

TECHNICAL MEMORANDUMS
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METHODS AND FORMULAS FOR CALCULATING THE STRENGTH OF
PLATE AND SHELL CONSTRUCTIONS AS USED IN AIRPLANE DESIGN

By O. S. Reck and H. Ebner

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PLATE AND SHELL CONSTRUCTIONS AS USED IN AIRPLANE DESIGN*

By O. S. Heck and H. Ebner

SUMMARY

This report is a compilation of previously published articles on formulas and methods of calculation for the determination of the strength and stability of plate and shell construction as employed in airplane design. In particular, it treats the problem of isotropic, orthotropic, and stiffened rectangular plates, thin curved panels, and circular cylinders under various loading conditions. The purpose of appending the pertinent literature references following the subjects discussed was to facilitate a comprehensive study of the treated problems.

I. INTRODUCTION

The purpose of this article is to gather the available simple formulas and methods of calculating the strength of thin-walled structures (plate and shell) that are increasingly important in airplane design into one comprehensive report and to facilitate the study of the original reports by appending an exhaustive list of pertinent literature. One essential characteristic of the treated designs is that the skin plating is called on to carry stresses as well as the stiffeners. Designs in which the sheet merely serves as covering and is not loaded to correspond to its strength are without the scope of this study.

The structural components of plate and shell structures (particularly, shell bodies and wings) are plain and stiffened plates and shells, usually of very light gage (about 0.5 to 1.2 mm (0.0197 to 0.047 in.)). In such

*"Formeln und Berechnungsverfahren für die Festigkeit von Platten- und Schalenkonstruktionen im Flugzeugbau." Luftfahrtforschung, February 6, 1935, pp. 211-222.

structures the solution of the stress problem is subordinate to the question of stability and particularly, to the strength of a structure after exceeding the stability limit. The question of the buckling strength of sections will not be included within the scope of this article, although the buckling stresses of the stiffeners alone, i.e., as self-contained compression members, are frequently used for the determination of the strength of a plate or shell design.

The material to be considered is divided according to the three most essential structural elements of plate or shell construction:

1. Strength of rectangular plate;
2. Strength of thin curved panels (segment of circular shell);
3. Strength of cylindrical shells (circular cylinder).

These structural elements are further divided into isotropic, orthotropic (orthogonally anisotropic) and stiffened plates or shells. Orthotropic plates and shells having unequal but constant stiffness in mutually perpendicular direction are, for example, plates and shells of plywood and, strictly speaking, those also of rolled plate with different moduli of elasticity in directions parallel and perpendicular to the direction of rolling.

In many cases stiffened flat plates and sheets stiffened by corrugating can be treated as orthotropic plates by substituting for the periodically changing stiffnesses in the directions (mutually perpendicular) of the stiffeners, constant mean stiffnesses. The kind of loading of the individual element affords yet another subdivision.

The theoretical formulas are, wherever possible, compared to test data. When applying theoretical formulas for the strength of thin-walled structures, it is particularly advisable to ascertain whether and to what extent they have been experimentally verified. The test data in many cases are quite different from the theoretical values. In stability investigations of very thin unstiffened plates and shells, unavoidable initial buckling - that, compared to the wall thickness used, can be regarded only as small - reduces the actual buckling load relative to the theoretical buckling load of the ideal plate or shell considerably. At times the experimental data manifest

an appreciable scatter, so that in many cases any conclusion as to the behavior of a structural component on the basis of the result of one or a few tests must be drawn with caution.

The present report reveals that there are still considerable gaps in our knowledge of the strength of plate and shell constructions. Notwithstanding the numerous formulas, there are still many cases not covered by stability investigations of plates and shells under combined stresses, as well as investigations of the stability of shells of other than circular or of variable section (such as conical shells).

Many problems still call for study in connection with the requirements of airplane design.

The DVL undertook the solution of these problems. Experiments were made on stiffened cylindrical panels as structural components of shell bodies together with experiments on complete, stiffened cylindrical shells (monocoque bodies), the results of which are recounted. The program further included a theoretical and experimental investigation of the behavior of cylindrical shells of elliptical section under bending stresses.

Notation

- E, modulus of elasticity in tension
- G, modulus of shear
- ν , Poisson's ratio (approximately = 0.3 for steel and duralumin)
- δ , wall thickness of plate or shell
- a, b, edges of rectangular plate
- r, radius of curvature of shell
- l, length of shell

II. STRENGTH OF RECTANGULAR PLATES

1. Isotropic Plates

a) Loaded in compression.- We first give the formulas defining the critical stresses for various important specific cases of rectangular plates under compressive load. (See references 15, 16, 22, 23, 24, 26, pp. 9 & 10.)

α) Uniform Compressive Load on Opposite Edges of the Plate

Let a represent the unloaded and b the loaded edges of the plate of thickness δ .

The critical compressive stress σ_{kr} (at the moment of buckling) follows from formula

$$\sigma_{kr} = k \frac{E}{1 - \nu^2} \left(\frac{\delta}{b} \right)^2 \quad (1)$$

The factor k in this formula (1) is dependent upon the aspect ratio a/b and the edge conditions of the plate. The values are given below for different edge conditions.

1) All edges of the plate simply supported*

$$\frac{a}{b} = 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 1.2 \quad 1.4 \quad 1.6 \quad 1.8 \quad 2.0 \quad 2.2$$

$$k = 6.92 \quad 4.23 \quad 3.45 \quad 3.29 \quad 3.40 \quad 3.68 \quad 3.45 \quad 3.32 \quad 3.29$$

3.32

$$\frac{a}{b} = 2.4 \quad 2.7 \quad 3 \quad \infty$$

$$k = 3.40 \quad 3.32 \quad 3.29 \quad 3.29$$

2) The four edges of the plate are clamped (reference 26)**

$$\frac{a}{b} = 1 \quad 2 \quad 3 \quad \infty$$

$$k = 7.7 \quad 6.7 \quad 6.4 \quad 6.0$$

*Simply "supported" is hereinafter assumed synonymous with hinged support.

**The limiting values of k for $\frac{a}{b} = \infty$ really should be the same for case 2 and 3. The value $k = 5.73$ is more accurate.

3) Edges a are clamped; edges b simply supported*

$\frac{a}{b} =$	0.4	0.5	0.6	0.7	0.8	1.0	1.2	1.4	1.6	1.8	2.1	∞
k =	7.76	6.32	5.80	5.76	6.00	6.32	5.80	5.76	6.0			
		5.8	5.76	5.73.								

