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# THE CRITICAL VELOCITY OF A BODY TOWED BY A CABLE FROM AN AIRPLANE

By C. Koning and T. P. DeHaas

Rijks-Studiedienst voor de Luchtvaart, Amsterdam

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# THE CRITICAL VELOCITY OF A BODY TOWED BY A CABLE FROM AN AIRPLANE*

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For the accurate measurement of the velocity in flight tests, static tubes are employed that are towed behind the airplane at the end of a cable. Fundamentally, the design of the instrument agrees with that of the usual pitot tube except that the dynamic pressure tube and the static tube are separated, the first one being fixed to the airplane and the second being towed below and behind the airplane at the end of a wire or cable about 20 meters (65 feet) long.

It has often occurred, in practice, that the static tube got lost by loosening from the cable or breaking off. At first this was ascribed to insufficient strength of the attachment of the instrument to the cable. But even after the attachment had been made strong enough to withstand a far greater load than would be expected in normal operation, static tubes continued to get lost. Upon closer observation, phenomena could often be observed that gave a strong indication of the setting up of unstable oscillations, a fact which was possibly the cause of the difficulty described above.

Some computations were carried out starting with the theory developed by Glauert (reference 1) on the stability of a body towed by an airplane. According to the results of these computations, however, a static tube of the usual design and at normal velocities should be stable. The above-mentioned theory which considers mainly the case of the stability of antenna weights, hung from the end of a very thin wire, is based on assumptions that all come down to the fact that the mass of the wire is neglected. The question was now whether these assumptions should be considered as valid in our present computations. Consideration of the problem showed that the mass of the wire should

*"De kritische snelheid van een lichaam, dat door een vliegtuig aan een kabel gesleept wordt." Rapport A 367, Rijks Studiedienst voor de Luchtvaart, Amsterdam.

not be left out of the computation. It was found possible to set up the system of equations of motion by introducing other and, in our case, justifiable simplifications. The solution of these equations, however,. and the determination of the stability of the system in a given case, led to difficulties that were only partially overcome.

A positive result of the investigation, however, was the finding of a restricted but simple stability criterion. This criterion is restricted insofar as it does not answer the question whether a given system at any given velocity will or will not be stable. It does give, however, a "critical velocity" as the upper limit of the range of velocities within which it may be positively stated that no unstable oscillations will occur.

As will appear in what follows, it is sufficient to consider only the equations of motion of the towed body whereas those of the cable may be left out of consideration. The result obtained makes it possible to determine which factors affect the critical velocity and what modifications of the instrument are necessary for extending the upper limit of that velocity. Tests conducted with several forms of static tubes, as described briefly in section 7, page 16, confirm the results obtained this way.

Although the discussion above deals mainly with the towed static tube, it is clear that what follows applies also to other cases (such as other towed instruments and antenna weights), provided the assumptions of section 2 (page 3) are applicable.

With regard to the part contributed by the authors in preparing this report, the following may be mentioned. The theoretical development is that of the first-named author. The computations necessary for determining the effect of the various factors and for the design of new forms of static tubes were chiefly carried out by the second-named author, who also conducted almost all the necessary flight tests. Some flight tests at very high velocities were conducted by Reserve First Lieutenant L. Th. van Hootegem. The airplanes used in the tests were kindly made available by the Aeronautics Division at Soesterberg. We here wish to express our thanks for all assistance given.

## 2. ASSUMPTIONS

The towed body is considered to perform small oscillations about its steady state condition in a vertical plane through the body and the airplane, the latter being assumed to fly with uniform velocity along a straight horizontal path. The towed body is symmetrical, both with respect to the vertical plane and with respect to a plane that is horizontal in steady flight, and is attached to the cable at the center of gravity. The cable is considered inextensible and perfectly flexible so that it can exert a force on the body in its own direction but no moment.

The magnitude and direction of the resultant of the aerodynamic forces acting on the body are assumed to depend only on magnitude of the relative velocity of the center of gravity with respect to the air and on the angle of attack, the latter being the angle between the second plane of symmetry mentioned above and the relative wind direction. The moment of the aerodynamic forces about the transverse axis through the center of gravity of the body is also affected, however, by the angular velocity with respect to that axis as well as by the two factors mentioned above. The body is assumed to be statically stable. The lift coefficient increases with increase in angle of attack and positive damping is set up in rotating about the transverse axis.

In addition to the above assumptions of a fairly general nature that require no further discussion, there are two more that will be gone into in more detail.

In the steady motion the form of the cable is generally taken to be a plane curve. In what follows, it will be assumed that this curve may be replaced by a straight line whose direction is the same as that of the tangent to the cable at the body. At first view this assumption appears to be wholly arbitrary. Although it must be admitted that a closer study of this point is desirable, the impression was gained after a consideration of various cases that this assumption does not deviate too far from the truth.

Another characteristic of the system that will here be introduced in the form of an assumption, is the following. If the towed body is removed leaving conditions unchanged, the wire may be allowed to perform harmonic oscillations.

