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THE BENDING OF BEAMS WITH THIN TENSION FLANGES*

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SUMMARY

The writer analyzes the action of a cantilever T beam with a tension flange so thin that it can carry only tensile stresses.

In airplane design we frequently find beams with thin-walled cover plates consisting of the same elements which serve as covering. To illustrate: The wing structure, consisting of spars and ribs, is very often covered with a thin sheet of wood or metal, which contributes materially to the stiffness of the whole. As far as torsion is concerned, the usual theory permits a safe enough analysis of the stresses and deformations. But for bending, the assumptions of the conservation of plane sections are no longer admissible; the calculations usually neglect the effect of the covering or we retain only a very small strip of it which, attached to the spar, bends solidly with it.

Metzer, referring the case to a T-section beam with infinite supports, resolved the problem for a strip of the cover plate, which may be assumed as contributing to the bending of the beams.** The resolution may equally be extended to include the case of cantilever beams, provided one of the conditions on the contour, i.e., multiplication of the shear along the outside edge, is disregarded. However, this theory proceeds from the assumption that the cover plate is suitable for transmitting even compressive stresses, something which leaves some uncertainty when we

*"La flessione delle travi con piastra sottile." (Laboratorio di Aeronautica della R. Scuola di Ingegneria.) Reprint from Atti della Reale Accademia delle Scienze di Torino, vol. 69, 1933-1934, pp. 171-187.

**Die Mittragende Breite. Luftfahrtforschung, vol. IV, no. 1, June 5, 1929.

consider that the sheets of the covering, not strong because of the curvature, already wrinkle under comparatively low loads.

In this study it is attempted to analyze the case in which the sheet, while wrinkling, is able to take up tension stresses. From the introduced hypothesis, discussed elsewhere in the report, that the wrinkles are parallel, we deduce the behavior of the stresses in the beams and from this the effective width of the sheet which, bending solidly with the beam, contributes to the bending of the whole in proportion to the amount supported by the sheet. Then we deduce the term for the work of deformation and find the slope of the wrinkling with respect to the beam axis, for which this is minimum.

Lastly, we deduce the simple formulas with which we can determine the effective sheet width accurately enough to apply to cases generally encountered in practice.

FUNDAMENTAL EQUATION

Consider (fig. 1) a beam of constant section, restrained at one end and free at the other, subjected to normal loads at its axis located in a plane containing a principal axis of the generic section. The resulting bending moment carries always the same sign over the whole length of the beam. At the stressed edge, symmetrically to the two parts, we attach two strips of sheet of constant width.

If we assume the sheet to be able alone to resist tension stresses, it follows that the path of the tension is straight. In fact, let x and z represent two perpendicular axes lying in the median plane of the plate of minimum thickness, so that only one of the principal tensions is other than zero, according to Mohr and, following the usual notations, the tensions in the sheet comply with the relations:

$$\sigma_x = \sigma \cos^2 \alpha \qquad \tau_{xz} = \sigma \sin \alpha \cos \alpha \qquad \sigma_z = \sigma \sin^2 \alpha$$

where σ is the maximum tension in a generic point, and

the angle formed with axis x.

Substituting these terms in the two equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0$$

Subtracting the first of these multiplied by $\sin \alpha$, then the second, multiplied by $\cos \alpha$, gives:

$$\frac{\partial \alpha}{\partial x} \cos \alpha + \frac{\partial \alpha}{\partial z} \sin \alpha = 0$$

and denoting with $d\zeta$ an element of the line parallel with the direction of σ, defined, that is, from

$$d\zeta = \frac{dx}{\cos} \alpha = \frac{dz}{\sin} \alpha$$

we obtain the equation $\frac{d\alpha}{d\zeta} = 0$, which proves our statement. Now assume edge AB of the sheet to be rigidly fixed. If d and f are free edges parallel to axes z and x, we have at d:

$$\sigma_x = \tau_{xz} = 0 \qquad \sigma \cos^2 \alpha = \sigma \sin \alpha \cos \alpha = 0$$

and at f:

$$\tau_{xz} = \sigma_y = 0 \qquad \sigma \sin \alpha \cos \alpha = \sigma \sin^2 \alpha = 0$$

If we should exclude the possibility of $\cos \alpha = 0$ along d, and of $\sin \alpha = 0$ at f, the result is $\sigma = 0$ over the contour. Seeing that the path of the tensions must be straight, it follows that the field outside of AD and BC must be unstressed. This condition exists even if f, instead of being the free contour line, delimits the symmetrical field.

