

$$\frac{aK^2}{2k}\int_{-\infty}^{2z+\ln(\sqrt{f}\cos\theta)} \frac{\cos\theta}{r\sqrt{f}} \quad z^3\, e^{2z}\, d\left[2z+\ln(\sqrt{f}\cos\theta) +\right.$$

$$+ \quad \frac{K^2}{2k}\int_{-\infty}^{z(2b-a)+b\ln\cos\theta} \frac{\cos\theta}{r\sqrt{f}} \qquad z^2 e^{2z} d[z(2b-a)+b\ln\cos\theta] \approx -$$

$$-\frac{aK^2}{2k}\int_0^z \frac{\cos\theta}{r\sqrt{f}}\, 2z^3 e^{2z} dz + \frac{aK^2}{2k}\int_0^z \frac{\cos\theta}{r\sqrt{f}}\, z^2 e^{2z}\left(\frac{2b}{a} - 1\right) dz =$$

$$= -\int_0^z \frac{aK^2\cos\theta}{2kr\sqrt{f}}\, e^{2z}\left[2z^3 - z^2\left(\frac{2b}{a} - 1\right)\right] dz$$

Here, too, as was done previously, we introduce for convenience in computation, the function $e^{2z}\left[2z^3 - z^2\left(\frac{2b}{a} - 1\right)\right]$ under the differential sign. We thus obtain:

$$-\int_0^z \left(\frac{a\cos\theta}{2kr\sqrt{f}}\right) e^{2z}\left[2z^3 - z^2\left(\frac{2b}{a} - 1\right)\right] dz =$$

$$= -\int_{-\frac{1}{2}\left(1+\frac{b}{a}\right)}^{e^{2z}\left[z^3-\left(1+\frac{b}{a}\right)\left(z^2-z+\frac{1}{2}\right)\right]} \left(\frac{a\cos\theta}{2kr\sqrt{f}}\right) d\left\{e^{2z}\left[z^3 - \left(1 + \frac{b}{a}\right)\left(z^2 - z + \frac{1}{2}\right)\right]\right\} =$$

$$= -\int_{\varphi_{5_0}}^{\varphi_5} \frac{a\cos\theta}{2kr\sqrt{f}}\, d\varphi_5$$

where we have set:

$$\varphi_5 = e^{2z}\left[z^3 - \left(1 + \frac{b}{a}\right)\left(z^2 - z + \frac{1}{2}\right)\right]$$

and the initial value:

$$\varphi_{5_0} = (\varphi_5)_{z=0} = -\left(1 - \frac{b}{a}\right)$$

The final equation for the determination of $z = f(s)$ will thus be:

$$\varphi_4 = k\int_0^s \sqrt{f}\,ds - \int_{-\infty}^{\ln\sqrt{f}} \varphi_2\, d\ln\sqrt{f} - \int_{-\infty}^{\ln r} \varphi_3\, d\ln r - \int_{\varphi_{5_0}}^{\varphi_5} \frac{a\cos\theta}{2kr\sqrt{f}}\, d\varphi_5 \tag{32}$$

The above expression is the fundamental working formula in the most convenient form.

Having the relation $z = f(s)$ the remaining functions of interest to us may readily be found. Thus, according to (22), the thickness of the boundary layer is expressed by

$$\delta = \frac{n}{K}\frac{ze^z}{u_\delta} = \frac{L}{Re\, C_2\sqrt{2}\, K}\frac{ze^z}{\sqrt{f}} \tag{33}$$

The shearing stress distribution is found with the aid of the expression

$$v_* = \frac{K\, u_\delta}{z}$$

whence

$$\tau_0 = \frac{\rho\, K^2\, V^2\, f}{z^2} \tag{34}$$

The ratio of the shearing stress to the dynamic impact at infinity is:

$$\frac{\tau_0}{\frac{1}{2}\rho\, V^2} = \frac{2K^2 f}{z^2} \tag{35}$$

The coefficient of total frictional drag for the entire hull based on its volume is

$$C_u = \frac{2X''}{\rho\, U^{2/3}\, V^2}$$

where $X''$ is the resultant of the frictional forces, and U, the volume of the body, is found by integrating the shearing stress distribution. We have:

$$X'' = \int_0^{s_L} \tau_0 \cos\theta \; 2\pi\, r\, ds = 2\pi\, \rho\, K^2 V^2 \int_0^{s_L} \frac{rf \cos\theta}{z^2}\, ds \qquad (36)$$

where $s_L$ is the length of the generating line of the figure of revolution and thus $C_u$ is given by

$$C_u = \frac{4\pi K^2}{U^{2/3}} \int_0^{s_L} \frac{rf \cos\theta}{z^2}\, ds \qquad (37)$$

We preferred to integrate with respect to $s$ rather than with respect to $x$ as being more convenient since we have the relation $z = f(s)$. If we had performed the integration with respect to $x$ from the very beginning, then it would have been convenient to substitute also in expression (37) $\cos\theta\, ds = dx$.

In concluding this first part, we may observe that if we had made use of the Nikuradse-Prandtl formula (6") in place of the Kármán formula (12) as the fundamental velocity distribution law, the computations all would have come out the same except that in place of the coefficients:

$$A = 5/6; \quad B = 14/9; \quad a = 23/60; \quad b = 601/1800$$

we should have obtained:

$$A' = 1; \quad B' = 2; \quad a' = \tfrac{1}{2}; \quad b' = \tfrac{1}{2}$$

# PART II

## 1. COMPUTATIONS FROM THE EQUATION DERIVED IN PART I

We decided to apply the computations from the equation derived in Part I to the hull of the airship "Akron" so as to compare the results with those obtained from the excellent tests for the determination of the boundary-layer thickness and shearing stress distribution conducted by Freeman (reference 6) on a 1/40-scale model at a mean Reynolds Number $Re = 15.88 \times 10^6$.

The computation program included the determination of the boundary-layer thickness, the shearing stress or skin friction per unit area distribution, and the coefficient of frictional drag at four Reynolds Numbers:

$15.88 \times 10^6$, $79.4 \times 10^6$, $251.0 \times 10^6$, and $684 \times 10^6$

The first number corresponds to the tests of Freeman, and the others were selected for the purpose of making an easy comparison of the results obtained with those given by K. K. Fediaevsky (reference 14) who, using the power law for the velocity distribution in the boundary layer, assumed that to the above Reynolds Numbers there corresponded the exponents 1/7, 1/8, 1/9, and 1/10 in the formula for the velocity distribution, and compared his results with those given by the tests of Freeman. Thus, considering the logarithmic law as more nearly expressing the actual velocity distribution than the power law, it was possible indirectly to check the correctness of the relation $n = f(Re)$ given by K. K. Fediaevsky.

The values of the universal constant $K$ and the number $C_2$ were taken as $K = 0.392$ $C_2 = 7.375$ which correspond to the form for the local friction formula

$$\frac{1}{\sqrt{C_f}} = 3.6 + 4.15 \log_{10} (R_\delta \sqrt{C_f})$$

The constants in the above formula are those of Kármán and are in very good agreement with experiment.

The relation $z = f(s)$ was found by solving the fundamental equation in the form (32) by the method of successive approximations, the data of K. K. Fediaevsky being assumed as the first approximation. The operation was continued until the difference in the values of $z$ given by two successive approximations differed only in the third decimal place, this corresponding to an accuracy in the determination of $z$ of 0.08 to 0.13 percent. This degree of accuracy, which may appear as too great, resulted from the fact that, in the first place, we decided to use great accuracy for the first calculations so as to have confidence in the results; and secondly, we wished to determine the boundary-layer thickness $\delta$ with the greatest possible precision. The values of $\delta$ change very sharply with a slight change in $z$ since they depend on the very "sensitive" function $ze^z$ (33). Thus the maximum possible errors in the determination of $z$ give errors in $\delta$ of 1.3 to 1.1 percent. Generally the required accuracy in the computation of $z$ was obtained after four or five approximations. The detailed procedure in carrying out the computations (for practical application) is given in a supplement.

## 2. DISCLOSURE OF DISCREPANCIES IN THE RESULTS OBTAINED FOR THE THICKNESS OF THE BOUNDARY LAYER

Having obtained the relation $z = f(s)$, we constructed a diagram $\delta = f(s)$, figure 2 (topmost curve). Expecting to obtain with the logarithmic law much better agreement with experiment than with the power law, we were very much surprised to obtain large discrepancies. For the entire length of hull the values of $\delta$ computed by our present theory came out approximately 50 percent greater than the corresponding value computed by the seventh-power law and the test data, the values of $\delta$ from the seventh-power law giving good agreement with the test results as stated by Freeman and as shown in figure 2.

Our greatest surprise was the very large deviation of results from those obtained on the basis of the seventh-power law. It would have been natural to consider what the computation results would be upon application of the velocity-distribution law of Prandtl-Nikuradse:

$$\frac{u - u_\delta}{\sqrt{\frac{\tau_o}{\rho}}} = - \frac{1}{K} \ln \frac{\delta}{y}$$

Since, however, using the above formula would have involved a large amount of computation work and taken up too much time, we decided to make the comparison of the computation results for $\delta = f(s)$ from the Kármán-Nikuradse formula and the seventh-power law, respectively, for a flat plate for which the integrals may be directly evaluated. There was every reason to expect that the relation between the positions of the curves $\delta = f(x)$ for the flat plate, according to the different laws, would be of the same character as obtained for the figure of revolution. For the flat plate the data chosen were: length $L = 5.99$ m, velocity $V = 38.45$ m/s, corresponding to a Reynolds Number $Re = 15.88$ m/s - that is, data approaching those of the "Akron". The computations were conducted according to the formulas obtained from the fundamental equation (26) after integrating, setting $r = \infty$ and $f = 1$.

With the velocity-distribution law of Kármán,

$$\int_0^z \varphi_1 dz = \varphi_4 = e^z [Az^2 - (2A+B)z + 2(A+B)] - 2(A+B) = \frac{K^3 C_2 \sqrt{2}}{L} Re\, x \tag{38}$$

and with that of Nikuradse-Prandtl:

$$\int_0^z \varphi'_1 dz = \varphi'_4 = e^z [z^2 - 4z + 6] - 6 = \frac{K^3 C_2 \sqrt{2}}{L} Re\, x \tag{39}$$

whence
$$\delta = \frac{L}{Re\, K\, C_2 \sqrt{2}} z e^z$$

and from the known formula for $\delta$, according to the seventh-power law (reference 13),

$$\delta = 0.37 L^{1/5} Re^{-1/5} x^{4/5} \tag{40}$$

The values of $C_2$ and $K$ were taken to be the same as for the "Akron".

The results are shown on figure 3. As may be seen, both logarithmic laws give values of $\delta$ much larger than are given by the seventh-power law, the Nikuradse-Prandtl formula giving results nearer to the seventh-power law. The attempt to vary the value of $C_2$ (Moore takes $C_2 = 10.84$) gave no particular results as may be seen from the figure. It was thus possible to conclude from the above that the reason for the large disagreement in the results of the present theory with the seventh-power law and hence also with the test results[6)] was to be looked for in the theory itself. Very recently we received the work of Moore (reference 7) where, in his figure 2, which we reproduce in our figure 4, we see a further confirmation of our conclusion. Here the results of the computation of $\delta$ for the "N.P.L. long model" at $R_u = \frac{VU^{1/3}}{\nu} = 10^8$ (where $U$ is the volume of the model) according to the theory in which the Nikuradse-Prandtl formula (6") is applied (denoted by $\delta_{z2}$) give the same order of discrepancies from the seventh-power law (denoted by $\delta_7$) as in our case. The results of Moore (fig. 5) on the "Akron" where we see a much better agreement with the tests and the seventh-power law than was obtained by us, therefore appear all the more strange. Since the identical theory was applied to the two models, there should have been obtained a discrepancy of the same order and this evidently was not so, as may be seen on comparing figures 4 and 5.

## 3. EXPLANATION OF THE DISCREPANCIES OBTAINED

Bearing in mind all that was said above, we decided to consider somewhat more carefully the reason for the unsatisfactory results obtained by our theory. Up to the present we have considered the agreement of the results with the experiment applying the theory of Kármán in its original form. We shall now consider the velocity distribution itself. We assumed at first that the velocity distribution in the boundary layer is computed by the logarithmic law:

$$\frac{u - u_\delta}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{1}{K}\left[\ln\left(1 - \sqrt{1 - \frac{y}{\delta}}\right) + \sqrt{1 - \frac{y}{\delta}}\right]$$

and the local friction by

$$\frac{1}{\sqrt{C_f}} = a + b \log\left(R_\delta \sqrt{C_f}\right)$$

---

6) Kempf (reference 10) has shown that for a flat plate the seventh-power law agrees very well with test results.