β) Linearly Varying Load (in the Plane of the Sheet)
Applied to Opposite Edges of the Plate

All edges are assumed simply supported and the load on the plate is:

$$\sigma = \sigma_o \left(1 - \frac{y}{\alpha b}\right) \quad \text{with} \quad \alpha = \frac{\sigma_o}{\sigma_o - \sigma_u} \quad (\text{fig. 2})$$

Then the critical stress follows from

$$\sigma_{o_{kr}} = k \frac{E}{1 - \nu^2} \left(\frac{\delta}{b}\right)^2 \quad (2)$$

$\frac{a}{b} =$	0.4	0.5	0.6	0.667	0.75	0.8	0.9	1.0	1.5
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$\alpha=0.5;$	k=23.9	21.1	19.8	19.7	19.8	20.1	21.1	21.1	19.8
0.75;	15.4		10.6		9.5	9.2		9.1	9.5
1.00;	12.4		8.0		6.9	6.7		6.4	6.9
1.25;	10.95		6.8		5.8	5.7		5.4	5.8
1.50;	8.9		5.8		5.0	4.9		4.8	5.0

γ) Uniform Compression Along Each Edge of the Plate

Edges b are under compressive load σ_1 and edges a under compression σ_2 (fig. 3). If all edges of the plate are simply supported the critical σ_1 and σ_2 follow from

$$\sigma_{1_{kr}} \frac{m^2}{a^2} + \sigma_{2_{kr}} \frac{n^2}{b^2} = 0.823 \frac{E}{1 - \nu^2} \delta^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 \quad (3)$$

Here m denotes the number of half waves of the buckled plate in direction x (parallel to edges a), and n

*The limiting values of k for $\frac{a}{b} = \infty$ really should be the same for case 2 and 3. The value k = 5.73 is more accurate.

those in direction y (parallel to edges b); m and n are integral and to be so chosen that σ_{1kr} and σ_{2kr} take on the smallest possible values.

In the special case of $\sigma_1 = \sigma_2 = \sigma$ the critical load is determined by

$$\sigma_{kr} = k \frac{E}{1 - \nu^2} \left(\frac{\delta}{b} \right)^2 \quad (4)$$

wherein k depends on the edge conditions and aspect ratio a/b of the plate.

1) All edges of the plate are simply supported

$\frac{a}{b} = 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5 \quad 0.6 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1.0$

$k = 83.1 \quad 21.4 \quad 9.96 \quad 5.96 \quad 4.11 \quad 3.11 \quad 2.50 \quad 2.11 \quad 1.84$
 1.645

2) All edges of the plate are clamped (reference 15)

$\frac{a}{b} = 1, \quad k = 4.36$

According to Cox (references 16, 26), the theoretical values for the buckling stresses of plates loaded in compression are in fairly close agreement with the experimental data for the edge conditions for which they have been experimentally checked.

Behavior after Buckling

In many cases plates loaded in compression are still able after buckling to carry considerable additional loads before failure. In a plate with simply supported or clamped edges and two opposite edges under compressive load, the stress distribution over the section ceases to be uniform once the buckling load has been exceeded, and the stress increases toward the corners. The plate reaches its strength limit when the maximum compressive stress reaches the yield point of the material, or when the sheet, supported by riveted stiffeners along the unloaded edges, buckles between the rivets or when in the lateral stiffeners which are compressed simultaneously with the sheet, the compressive stresses have reached critical values. The

ultimate load P_B of a plate under compression can be calculated from the effective width $2w$ by the following formula:

$$P_B = 2w \delta \sigma \quad (5)$$

in which the value of σ is to be taken as (according to the yield point of the material) either the stress at which the plate buckles between stiffener rivets or the buckling stress of the stiffeners themselves. For the effective width $2w$ with simply supported edges, v. Karman (references 14, 25) gives the formula:

$$2w = C \sqrt{\frac{E}{\sigma}} \delta \quad (6)$$

where σ has the same significance as in (5), and the value $C = \frac{\pi}{\sqrt{3(1-\nu^2)}} = 1.90$ at $\nu = 0.3$ replaces the

constant C . Steel with a yield point of 30 kg mm^{-2} ($42870.5 \text{ lb. per sq.in.}$) has an effective width $2w$, approximately equal to 50δ , and duralumin with a yield point of 27 kg mm^{-2} ($38403.5 \text{ lb. per sq.in.}$) has an effective width of $2w$ approximately equal to 30δ , assuming the yield point of the material as determining the failure.

For sheet free at one edge and simply supported at the other three edges, v. Karman computes the effective width at

$$w = 0.68 \sqrt{\frac{E}{\sigma}} \delta \quad (7)$$

Cox's formula (references 16, 26) for $2w$ is:

$$2w = C \sqrt{\frac{E}{\sigma}} \delta + Db \quad (8)$$

i.e., the effective width varies slightly with the width b of the plate. C and D are constants dependent on the edge conditions of the plate. When all sides of the plate are simply supported: $C = 1.52$, $D = 0.09$. When the loaded edges are simply supported, the others clamped: $C = 2.18$, $D = 0.14$.

According to Cox, the experiments available (refer-

ences 16, 17) agree quite well with formula (8). Other pertinent articles on the subject are the reports by G. Schnadel (references 9, 10, 11), which are likewise concerned with the behavior of plates loaded in compression after exceeding the limits of stability.

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b) Buckling stresses.— The critical shearing stress τ_{kr} of a rectangular plate, with edges a and b and wall thickness δ , loaded in shear, is:

$$\tau_{kr} = k \frac{E}{1 - \nu^2} \left(\frac{\delta}{b}\right)^2 \quad (9)$$

The coefficient k in (9) depends on the edge conditions and on the aspect ratio a/b of the plate.

- 1) When all edges of the plate are simply supported (references 11, 12, 13)

$\frac{a}{b}$	= 1.0	1.2	1.4	1.5	1.6	1.8	2.0	2.5	3.0	∞
k	= 7.75	6.58	6.00	5.84	5.76	5.59	5.43	5.18	5.02	4.4

- 2) When all edges are clamped (reference 14)

$\frac{a}{b}$	= 1	2	∞
k	= 12.7	9.5	7.4

Cox (reference 14), while conceding fair agreement between theory and test data for approximately rectangular plates, observes that plates having an aspect ratio a/b that differs much from unity, can buckle long before the theoretical critical load is reached. Seydel (reference 8) points to the markedly disturbing effect of initial buckles in thin plates. Bollenrath (reference 6) states that in his investigations with celluloid plates, the experimental values for critical shear stresses are approximately 43 percent below Southwell and Skan's (reference 2) theoretical figures.