For this it is necessary that a periodic force act at the outer end of the cable normal to its initial direction. The work that must be performed by this cable will be positive; i.e., work is performed on the wire in order to maintain the state of motion. This property can be derived from the equation of motion of the cable, which equation will not be further considered here and which is set up with the aid of the conventional assumption that the aerodynamic force on an element of the wire depends only on the magnitude of the relative wind velocity and on the angle of attack of the element. It also seems not improbable that the aerodynamic forces on the cable have a damping effect. There are indications, however, that in particular cases a vibrating cable may take up energy from the air. The assumption here discussed does not deny the possibility of exceptional cases such as these but only excludes them from consideration.

## 3. THE EQUATIONS OF MOTION OF THE TOWED BODY

a) Notation.- The meaning of each symbol used will be indicated at the place where it is first introduced. A list of the symbols used is also given at the end of the report, so that it will here be sufficient to call attention to a few points of a general character.

The coefficients for the aerodynamic forces and moments acting on the body were obtained by dividing the corresponding magnitudes by $\frac{1}{2}\rho V^2$ (for the damping moment by $\frac{1}{2}\rho V \dot{\vartheta}$) where $\rho$ is the mass density of the air, $V$ the relative velocity of the body with respect to the air, $\dot{\vartheta}$ the angular velocity about the transverse axis through the center of gravity of the body. The coefficients are therefore, in contrast to those conventionally employed, not nondimensional. They were here chosen in this manner because the dimensions of the body in practical applications are generally established and there is little significance therefore in always carrying along one scale of length or some other throughout the computations.

Derivatives with respect to time are indicated in the usual manner by one or more dots above the magnitude under consideration, while those with respect to the angle of attack are indicated by a prime.

b) General.- The assumptions discussed in section 2 of inextensibility and rectilinearity of the cable lead to a simplification in the treatment of the problem. Since only motion in the vertical plane is considered, the body possesses only two degrees of freedom, and its motion may therefore be fully described by assigning two coordinates. These were chosen as the displacement $\xi$ of the center of gravity in the direction at right angles to the initial direction of the cable and the angle of rotation $\vartheta$ about the transverse axis through the center of gravity. Both were measured from the steady state, their positive directions being indicated on figure 1.

The system of coordinates was so chosen (see fig. 1) that the x-axis was parallel to the direction of flight with the y-axis lying in the vertical plane. In the steady state the wire makes an angle $\varphi$ with the latter axis. The computation of this angle is considered in section 3e.

The external forces on the body are the tension of the cable, the weight, and the aerodynamic forces. To determine the latter, the magnitude of the relative velocity $V$ and the angle of attack $\alpha$ must be known. The relative velocity is the resultant of the flight velocity $V_0$ and the velocity $\dot{\xi}$ of the center of gravity of the body resulting from the oscillation. As may be seen from figure 2, the components of $\dot{\xi}$ in the x- and y-directions, respectively, are:

$$v_x = V_0 + \dot{\xi} \cos \varphi$$

$$v_y = - \dot{\xi} \sin \varphi$$

Hence it follows that the absolute value of the relative velocity is given by:

$$V^2 = v_x{}^2 + v_y{}^2 = V_0{}^2 + 2V_0 \dot{\xi} \cos \varphi + \dot{\xi}^2 \qquad (1)$$

and the angle its direction makes with the x-axis by

$$\beta = \text{arc tan} \frac{-v_y}{v_x} = \text{arc tan} \frac{\dot{\xi} \sin \varphi}{V_0 + \dot{\xi} \cos \varphi} \qquad (2)$$

The angle the plane of symmetry of the body makes with the x-axis is $\vartheta$ so that the angle of attack is:

$$\alpha = \vartheta + \beta = \vartheta + \text{arc tan} \frac{\dot{\xi} \sin \varphi}{V_0 + \dot{\xi} \cos \varphi} \qquad (3)$$

The results of (1), (2), and (3) hold good for any values of $\dot{\xi}$. Taking account of the fact, however, that only small oscillations are considered, so that the second and higher powers of $\dot{\xi}$ may be neglected, the above equations become:

$$V^2 = V_0^2 + 2V_0 \dot{\xi} \cos \varphi \qquad (4)$$

$$\beta = \frac{\dot{\xi}}{V_0} \sin \varphi \qquad (5)$$

$$\alpha = \vartheta + \frac{\dot{\xi}}{V_0} \sin \varphi \qquad (6)$$

c) Equation of motion of the center of gravity.- The product of the mass of the body by the acceleration of the center of gravity in the $\xi$-direction must be equal to the sum of the components of all external forces:

$$m \ddot{\xi} = X_1 + X_2 + X_3 \qquad (7)$$

As indicated above, this sum is made up of the contributions of three different forces, namely, the weight, the cable tension, and the aerodynamic forces. The weight $mg$ acts in the direction of the negative y-axis, so that its component in the $\xi$-direction is:

$$X_1 = + mg \sin \varphi \qquad (8)$$

The tension $T$ of the cable acts along the direction of the latter but does not coincide with its initial position when the system is in motion. If the angle between the two directions is $\theta$ then the component of the cable tension in the $\xi$-direction, since only small deviations are considered, is

$$X_2 = + T \theta \qquad (9)$$

Strictly speaking, the value of $T$ will also be affected by the motion. Since, however, in the expression for $X_2$ the tension $T$ is multiplied by the small quantity $\theta$, the differences may be neglected and $T$ thus considered as constant.