We shall now attempt, using the excellent tests of Freeman, to check the correctness of these laws. We have at our disposal the tables of $u/u_\delta = f(y)$ for the various sections of the hull (placed at the end of the report of Freeman) and also the experimental curve of shearing stress distribution (the curve $C = \tau_o/q_o$ on fig. 8 of Freeman's report). The change-over in the computation from the values $u/u_\delta$ to the values $\dfrac{u - u_\delta}{\sqrt{\dfrac{\tau_o}{\rho}}}$ is easily effected thus:

$$\frac{u - u_\delta}{\sqrt{\dfrac{\tau_o}{\rho}}} = - \frac{\sqrt{f}\left(1 - \dfrac{u}{u_\delta}\right)\sqrt{2}}{\sqrt{\dfrac{\tau_o}{q_o}}}$$

where $$q_o = \frac{1}{2} \rho V^2$$

If the formula for the velocity distribution is correct, then by drawing the test curves, using as coordinates $\dfrac{u - u_\delta}{\sqrt{\dfrac{\tau_o}{\rho}}}$ and $\ln\left(1 - \sqrt{1 - \dfrac{y}{\delta}}\right) + \sqrt{1 - \dfrac{y}{\delta}}^{\,2}$ we should obtain a grouping of the points about a straight line passing through the origin of coordinates and having a slope determined by the magnitude of the universal constant K. We pointed out at the beginning that the current method of determining K was by tests on the drag. For this reason the method proposed by us for the determination of K from the velocity distribution appears to us to be original and, as we shall see below, leads to results of extreme importance.

If we consider further the formula for the local frictional intensity distribution as true, then we should obtain a straight line on constructing the friction curve using as coordinates:

$$\frac{1}{\sqrt{C_f}} = \frac{\sqrt{f}}{\sqrt{\dfrac{\tau_o}{q_o}}}$$

and $$\log (R_\delta \sqrt{C_f}) = \log \left(\frac{V\delta}{\nu} \sqrt{\frac{\tau_o}{q_o}}\right)$$

and having a slope that determines the coefficient b and hence also the same universal constant K.

Figures 6 and 7 show the above curves thus plotted. The points $\dfrac{u - u_\delta}{\sqrt{\dfrac{\tau_o}{\rho}}}$ do not lie on one straight line but arrange themselves along some curve showing that law (6') is in general not strictly true as, of course, is to be expected for the regions near the wall (at the right of the diagrams) where the viscosity has not been taken into account and where the curve very appreciably slopes downward. In the region, however, sufficiently far removed from the wall, it appears possible to draw a straight line through the points, which fact shows that in this region the universal law is valid.

The result of the computation of the universal constant from the slope of this straight line led to the value[7] K = 0.214 in place of the value 0.4 generally assumed. Considering now, figure 7, where out of a very small number of test points many dropped out (in the region from 9 to 12 feet from the nose, according to Freeman (x/L from 0.4 to 0.6), where the curve $\tau_o/q_o = f(x)$ takes an improbable downward slope). Five of the points, however, aline themselves sufficiently well on a single straight line, from whose slope and intercept the values a = 3.4, b = 3.77 are obtained, in very good agreement with the values assumed by us in the computations, namely: a = 3.6, b = 4.15 and from which the value K = 0.389, which is generally accepted, is obtained.

Having obtained these results from the tests on the "Akron", it was natural to reconsider the classical tests of Nikuradse on pipes, in the light of these results these tests being the only generally accepted source for furnishing the universal constants. We made use of the data of table VII of Nikuradse on smooth pipes (reference 3). The plot is given on figure 8. Contrary to what was expected, in the region sufficiently far removed from the

---

[7]The curve corresponding to K = 0.392 is given on the diagram.

wall and for sufficiently large Reynolds Numbers, the same smaller value of $K$, namely, $K = 0.306$, that is, much less than what is currently assumed, was again obtained also in this case. From the same test data, but applied to finding the drag coefficient, values of $K$ were obtained near 0.4 (the corresponding straight velocity-distribution lines are shown on fig. 8). We are thus led to the conclusion that the "universal constant" by no means appears to be universal. In all cases of tests on the velocity distribution the constant comes out much less than in the tests on the drag.

Besides the tests of Nikuradse, we have also employed the data of Hansen (reference 11) on a flat plate. In this case an unusually low value of $K = 0.175$ was obtained.

It was also of interest, naturally, to work over the tests of Freeman, Nikuradse, and Hansen in terms of $\ln \delta/y$, that is, to check the Nikuradse-Prandtl formula. The curves have a form analogous to those just analyzed, and the values of the constant in the region far removed from the wall, according to the tests of Freeman, Nikuradse, and Hansen are, respectively, 0.267, 0.360, and 0.22. An excellent explanation of this phenomenon after it was disclosed, has been furnished by Professor L. G. Loitsansky, who states that the reason for the considerable variation in $K$ is the result of neglecting the effect of the viscosity in the theory. Actually, the application of the theory of similitude gives a logarithmic law only under the conditions where the effect of viscosity is negligible. In those regions far removed from the wall, where the effect of viscosity is negligible, the points follow the logarithmic law with sufficient accuracy with corresponding actual value of $K$ (0.2 - 0.3). As the wall is approached, the effect of the viscosity increases and the points begin to deviate from the straight line in such a manner that the derivative

$$\frac{d\,\dfrac{u - u_\delta}{v_*}}{d\left[\ln\left(1 - \sqrt{1 - \dfrac{y}{\delta}}\right) + \sqrt{1 - \dfrac{y}{\delta}}\right]} = \frac{1}{K}$$

decreases continuously (corresponding to a continuous increase in $K$) and in the regions near the wall the inclination of the curve (as may be seen on the diagrams) corresponds to a value of $K \simeq 0.4$.

The latter circumstance is particularly noteworthy. According to the classical theory of Kármán, the frictional drag law is obtained from that of the velocity distribution by applying the latter to the conditions on the boundary of the laminar sublayer (which process, of course, is formally unjustified since the effect of the viscosity is still very appreciable) - i.e., to the region where the value $K \approx 0.4$ should be taken. Since the value of $K$ has, up to the present, been determined exclusively from tests on drag, the value always obtained was near 0.4, i.e., this value represented conditions at the boundary of the laminar sublayer in the region very near the wall. This result was likewise obtained by us in figure 7.

There was thus obtained a paradox: The usual method of the determination of $K$ by studying the variation of the resistance of pipes with Reynolds Number gave values of $K$ for a region for which the logarithmic law, strictly speaking, was not applicable and the region where the law was applied - that is, at some distance from the wall - was not assigned the corresponding value of $K$ which, as we now see, is much less than the generally accepted one.

## 4. IMPROVEMENT OF THE LOGARITHMIC THEORY

From all that has been said above, the following practical conclusion may be derived. Since we do not at the present time possess any velocity-distribution law that takes into account the effect of viscosity, we are constrained to use the logarithmic law of Kármán, rendering it more accurate by assuming for the region far removed from the wall a value of $K \approx 0.2$, and near the wall $K \approx 0.4$. We cannot, however, determine definitely just where the value of $K$ undergoes a discontinuity (corresponding to a hypothetical limit where the viscosity ceases to have any effect).[8] Moreover, the breaking up of the interval $y/\delta$ from 0 to 1 into two parts would introduce complications in the computation. We therefore decided that, with very

---

[8] It is interesting that, according to the form of the distribution of the points $u-u_\delta/v_*$ for the "Akron", a sharp discontinuity is observed at the value $y/\delta = 0.19$ (fig. 6).

little error, we may arrive at the following compromise: for the velocity-distribution formulas to consider for the whole region $K_1 = 0.214$, and for the resistance formula $K = 0.392$. There is no doubt that the value $K_1$ is affected also by the not entirely correct procedure in passing from the internal to the external problem. It is quite possible that for other cases of the external problem different values of $K_1$ would be obtained. The only thing that may be affirmed with any certainty is, that in every case $K_1$ will be less than $K$. This is confirmed in all three cases of flow considered here.

Having assumed that the reason for the very large deviations in the computed values of $\delta$ from the test values is the incorrect choice of $K$ in the formula for the velocity distribution, we decided to reconsider the theory on the basis of the above compromise assumption that leads to a choice of two values for $K$.

Assuming the value $K_1 = 0.214$, we obtain the following expressions for the integrals

$$\int_0^\delta u\,dy,\quad \int_0^\delta u^2\,dy,\quad \int_0^\delta uy\,dy,\quad \int_0^\delta u^2 y\,dy$$

$$\left.\begin{aligned}
\int_0^\delta u\,dy &= \delta\left(u_\delta - A\frac{v_*}{K_1}\right) \frac{n}{K}\, e^z\left(z - A\frac{K}{K_1}\right)\\
\int_0^\delta u^2\,dy &= \delta\left(u_\delta^2 - 2A\frac{u_\delta v_*}{K_1} + B\frac{{v_*}^2}{{K_1}^2}\right) = \frac{n}{K}\, u_\delta\, e^z\left(z - 2A\frac{K}{K_1} + B\frac{K^2}{{K_1}^2}\frac{1}{z}\right)\\
\int_0^\delta uy\,dy &= \frac{1}{2}\,\delta^2\left(u_\delta - a\frac{v_*}{K_1}\right) = \frac{n^2}{2K^2}\,\frac{e^{2z}}{u_\delta}\left(z^2 - a\frac{K}{K_1}\,z\right)\\
\int_0^\delta u^2 y\,dy &= \frac{1}{2}\,\delta^2\left(u_\delta^2 - 2a\frac{u_\delta v_*}{K_1} + b\frac{{v_*}^2}{{K_1}^2}\right) = \frac{n^2}{2K^2}\, e^{2z}\\
&\qquad\left(z^2 - 2a\frac{K}{K_1}\,z + b\frac{K^2}{{K_1}^2}\right)
\end{aligned}\right\}\quad(41)$$

i.e., we finally, after substituting $\delta$ from expression (22), obtained different coefficients in place of A, B, a, and b, as follows:

$$A_1 = \frac{AK}{K_1}, \quad B_1 = \frac{BK^2}{K_1^{\,2}}, \quad a_1 = \frac{aK}{K_1}, \quad b_1 = \frac{bK^2}{K_1^{\,2}}$$

The functions $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$, and $\varphi_5$ in the fundamental equation will then have different coefficients: Denoting the new functions by $\varphi_{1_1}$, $\varphi_{2_1}$, $\varphi_{3_1}$, $\varphi_{4_1}$, and $\varphi_{5_1}$ we obtain:

$$\left.\begin{aligned}
\varphi_{1_1} &= e^z (A_1 z^2 - B_1 z + B_1) = e^z \left(A \frac{K}{K_1} z^2 - B \frac{K^2}{K_1^{\,2}} z + B \frac{K^2}{K_1^{\,2}}\right) \\
\varphi_{2_1} &= e^z (2A_1 z^2 - B_1 z) = e^z \left(2A \frac{K}{K_1} z^2 - B \frac{K^2}{K_1^{\,2}} z\right) \\
\varphi_{3_1} &= e^z (A_1 z^2 - B_1 z) = e^z \left(2A \frac{K}{K_1} z^2 - B \frac{K^2}{K_1^{\,2}} z\right) \\
\varphi_{4_1} &= e^z \; A_1 z^2 - (2A_1 + B_1) z + 2(A_1 + B_1) \; - 2(A_1 + B_1) = e^z \left[A \frac{K}{K_1} z^2 - \right. \\
&\quad \left. - \left(2A + B \frac{K}{K_1}\right) \frac{K}{K_1} z + 2 \left(A + B \frac{K}{K_1}\right) \frac{K}{K_1}\right] - 2 \left(A + B \frac{K}{K_1}\right) \frac{K}{K_1} \\
\varphi_{5_1} &= e^{2z} \left[z^3 - \left(1 + \frac{b_1}{a_1}\right) \left(z^2 - z + \frac{1}{2}\right)\right] = \\
&= e^{2z} \left[z^3 - \left(1 + \frac{bK}{aK_1}\right) \left(z^2 - z + \frac{1}{2}\right)\right]
\end{aligned}\right\} \quad (42)$$

The final equation, analogous to (32), for the determination of $z = f(s)$, applying the law with two values of K, will be:

$$\begin{aligned}
\varphi_{4_1} &= \int_0^z \varphi_{1_1} \, dz = K^3 C_2 \sqrt{2} \frac{Re}{L} \int_0^s \sqrt{f} \, ds - \int_{-\infty}^{\ln \sqrt{f}} \varphi_{2_1} \, d \ln \sqrt{f} - \\
&- \int_{-\infty}^{\ln r} \varphi_{3_1} \, d \ln r - \frac{La}{2K_1 C_2 \sqrt{2} \; Re} \int_{\varphi_{5_1}(z=0)}^{\varphi_{5_1}} \frac{\cos \theta}{r \sqrt{f}} \, d\varphi_{5_1}
\end{aligned} \quad (43)$$

where $\varphi_{5_1(z=0)}$ is the value of $\varphi_{5_1}$ at $z = 0$ and is equal to

$$\varphi_{5_1}(z=0) = \frac{1}{2}\left(\frac{bK}{aK} - 1\right)$$

Since in the above equation the functions $\varphi$ enter with different coefficients (in accordance with (42)), it would appear necessary in solving it, for us to recompute all our functional scales of which we made use for rapidity of calculation. This, in the first place, would have very much delayed the completion of the work, but what is more important, would not have been justified since the value $K_1 = 0.214$ has not yet been established as a universal constant, and on changing it the recomputed scales $\varphi_{2_1}$, $\varphi_{3_1}$, $\varphi_{4_1}$, and $\varphi_{5_1}$ would have become useless.