With the proviso that the field ABCD may enter in tension, we introduce the assumption that the formation of the stress trajectories inside this field is so constituted by the tension lines which emanate from the beam as to form with it an angle which is constant over its whole length. This inclination will be discussed by means of the equation of least work.

The introduced hypothesis is equivalent to assuming a series of infinite parallel cuts in the sheet which reduce it to a continuous system of elastic tension members; it is fairly obvious that this system is not as stiff as the actual one.

Direct research to determine the law of variation of the slope of the tension lines in a manner satisfactory to the conditions of least work is too complicated to permit application to the multiplicity of cases encountered in actual practice. In fact, it would preclude making allowance for the variability of the beam section, the effect produced by the elements which usually are employed to stiffen the sheet, etc.

Now let α be the angle formed by the tension lines with the axis of the beam, x the distance of the generic section from the point of fixity and l the length of the beam. The sheet is assumed to be so thin as to allow its section to be identified with the straight line tangent over the section of the beam, and to let N represent the antipole of this straight line with respect to the ellipsoid of inertia of the section. Further, let M_n be the moment with respect to this point, of the external load and of the tension applied by the sheet between the section considered and the tip of the beam.

The unit elongation of the fiber of the beam in contact with the sheet is

$$\frac{M_n\, y}{EJ}$$

where y is the distance of fiber from c.g., G, of the section.

J, moment of inertia with respect to the horizontal c.g. axis.

E, modulus of elasticity of the material.

The point X (fig. 1) will now be displaced through the effect of the deformation of the frustum which precedes it, to the amount

$$\delta = \int_0^x \frac{M_n\, y}{EJ}\, dx \qquad (0)$$

The elongation of the fiber HX is given from $\delta \cos \alpha$; and since its length is $x/\cos \alpha$, the tension for the beam and sheet of the same material is:

$$\sigma = \frac{\cos^2 \alpha}{x} \int_0^x \frac{M_n y}{J} dx$$

From either side, along a path dx, to which corresponds a section of the sheet normal to the tensions $s\, dx \sin \alpha$ (s = sheet thickness), we apply two loads dF symmetrical with respect to the axis of the beam, which give:

$$\frac{2s \sin \alpha \cos^3 \alpha\, dx}{x} \int_0^x \frac{M_n y}{J} dx$$

The moment M_n then is:

$$M_n = M_e - \int_x^l \frac{2s\, y_n \sin \alpha \cos^3 \alpha}{x} dx \int_0^x \frac{M_n y}{J} dx$$

with M_e the moment of the external load applied between C and X; y_n, the distance of point N with respect to which the moments of the plane of the plate are measured.

Assuming the section to be constant, we put

$$k = 2s\, l\, y\, y_n \sin \alpha \cos^3 \frac{\alpha}{J}$$

so that

$$M_n = M_e - \frac{k}{l} \int_x^l \frac{dx}{x} \int_0^x M_n\, dx \qquad (1)$$

Quantity k introduced here is a numerical constant which completely defines the characteristics of the whole insofar as concerns the processes under consideration.