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c) Combined loading.— For a long rectangular plate uniformly loaded under combined compression (σ_x), or tension (σ_y), and shear (τ) (fig. 4), Wagner has established two formulas for calculating the critical stress τ_{kr} with given values of σ_x and σ_y (reference 1):

With simply supported edges,

$$\tau_{kr}^2 = \kappa^2 \left(2 \sqrt{\frac{\sigma_y}{\kappa} + 1} + 2 + \frac{\sigma_x}{\kappa} \right) \left(2 \sqrt{\frac{\sigma_y}{\kappa} + 1} + 6 + \frac{\sigma_x}{\kappa} \right) \quad (10)$$

With clamped edges,

$$\tau_{kr}^2 = \kappa^2 \left(\frac{4}{\sqrt{3}} \sqrt{\frac{\sigma_y}{\kappa} + 4} + \frac{4}{3} + \frac{\sigma_x}{\kappa} \right) \left(\frac{4}{\sqrt{3}} \sqrt{\frac{\sigma_y}{\kappa} + 4} + 8 + \frac{\sigma_x}{\kappa} \right)$$

$$\kappa = \frac{\pi^2}{12} \frac{E}{1 - \nu^2} \left(\frac{\delta}{b} \right)^2 \quad (11)$$

He also represents these formulas (10) and (11) graphically by plotting curves $\tau_{kr} = \text{constant}$ with values of σ_x and σ_y as coordinates.

The stability of a rectangular plate under combined bending and shear (fig. 5) is treated in an article by O. Stein (reference 2).

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2. Orthotropic Plates

When the stiffeners are spaced sufficiently close, such a plate may in many cases be considered as orthotropic, i.e., as a plate with unequal stiffness in two mutually perpendicular directions. This is particularly valid in the investigation of the stability of corrugated plate.

a) Loaded in compression.— According to Dean (references 1, 5), the critical compressive stress σ_{kr} of a corrugated plate with edges a and b (a is parallel to the corrugations), when the edges b are uniformly loaded in compression (fig. 1), is:

$$\sigma_{kr} = k \frac{E}{1 - \nu^2} \left(\frac{\delta}{a} \right)^2 \quad (12)$$

When s = chord length of half wave of the corrugation and f = wave depth of corrugation (distance of highest point from median plane), and if $\frac{s}{a} < 0.2$, then the value of k is given approximately by

$$k = 5.8 \frac{f^2}{\delta^2} \quad (13)$$

Consequently the buckling stress σ_{kr} is unaffected by the sheet thickness δ ; k also changes probably with the half-wave length λ of the sheet, although it does not appear from (13).

Yamana's (reference 3) formula for the critical compressive load s_{kr} (loading per unit length along the edge) for an orthotropic plate with simply supported edges is:

$$s_{kr} = 2 \frac{\pi^2}{b^2} (\sqrt{D_1 D_2} + D_3) \quad (14)$$

D_1 and D_2 denote the bending stiffness of the orthotropic plate, D_3 depends on D_1 and D_2 and on the torsional stiffness of the plate. The calculation of these quantities is given on a subsequent page.

Formula (14) agrees with one of Wagner's (reference 2) for the stability of a stiffened plate loaded in compression.

If the unloaded edges of the orthotropic plate are clamped, the critical compressive load follows from

$$s_{kr} = 2 \frac{\pi^2}{b^2} \left(2D_1 \frac{b^2}{a^2} + D_3 \right) \quad (15)$$

in which, however, D_2 is assumed small with respect to D_1 .

Regarding the results of U.S. investigations on corrugated sheet loaded in compression, see reference 2.

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b) Buckling stresses.— The formula for the critical shear load t_{kr} (per unit length of plate edge) of an orthogonally anisotropic (orthotropic) rectangular plate (see fig. 6) with edges a (in direction of x axis) and b (in direction of y axis), of bending stiffness D_1 in x direction (i.e., by bending about y axis) and bending stiffness D_2 in y direction (i.e., by bending about x axis) (reference 8) is:

$$t_{kr} = c_a \frac{\sqrt[4]{D_1 D_2^3}}{\left(\frac{b}{2}\right)^2} \quad (16)$$

where c_a is a coefficient dependent on parameters

$$\vartheta = \frac{\sqrt{D_1 D_2}}{D_3} \quad \text{and} \quad \beta_a = \frac{b}{a} \sqrt[4]{\frac{D_1}{D_2}}$$

and which may be taken, according to Seydel, from figure 7 for ϑ values from 1 to ∞ and for β_a values from 0 to 1. The c_a values of figure 7 hold for the case of simple support on all four sides. Quantity D_3 depends on D_1 , D_2 and the torsional stiffness of the plate. The formulas for calculating D_1 , D_2 , and D_3 are given on a subsequent page.

The critical shear load t_{kr} of very long orthotropic plates (strictly speaking, infinitely long compared to width b) may be determined from formula (16); for ϑ values between 1 and ∞ with simply supported or clamped edges, c_a is taken from the following table (reference 4):

	$\vartheta = 1$	2	3	5	10	20
Edges simply supported:	$c_a = 13.17$	10.8	9.95	9.25	8.7	8.4
Edges clamped:	$c_a = 22.15$	18.75	17.55	16.6	15.85	15.45
	$\vartheta = 40 \infty$					
Edges simply supported:	$c_a = 8.25$	8.125				
Edges clamped:	$c_a = 15.25$	15.07				

If the ϑ values of the infinitely long orthotropic plate range from 0 to 1, the critical shear load t_{kr} is computed from:

$$t_{kr} = c_b \frac{\sqrt{D_2 D_3}}{\left(\frac{b}{2}\right)^2} \quad (17)$$

the values of c_b being taken from the following table:

	$\vartheta = 0$	0.2	0.5	1.0
Edges simply supported:	$c_a = 11.71$	11.8	12.2	13.17
Edges clamped:	$c_a = 18.59$	18.85	19.9	22.15

The half-wave length of the buckled plate in the direction of the x-axis follows from:

$$l_{kr} = c_a' \frac{b}{2} \sqrt[4]{\frac{D_1}{D_2}} \quad \text{for } \vartheta \geq 1 \quad (18)$$

$$l_{kr} = c_b' \frac{b}{2} \sqrt[2]{\frac{D_3}{D_2}} \quad \text{for } \vartheta \leq 1 \quad (19)$$

The factors c_a' and c_b' are taken from the following tables:

	$\vartheta =$	1	2	3	5	10	30
Edges simply supported:	$c_a' =$	2.49	2.28	2.16	2.13	2.08	2.05
Edges clamped:	$c_a' =$	1.66	1.54	1.48	1.44	1.41	1.38
	$\vartheta =$.05	.2	.5	1		
Edges simply supported:	$c_b' =$	1.92	1.94	2.07	2.49		
Edges clamped:	$c_b' =$	1.16	1.20	1.36	1.66		

The validity of considering a plate with periodically changing stiffness in two mutually perpendicular directions (parallel to plate edges) as orthotropic is governed by the half-wave length. This length l_{cr} must be several times larger than the spacing of the plate stiffeners or the width of corrugation before the plate may be treated approximately as orthotropic with regard to its critical behavior.