We therefore decided to reduce equation (43) to a form in which it may be used with the original functions $\varphi_2$, $\varphi_3$ $\varphi_4$, and $\varphi_5$ as may be done, as we shall see, by introducing in it one extra term.

We find the expressions $\varphi_{1_1}$, $\varphi_{2_1}$, $\varphi_{3_1}$, $\varphi_{4_1}$, from $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$. From (42) and (28) we obtain:

$$\varphi_{1_1} = e^z \frac{K}{K_1}\left(Az^2 - B\frac{K}{K_1}z + B\frac{K}{K_1}\right) = \frac{K^2}{K_1^2}\left[e^z(Az^2 - Bz + B) + Az^2 e^z\left(\frac{K_1}{K} - 1\right)\right] =$$

$$= \frac{K^2}{K_1^2}\left[\varphi_1 + Az^2 e^z\left(\frac{K_1}{K} - 1\right)\right]$$

and similarly for the remaining functions:

$$\varphi_{2_1} = \frac{K^2}{K_1^2}\left[\varphi_2 + 2A\, z^2 e^z\left(\frac{K_1}{K} - 1\right)\right]$$

$$\varphi_{3_1} = \frac{K^2}{K_1^2}\left[\varphi_3 + Az^2 e^z\left(\frac{K_1}{K} - 1\right)\right]$$

The function $\varphi_{4_1}$ we define as $\int_0^z \varphi_{1_1}\, dz$, that is:

$$\varphi_{4_1} = \int_0^z \varphi_{1_1}\, dz = \frac{K^2}{K_1^2}\left[\int_0^z \varphi_1\, dz + A\left(\frac{K_1}{K} - 1\right)\int_0^z e^z\, z^2\, dz\right] =$$

$$= \frac{K^2}{K_1^2}\left[\varphi_4 + A\left(\frac{K_1}{K} - 1\right)\int_0^z e^z\, z^2\, dz\right]$$

To obtain a suitable expression for the graphical integration, it is better not to compute the integral on the right-hand side of the above expression. With regard to the last member of the equation which takes account of the effect of the term $y \cos \theta$ we decided, instead of replacing $\varphi_{5_1}$ by $\varphi_5$, to compute the scale for $\varphi_{5_1}$ directly (setting $K_1 = 0.214$). At some other value of $K$ (for a different case) this would have a negligible change in the final results since, as we shall see below, the effect of the last term of equation (43) is in general very small.

Let us now substitute all the expressions obtained in (43) and split the integrals with respect to $\ln \sqrt{f}$ and $\ln r$ into two. We then obtain:

$$\frac{K^2}{K_1^2}\left[\varphi_4 + A\left(\frac{K_1}{K} - 1\right)\int_0^z e^z z^2\, dz\right] = K^3 C_2 \sqrt{2}\,\frac{L}{Re}\int_0^s \sqrt{f}\, ds -$$

$$- \frac{K^2}{K_1^2}\int_{-\infty}^{\ln\sqrt{f}} \varphi_2 d \ln \sqrt{f} - \frac{K^2}{K_1^2}\, 2A\left(\frac{K_1}{K} - 1\right)\int_{-\infty}^{\ln\sqrt{f}} e^z z^2 d \ln \sqrt{f} -$$

$$- \frac{K^2}{K_1^2}\int_{-\infty}^{\ln r} \varphi_3 d \ln r - \frac{K^2}{K_1^2}\, A\left(\frac{K_1}{K} - 1\right)\int_{-\infty}^{\ln r} e^z z^2 d \ln r -$$

$$- \frac{La}{2K_1\, C_2 \sqrt{2}\, Re}\int_{\varphi_{5_1}(z=o)}^{\varphi_{5_1}} \frac{\cos \theta}{r\sqrt{f}}\, d\varphi_{5_1}$$

Multiplying the entire expression by $K_1^2/K^2$, rearranging the terms and combining the integrals of the functions $e^z z^2$ into a single integral with a complex differential, we find:

$$\varphi_4 = KK_1^2 \, C_2 \sqrt{2} \, \frac{Re}{L} \int_0^s \sqrt{f} \, ds - \int_{-\infty}^{\ln \sqrt{f}} \varphi_2 \, d \ln \sqrt{f} - \int_{-\infty}^{\ln r} \varphi_3 \, d \ln r +$$

$$+ A \left(1 - \frac{K_1}{K}\right) \int_{-\infty}^{z+\ln rf} e^z z^2 \, d(z+\ln rf) -$$

$$- \frac{La_1 \; K_1}{2K^2 \, C_2 \sqrt{2} \; Re} \int_{\varphi_{5_1}(z=0)}^{\varphi_{5_1}} \frac{\cos \theta}{r \sqrt{f}} \, d\varphi_{5_1} \qquad (44)$$

We have thus obtained an expression that differs from (32) by one additional integral and has different constants before the first and last terms on the right-hand side. The presence of still another integral prolongs the computation somewhat (by about 1/5) but on the other hand the new expression, using the same scales of the functions $\varphi$ permits computation with any value of $K_1$.

The above method of introducing into the equation an extra term that takes into account the variation in the coefficients $\varphi$ is of advantage not only when the change is brought about by the equality of $K$ and $K_1$, but in other cases as well. In particular, as has been pointed out in Part I, the coefficients may vary as a result of the application of a different logarithmic formula for the velocity distribution. Let us put

$$A_1' = \alpha \frac{K}{K_1} A \qquad B_1' = \beta \frac{K^2}{K_1^2} B$$

Proceeding with the functions $\varphi$ as above, we obtain finally:

$$\varphi_4 = \frac{K^2 K_1 C_2 \sqrt{2}Re}{\beta L} \int_0^s \sqrt{f}\, ds - \int_{-\infty}^{\ln\sqrt{f}} \varphi_2\, d\, \ln\sqrt{f} - \int_{-\infty}^{\ln r} \varphi_3 d\, \ln r +$$

$$+ A\left(1 - \frac{\alpha K_1}{\beta K}\right) \int_{-\infty}^{z+\ln rf} e^z z^2\, d(z+\ln rf) -$$

$$- \frac{La'_1\; K_1}{2K^2\; \beta\; C_2 \sqrt{2}Re} \int_{\varphi'_{5_1}(z=0}^{\varphi'_{5_1}} \frac{\cos\theta}{r\sqrt{f}}\, d\varphi'_{5_1}$$

where the change of coefficients in $\varphi'_{5_1}$ may likewise be neglected.

The above expression is suitable for computation with our scales using any coefficients A and B. In particular, if it is desired to use the velocity-distribution formula:

$$\frac{u - u_\delta}{v_*} = - \frac{1}{K_1} \ln \frac{\delta}{y}$$

we must take:

$$\alpha = 1 : \frac{5}{6} = 1.2 \qquad \beta = 2 : \frac{14}{9} = 1.286$$

## 5. REPEATED COMPUTATIONS USING EQUATION (44)

The computations from equation (44) were carried out for the same Reynolds Numbers:

$$15.88 \times 10^6 \quad 79.4 \times 10^6 \quad 251 \times 10^6 \quad 684 \times 10^6$$

with and without the last term that takes into account $y \cos\theta$. The method of computation was the same combined graphic and analytical one, successive approximations in the values of $z$ being taken until a difference in the third decimal place only was obtained.

On figure 9 are shown plotted the changed values of $z$ for different Reynolds Numbers against the ratio $x/L$ along the "Akron" hull. As may be seen, the effect of taking account of the last term with $\cos\theta$ shows up, as also may be expected, only at the tail itself where the error in rejecting the term $y\cos\theta$ in the expression

$$r' = r + y\cos\theta$$

is a maximum as has also been pointed out by K. K. Fediaevsky.

The strange bends in the curves $z = f(x/L)$ on figure 9 in the region of the stern result from the fact that the function $z + \ln rf$, with respect to which the integration in the next to the last term was performed has a wavelike form in the region of the stern.

Having obtained $z = f(s)$, we plotted on figure 2 the changed thicknesses of the boundary layer $\delta = f(s)$ for $Re = 15.88 \times 10^6$. As may be seen, excellent agreement was obtained with the test results - much better than was given by the seventh-power law. Particularly noteworthy is the good agreement in the region of the stern $(s = 4 - 5\ m)$ due in large part to the fact that $y\cos\theta$ was taken into account as may be seen from a comparison of the curves $\delta = f(s)$ with and without the corresponding term accounted for.

A comparison of the shearing stress computed from the formula:

$$\frac{\tau_0}{\frac{1}{2}\rho V^2} = \frac{2K^2 f}{z^2}$$

according to the original solution $K = 0.392$, the "compromise" solution with $K = 0.392$ and $K_1 = 0.205$, and the power law, with the results of the Freeman tests, is given on figure 10.

We see that our compromise solution gives the best agreement with experiment. In the center portion, without considering the doubtful bend in Freeman's curve, our curve almost coincides with the test curve. In the tail region it may again be observed that taking into account $y\cos\theta$ gives a better agreement with experiment. As regards the very great discrepancy in the nose region, this phenomenon is unavoidable. In the first place, due to the fact that the hull contour curvature, which is considerable near the nose,[9] was neglected, the theory is not entirely true, and in the second place (and this is the more important reason) the accuracy of the experimental determination of the skin friction, due to the smallness of the boundary layer, is not very large.

From the results obtained above, we could plot the drag coefficient against the Reynolds Number. The value of the coefficient for each Reynolds Number is obtained by computing the expression:

$$C_u = \frac{4\pi \ K^2 \ L}{U^{2/3}} \int_0^1 \frac{rf}{z^2} \ d \ \frac{x}{L}$$

The results are plotted on figure 11, which shows the test curves of different models of the "Akron" and also the computed curves of Moore and K. K. Fediaevsky.[10]

We are justified in saying that our curve more closely represents actual conditions than the other theoretical curves since it agrees better with the results of the most reliable tests conducted on a metal model in the variable density tunnel. The form drag, i.e., the integral of the excess aerodynamic pressure along the hull was not taken into account since, according to repeated calculations (reference 8), this drag component appears to be very near zero. We may thus consider the coefficient $C_u$ as being that of the total drag.

Figure 11 also shows two points computed by our theory with $K = K_1 = 0.392$. They lie considerably below the curve with $K = 0.392$ and $K_1 = 0.214$ as should be expected. The curve of Moore lies between the latter curve and the points. This apparently shows that the velocity distribution of Nikuradse-Prandtl:

$$\frac{u - u_\delta}{\sqrt{\frac{\tau_0}{\rho}}} = - \frac{1}{K} \ln \frac{\delta}{y}$$

---

[9] In introducing the function $r = f(s)$, we introduce not the curvature of the contour but the law of variation of the surface at which the friction is developed.

[10] The curves are given as functions of the Reynolds Number based on the volume to the 1/3 power: $R_u = \frac{VU^{1/3}}{\nu}$.

for the condition $K = K_1 \approx 0.4$ corresponds more nearly to the truth than the Kármán formula under the same conditions. This is likewise confirmed by the fact that the plot of the points $\frac{u - u_\delta}{\sqrt{\frac{\tau_o}{\rho}}}$ against $\ln \frac{\delta}{y}$ gives a larger value for $K_1$ than $K_1 = 0.214$. Thus Moore, choosing the Nikuradse-Prandtl formula and not taking into account any difference between $K$ and $K_1$, committed a smaller error than we did in our first computations. The curve lying lowest is that computed by the method of K. K. Fediaevsky with $n$ = var.

Figure 12 shows a comparison of the variation of the relative thickness of the boundary layer with distance along the hull according to the present theory using $K = 0.392$ and $K_1 = 0.214$, with the computations of K. K. Fediaevsky. Although the curves agree sufficiently well for $Re = 15.88 \times 10^6$, the agreement is far less at the higher Reynolds Numbers. From this it may be concluded that the relation $n = f(Re)$ proposed by K. K. Fediaevsky, does not entirely correspond to an approximation to the logarithmic law. We found by integrating with respect to the curves $\delta = f(n)$ and averaging the results for different sections, a new relation $n = f(Re)$ (fig. 13), which should more nearly bring the results of the computations of $\delta$ by the power law closer to the analogous computations by the logarithmic law.

In conclusion, we may note that with very considerable accuracy, for more rapid computation of the relation $\frac{\tau_o}{q_o} = f\left(\frac{x}{L}\right)$ and $C_u = f(Re)$, a "mixed" method may be applied, computing the boundary-layer thickness by the power law (using the corrected curve $n = f(Re)$) and the shearing-stress distribution and drag coefficient by the logarithmic law. For this purpose, having obtained the values of $\delta$, the values of $z$ are next computed by solving equation (22) for $ze^z$:

$$ze^z = \frac{Re\ K\ C_2\ \sqrt{2}}{L}\ \delta\ \sqrt{f} \qquad (45)$$

after which $\tau_o/q_o$ and $C_u$ are computed by formulas (35) and (37). This method of computation gave curve $C_u = f(Re)$

lying only slightly above the more accurate curve. For practical purposes this method appears entirely acceptable.