SOLUTION OF FUNDAMENTAL EQUATION

Equation (1) is readily reduced to a Fredholm equation of the second order with symmetrical nucleus. Its resolution is effected with Bessel's function J of degree 0 with imaginary arguments, which may be expressed with the development

$$I(x) = 1 + \frac{x}{\underline{|1}\ \underline{|1}} + \frac{x^2}{\underline{|2}\ \underline{|2}} + \ldots..$$

In fact, equation (1) may, when $\xi = \frac{x}{l}$, be written as

$$M_n = M_e - k \int_{\xi}^{1} \frac{d\xi}{\xi} \int_{0}^{\xi} M_n \, d\xi$$

Subtracting once, then multiplying both terms by ξ and deducting a second time, we obtain the unhomogeneous differential linear equation:

$$\xi \frac{d^2 M_n}{d\xi^2} + \frac{d M_n}{d\xi} - k M_n = \frac{d}{d\xi}\left(\xi \frac{d M_e}{d\xi}\right) \qquad (2)$$

The homogeneous equation is integrated with the Bessel function $I(k\xi)$. If the external moment M_e is represented by means of a polynomial in ξ:

$$M_e = \sum_{0}^{n} \alpha_r \xi^r$$

A particular integral of the homogeneous equation is given by the polynomial

$$a_0 + a_1 \xi + \ldots.. + a_{n-1} \xi^{n-1}$$

whose constants $a_0, a_1, \ldots a_{n-1}$, substituting in (2) and equating the coefficients of each power of ξ in the two terms give:

$$a_r - \alpha_r = \frac{k\, a_{r-1}}{r^2} \text{ (da } r = 1 \text{ a } r = n, \text{ with } a_n = 0) \qquad (3)$$

The general integral of (2) follows from

$$M_n = C I(k\xi) + a_0 + \ldots. + a_{n-1} \xi^{n-1}$$

in which C is a constant defined by means of limiting condition $(M_n = M_e)_{\xi = 1}$.

Now let us apply the above equation to the case of uniform load. The moment of the external load may be expressed with

$$M_e = M_o (1 - \xi)^2$$

where M_o is a constant.

Then we have: $n = 2$ $\alpha_0 = M_o$ $\alpha_1 = -2M_o$ $\alpha_2 = M_o$

Equation (4) may be written as

$$M_n = C\ I\ (k\xi) + a_o + a_1\ \xi \tag{5}$$

The recurrent formula (3) gives the equations in a_o and a_1:

$$a_1 + 2M_o = k\ a_o \qquad -M_o = \frac{k\ a_1}{4}$$

which yields a by insertion in (5).

Putting $\xi = 1$, $M_n = M_e = 0$, we have:

$$0 = C\ I(k) - \frac{2M_o}{k} - \frac{4M_o}{k^2} \tag{5'}$$

Lastly, we substitute the value of C given from the above formula in (5') and obtain:

$$M_n = M_o\ \frac{4}{k^2}\left[\left(1 + \frac{k}{2}\right)\frac{I(k\xi)}{I(k)} - 1 + \frac{k}{2} - k\xi\right] \tag{6}$$

By the same argument we find for the case of the beam under constant moment M_o:

$$M_n = M_o\ \frac{I(k\xi)}{I(k)} \tag{7}$$

With a load applied at its free end, that is, an external moment $M_e = M_o\ (1 - \xi)$, we have

$$M_n = M_o\ \frac{I(k) - I(k\xi)}{k\ I(k)} \tag{8}$$

In any case whatever, the course of M_n is readily computed when the values of the function $I(x)$ are known. Such functions are to be found in tables and diagrams.* We reproduce some figures which should be sufficient for such application in the majority of cases.

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*Jahnke, Eugen, and Emde, Fritz: Funktionentafeln mit formeln und kurven. 1933.

1 (0) = 1.0000	1 (0.1) = 1.1025	1 (0.2) = 1.2102
1 (0.3) = 1.3233	1 (0.4) = 1.4418	1 (0.5) = 1.5661
1 (0.6) = 1.6964	1 (0.7) = 1.8325	1 (0.8) = 1.9750
1 (0.9) = 2.1239	1 (1) = 2.2796	

The expression (6) of the moment M_n in the uniform load case, may also be written:

$$M_n = \frac{M_o}{I(k)} \left[\frac{4}{k^2} I(k\xi) + \frac{2}{k} I(k\xi) - \left(\frac{4}{k^2} - \frac{2}{k} + \frac{4\xi}{k}\right) I(k)\right];$$

giving the approximation with which the usual tables give the values of I, for the small values of k, does not permit an exact calculation.