The coefficients of the components L and W of the resultant aerodynamic force normal and parallel, respectively, to the direction of the relative velocity, may be introduced in the form $k_L'$ and $k_W$, both being positive constants. It may be observed here that only small oscillations about the position of equilibrium and therefore also only small angles of attack are considered and, as a result of the symmetry of the body, $k_L$ at $\alpha = 0$ and $k_W'$ are zero.

The above aerodynamic components are therefore:

$$L = k_L' \alpha \frac{1}{2} \rho V^2 \tag{10}$$

$$W = k_W \frac{1}{2} \rho V^2 \tag{11}$$

As has already been noted, the relative wind direction makes an angle $\beta$ with the x-axis, while the angle between the latter and the $\xi$-direction is $\varphi$. The angle between the relative velocity and the $\xi$-direction is therefore $(\varphi - \beta)$ (see fig. 3), and the component of the aerodynamic force in the latter direction is thus,

$$X_3 = - L \sin(\varphi - \beta) - W \cos(\varphi - \beta)$$

Substituting the values of L and W from equations (10) and (11) and then the values of $V^2$, $\beta$, and $\alpha$ from (4), (5), and (6) and again neglecting the higher powers of the small quantities, we obtain:

$$X_3 = - \frac{1}{2} \rho V_0^2 k_W \cos\varphi - \frac{1}{2} \rho V_0 \left\{ k_W(1+\cos^2\varphi) + k_L' \sin^2\varphi \right\} \dot{\xi} - \frac{1}{2} \rho V_0^2 k_L' \sin\varphi\, \vartheta \tag{12}$$

Substituting the values from (8), (9), and (12), equation (7) becomes:

$$m \ddot{\xi} - mg \sin\varphi + \frac{1}{2} \rho V_0^2 k_W \cos\varphi + \frac{1}{2} \rho V_0 \left\{ k_W(1+\cos^2\varphi) + k_L' \sin^2\varphi \right\} \dot{\xi} + \frac{1}{2} \rho V_0^2 k_L' \sin\varphi\, \vartheta - T \epsilon = 0 \tag{13}$$

d) Equation of motion about the center of gravity.- The product of the moment of inertia I about the transverse axis through the center of gravity by the angular acceleration $\ddot{\vartheta}$ is equal to the sum of the external moments about this axis

$$I \ddot{\vartheta} = M_1 + M_2 \tag{14}$$

The lines of action of the weight and the tension pass through the center of gravity so that the aerodynamic forces alone produce a moment. The latter may be considered as consisting of two parts, the "pitching moment" $M_1$ and the "damping moment" $M_2$. The first is taken equal to the moment which would be exerted if the angle of attack were the same but the angular velocity zero. The second is proportional to the angular velocity $\dot{\vartheta}$ but independent of the angle of attack. On the basis of the same considerations as given above with respect to $k_L$, the coefficient of the pitching moment may be taken in the form $k_M' \alpha$. The two moments will then be

$$M_1 = -\tfrac{1}{2} \rho V^2 k_M' \alpha$$

$$M_2 = -\tfrac{1}{2} \rho V k_D \dot{\vartheta}$$

The sign is so chosen that for the usual case of statically stable body with positive damping $k_M'$ and $k_D$ have positive values. Substituting the values of $V$ and $\alpha$ from (4) and (6), we obtain after neglecting as usual the higher powers of the small quantities:

$$M_1 = -\tfrac{1}{2} \rho V_o k_M' \sin \varphi \dot{\xi} - \tfrac{1}{2} \rho V_o^2 k_M' \vartheta$$

$$M_2 = -\tfrac{1}{2} \rho V_o k_D \dot{\vartheta}$$

so that equation (14) for the motion of the body about the center of gravity becomes:

$$I \ddot{\vartheta} + \tfrac{1}{2} \rho V_o k_D \dot{\vartheta} + \tfrac{1}{2} \rho V_o^2 k_M' \vartheta + \tfrac{1}{2} \rho V_o k_M' \sin \varphi \dot{\xi} = 0 \tag{15}$$

e) Equilibrium of the steady state condition.- For the steady state the condition of equilibrium of the force in the $\xi$-direction is:

$$- mg \sin \varphi + \tfrac{1}{2} \rho V_o^2 k_W \cos \varphi = 0 \tag{16}$$

and the angle $\varphi$ is therefore given by

$$\tan \varphi = + \frac{\tfrac{1}{2} \rho V_o^2 k_W}{mg} \tag{17}$$

<u>f) Equations of motion for small oscillations</u>.- Equation (13) may be simplified by taking account of equation (16). Keeping (15) unchanged, the equations of the body for small oscillations about the steady state are:

$$m \ddot{\xi} + \frac{1}{2} \rho V_0 \frac{1}{B} k_L' \sin^2 \varphi \, \dot{\xi} + \frac{1}{2} \rho V_0^2 k_L \sin \varphi \, \vartheta - T \theta = 0 \tag{18}$$

$$I \ddot{\vartheta} + \frac{1}{2} \rho V_0 k_D \dot{\vartheta} + \frac{1}{2} \rho V_0^2 k_M' \vartheta + \frac{1}{2} \rho V_0 k_M' \sin \varphi \, \dot{\xi} = 0 \tag{19}$$

with

$$B = \frac{k_L' \sin^2 \varphi}{k_W (1 + \cos^2 \varphi) + k_L' \sin^2 \varphi} \tag{20}$$

$$\tan \varphi = \frac{\frac{1}{2} \rho V_0^2 k_W}{mg} \tag{21}$$

If the term $T \theta$ did not occur in the first equation the solutions of these equations would offer no difficulties. This term denotes, however, the coupling with another part of the system, namely, that of the cable, so that to obtain the solution in question the motion of the latter system must also be known or at least be determined. In what follows, however, it will appear that without going into any deeper analysis, an important result may be nevertheless derived from the equations above.

## 4. ENERGY CONSIDERATIONS ON THE STABILITY

In order to determine the character of the stability for small oscillations of the body, let the term $T \theta$ of the tension in equation (18) be replaced by an external force $P = P_1 \sin \lambda t + P_2 \cos \lambda t$. The original system is thus tentatively replaced by another whose equations of motion are the same, except for the term $T \theta$, as equations (18) and (19) and whose motion is now considered under the effect of the force $P$. The body will then perform an oscillation of frequency $\lambda$. By a suitable choice of $P_1$ and $P_2$, it is possible to write $\xi = \xi_0 \sin \lambda t$. From equation (19) we now have:

$$\vartheta = - \frac{(\frac{1}{2}\rho V_0 \lambda)^2 k_D k_M' \sin\varphi}{(I\lambda^2 - \frac{1}{2}\rho V_0^2 k_M')^2 + (\frac{1}{2}\rho V_0 k_D \lambda)^2} \xi_0 \sin\lambda t +$$

$$+ \frac{\frac{1}{2}\rho V_0 \lambda (I\lambda^2 - \frac{1}{2}\rho V_0^2 k_M') \sin\varphi}{(I\lambda^2 - \frac{1}{2}\rho V_0^2 k_M')^2 + (\frac{1}{2}\rho V_0 k_D \lambda)^2} \xi_0 \cos\lambda t$$

Substituting the values of $\xi$ and $\varphi$ in equation (18), we have the "components" $P_1$ and $P_2$ of the external force

$$P_1 = - \xi_0 \lambda^2 \left\{ m + \frac{(\frac{1}{2}\rho V_0)^3 V_0 k_D k_M' k_L' \sin^2\varphi}{(I\lambda^2 - \frac{1}{2}\rho V_0^2 k_M')^2 + (\frac{1}{2}\rho V_0 k_D \lambda)^2} \right\} \quad (22)$$

$$P_2 = + \frac{1}{2}\rho V_0 \xi_0 \lambda k_L' \sin^2\varphi \left\{ \frac{1}{B} + \right.$$

$$\left. + \frac{\frac{1}{2}\rho V_0^2 (I\lambda^2 - \frac{1}{2}\rho V_0^2 k_M') k_M'}{(I\lambda^2 - \frac{1}{2}\rho V_0^2 k_M')^2 + (\frac{1}{2}\rho V k_D \lambda)^2} \right\} \quad (23)$$

It is evident that when, for some frequency $\lambda$ of the system, both of these magnitudes become zero the motion can take place without the application of any external force, i.e., a free harmonic oscillation of frequency $\lambda$ is possible. In general, this will not be the case, so that the question of the meaning of these magnitudes must be gone into more closely. To obtain further insight into this question, let us consider the work performed by the outside force during a complete oscillation. The work done in time element $dA = P \dot{\xi} dt$, so that for a complete period we have:

$$A = \int_0^{\frac{2\pi}{\lambda}} P \dot{\xi} dt =$$

$$= \int_0^{\frac{2\pi}{\lambda}} (P_1 \sin\lambda t + P_2 \cos\lambda t) \xi_0 \lambda \cos\lambda t \, dt = + P_2 \xi_0 \pi \quad (24)$$

The work thus appears to be independent of $P_2$. As may be seen from equation (23), $P_2$ depends on $\lambda$ and is proportional to $\xi_0$.

If, for all real positive values of $\lambda$, the work given by (24) is positive, this means that oscillations with constant amplitude are possible only when work is supplied from the outside. With the system left to itself, no oscillations of this kind can occur and certainly no oscillations with increasing amplitude. For the modified system considered at the beginning of this section the stability condition thus obtained is that, for all positive values of $\lambda$, $\frac{P_2}{\xi_0} > 0$.