## 6. CONCLUSIONS

The results of our investigation lead to the following conclusions:

a) The logarithmic law of velocity distribution of Kármán, applied to a figure of revolution and to a flat plate with the generally accepted value of $K \approx 0.4$, gives too sharp an increase in the boundary-layer thickness and too small values for the drag.

b) This is due to the fact that a much smaller value of the constant $K$ $(K_1 \approx 0.2 - 0.3)$ corresponds to that region where the logarithmic velocity-distribution law is actually applicable (i.e., where the effect of viscosity is negligible).

c) If the logarithmic law is applied to the region near the wall where the effect of the viscosity shows up strongly and where this law, strictly speaking, is not applicable, an error in principle is made which it is possible to compensate by taking $K \approx 0.4$.

d) Since a variable $K$ cannot be introduced, it is proposed that in the formula for the velocity distribution a value $K_1 \approx 0.2 - 0.3$ be taken, and in the formula for the drag, $K \approx 0.4$.

e) With the above values assumed for $K_1$ excellent agreement is obtained with experiment in all respects (thickness of the layer, shear stress distribution, total drag).

f) Further light on the significance of $K$ for the velocity-distribution law can be obtained only by more accurate tests on plates, wings, and figures of revolution.

g) The method proposed by us of employing two values of $K$ for the computation by the logarithmic law is of course a temporary expedient to be applied until such time as a more accurate law that takes into account the effect of viscosity - i.e., one correct for all regions of the flow, is substituted for the logarithmic law. Obtaining such a law is one of the most urgent of present-day problems.

## APPENDIX

### Some Practical Suggestions with Regard to the Computation Procedure

For carrying out the computation using the complete equation (44), we may here present some practical suggestions, gained from our experience, to those who wish to make use of our method.

(a) Choice of initial conditions.- It is best to begin the integration from the end of the laminary portion of the boundary layer. For determining the transition point from the laminary to the turbulent layer, it is necessary to make use of the critical Reynolds Number (based on the boundary-layer thickness) $R_\delta = \frac{u_\delta \delta}{\nu}$ and which lies within the range $1,600 < R_\delta < 10,000$. From the corresponding value of $R_\delta$, $\delta_{lam} \approx \frac{\nu R_\delta}{u_\delta}$ is determined; whence according to the transition hypothesis of Kármán (reference 13), $\delta_{turb} \approx 1.4\ \delta_{lam}$.

After this, proceeding to $z$ according to formula (45), the integration is begun, the integrals being taken from the section chosen and the corresponding initial value of the function $\varphi_4$ found.

As may be seen, however, the choice of the critical Reynolds Number is subject to a certain degree of indefiniteness. For this reason it is possible, with very little error, to take the lower limit of integration at the nose itself, setting $z = 0$ at $s = 0$; that is, consider the turbulent layer to start at the nose itself. This procedure will be more nearly justified the larger the Reynolds Number, $Re = VL/\nu$, taken. Here a new difficulty arises, however. With the initial conditions $z = 0$ when $s = 0$, the integrals with respect to $\ln \sqrt{f}$ and $\ln r$ must be evaluated from $-\infty$ (since when $s = 0$, $f = 0$ and $r = 0$) which, of course, cannot be done graphically. However, as computations have shown, the values of these integrals contribute very little near the nose on account of the "weakness" of the logarithmic infinity. For this reason the lower limits of these integrals may, with very little error, be taken at any point sufficiently near the nose (for example, the point corresponding to $\ln \sqrt{f} \approx -3.0$) neglecting the integration up to this point. The integration may also be begun from any point sufficiently near the nose, having determined the variation of $\delta$ from the nose up to the point by the method of K. K. Fediaevsky, with $n$ = var. This is, in fact, the most accurate method of integration assuming $z = 0$ when $s = 0$.

(b) Choice of the first approximation.- As a first approximation, the relation between $z$ and $s$ obtained from the boundary distribution by formula (45) according to the method of K. K. Fediaevsky, may be taken, as was done by us. It is also possible to take as the first approximation the solution of equation (44), neglecting all the integrals on the right-hand side except the first, i.e.,

$$\varphi_4' = K\ K_1^2\ C_2 \sqrt{2}\ \frac{Re}{L} \int_0^s \sqrt{f}\ ds$$

as was done by Moore. Having obtained the value of $\varphi_4'$, $z'$ is found by using the corresponding scale.

(c) Obtaining the succeeding approximations.- After obtaining the relation $z = f(s)$ from some one or other first approximation, we find the values of the functions $\varphi_2$, $\varphi_3$, $\varphi_{5_1}$, plot the functions to be integrated against the corresponding expressions under the differential sign, i.e.:

$$\begin{aligned} \varphi_2 &= f(\ln \sqrt{f}) & \varphi_3 &= f_1(\ln r) \\ e^z z^2 &= f_2(z + \ln rf) \quad \text{and} & \frac{\cos\theta}{r\sqrt{f}} &= f_3(\varphi_{5_1}) \end{aligned} \tag{46}$$

and integrate the curves graphically between the limits chosen on the meridian line $s$. Having obtained the values of the integrals, we substitute them in expression (44), determine a new value $\varphi_4''$ and a new value $z''$ and repeat the operation until two successive values of $z$ differ by as little as required for the accuracy desired.

In performing the integration outright along the entire length of the meridian, however, the following difficulties are met with. The functions $\ln \sqrt{f}$ and $\ln r$

generally vary sharply near the nose and tail ($\sqrt{f} \longrightarrow 0$, $r \longrightarrow 0$), whereas in the center portion the change is very small. The $\varphi$ functions, on the contrary, vary much more sharply at the center portion than near the nose. For this reason it is quite impossible to choose for the entire length of hull the same scales for plotting (46) that give more or less the same accuracy in the determination of $z$ in the different regions. It is therefore suggested that the distance $s$ be broken up into three intervals[11] and the integration performed for each of these separately. It may appear that this device would prolong the computation. The computation, however, will, on the contrary, be shortened while the accuracy will at the same time be increased. The shortening of the amount of labor will be accomplished in the following manner:

Having integrated over the first interval and finally obtained the real value $z_1$ at the end of the interval, the values $z^{(')}$ that were obtained from the first approximation are corrected in the second interval by multiplying them by the ratio $z_1/z^{(')}$, that is, by the ratio of the true value at the end of the first interval to the value of the first approximation at the same point. We thus obtain at once for the second interval a better approximation to the actual curve $z = f(s)$. Computations have shown that by such a procedure good values of $z$ were obtained for the second interval after two or three approximations, while for the first interval it was necessary to take five or six approximations. The same method is applied in passing from the second to the third interval. A better result may also be obtained in passing from one interval to another by multiplying the function $e^z$ or $\varphi_4$ corresponding to $z$ of the first approximation by the ratios of the values of these functions at the end of the first interval.

To give an idea of how the plot of (46) looks for the complete graphical integration, we present figures 14, 15, and 16, showing these plots for the first, second, and third intervals of the model of the "Akron" at a Reynolds Number $Re = 15.88 \times 10^6$ and corresponding to the data of the last approximation $z = f(s)$. The points are numbered

---

[11] We divided the length of the "Akron" into three portions, thus: the first from $x/L = 0$ to $x/L = 0.1$; the second from $x/L = 0.1$ to $x/L = 0.7$; and the third from $x/L = 0.7$ to $x/L = 1.0$.

according to the numbers of the sections chosen. The table (p. 56) gives for each of the numbered points the distance from the nose along the hull and the values of f, r, and cos θ.

The graphical integration of the curves of a given type was performed using the integraph of the Coradi firm of Zurich - an apparatus constructed on the principle of Abdank-Abakanovitch and which by following along the curve, gave the area under the curve between the initial point and any point on the curve. If the above apparatus is not available the usual planimeter may be used or the area found by summing up the squares under the curve. For convenience in computing by our method, we give in the appendix the functional scales of $\varphi$ as functions of z.

Our scales, however, do give not the functions $\varphi_2$, $\varphi_3$, $\varphi_4$, $\varphi_{5_1}$, but these functions divided by $K = 0.392$; that is,

$$\frac{\varphi_2}{0.392}, \quad \frac{\varphi_3}{0.392}, \quad \frac{\varphi_4}{0.392}, \quad \text{and} \quad \frac{\varphi_5}{0.392}$$

Thus, in using these scales, it is necessary to divide the terms of the original equation by 0.392.

In conclusion we may observe that, in accordance with our computations, the term containing $\cos\theta$ may be entirely neglected in the first interval and at the beginning of the second. Only farther along the hull does this member show any effect, the latter showing up only in the thickness of the boundary layer and in the distribution of the shearing stress but having practically no effect on the integrated drag. If the computations therefore are made merely for the purpose of determining the curve $C_u = f(Re)$, the last term may be entirely neglected. We may also point out that for rapidity of computation we may, in the fourth term on the right-hand side of the last equation, write:

$$Az^2 e^z = \varphi_2 - \varphi_3$$

i.e., in the form of a difference in the functions used in the two preceding terms.

Translation by S. Reiss,
National Advisory Committee
for Aeronautics.

## REFERENCES

1. Miller, Horst: Der Reibungswiderst und unströmter Korper. Werft-Reederei-Hafen 4, 1932.

2. Millikan, Clark B.: The Boundary Layer and Skin Friction for a Figure of Revolution. A.S.M.E. Trans., January 30, 1932.

3. Nikuradse, J.: Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren. Forschungsheft 356, Forschung auf dem Gebiete des Ingenieurwesens, edition B, September-October, 1932.

4. von Kármán, Th.: Aehnlichkeit und Turbulenz. Proc. Third International Congress for Applied Mechanics, I.

5. Lrandtl, L.: Recent Results of Turbulence Research. T.M. No. 720, N.A.C.A., 1933.

6. Freeman, Hugh B.: Measurements of Flow in the Boundary Layer of a 1/40-Scale Model of the U.S. Airship "Akron". T.R. No. 430, N.A.C.A., 1932.

7. Moore, Norton B.: Application of Kármán's Logarithmic Law to Prediction of Airship-Hull Drag. Jour. of the Aeronautical Sciences, vol. II, no. 1, 1935.

8. Freeman, Hugh B.: Pressure Distribution Measurements on the Hull and Fins of a 1/40-Scale Model of the U.S. Airship "Akron". T.R. No. 443, N.A.C.A., 1932.

9. von Kármán, Th.: Turbulence and Skin Friction. Jour. of the Aeronautical Sciences, vol. 1, no. 1, 1934.

10. Kempf, G.: Weitere Reibungsergebnisse in ebenen glatten und rauhen Flächen. Hydromechanische Probleme des Schiffsantrieb, 1932.

11. Hansen: Die Geschwindigkeitsverteilung in der Grenzschicht an einer eingetauchten Platte. Abh. a. d. Aerod. Inst. an der Tech. Hochs., Aachen, Heft 8, 1928.

12. Lyon, Hilda M.: The Drag of Streamline Bodies. The Relative Importance of Skin Friction and Pressure in Relation to Full-Scale Design. Aircraft Engineering, September, 1934.