The development of I in parentheses in (18) and ordination according to the power of k, give

$$M_n = \frac{M_o}{I(k)} (1 - \xi) (1 - \xi + c_1 k + c_2 k^2 + \ldots..) \qquad (6')$$

The c values, functions of ξ only, are given in terms of such variables in figure 2.

NUMERICAL EXAMPLE

The developed equations are now applied to a system having the following characteristics:

$sl = F$ (section of beam) $\qquad F\, y\, y_n = 2.5\, J$

assuming $\tan \alpha = 0.1$. The result is $k = 0.5$.

This value, in the case of a load applied at the tip of the beam, gives for moment M_n the diagram a of figure 3. The curve b of the graph itself gives the course of the moment M_n for the same value of k when the beam is subjected to a uniform load. In the remaining space of this diagram the moments change signs: In fact, in this space the stresses in the sheet are severe because it feels the effects of the expansion of the fibers of the beam over the entire length of the beam and their moment exceeds, as a result, that of the external loads.

Another result of the calculation is that the aid given by the sheet depends upon the type of external stresses. In the case in point, under a constant moment, the moment at the point of fixity has dropped 36 percent; for a load applied at the tip, the reduction amounted to 27 percent; for uniform load, it decreased 23 percent.

DETERMINATION OF THE EQUIVALENT WIDTH

The equivalent width of the sheet is the dimension of the ideal strip which, conforming to the law of conservation of flat sections, bends solidly with the beam and gives the same stresses in it as the actual system. This quantity generally varies for each section and gives a direct idea of the stresses between sheet and beam.

In the foregoing analysis we deduced the term M_n which gave the stress σ in the stretched edge:

$$\sigma = \frac{M_n\, y}{J}$$

If a strip of the sheet of width $2b_i$ bends together with the beam, the baricenter of the T-section beam resulting from it, will be, besides that of the original section:

$$\Delta = \frac{2b_i\, s}{F + 2b_i\, s}\, y$$

and the moment of inertia of the c.g. of the section is complete with

$$J_1 = J + 2b_i\, s\, y^2 - (F + 2b_i\, s)\Delta^2$$

which, substituting

$$\sigma = M_e\, \frac{y_1}{J_1}$$

allowing for $y_1 = y - \Delta$, gives

$$\sigma\left(\frac{J}{y} + 2b_i\, s\, y_n\right) = M_e$$

with y_n, already defined, linked to other quantities of the relation $F\ y\ y_n = J + F\ y^2$.

If it is desired to obtain σ under these conditions, the same as is verified in the effective system as defined by (9), we must put

$$\left(\frac{J}{y} + 2b_i\ s\ y_n\right)\frac{M_n\ y}{J} = M_e$$

and, making $\lambda = \frac{b_i}{b}$ with $b = l \tan \alpha$ the apparent width,

$$\lambda = \frac{M_e - M_n}{k\ M_n} \cos^4 \alpha \qquad (10)$$

With this equation for the same value of k, referred to in the diagram of figure 3, we computed the equivalent width for the case of a load applied at the free end and for the case of uniform load. The following values were obtained:

For ξ =	1	0.8	0.6	0.4	0.2	0	
λ =	0.46	0.52	0.58	0.64	0.70	0.75(0.87)	end load.
λ =	-	2.61	1.04	0.687	0.62	0.60 (0.80)	distributed load.

The figures in parentheses are those obtained with Metzer's theory.