The formulas for calculating quantities: D_1 , D_2 , and D_3 (reference 4) are as follows:

$$D_1 = \frac{(EJ)_x}{1 - \nu_x \nu_y} \quad (20)$$

$$D_2 = \frac{(EJ)_y}{1 - \nu_x \nu_y} \quad (21)$$

$$D_3 = \frac{1}{2} (\nu_x D_2 + \nu_y D_1) + 2 (GJ)_{xy}$$

where D_1 is the bending stiffness of the plate in x direction (i.e., in bending about y axis);

D_2 , the bending stiffness of the plate in y direction (i.e., in bending about x axis);

$4(GJ)_{xy}$ the torsional stiffness of the orthotropic plate;

ν_x, ν_y are Poisson's ratios belonging to the bending stiffnesses in x and y direction of the plate (definition of ν_x, ν_y given in reference 8).

The following relation holds

$$\nu_y D_1 = \nu_x D_2$$

by use of which it follows that

$$D_3 = \nu_y D_1 + 2 (GJ)_{xy} = \nu_x D_2 + 2 (GJ)_{yx} \quad (22)$$

For sheet with corrugations running parallel to the y-axis (see fig. 8):

$$D_1 = \frac{l}{s} \frac{E \delta^3}{12} \frac{1}{1 - \nu^2}$$

l = arc length of half wave

s = chord length of half wave

$$D_2 = EJ$$

whereby J is the mean moment of inertia per unit of length of a section parallel to the x-axis, with respect to the axis lying in the median plane of the sheet, and

$$D_3 = \frac{s}{l} \frac{G \delta^3}{6}$$

neglecting $\nu_y D_1$. Further details are given in reference 7.

The experimental results for corrugated sheet loaded in shear are in accord with theory (reference 7).

A simple formula for appraising the critical shear load of an orthotropic stiffened plate has also been given by Wagner (reference 5).

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3. Stiffened Sheet

With regard to the strength of stiffened plates, it is important to know the stability limit of the stiffened plate in compression and in shear. But in most cases its strength, rather than being exhausted upon reaching this limit, still permits considerable load increase before failure takes place.

a) Loaded in compression.- When the stiffeners are sufficiently close together the stability investigation of

* Compare also, J. Jenissen: Investigations of Corrugated Sheet Loaded in Shear. Thesis, Technische Hochschule, Aachen, 1932.

a stiffened plate may in many cases proceed on the assumption of an orthotropic plate, and the formulas given in section II,2 may be employed. For determining the buckling load of a panel between stiffeners spaced farther apart, formula (1) may be used, but the bending of the stiffeners must be assumed small enough to permit the sides of the panel to be considered as straight. The effect of flexibility in bending of stiffeners on the stability of the plate panels between the stiffeners is treated in a report by Timoschenko (reference 1). The critical compressive load of the stiffened sheet is obtained from formula (1); but the coefficient k depends, aside from the edge conditions of the plate and its aspect ratio a/b , on the number of stiffeners, the ratio of the bending stiffness of the stiffeners and sheet, and on the ratio of sectional areas of stiffeners and sheet. Timoschenko has calculated the coefficient k for a number of cases and has given the values in tabular form (references 1, 7, 8).

A most important factor in airplane design is the knowledge of ultimate load of a stiffened plate stressed in compression parallel to the stiffeners. Three methods have been suggested for approximate calculation of the ultimate load (reference 3):

1. Determine the ultimate loads that can be carried by the stiffeners alone and the ultimate load of the sheet between two stiffeners (calculation or experiment), the lateral sheet edges being assumed as rigidly supported; then add the ultimate load of sheet and stiffeners, which gives the ultimate load of the stiffened plate (reference 5).
2. Determine the ultimate load of one stiffener (by test) and compute the effective width $2w_e$, or w_e , of the sheet from the ultimate load σ of the stiffener, according to formulas (6) and (7). The total area of the stiffeners and of the effective plate multiplied by the ultimate stress σ then gives the ultimate load of the stiffened plate.

Lundquist (reference 3) concludes from experiments by Schuman and Back (reference 4) that it would be better to use the coefficients 1.70 and 0.60 than 1.90 and 0.68 in equations (6) and (7) of section II,1, so as to give the following formula for the effective width of the sheet of

a stiffened structure:

$$2w = 1.70 \sqrt{\frac{E}{\sigma}} \delta \quad (23)$$

for effective width of sheet between two stiffeners and

$$w = 0.60 \sqrt{\frac{E}{\sigma}} \delta \quad (24)$$

for effective width of sheet at an unsupported edge.

3. Plot against slenderness ratio the ultimate stress σ of the stiffeners alone. Formula (23) or (24) affords the effective width of the skin with an estimated ultimate stress. Define the slenderness ratio of the combination, stiffener plus skin, and determine from the originally plotted curve (which strictly is valid only for the profile) the pertinent ultimate stress σ . If this σ does not agree with the assumed σ , the method is repeated. Two to three iterations suffice as a rule. The final ultimate stress and the section of the stiffeners and skin give the ultimate load of the stiffened plate.

According to American experiments (reference 3), this last method recommends itself particularly for calculating the ultimate load of a stiffened shell. Attention should be called to the fact that, in determining the strength of the stiffener by test, care must be taken to assure failure of the stiffeners in the same manner as it would occur in the combination of plate and stiffener; that is, buckling perpendicular to the shell.

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b) Strength in shear.- Wagner (references 1 and 2) made an elaborate investigation of the stability and strength of a stiffened plate in shear under the assumption that the thin skin plating buckles under very small loads. Disregarding the bending stresses due to wrinkling, a one-dimensional state of stress develops in the skin, the so-called "diagonal-tension field," whose properties have also been described in detail by Wagner.

Two cases of practical importance are analyzed with Wagner's simple method. The edge members (i.e., flanges or chords and web stiffeners) are assumed stiff enough to

be considered rigid in bending, an assumption which is permissible for many practical cases. With regard to allowance for the effect of flexibility in bending of the flanges, we refer to Wagner's report (reference 2).

1) Plate Girder with Parallel Chords and Vertical Stiffeners

A plate girder having parallel chords (spacing h) and vertical stiffeners (spacing t) is clamped at one end and loaded at the other with a load Q (fig. 9). If the edge members are stiff in bending, the tensile stress σ in the sheet and the direction α of the wrinkles are constant. With δ = sheet thickness, the tensile stress in the sheet is:

$$\sigma = \frac{2Q}{h\delta} \frac{1}{\sin 2\alpha} \quad (25)$$

the compression in an upright,

$$-V = Q \frac{t}{h} \tan \alpha \quad (26)$$

the load in the upper chord,

$$H_o = \frac{Q \cdot x}{h} - \frac{Q}{2} \cot \alpha, \quad (27)$$

and in the lower chord,

$$H_u = -\frac{Q \cdot x}{h} - \frac{Q}{2} \cos \alpha \quad (28)$$

In the above, x = distance of section in which chord loads H_o and H_u are to be determined from the point of application of the load Q .