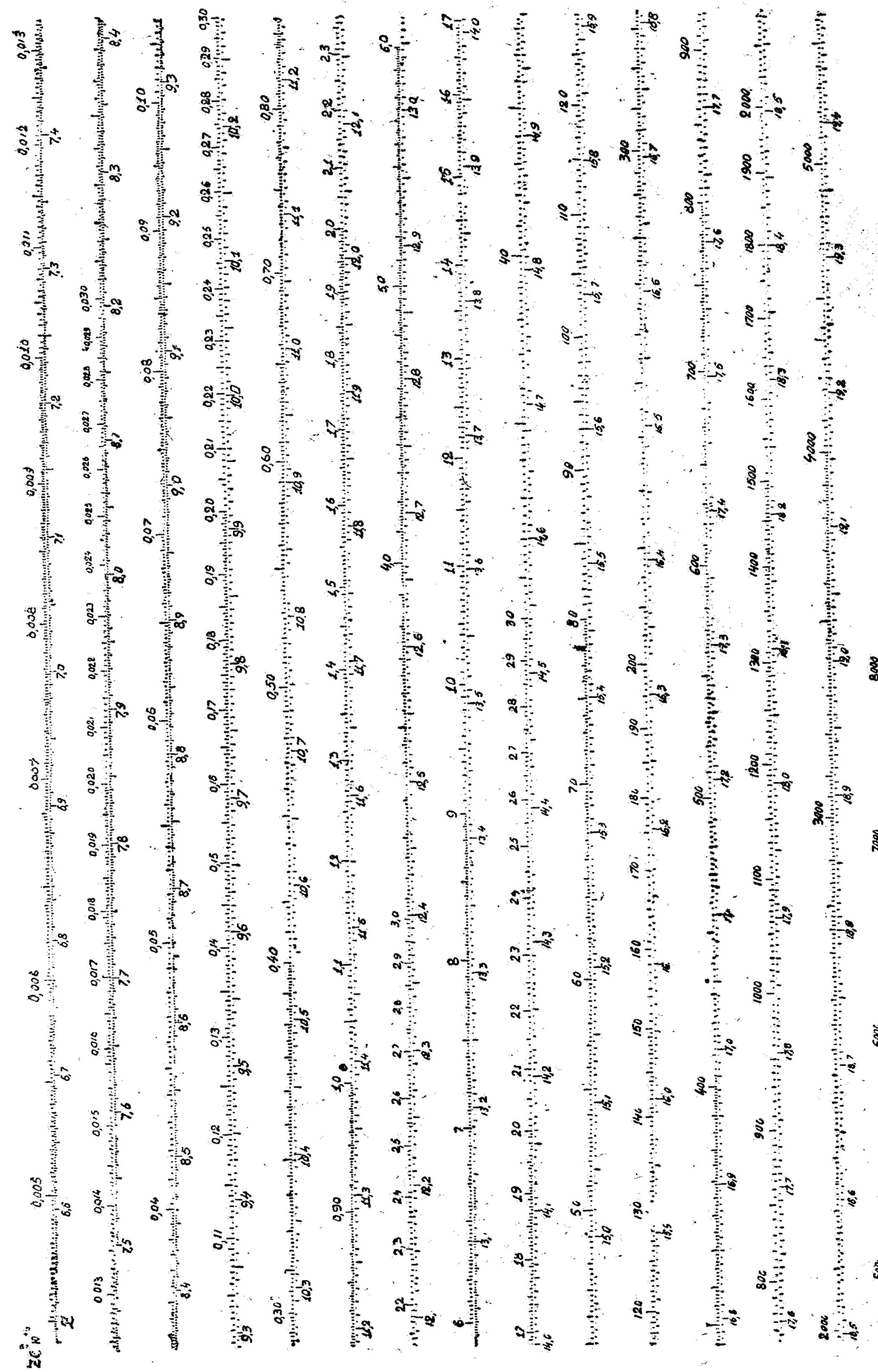
REFERENCES (Cont.)

13. Fediaevsky, K. K.: Boundary Layer and Frictional Drag of Streamline Bodies. CAHI Technical Note No. 30, 1932. (Russian)

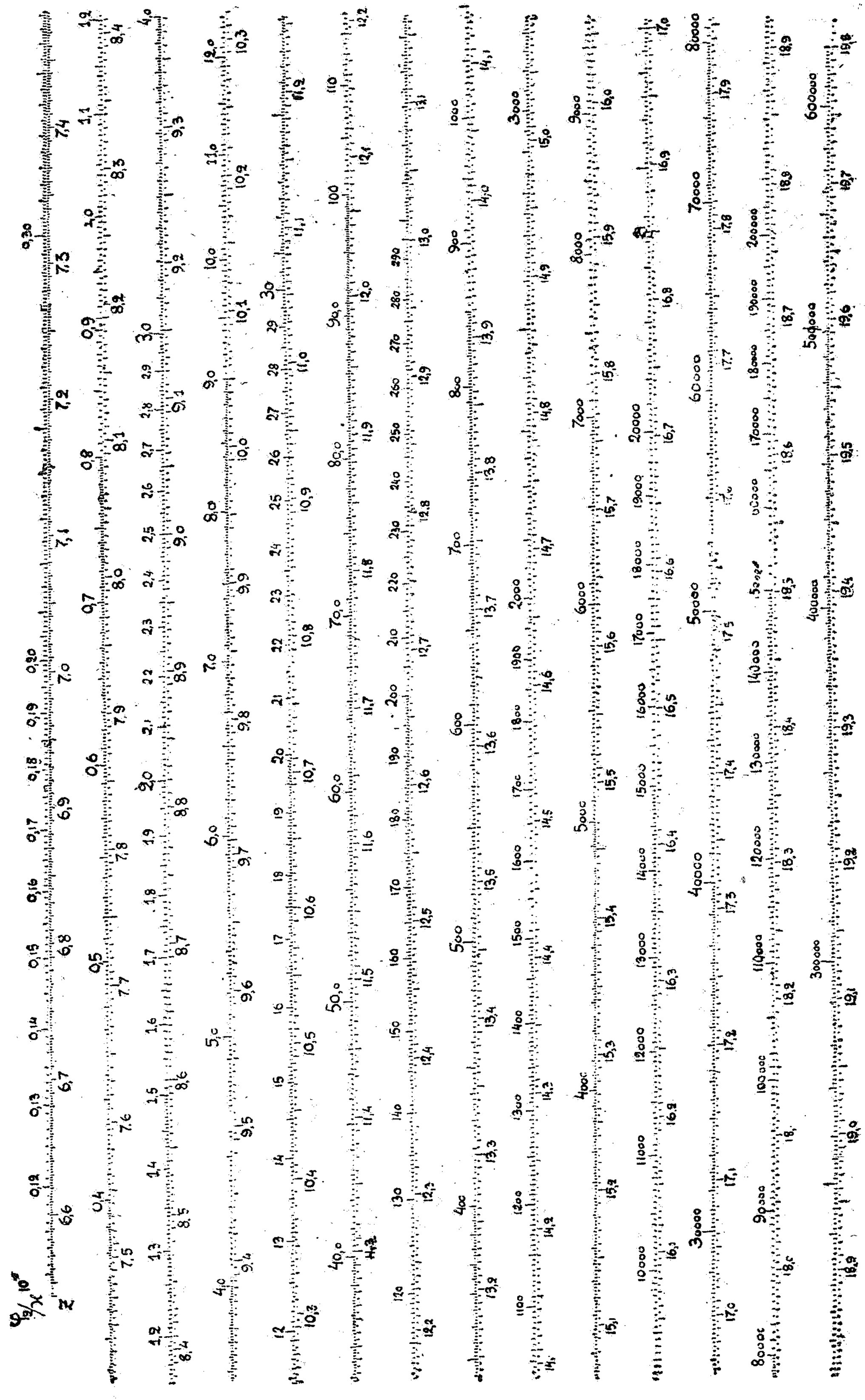
14. Fediaevsky, K. K.: Boundary Layer and Frictional Drag of Figures of Revolution at Large Reynolds Numbers. CAHI Report No. 179, 1934. (Russian).

TABLE I

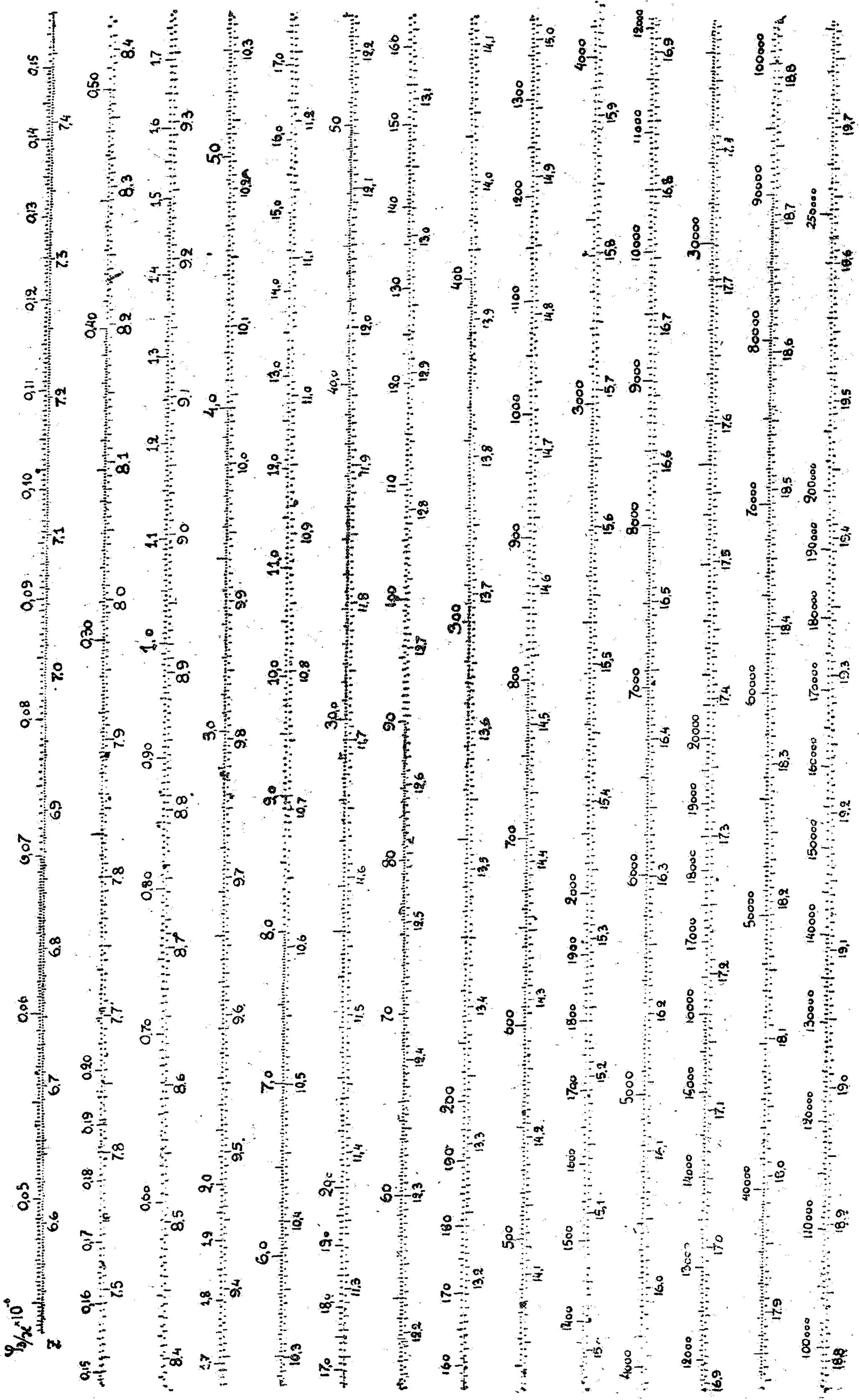
| Intervals | Number of points | s (m) | $\frac{x}{L}$ | f | r (m) | cos θ |
|---|---|---|---|---|---|---|
| First | 0 | 0 | 0 | 0 | 0 | 0.6 |
| | 1 | 0.08 | 0.0098 | 0.247 | 0.056 | 0.640 |
| | 2 | 0.16 | 0.0186 | 0.497 | 0.1168 | 0.685 |
| | 3 | 0.32 | 0.039 | 0.870 | 0.216 | 0.848 |
| | 4 | 0.56 | 0.0753 | 1.100 | 0.316 | 0.941 |
| Second | 5 | 0.72 | 0.1007 | 1.160 | 0.362 | 0.965 |
| | 6 | 1.00 | 0.1465 | 1.195 | 0.420 | 0.985 |
| | 7 | 1.20 | 0.1798 | 1.185 | 0.450 | 0.993 |
| | 8 | 1.6 | 0.246 | 1.160 | 0.489 | 0.999 |
| | 9 | 2.0 | 0.306 | 1.138 | 0.500 | 1.000 |
| | 10 | 2.4 | 0.380 | 1.112 | 0.503 | 1.000 |
| | 11 | 2.8 | 0.446 | 1.110 | 0.504 | 1.000 |
| | 12 | 3.2 | 0.514 | 1.115 | 0.502 | 1.000 |
| | 13 | 3.6 | 0.580 | 1.115 | 0.4 | 0.999 |
| | 14 | 4.0 | 0.647 | 1.117 | 0.470 | 0.998 |
| Third | 15 | 4.4 | 0.714 | 1.120 | 0.436 | 0.995 |
| | 16 | 4.8 | 0.780 | 1.090 | 0.384 | 0.989 |
| | 17 | 5.2 | 0.895 | 1.047 | 0.317 | 0.980 |
| | 18 | 5.6 | 0.907 | 0.960 | 0.230 | 0.966 |
| | 19 | 5.8 | 0.940 | 0.890 | 0.179 | 0.939 |
| | 20 | 5.88 | 0.952 | 0.825 | 0.155 | 0.848 |