EQUATIONS OF THE WORK OF DEFORMATION

Let $N(x)$ be the end load produced in the beam by the stress applied at the sheet in the space between the tip of the beam and the generic section X (fig. 1). This gives:

$$M_e - M_n = N\ y_n$$

The work of deformation of the beam is:

$$L_1 = \frac{1}{2EJ}\left(\int_0^l M_e^2\ dx + y\ y_n \int_0^l N^2\ dx\right)$$

Consider further the portion of the sheet constituted by the two strips which have a beam component along dx; the sudden elongation of the two fibers is $\cos\alpha\,\delta$ (formula (0)) and with $2\,dx\,s\,\sin\alpha$ as the section and $x/\cos\alpha$ as the length of this fiber, the relative work is:

$$dL_2 = \frac{E}{2}\ \frac{2s\sin\alpha\cos^3\alpha}{x}\ \delta^2\ dx$$

For the whole field of the sheet the work of deformation then is:

$$L_2 = \frac{yk}{2EJ\ y_n\ l}\int_0^l \frac{dx}{x}\left[\int_0^x M_n\ dx\right]^2$$

Summarizing the terms for L_1 and L_2, we obtain the total work of deformation. It is readily proved that the expression is identical with that obtained for the system, subjected to pure bending, consisting of beam and strip of sheet of width $b_1 = \lambda b$, which bends with it, according to the law of conservation of plane sections.

What is of greater interest, however, is that the term expressing the work of deformation does not directly contain α; this results as function of α through k. Therefore, we may write:

$$\frac{\partial L}{\partial\alpha} = \frac{\partial L}{\partial k}\,\frac{\partial k}{\partial\alpha}$$

L is a decreasing function of k; 0 may be inferred from the fact that k is proportional to the sheet thickness and that an increase in it must in any case carry a decrease in the total deformations and consequently, in the work of deformation. Its minimum value, then, corresponds to that value of α for which k becomes maximum. In that case, we have, in fact,

$$\frac{\partial L}{\partial\alpha} = 0 \qquad \frac{\partial^2 L}{\partial\alpha^2} = \frac{\partial L}{\partial k}\,\frac{\partial^2 k}{\partial\alpha^2} > 0.$$

It is readily ascertained what happens when $\alpha = 30^\circ$. Hence, we may state that, provided the wrinkles start at a constant angle at every point of the sheet, it tends to assume the maximum inclination permitted by the length of the sheet, not exceeding, however, the value of 30°.

For the calculation of k and of the contributory effect of the sheet, we must therefore assume $\alpha = \text{arc tan } \frac{b}{l}$ if the results are less than 30°; in the contrary case, we assume $\alpha = 30°$.

ANALYSIS OF THE RESULTS

The developed theory allows us to compute the effect of collaboration of the sheet with the beam on the premise that the wrinkling sheet can resist the tensile stresses alone and that the waves run parallel. The obtained results are qualitatively in accord with Metzer's data, stipulating that the plate be resistant to compression also. The ideal widths in our hypothesis are, however, markedly inferior.

We shall give some approximate terms which should make the application of the developed theory quite easy.

For the case of a single load applied at the tip of the beam, we have, according to (8), the moment M_n in the restrained section $\xi = 0$.

$$M_n = M_o \frac{I(k) - 1}{kI(k)}.$$

Substituting in (10) and reflecting that in our case $M_e = M_o$, we have:

$$\frac{\lambda}{\cos^4 \alpha} = 1 - \frac{I(k) - 1 - k}{kI(k) - k} \tag{11}$$

The fraction in the last term tends toward 0 when k increases, $I(k)$ being an increasing function of k; it is therefore convenient to expand in series k as denominator. This expansion, followed by division of numerator and denominator by $I(k) - 1 - k$, gives the fraction itself:

$$\frac{1}{4} (1 + 0.14\, k + 0.005\, k^2 + \ldots\ldots)^{-1}$$

Equation (11) is closely approximated at:

$$\frac{\lambda}{\cos^4 \alpha} = 1 - \frac{0.25}{1 + 0.14\, k}.$$

For the load cases in question the effective width decreases from the restrained section to the tip of the beam. The curve a in figure 4, illustrates the course of λ over the beam for $k = 0.5$. Curve b represents Metzer's figures.