This condition is sufficient but not necessary. Unstable oscillations are impossible whenever this condition is satisfied but such oscillations will not necessarily be set up in all cases where the condition is not satisfied. With regard to the latter statement, it may be observed that harmonic oscillations, which may be considered as the limiting case of unstable oscillations, can be set up not only when $P_2 = 0$, but also when the condition is satisfied that $P_1 = 0$.

Considering now the significance of this result in the case of the original system - that is, for the body towed by the cable - the following may be observed. As has already been discussed under section 2, in order to maintain a state of oscillation with constant amplitude, work must be supplied to the cable from the outside. Under these circumstances it need not be feared that the cable will supply work to the body. If the cable is now considered to take over the function of the external force in the above system, no unstable oscillations can occur if the above condition is satisfied. Substituting the value of $P_2$ from (23) and leaving out all factors that will surely be positive, the condition becomes: No unstable oscillations can be set up on a body towed by a cable when, for all real values of $\lambda$,

$$\frac{1}{B} + \frac{\frac{1}{2}\rho V_0^2 k_M' (I\lambda^2 - \frac{1}{2}\rho V_0^2 k_M')}{(I\lambda^2 - \frac{1}{2}\rho V_0^2 k_M')^2 + (\frac{1}{2}\rho V_0 k_D \lambda)^2} > 0 \qquad (25)$$

## 5. THE "CRITICAL" VELOCITY

The denominators of both fractions occurring in (25) are always positive. The condition may thus be written in the form:

$$\left(I\,\lambda^2 - \tfrac{1}{2}\,\rho\,V_0^{\,2}\,k_M'\right)^2 + \left(\tfrac{1}{2}\,\rho\,V_0\,k_D\,\lambda\right)^2 +$$

$$+\ \tfrac{1}{2}\,\rho\,V_0^{\,2}\left(I\,\lambda^2 - \tfrac{1}{2}\,\rho\,V_0^{\,2}\,k_M'\right) B\,k_M' > 0 \qquad (26)$$

Expressed in powers of $\lambda$ this may be written as

$$f(\lambda) = a_4\,\lambda^4 + a_2\,\lambda^2 + a_0 > 0 \qquad (27)$$

where

$$a_4 = +\,I^2$$

$$a_2 = +\,\tfrac{1}{4}\,\rho^2\,V_0^{\,2}\,k_D^{\,2} - \tfrac{1}{2}\,\rho\,V_0^{\,2}\,I\,k_M'\,(2 - B)$$

$$a_0 = +\,\tfrac{1}{4}\,\rho^2\,V_0^{\,4}\,k_M'^{\,2}\,(1 - B)$$

Since, as may be seen from (20), B is less than 1, $a_4$ and $a_0$ are always positive.

Considering now the values of $\lambda$ that satisfy (27), it is immediately evident that $\lambda = 0$ is one such value

$$f(0) = +\,a_0 > 0$$

Since $f(\lambda)$ is a continuous function of $\lambda$, it may become less than zero only if the equation

$$f(\lambda) = a_4\,\lambda^4 + a_2\,\lambda^2 + a_0 = 0 \qquad (28)$$

has real roots. The roots are:

$$\lambda = \pm\sqrt{\frac{-\,a_2 \pm \sqrt{a_2^{\,2} - 4a_0\,a_4}}{2a_4}}$$

None of these roots will be real when either $a_2 > 0$ or $a_2^{\,2} - 4a_0\,a_4) > 0$. In the first case, provided the second inequality is not satisfied at the same time, the numerator will always be negative due to the fact that $a_0\,a_4$ is positive, so that all the roots are purely imaginary. In the second case, independently of the sign of $a_2$, all roots will be complex. Both inequalities lead to a stability criterion. From the first it follows that the body will not perform any unstable oscillations whenever, as may be seen by substituting the value of $a_2$ given in (27)

$$a_2 = \frac{1}{4} \rho^2 V_0^2 k_D^2 - \frac{1}{2} \rho V_0^2 I k_M' (2 - B) > 0 \qquad (29)$$

Since $B$ is always positive (see (20)), $a_2$ will always be greater than $(\frac{1}{4} \rho^2 V_0^2 k_D^2 - \rho V_0^2 I k_M')$. Thus condition (29) will certainly be satisfied when the latter expression is greater than zero or

$$I k_M' < \frac{1}{4} \rho k_D^2 \qquad (30)$$

In the above condition $V_0$ does not occur, which fact signifies that whenever the condition is satisfied, no unstable oscillations can be set up at any velocity. As will be seen in section 6, this criterion appeared to be of little practical value.

Of greater practical importance is the second condition:

$$a_2^2 - 4a_0 a_4 < 0 \qquad (31)$$

Substituting the values of $a_4$, $a_2$, and $a_0$ from (27), the above inequality becomes, after discarding the positive factor $\rho^2 V_0^4$.

$$F(V_0) = \left\{ \frac{1}{4} \rho k_D^2 - \frac{1}{2} I k_M'(2-B) \right\}^2 - I^2 k_M'^2 (1-B) < 0 \qquad (32)$$

As follows from (20) and (21), B, and hence also the above expression, is a function of $V_0$. This indicates the possibility that condition (32) may be satisfied for some values of $V_0$ and not for others. For the first range of velocities the body will certainly perform no unstable oscillations; for the second range no certainty exists with regard to unstable oscillations.