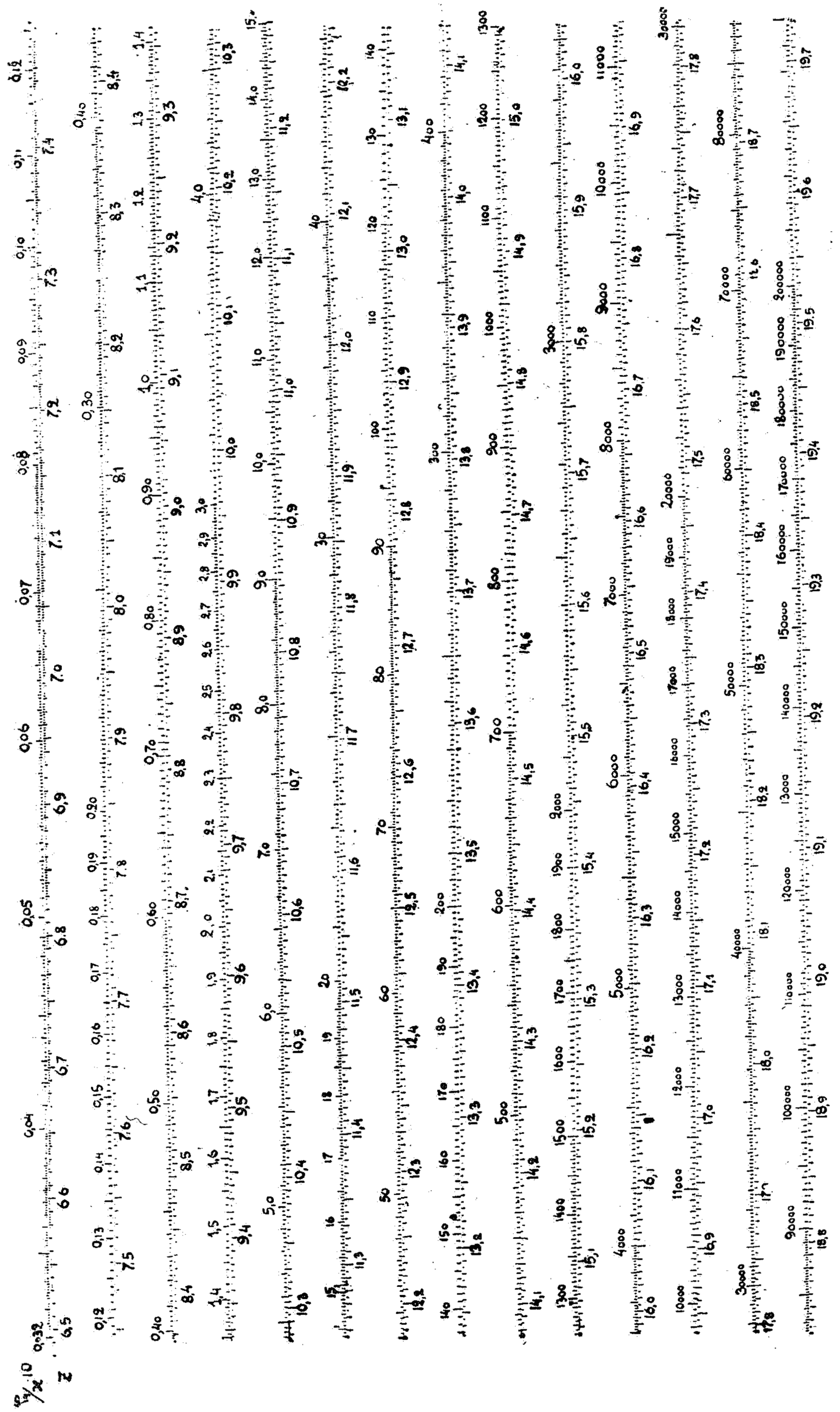
Scale of function $Ze^{Z}$.



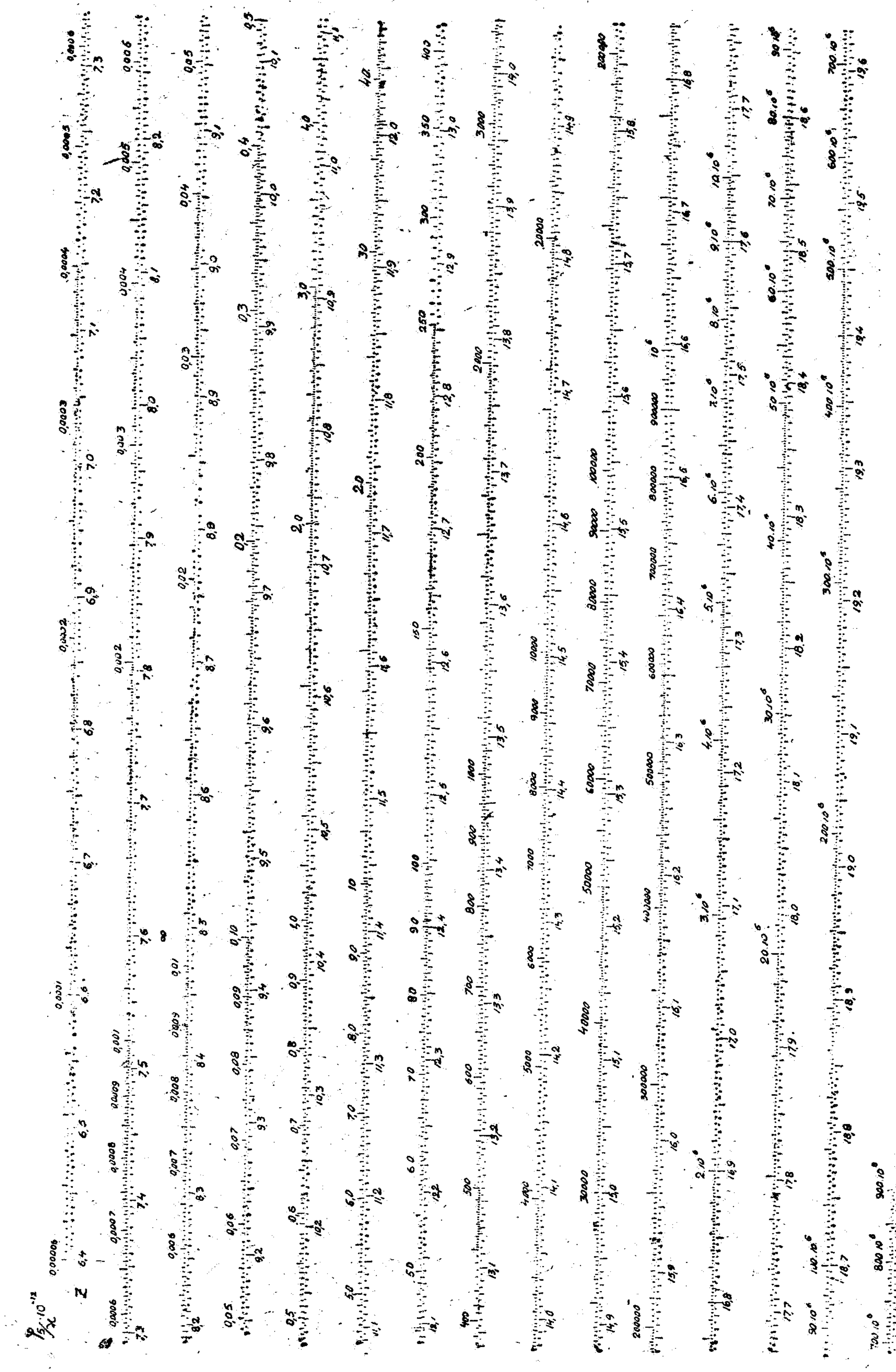
Scale of function $\varphi_2/\varkappa = e^Z (4.2516\,Z^2 - 3.9682\,Z = f(Z)$ for $\varkappa = 0.392$.



Scale of function $\varphi_B/\varkappa = e^Z(2.1258\,Z^2 - 39682\,Z)$ for $\varkappa = 0.392$.



**Scale of function** $\varphi_4/x = \int_0^Z \varphi_1/x \, dz = e^Z (2.1258\, Z^2 - 8.2198\, Z + 12.1880) - 12.1880$ **for** $x = 0.392$.



Scale of function $\varphi_5/\varkappa = e^{2Z}[Z^3 - 2,664(Z^2 - Z + 1/2)]$ for $\varkappa = 0.392$.

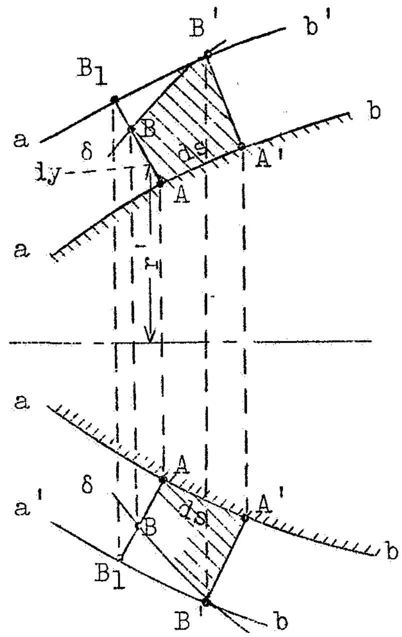



Figure 1.

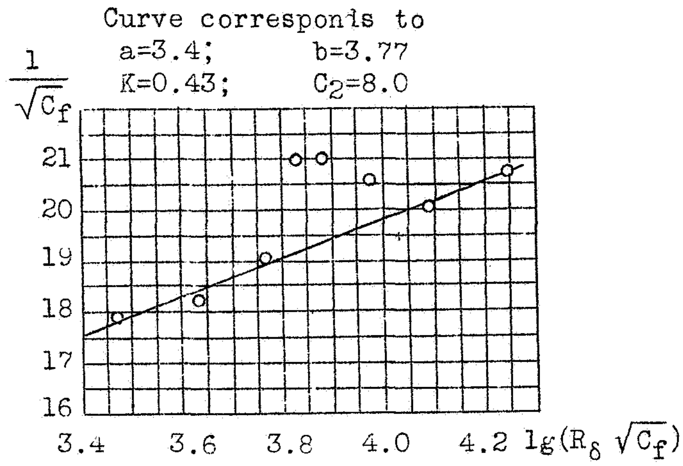



Figure 7.- Check of the Karman relation $\frac{1}{\sqrt{C_f}}$ =a+b lg ($R_\delta \sqrt{C_f}$) from Freeman's data on the airship model "Akron.

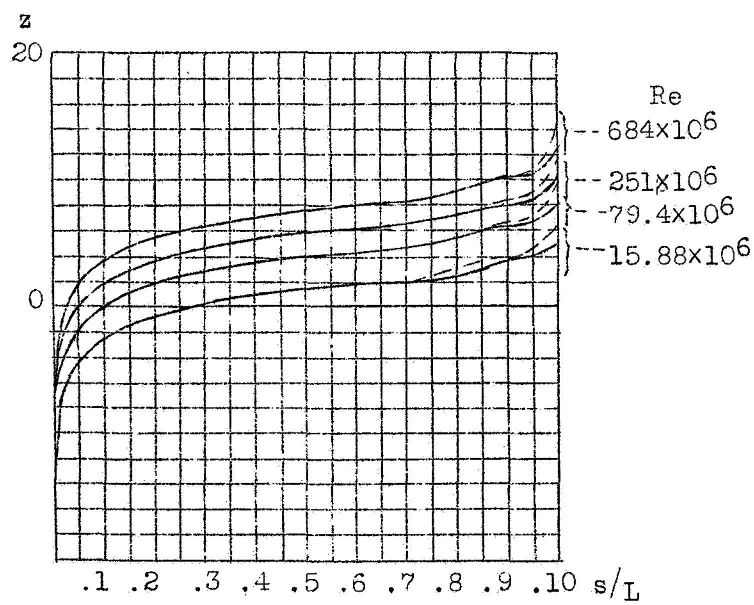



Figure 9.- Variation of z along the hull of airship model "Akron" for different Reynolds Numbers (our theory with K = 0.392; $K_1$ = 0.214).