In the extreme section, λ decreases as k increases and approaches 0 as $k \rightarrow \infty$. For simplicity we may, in the case of a load applied at the beam end, have a linear course for λ, which from the value in (12) for the restrained section, approaches 0 in the extreme section (line a_1 in fig. 4).

In the case of uniform load, we have for the restrained section, the approximation:

$$\frac{\lambda}{\cos^4 \alpha} = 0.5 \left(1 + \frac{2}{9 + 3k}\right)$$

Here the value of $\lambda = 0.61 \cos^4 \alpha$ for $k = 0$ decreases as k increases and approaches the asymptotic value $0.5 \cos^4 \alpha$. Metzer's theory arrives at the same conclusion. In the case in question, the effective width decreases as the sheet thickness increases.

In our case there is always an increase of λ along the beam, so that the value for the whole length of the beam may be taken from (13) for the restrained section.

If the load instead of being uniform decreases toward the beam tip, the theory stipulates lower λ values. But the minimum value in the case of linear load reduction approaching 0 at the tip, is $0.33 \cos \alpha^4$ for the restrained section.

This theory lends itself to practical application in simple fashion, but the assumed parallelism of the tension curves remains to be proved experimentally; that is, to say, it remains to be proved that the interruption of the continuity of the sheet by means of a system of infinite parallel cuts puts it in worse conditions than those actually encountered.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

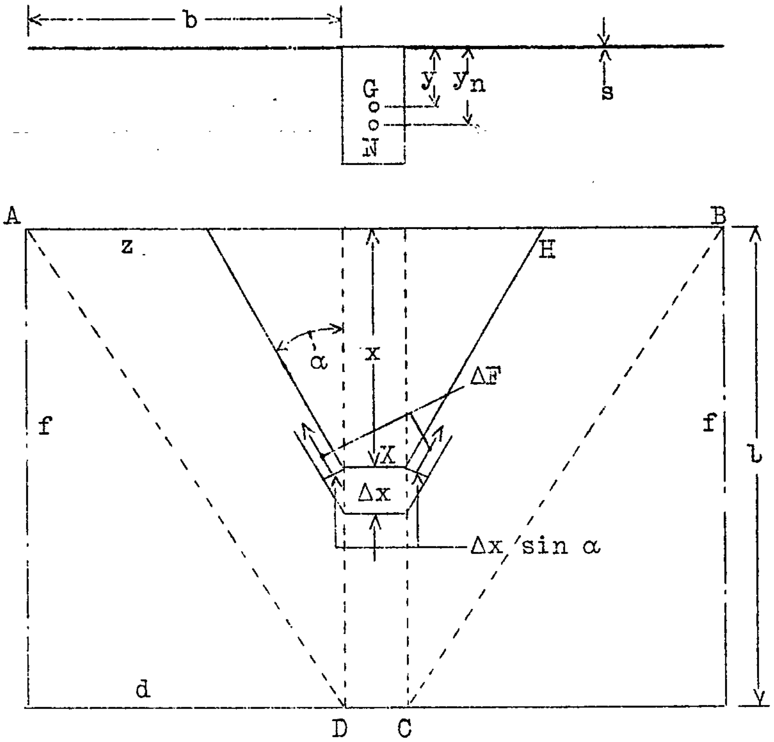

Figure 1.