In order to see whether this is actually the case it is necessary to consider more closely the variation of $F(V_0)$ with $V_0$, the line of reasoning followed being similar to that employed in discussing $f(\lambda)$.

When $V_0 = 0$, $B$ is likewise zero, and

$$F(0) = + \frac{1}{16} \rho k_D^2 (\rho k_D^2 - 8 I k_M')$$

In case the above expression is positive criterion (30) is satisfied and will not be further considered. If it is negative, however, then condition (32) is satisfied for $V_0 = 0$. Unstable velocities at other values of $V_0$, for instance $V_1$, will then only occur when $F(V_1) > 0$. Between $V_0 = 0$ and $V_0 = V_1$, however, there must be a value of $V_0$ that satisfies the equation

$$F(V_0)=\left\{\frac{1}{4}\rho\ k_D^2-\frac{1}{2}\ I\ k_M'(2-B)\right\}^2-I^2\ k_M'^2(1-B)=$$

$$=\frac{1}{4}\ I^2\ k_M'^2\ B^2+\frac{1}{4}\rho\ k_D^2\ I\ k_M'\ B+\frac{1}{16}\rho\ k_D^2(\rho\ k_D^2-8I\ k_M')=0 \tag{33}$$

Solving the above for $B$, we have, since only a positive root has any significance here,

$$B_k = \frac{-\rho\ k_D^2 + 2k_D\sqrt{2\rho\ I\ k_M'}}{2I\ k_M'} \tag{34}$$

From (20) and (21) the corresponding values of $\varphi$ and $V$ are:

$$\sin^2\varphi_k = \frac{2B_k\ k_W}{(1-B_k)\ k_L' + B_k\ k_W} \tag{35}$$

$$V_k^2 = \frac{2\ mg\ \tan\varphi_k}{\rho\ k_W} \tag{36}$$

Summing up, it may be said that $B_k$ and hence also $V_k$ give the condition at which the value of $F(V_0)$ goes over from negative to positive, $V_k$ thus forming the boundary between the regions of lower and higher velocities for which condition (31) is and is not satisfied, respectively. This value of the velocity was therefore denoted as the "critical velocity." At velocities $V_0$ lower than $V_k$, no unstable oscillations will occur, while at velocities higher than $V_k$ such oscillations may or may not occur. The limiting velocity at which these unstable oscillations will start to occur may thus be greater but certainly not smaller than $V_k$.

## 6. SOME GENERAL DEDUCTIONS

In the applications of the theory that have been considered up to the present condition (30) appeared to set such a high requirement that it was practically impossible to satisfy this criterion. This is bound up with the fact that a certain degree of static stability is always desirable.

More favorable results are obtained by the use of the second criterion. The latter makes it possible to estimate the effect of various factors on the critical velocity and also compare the latter for various forms. As may be seen from equations (34) to (36), other conditions remaining the same, an increase in the weight (mg) and in the damping $(k_D)$ and a decrease in the lift slope $(k_L')$ increase the critical velocity $V_k$. It may also be observed that an increase in $B_k$ increases $\varphi_k$ and therefore also $V_k$.

The effect of the remaining factors is not so evident from the formulas. From the computations that have been carried out, the result was obtained that in the range of velocities considered, an increase in the moment of inertia (I) and in the static stability $k_M'$ produced a decrease in the value of $V_k$. On the effect of the drag $k_W$ very little in general can be said since $\varphi_k$ increases with the drag and therefore both the numerator and denominator of the fraction in (36) increase. From the computation of a particular case it appeared that an increase in the drag results in a decrease in $V_k$ but the effect was not conclusive. Finally, it was found, likewise for a particular case, that a decrease in the density $(\rho)$ led to an increase in $V_k$ but the corresponding dynamic pressure $(\frac{1}{2}\rho V_k^2)$ decreased.

The means by which the critical velocity may be raised thus appear to be: by an increase in the weight (mg) and the damping $(k_D)$, a decrease in the lift coefficient with angle of attack $k_L'$ and a decrease in the moment of inertia (I) and static stability $(k_M')$. This conclusion is valid generally insofar as concerns the first three magnitudes. For the two last-named factors the conclusion holds for the rather wide range within which the practical forms of static tubes are employed. For cases that fall far outside the range the correctness of the above statement must be checked by computation.

The values of $k_L'$, $k_M'$, and $k_D$ for bodies such as static tubes depend mainly on the size of the tail surface and, for the two last-named coefficients, on the distance of the tail surface from the center of gravity. It thus appears desirable to make the tail surface small and the distance from tail to center of gravity large.