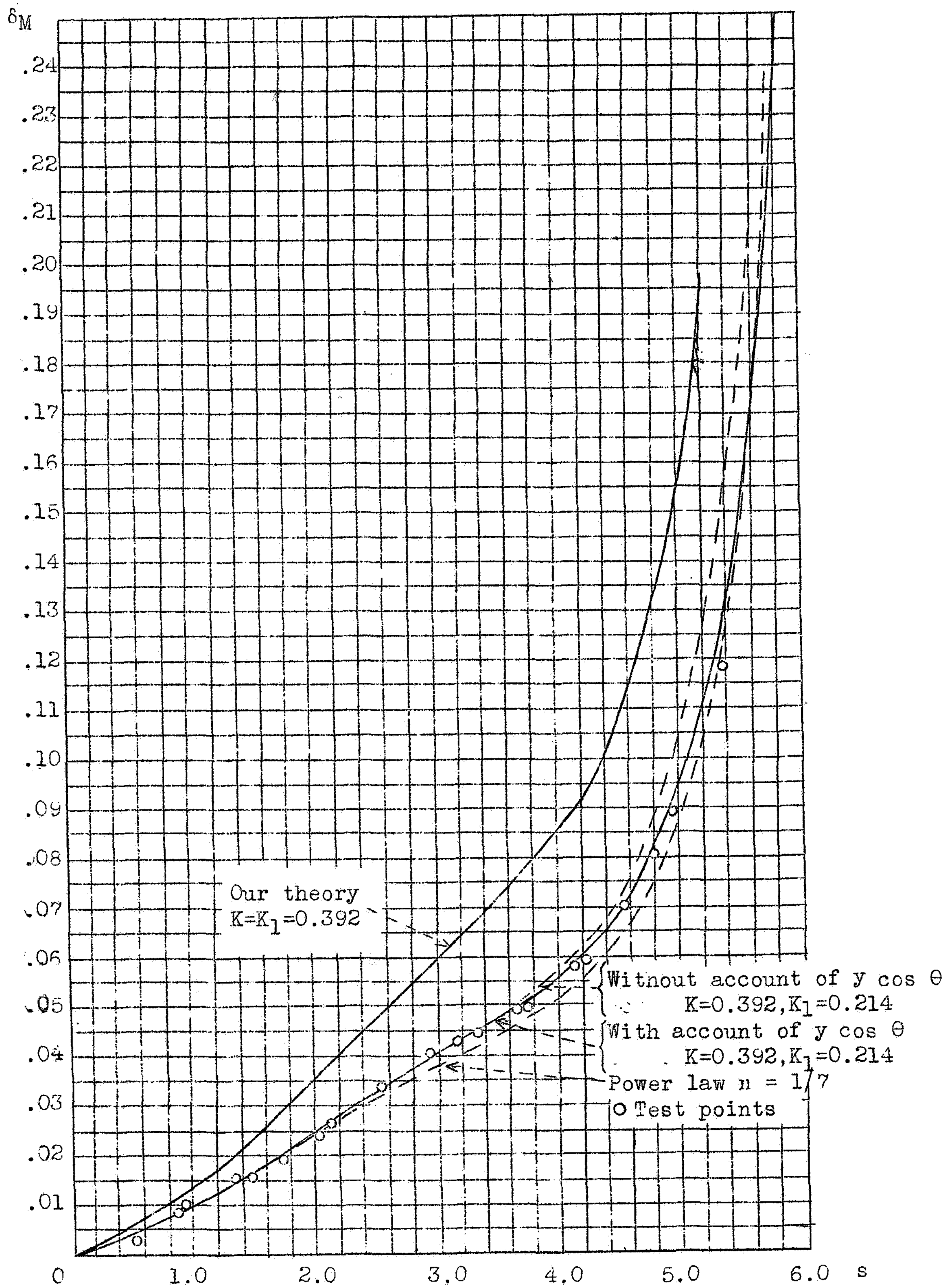



Figure 2.- Boundry-layer thickness δ in meters along hull for the 1/40-scale model of the "Akron" at Re = $vL/\nu$ = $15.88\times10^6$.

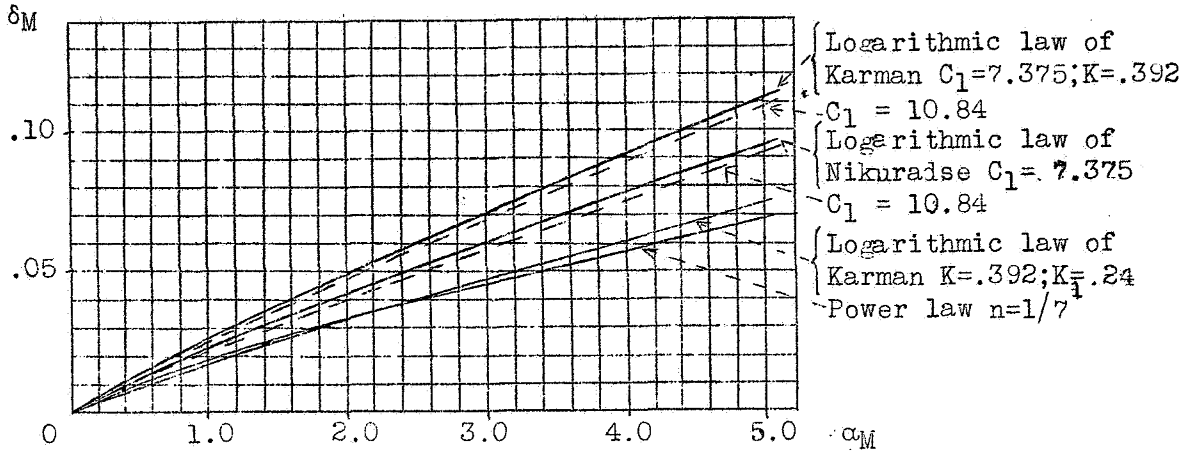



Figure 3.- Boundary-layer thickness along a flat plate according to the various theories, velocity of flow V = 38.45 m/sec.

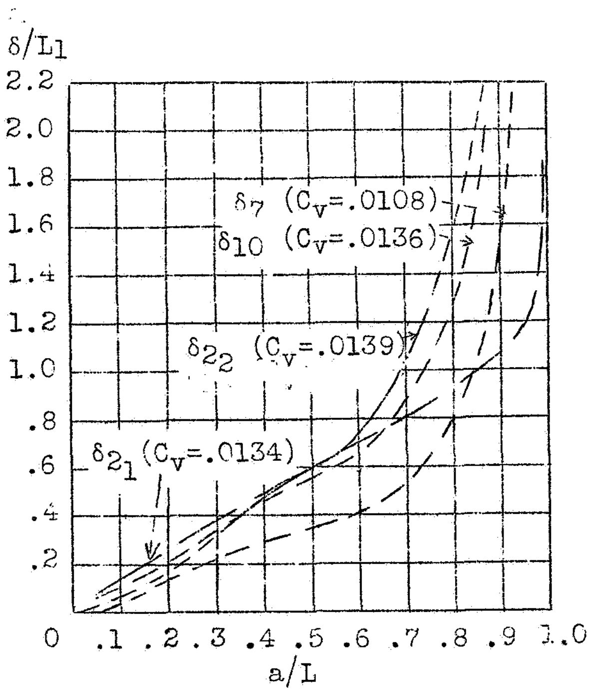



Figure 4.

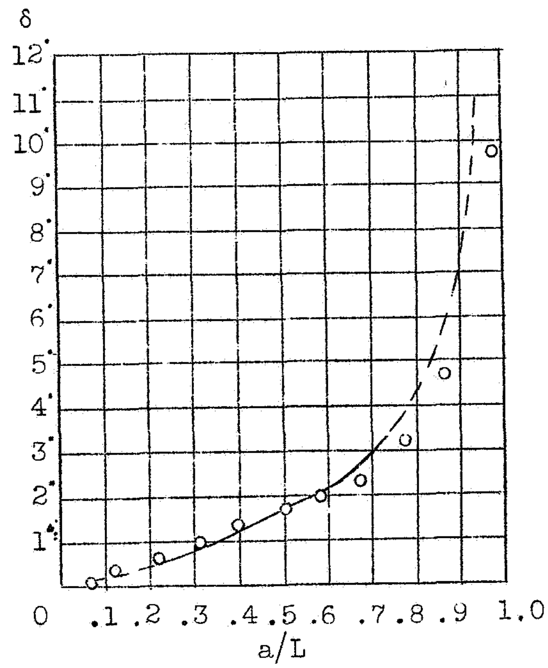



Figure 5.

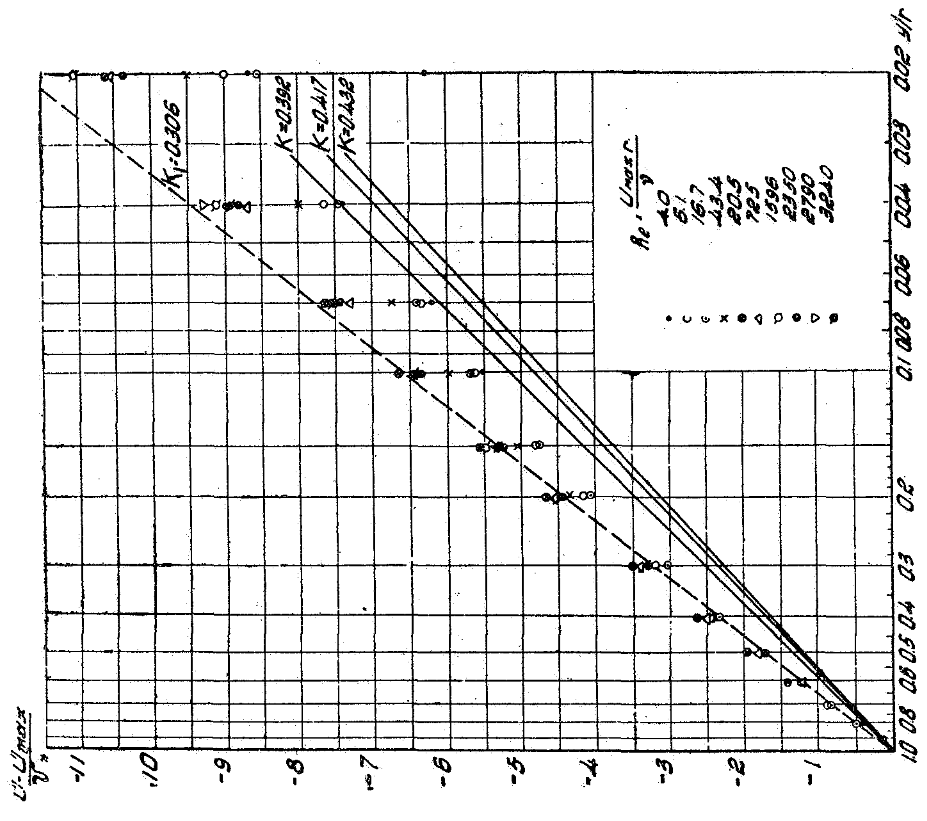



Figure 8.- Plot of $\frac{u - u_{max}}{u_*}$ against $l_n(1-\sqrt{1-\frac{y}{r}})+\sqrt{1-\frac{y}{r}}$ from Nikuradse's on smooth pipes.

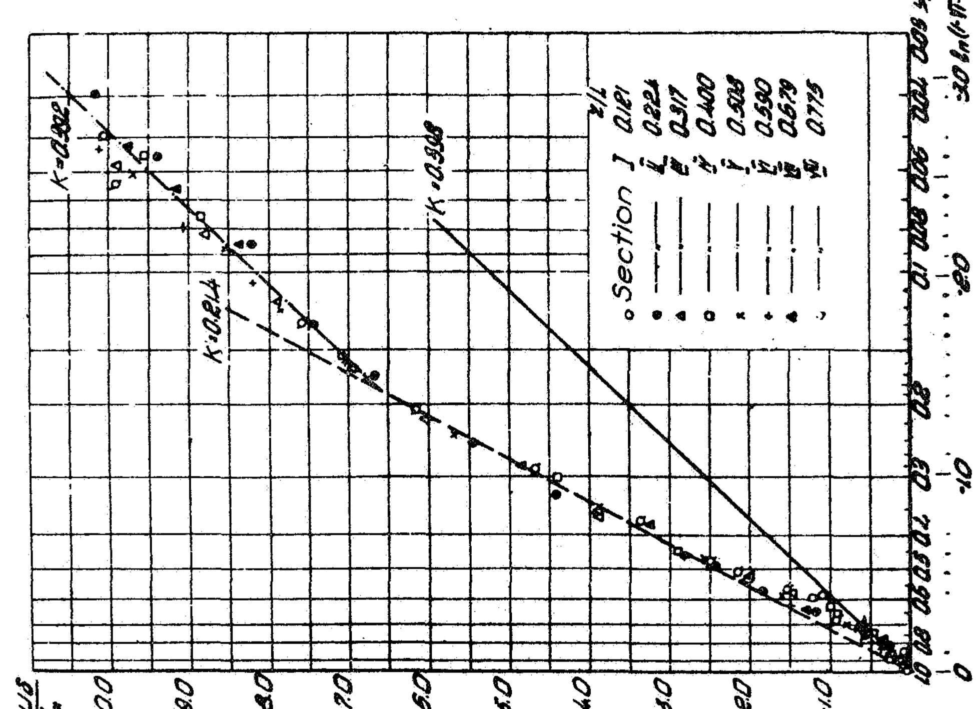



Figure 6.- Plot of $\frac{u - u_\delta}{v_*}$ against $l_n(1-\sqrt{1-\frac{y}{\delta}})+\sqrt{1-\frac{y}{\delta}}$ from Freemans tests, Re = 15.88 × $10^6$.