The maximum local bending moment in the chords (at points of uprights) is:

$$B_{H_{\max}} = \frac{Q}{h} \frac{t^2}{12} \tan \alpha \quad (29)$$

The direction of the wrinkles is given by:

$$\sin^2 \alpha = +\sqrt{a^2 + a} - a \quad \text{with} \quad a = \frac{1 + \frac{h\delta}{2F_H}}{\frac{t\delta}{F_V} - \frac{h\delta}{2F_H}} \quad (30,31)$$

(F_H = sectional area of chord, F_V = sectional area of upright.) For most cases it will be sufficient to estimate the value of α as 40° to 42° , giving the simplified formulas

$$\sigma = \frac{2Q}{h\delta} \quad (32)$$

$$-V = 0.9 Q \frac{t}{h} \quad (33)$$

$$H_{o,u} = \pm \frac{Q \cdot x}{h} - 0.6 Q \quad (34)$$

Wagner proves that the effect of fixity of the sheet at the edges where the wrinkles are interrupted, is negligibly small when the sheet is very thin.

2) Plate Girder with Nonparallel Rigid Flanges

Assuming constant shear Q and constant direction of wrinkles α in a section at distance x from the point of application of the load Q , the tensile stress in the sheet at the mid-height of the web is (fig. 10):

$$\sigma_m = \frac{Q}{\delta} \frac{h_{Q_x}}{h_x^2} \frac{1}{\sin \alpha \cos \alpha} \quad (35)$$

The tensile stress at the upper chord is:

$$\sigma_o = \sigma_m \frac{1}{(1 - \cot \alpha \tan \vartheta)^2} \quad (36)$$

and at the lower chord,

$$\sigma_u = \sigma_m \frac{1}{(1 + \cot \alpha \tan \vartheta)^2} \quad (37)$$

The angle ϑ is the angle between the axis of the beam and the chords. For the notations h_x and h_{Q_x} , see figure 11. The compressive load in an upright is:

$$-V = Q \frac{t}{h_x} \tan \alpha \quad (38)$$

However, the accuracy of these formulas is only approximate.

In a plate girder in which the uprights are eccentric to the web and riveted to it, the uprights are loaded in eccentric compression uniformly along their entire length. In contrast to ordinary compression members loaded eccentrically, the eccentric uprights of a plate girder have a true buckling load in combination with the web (stability problem). Wagner's investigations on this subject are cited in reference 2.

Experiments (references 1 and 3) have shown Wagner's simple calculations to be in close agreement with fact.

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III. STRENGTH OF THIN CURVED PANELS

1. Isotropic Panels

a) Loaded in axial compression.- Redshaw's (reference 1) theoretical formula of the critical compressive stress in an axially compressed isotropic thin curved panel is:

$$\sigma_{kr} = \frac{1}{6} \frac{E}{1 - \nu^2} \left\{ \sqrt{12 (1 - \nu^2) \left(\frac{\delta}{r}\right)^2 + \left(\frac{\pi\delta}{b}\right)^4} + \left(\frac{\pi\delta}{b}\right)^2 \right\} \quad (39)$$

where δ is wall thickness of panel

r , radius of curvature of panel

b , arc length of section of shell

The edges of the shell are assumed to be simply supported. If $(\delta/b)^2$ is negligibly small with respect to δ/r , formula (39) becomes (41) for the buckling stress of the axially compressed circular full shell; in the contrary case $\left(\frac{\delta}{r}\right)$ small compared to $\frac{\delta^2}{b^2}$ formula (39) gives the relation (1) for the critical compressive stress in a plate supported on all edges and compressed on two opposite edges of width b .

According to Redshaw, formula (39) is confirmed by test data, although additional investigations in this direction appear desirable.

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b) Buckling stresses.- Wagner's (reference 1) formulas for the critical shearing stress τ_{kr} of thin curved panels without stiffeners are:

$$\tau_{kr} = k_s E \frac{\delta}{r} + 5E \left(\frac{\delta}{b}\right)^2$$

when the sheet is simply supported, and

$$\tau_{kr} = k_s E \frac{\delta}{r} + 7.5 E \left(\frac{\delta}{b}\right)^2$$

for clamped edges; (b = arc length of curved edges of panel, k_s is approximately equal to 0.3).* G. M. Smith (reference 2) found from his experiments with panels curved to form the quadrant of a circle:

$$\tau_{kr} = k E \frac{\delta}{r} \tag{40}$$

where $k = 0.75$; the wall thickness δ of the test sheets ranged between 0.25 mm (0.010 in.) and 0.81 mm (0.032 in.); and the radii r , between 11 cm (4.3 in.) and 18 cm (6.9 in.).

*According to recent investigations by Wagner, it is k_s approximately equal to 0.1.

Formula (40) is equally applicable for the buckling shear, according to Smith, when the value of k is taken as 0.20. However, the experimental figures for the ultimate loads manifest considerable scattering.

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2. Thin Curved Sheet with Stiffeners Loaded in Axial Compression*

According to Lundquist (reference 1), the compressive strength of a longitudinally stiffened curved sheet can be calculated in the same fashion as that of a stiffened plate (section II, 3). Here again the last of the three methods recommends itself for calculating the strength. But, according to Newell's experiments (reference 2), the strength of a stiffened slightly curved sheet at high r/δ values (about 1000) is lower than that of a flat plate with stiffeners. Hence Lundquist's recommendation of a 10 to 15 percent reduction.**

*See also the article by Dji-Djuan Dschou, published in this issue of Luftfahrtforschung, p. 223.

**It was subsequently discovered that these experimental results at small curvatures were due to peculiarities of the test specimens. See J. S. Newell, Airway Age, November 1930, pp. 1422-23.

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IV. STRENGTH OF CYLINDRICAL SHELLS OF CIRCULAR SECTION

1. Isotropic Shells

a) Loaded in axial compression.— The stability failure of an axially compressed, unstiffened cylindrical shell may occur, according to the length of the shell, in either of the following ways:

1. Through lateral buckling of the shell as a whole (Euler case), or
2. Through local instability (buckling) of the cylinder wall.