## 7. APPLICATION TO TOWED STATIC TUBES

The above theory was advantageously applied to improving the dynamic stability of the towed static tube. The original form of the static tube is denoted by a in figure 4. Its characteristics were large tail surface, large moment of inertia, and relatively small weight - all of which must be considered as the reasons for the low critical velocity and hence, as undesirable. Flight tests were conducted with several modified forms applying to a greater or less degree the modifications mentioned in section 6. The intermediate forms and the results obtained with them will not be further discussed here, with the exception of a single case which very clearly brought out the effect of the moment of inertia. For this purpose a static tube was used, of which the moment of inertia could be varied by the shifting of movable weights. With small moments of inertia no oscillations were observed up to the highest velocity used in the test - about 180 kilometers per hour (112 miles per hour). In a subsequent test, however, using a large moment of inertia, energetic vibrations were set up even at a low velocity.

The final forms of static tubes developed are indicated by b and c in figure 4. They differ only in the dimensions of the center portion, which in both cases is massive. The first form (b) was very satisfactory but on account of its great weight appeared to be too inconvenient. For normal operation, therefore, the second form (c) was employed.

The data for the three types of tubes are given in table I (with fig. 4). The aerodynamic coefficients for static tube (a) were obtained from the results of measurements in the wind tunnel, and those for the others were estimated from these results and other available data. The results that were obtained with the three forms of static tubes in flight tests, were the following: The original static tube (a) showed unfavorable oscillations at velocities higher than about 180 kilometers per hour (112 miles per hour). The static tube b, however, remained completely undisturbed up to the highest velocity at which it was tested (about 450 kilometers per hour (280 miles per hour)). Similarly, no instability was observed with the tube c which has been used continuously for some years at velocities up to 300 kilometers per hour (186 miles per hour).

Bearing in mind the significance of the critical velocity considered in section 5 as the lower limit of the range of velocities within which unstable oscillations can occur, though not necessarily, then the agreement between theory and experiment may be considered satisfactory.

## 8. CONCLUSIONS

1. On the basis of the assumptions discussed in section 2, two stability criteria may be derived for a body towed by a cable. Both of these are sufficient though not necessary and refer to conditions under which it may be positively asserted that no unstable oscillations will occur.

2. The first criterion indicates that whenever inequality (30) is satisfied no unstable oscillations can be set up at all possible velocities.

3. The second criterion leads to the critical velocity $V_k$ which may be computed with the aid of formulas (34) to (36). This critical velocity forms the upper limit of the range of velocities within which no unstable oscillations can occur.

4. In the applications that have up to the present been considered, the first criterion set such a high requirement that it was not possible to apply it in practice.

5. The critical velocity $V_k$ can always be raised by increasing the weight (mg) and the damping ($k_D$) and by decreasing the lift coefficient slope ($k_L{}'$).

6. In the range here considered, an increase in the static stability ($k_M{}'$) and in the moment of inertia (I) lead to a decrease in the critical velocity.

7. By utilizing the results of the theory, the dynamic stability of the towed static tube could be considerably improved. The agreement between the results obtained with various forms of tubes and the computed values may be considered as very satisfactory.

## NOTATION

| | |
|---|---|
| $a_0$, $a_2$, $a_4$, | are coefficients (see section 5, (27)). |
| $g$, | acceleration of gravity. |
| $k_D$, $k_L$, $k_M$, $k_W$, | coefficients (see $M_2$, L, $M_1$, W). |
| m, | mass of body. |
| t, | time. |
| $v_x$, $v_y$, | components of the relative velocity V. |
| x, y, | coordinates (see fig. 1). |
| A, | work done by external force P (see section 4). |
| B, | (see section 3f, (20)). |
| I, | moment of inertia of body about transverse axis through its center of gravity. |
| $L = k_L \frac{1}{2} \rho V^2$, | lift component of the aerodynamic force at right angles to the relative velocity (see fig. 3). |
| $M_1 = -k_M \frac{1}{2} \rho V^2$, | pitching moment (see section 3d). |
| $M_2 = -k_D \frac{1}{2} \rho V \dot{\vartheta}$, | damping moment (see section 3d). |
| P, | external force (for subscripts see section 4). |
| T, | tension of the cable. |
| V, | relative velocity of body with respect to the air. |

$V_o$, flight velocity.

$V_k$, critical velocity (see section 5).

$W = k_W \frac{1}{2} \rho V^2$, drag component of the aerodynamic force in the direction of the relative velocity (see fig. 3).

$X$, component of external force in the $\xi$-direction.

$\alpha$, angle of attack of body (see fig. 2),

$\beta$, angle between relative velocity and x-axis.

$\vartheta$, angle of rotation about the transverse axis through the center of gravity (see fig. 1).

$\lambda$, frequency.

$\rho$, mass density of the air.

$\varphi$, angle between the direction of the cable in the position of equilibrium and the y-axis (see fig. 1).

$\xi$, displacement of the center of gravity of the body from the position of equilibrium normal to the initial cable direction (see fig. 1).

$\xi_o$, amplitude of forced oscillations (see section 4).

$\theta$, angle between the initial cable direction and that made by the body in motion (see fig. 1).

The subscript k denotes that the magnitude has reference to the critical velocity.