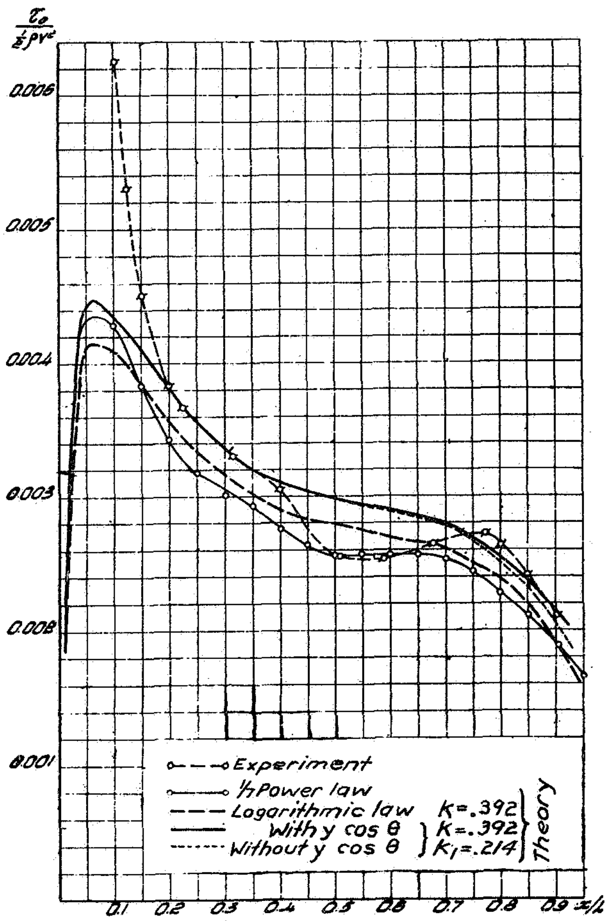



Figure 10.- Shearing stress distribution along hull of airship model "Akron" at $Re = vL/\nu = 15.88 \times 10^6$.

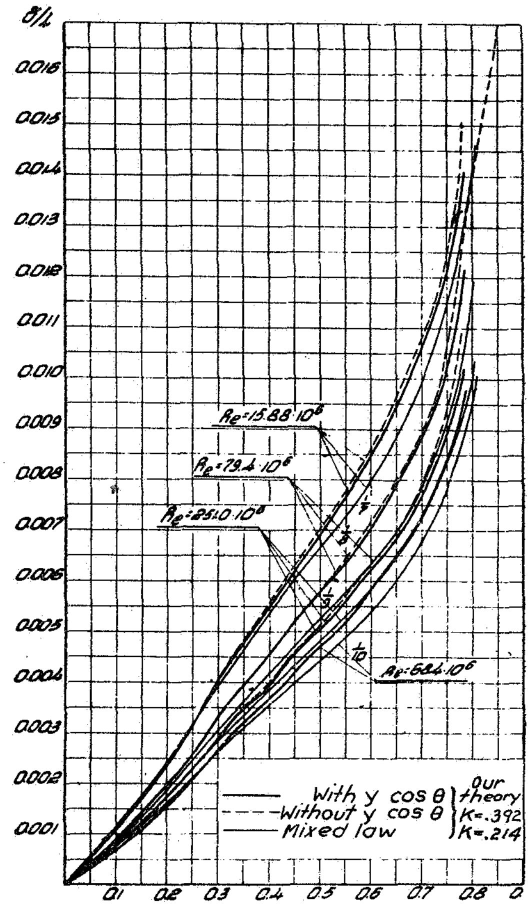



Figure 12.- Variation of relative boundary-layer thickness along hull of airship model "Akron" at various Reynolds Numbers according to the logarithmic and exponential laws.

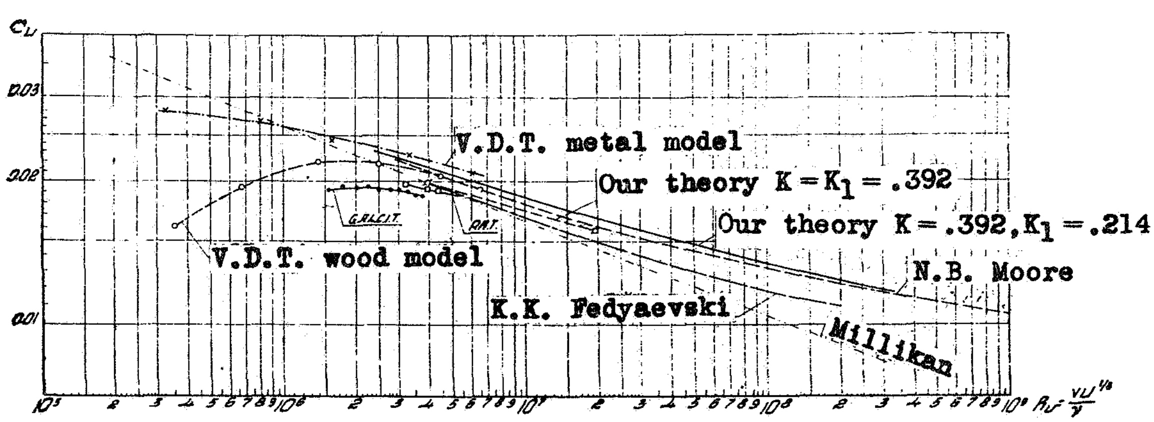



Figure 11.- Theoretical and experimental variation of frictional drag of hull of airship model "Akron" with Reynolds Number.

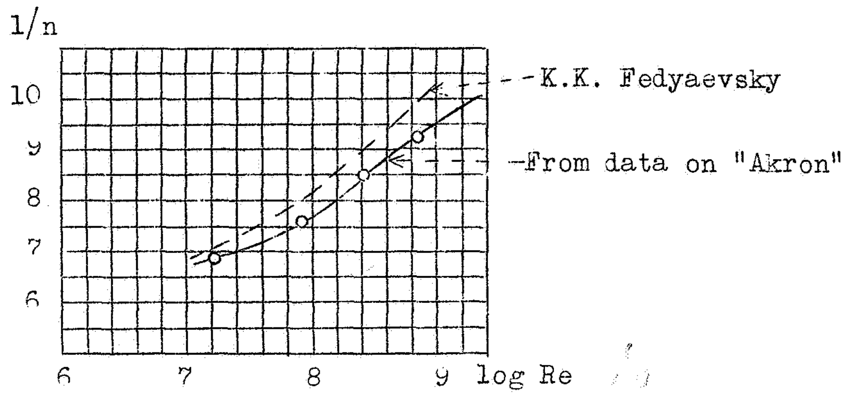



Figure 13.- Variation of the exponent in the power law of velocity distribution with Reynolds Number.

$$\frac{u}{u_\delta} = \left(\frac{y}{\delta}\right)^n$$

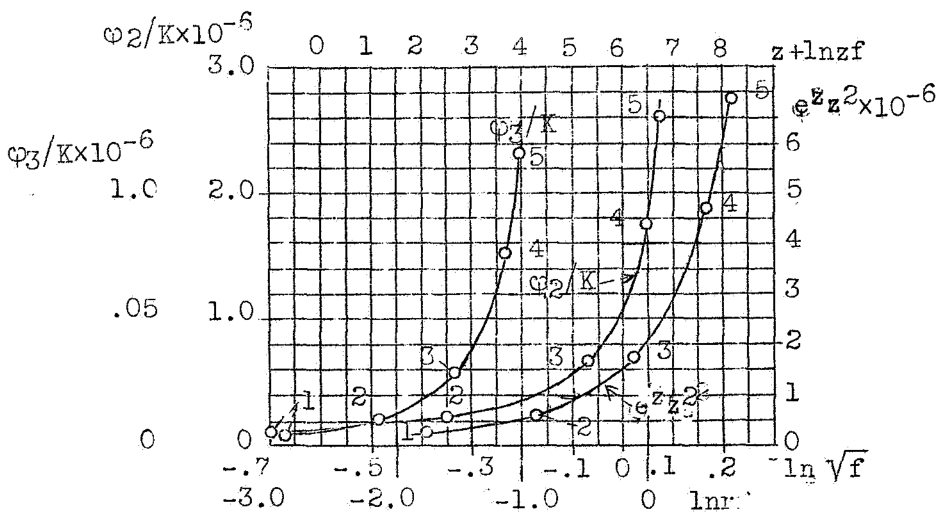



Figure 14.- Integral curves for interval I. Model of airship "Akron", Re=15.88×10$^6$.

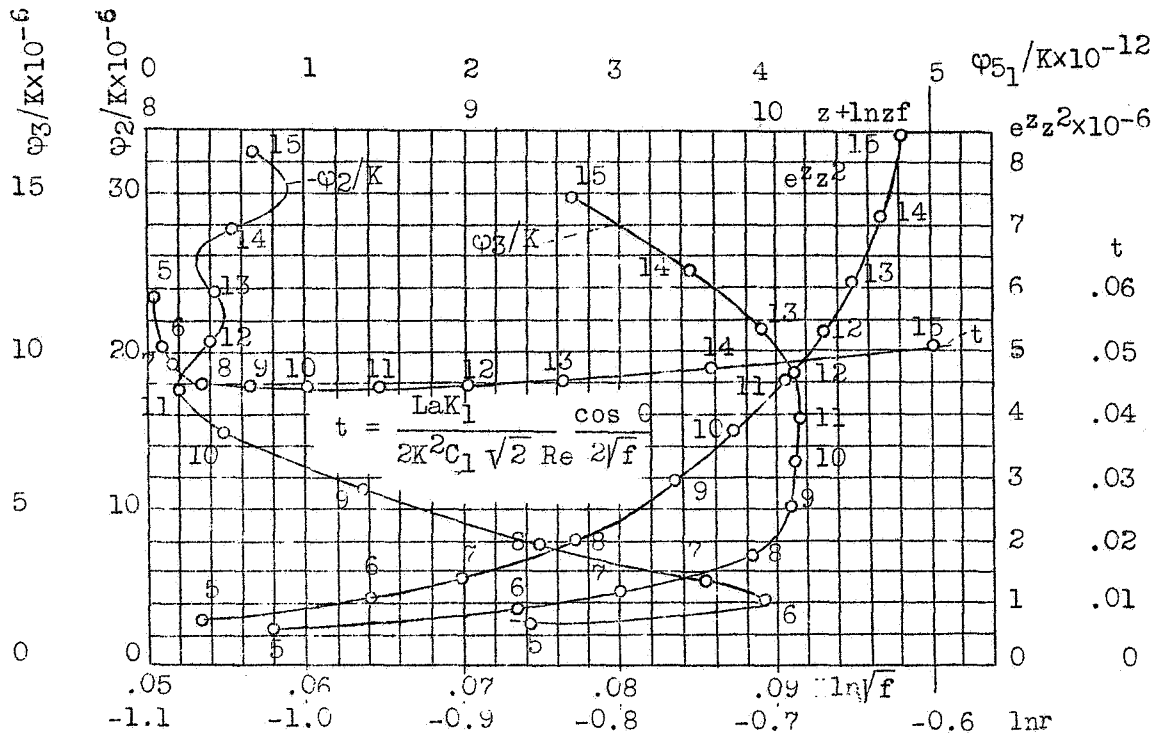


Figure 15.- Curves for interval II. Model of airship "Akron", Re $15.88 \times 10^6$

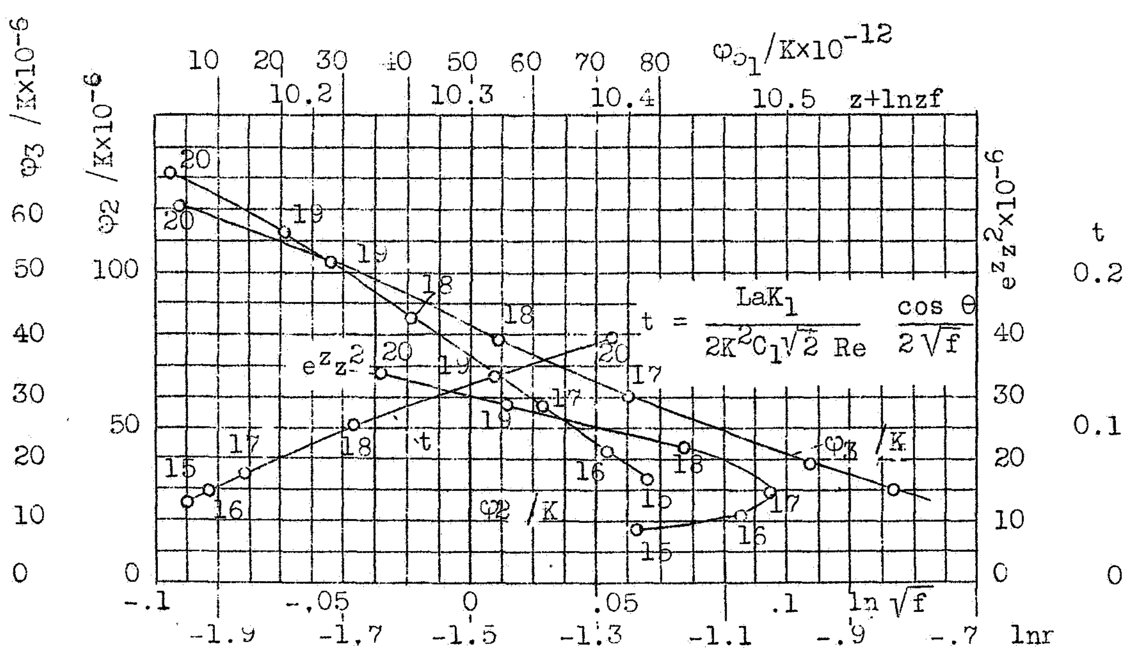


Figure 16.- Integral curves for interval III. Model of airship "Akron.