The local buckling consists either in the formation of circular lobes, whereby all surfaces wrinkle symmetrically and the shell sections remain circular (axially symmetrical buckling), or in forming lobes, whereby the cross sections assume a wavy shape. In both cases the theoretical formula for calculating the buckling stress has been derived theoretically* as:

$$\sigma_{kr} = \frac{1}{\sqrt{3}} \frac{E}{\sqrt{1-\nu^2}} \frac{\delta}{r} \quad (41)$$

*One frequently encounters in literature a formula by Southwell for σ_{kr} (references 4, 5, 14, 17), in which the factor $\frac{k^2-1}{k^2+1}$ (k = number of waves in peripheral direction) occurs and which is to be applicable in other than axially symmetrical buckling, despite its disapproval by Robertson (reference 11) and by v. Sanden and Tolke (reference 8), and its omission in more recent English articles (references 17, 18).

Strictly speaking, formula (41) is valid only for the infinitely long shell. But the length effect disappears almost completely for lengths greater than a few half waves of the buckling measured in the axial direction. The half-wave length in the case of axially symmetrical buckling of the shell is:

$$\frac{\lambda}{2} = 1.72 \sqrt{r \delta} \quad (42)$$

Formulas (41) and (42) assume the validity of Hooke's law. When σ_{kr} lies above the proportional limit of the material, these formulas are to be replaced by (reference 7):

$$\sigma_{kr} = \frac{1}{\sqrt{3}} \sqrt{\frac{E K}{1 - \nu^2}} \frac{\delta}{r} \quad (43)$$

and

$$\frac{\lambda}{2} = 1.72 \sqrt{r \delta} \sqrt{\frac{K}{E}} \quad (44)$$

in which the "buckling modulus" K is calculated from

$$K = \frac{4 E E'}{(\sqrt{E} + \sqrt{E'})^2} \quad (45)$$

In formula (45) $E' = \frac{d\sigma}{d\epsilon}$ (slope of stress-strain curve) must be determined from the stress-strain diagram at a stress equal to σ_{kr} .

Various experiments to check formula (41) have been made, but they all give lower values than correspond to this formula; the majority of the experimental values for the buckling stress are from 40 to 60 percent of the theoretical values. As causes of this discrepancy, two explanations have been advanced - both of which would reduce the buckling load and for which no allowance has been made in the derivation of the theoretical formula. First, a test specimen always deviates more or less from the exact cylindrical shape (references 9, 10); second, owing to the radial displacement of the compressed shell which is prevented at the cylinder ends by the clamping or friction at the loading plates, the cylindrical surface is deformed locally, and this may also cause a reduction in buckling load (references 6, 7, 8, 9).

The analogy existing between the stability problem of the axially compressed cylinder shell and the buckling of an elastically supported member has been pointed out at various times. (For example, see reference 13.)

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b) Loaded in bending.— The theoretical formula (references 1, 2) developed for the bending moment B_{kr} at stability failure of a thin-walled cylindrical shell is:

$$B_{kr} = k \frac{E}{1 - \nu^2} r \delta^2 \quad (46)$$

Brazier's figure for k is $k = 0.99$. The mean value of k taken from 77 tests is $k_{\text{mean}} = 1.14$, the minimum value of k in these tests being 0.72. Brazier's theory applies directly only to the case of pure bending and long cylindrical shells. On the other hand, the experiments thus far have shown that, apart from very short shells, the length effect is negligible (reference 4) and that, moreover, a small shearing force fails to produce an appreciable reduction in critical bending moment (reference 3).

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On the Strength of Isotropic Cylindrical Shells in Bending

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2. Chwalla, E.: Reine Biegung schlanker, dünnwandiger Rohre mit gerader Achse. Z.f.a.M.M., Bd. 13 (1933), no. 1, pp. 48-53. (T)
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4. Lundquist, Eugene E.: Strength Tests of Thin-Walled Duralumin Cylinders in Pure Bending. T.N. No. 479, N.A.C.A., 1933. (V)
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c) Torsional loading.- A cylindrical shell twisted by end couples may fail through local buckling of the cylinder wall if the latter is thin enough. Donnell (reference 5) gives the following theoretical formulas for calculating

the critical shearing stress τ_{kr} with clamped or simply supported ends:

Theoretical critical shearing stresses,

$$\tau_{kr} = \frac{E}{1 - \nu^2} \left(\frac{\delta}{l}\right)^2 [4.6 + \sqrt{7.8 + 0.590 H^{3/2}}] \quad (47)$$

for clamped ends:

$$\tau_{kr} = \frac{E}{1 - \nu^2} \left(\frac{\delta}{l}\right)^2 [2.8 + \sqrt{2.6 + 0.494 H^{3/2}}] \quad (48)$$

$$H = \sqrt{1 - \nu^2} \frac{l^2}{\delta r} \quad \text{for simply supported ends.}$$

Where l = length, r = radius, and δ = plate thickness.

Formulas (47) and (48) are applicable only when the following inequalities hold:

$$\frac{l}{r} > 7.9 \sqrt[4]{1 - \nu^2} \sqrt{\frac{r}{\delta}} \quad (\text{clamped ends}) \quad (49)$$

$$\frac{l}{r} < 6.6 \sqrt[4]{1 - \nu^2} \sqrt{\frac{r}{\delta}} \quad (\text{simply supported ends}) \quad (50)$$

Otherwise (i.e., very long cylinders), the following formula must be resorted to for both cases of clamped and simply supported ends:

$$\tau_{kr} = 0.272 \frac{E}{(1 - \nu^2)^{3/4}} \left(\frac{\delta}{r}\right)^{3/2} \quad (51)$$

After evaluation of all available data on this subject, Donnell found that the critical shearing stresses determined by experiment averaged about 75 percent of the theoretical values. He attributes this discrepancy to inaccuracies in the shape of the test specimens. The lower limit of the experimental values amounts to about 60 percent of the theoretical. Multiplication of the right-hand sides of (47) and (48) by 0.6 and taking $\nu = 0.3$ gives the following formulas in place of (47) and (48), the values so obtained being on the safe side under all normal conditions.