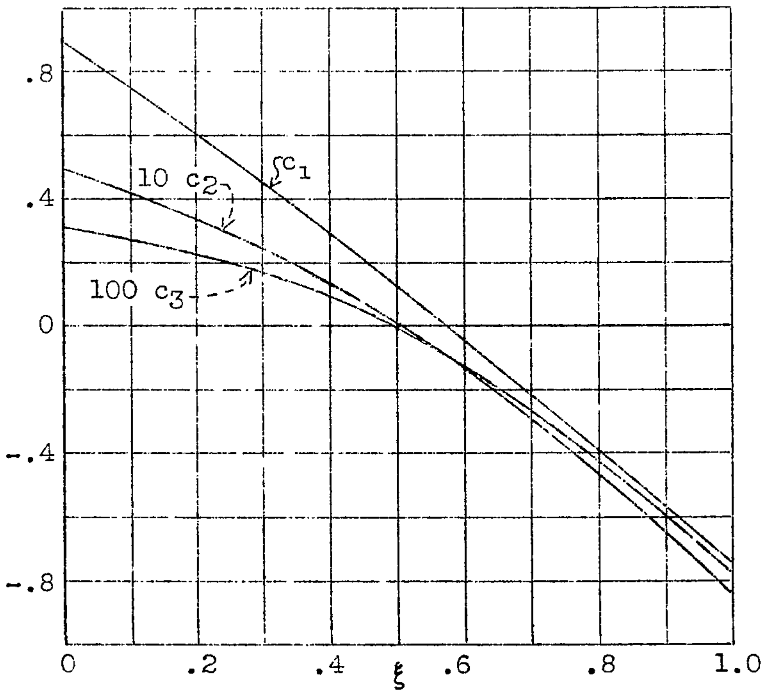

Figure 2.-Diagram of c.

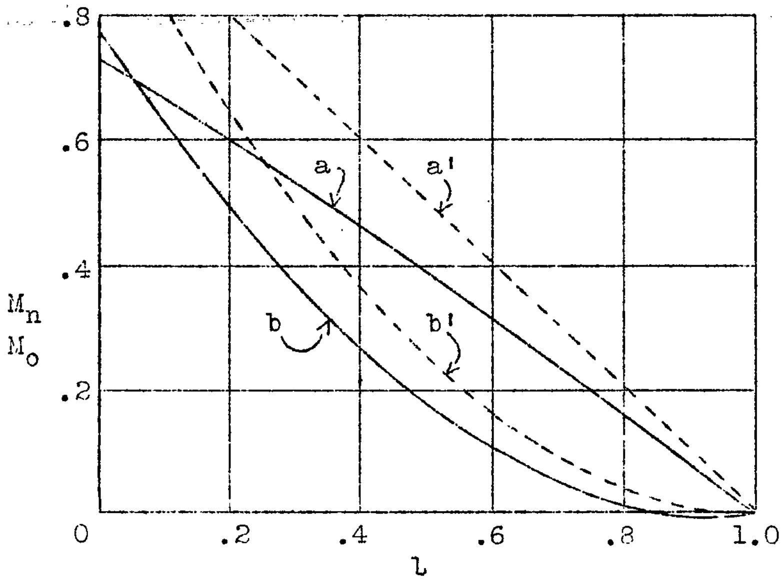

Figure 3.-Moment M_n: a,a' with load at free end, respectively with and without sheet; b,b' with uniform load.

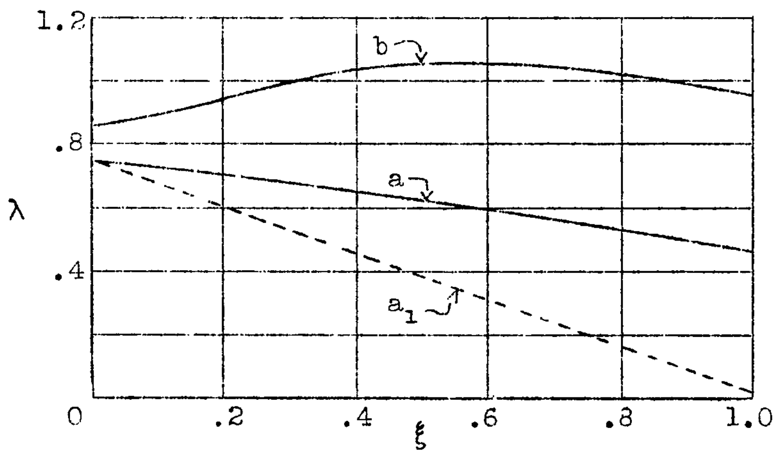

Figure 4.-Effective width in the case of end load for k=0.5 .