Derivatives with respect to time are denoted by dots above the symbol; those with respect to the angle of attack, by a prime.

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## LEGENDS

FIGURE 1.- Full line denotes position of body in motion; dotted line denotes equilibrium position.
1 = line of symmetry of body.
2 = cable.

FIGURE 2.- Relative velocity $V$ and angle of attack $\alpha$.

FIGURE 3.- Forces acting on the body.

FIGURE 4.- Towed static tubes.
a - original unstable form.
b, c - very stable forms.

## REFERENCE

1. Glauert, H.: The Stability of a Body Towed by a Light Wire. R. & M. No. 1312, British A.R.C., 1930.

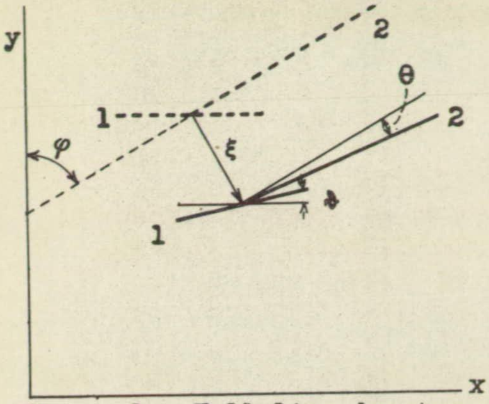


Figure 1.- Full line denotes position of body in motion, dotted line denotes equilibrium position. 1=line of symmetry of body; 2 = cable.

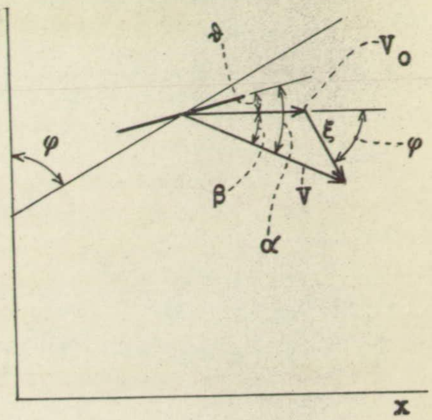


Figure 2.- Relative velocity V and angle of attack α.

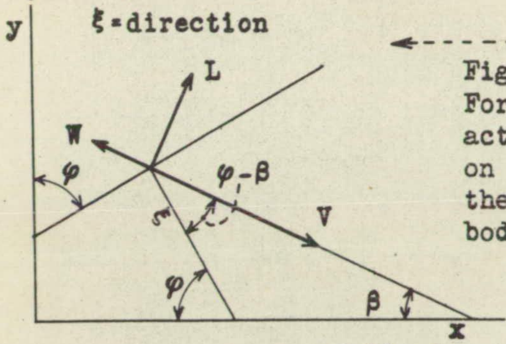


Figure 3.- Forces acting on the body.

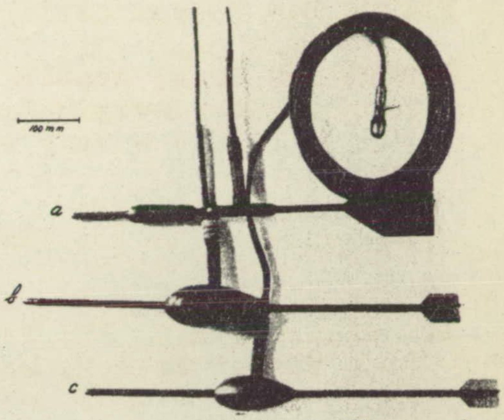


Figure 4.- Towed static tubes a: Original unstable form, b,c: very stable forms.

Table I

| Static tube | | a | b | c |
|---|---|---|---|---|
| mg | kg<br>lb | 1.445<br>3.18567 | 4.092<br>9.0213 | 2.274<br>5.0133 |
| I | $kg \cdot m \cdot sec^2$<br>lb·ft " | 0.00331<br>0.02394 | 0.00345<br>0.02495 | 0.00293<br>0.02119 |
| $k_{L'}$ | $m^2$<br>$ft^2$ | 0.0103<br>0.11086 | 0.0029<br>0.03121 | 0.0029<br>0.03121 |
| $k_W$ | $m^2$<br>$ft^2$ | 0.0063<br>0.0678 | 0.0080<br>0.0861 | 0.0063<br>0.0678 |
| $k_{M'}$ | $m^3$<br>$ft^3$ | 0.00424<br>0.14973 | 0.00172<br>0.06074 | 0.00172<br>0.06074 |
| $k_D$ | $m^4$<br>$ft^4$ | 0.00106<br>0.12281 | 0.00061<br>0.07067 | 0.00061<br>0.07067 |
| ρ | $kg \cdot m^{-4}\ sec^2$<br>$lb \cdot ft^{-4}$ " | 0.125<br>0.00237 | 0.125<br>0.00237 | 0.125<br>0.00237 |
| $V_k$ | $m \cdot sec^{-1}$<br>ft " | 41<br>134.514 | 96<br>314.960 | 75<br>246.062 |