Critical Shearing Stresses to Be Expected in Practice

Ends clamped:

$$\tau_{kr} = E \left(\frac{\delta}{l} \right)^2 \left[3.0 + \sqrt{3.4 + 0.240 \frac{l^3}{(\delta r)^{3/2}}} \right] \quad (52)$$

Ends simply supported:

$$\tau_{kr} = E \left(\frac{\delta}{l} \right)^2 \left[1.8 + \sqrt{1.2 + 0.201 \frac{l^3}{(\delta r)^{3/2}}} \right] \quad (53)$$

Lundquist (reference 6) gives a purely empirical formula for critical shearing stress:

$$\tau_{kr} = k E \left(\frac{\delta}{r} \right)^{1.35} \quad (54)$$

The factor k depends upon the ratio l/r and may be taken from the appended tabulation:

$\frac{l}{r} =$	0.2	0.25	0.3	0.4	0.5	0.75	1.0	1.5	2.0	3.0
	4.0	5.0								
$k =$	3.3	2.75	2.45	2.02	1.78	1.45	1.27	1.06	0.94	
	0.78	0.68	0.61							

The first theoretical investigation of the torsional stability of thin cylindrical shells is that by Schwerin (reference 1). His formula for infinitely long cylinder is:

$$\tau_{kr} = 0.248 E \left(1 + 0.45 \frac{\delta}{r} \right) \left(\frac{\delta}{r} \right)^{3/2} \quad (55)$$

He also analyzes the case of finite cylindrical shells with simply supported edges for ratios r/δ between 25 and 50 and represents his data in graphical form. But since the available experiments have been made on shells with substantially higher r/δ values, no direct comparison between Schwerin's analysis and the experimental data can be made. Using Schwerin's theory, the DVL made the calculation for the case of $\frac{r}{\delta} = 1000$ and simply supported

shell edges. The results of this calculation are fairly accurate at 75 percent of Donnell's theoretical values.

Sezawas' articles (references 2, 4) are noteworthy from the point of view of experimental technique, although his theory is inaccurate because of the omission of an important term in his equilibrium equations. The results of this theory are therefore at variance with Sezawas' experiments.

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d) Combined axial compression and torsion.— If σ_0 is the critical compressive stress of an isotropic cylindrical shell under axial compression alone, and τ_0 is

the critical shearing stress in torsion and σ and τ the critical stresses under combined load, then the following relation is approximately true:

$$1 - \frac{\sigma}{\sigma_0} = \left(\frac{\tau}{\tau_0} \right)^n \quad (56)$$

With predetermined values of σ_0 and τ_0 the equation furnishes the critical σ and τ under combined loading. Equation (56) is illustrated in figure 11. From experiments on cylindrical shells under combined torsion and axial compression or tension, the value of n was found to be approximately 3 (reference 1).

REFERENCE

On the Strength of Isotropic Cylindrical Shells Under Combined Loading

1. Bridget, F. J., Jerome, C. C., and Vosseller, A. B.:
Some New Experiments on Buckling of Thin-Wall Con-
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2. Orthotropic Shells

Combined loading in axial compression and external pressure torsional loading

Yamana (reference 1) explored the stability of orthotropic cylindrical shells under combined axial compression, radial external pressure and torsion. But his results do not permit direct determination of the critical loads from the final formulas. To prove his theory, he made experiments on cylindrical shells of corrugated sheet. The load tests in axial compression averaged only about 50 percent of the theoretical critical values. He made only one test under pure torsional load, and in this the discrepancy between theory and test amounted to around 12 percent. However, the number of buckles n predicted by theory corresponded in every case with those found by experiment.

REFERENCE

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Under Combined Loading

1. Yamana, M.: On the Elastic Stability of Aeroplane Structures. Jour. Fac. Eng'g, Tokyo Univ., vol. 20 (1933), no. 8, pp. 163-224. (TV)

3. Stiffened Cylinders

Approximate calculation for torsion

Various formulas are given below for the approximate calculation of the stresses in the skin and in the stiffeners of a stiffened circular cylinder under torsional moment T . Figure 12 illustrates the identifying features of the structure. The transverse stiffening members (frames) carry continuous stringers uniformly distributed over the periphery and are riveted to the skin plating. The spacing of the stringers is assumed to be small.

If the skin is very thin it will buckle under fairly small loads and will form diagonal tension fields. The analysis will be restricted to this case alone.

Assume the tension wrinkles are inclined at an angle α with respect to the cylinder axis; α can be approximated by a formula established by Wagner:

$$\tan^2 \alpha = \frac{1 - \frac{\sigma_x}{\sigma}}{1 - \frac{\sigma_y}{\sigma} + \frac{E}{\sigma} \frac{\varphi^2}{24}} \quad (57)$$

in which

σ is the diagonal tensile stress in the skin

σ_x , axial stress in the stringers

σ_y , axial stress in the frames

φ , angle at center subtended by two adjacent stringers

Formula (57) assumes the prior knowledge of the values of the stresses σ , σ_x , and σ_y . Using an estimating value of α , one first computes σ , σ_x , and σ_y from the subsequent equations (58), (59), and (61), and then obtains a more accurate value for α by the use of (57). The process is repeated if necessary.

The tensile stress σ in the skin along the wrinkles is given by

$$\sigma = \frac{T}{\pi r_H^2 \delta} \frac{1}{\sin 2\alpha} \quad (58)$$

where r_H = radius of curvature and δ = thickness of skin plating.

The tension field in the sheet produces the following axial stress in the stringers:

$$\sigma_x = - \frac{T}{n r_H F_x} \cot \alpha \quad (59)$$

where n = number of stringers and F_x = sectional area of a stringer.

Owing to the change in direction of the tension wrinkles at the stringers, the skin exerts a radial load inward on each stringer.

Designating this load per unit length of stringer with p and assuming p constant over the stringers, p is given by

$$p = \frac{T}{n r_H^2} \tan \alpha \quad (60)$$

The stringers must also be checked for buckling under axial load.

At the points of attachment between frames and stringers, the latter transfer their load $p \cdot t$ to the frames (t = frame spacing). These loads on the frames are directed radially inward, and give rise to an axial stress σ_y and a bending stress of maximum local value σ_b , equations for which follow:

$$\sigma_y = - \frac{T t}{2\pi r_H^2 F_y} \tan \alpha \quad (61)$$

and

$$\sigma_b = \frac{\pi}{6} \frac{T t}{n^2 k F_y} \frac{r_s}{r_H^2} \tan \alpha \quad (n \geq 6) \quad (62)$$

where F_y is the sectional area of frame

r_s , radius of centroidal axis of frame

$k = \frac{W}{F_y}$ (W , section modulus of frame section)

When the critical torque T_{kr} at which the sheet buckles between the stiffeners is of the same order of magnitude as T , T must be replaced by $T - T_{kr}$ in (58) to (61).

The ratio between bending and axial stress is:

$$\sigma_b : \sigma_y = \frac{\pi^2}{3} \frac{r_s}{n^2 k} \quad (63)$$

If the frames themselves are riveted to the skin, a part of the radially inward load exerted by the skin on the structure of the stiffeners, is directly transferred to the frames. In this case the use of the above formulas for calculating the stiffeners leaves one on the safe side.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

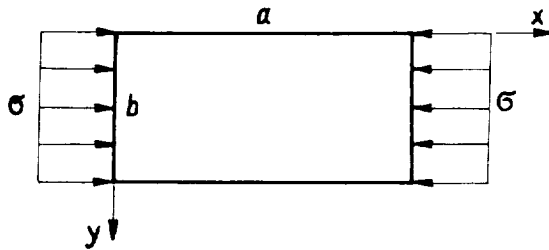


Figure 1.- Rectangular plate, two edges under uniform compression.

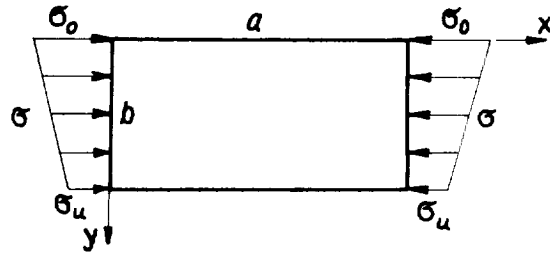


Figure 2.- Rectangular plate, two edges simultaneously loaded in bending and compression.

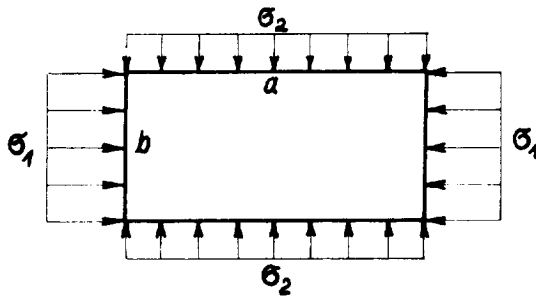


Figure 3.- Rectangular plate, all edges loaded in compression.

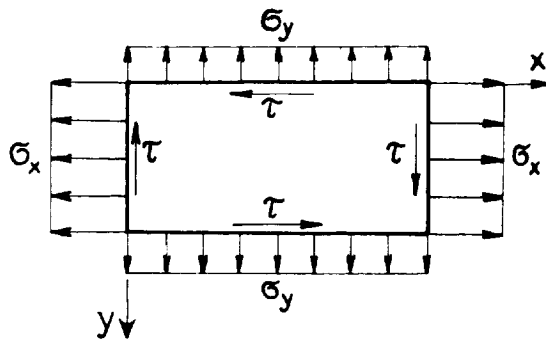


Figure 4.- Rectangular plate under combined compression and shear.

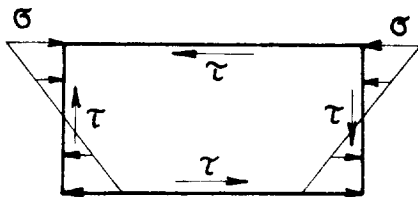


Figure 5.- Rectangular plate under combined bending and shear.

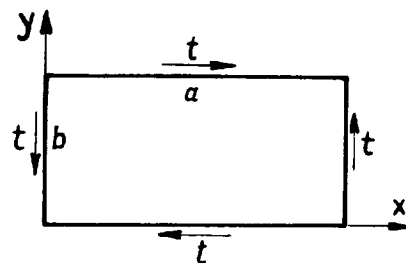


Figure 6.- Rectangular orthotropic plate loaded in shear.



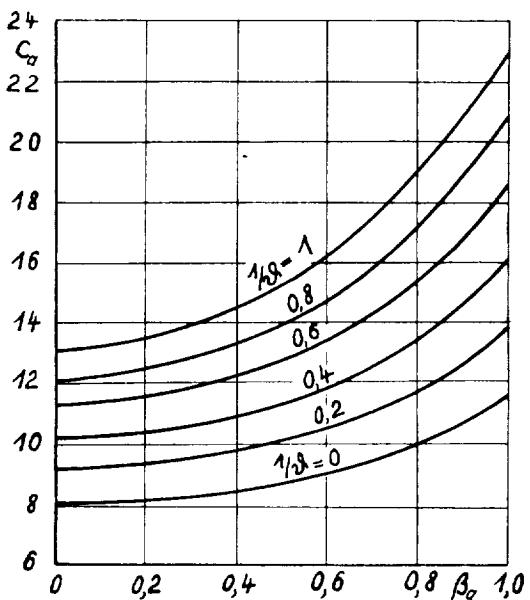


Figure 7.- Coefficient c_a versus β_a and $1/\delta$

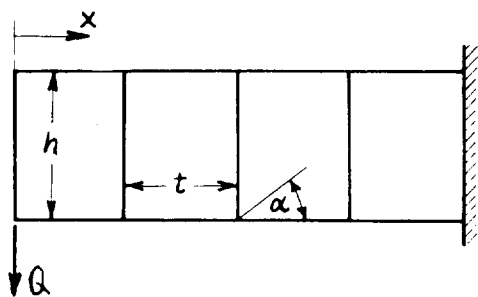


Figure 9.- Plate girder with parallel chords.

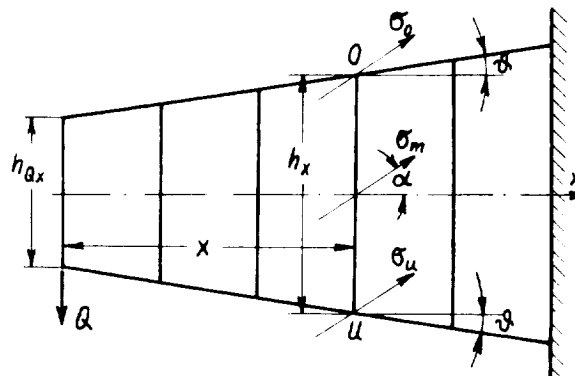


Figure 10.- Plate girder with non-parallel chords.

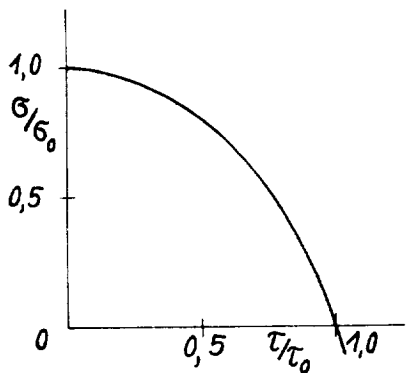


Figure 11.- Values σ/σ_0 versus τ/τ_0 .

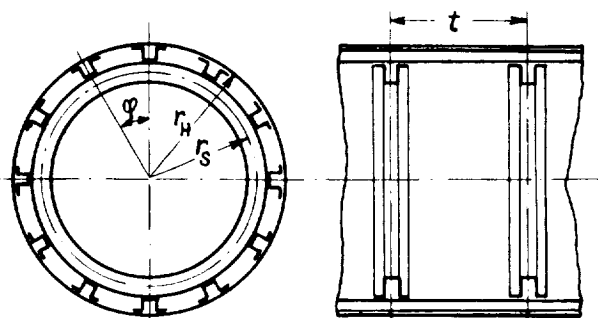


Figure 12.- Circular cylinder with stringers and frames.

