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IMPACT BUCKLING OF THIN BARS IN THE ELASTIC RANGE
HINGED AT BOTH ENDS

By Carel Koning and Josef Taub

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IMPACT BUCKLING OF THIN BARS IN THE ELASTIC RANGE HINGED AT BOTH ENDS*

By Carel Koning and Josef Taub

SUMMARY

Following the development of the well-known differential equations of the problem and their resolution for failure in tension, the bending (transverse) oscillations of an originally not quite straight bar hinged at both ends and subjected to a constant longitudinal force (shock load) are analyzed. To this end the course of the bar form is expanded in a sinusoidal series, after which the investigation is carried through separately for the fundamental oscillation and the (n-1)th higher oscillations.

The analysis of the fundamental oscillation distinguishes three cases: shock load lower, equal to, or higher than the Eulerian load.

The investigation of the (n-1)th higher oscillation also distinguishes between shock load smaller, equal to, or greater than the (n-1)th stability limit, although only the first case is of practical significance.

Shock loads in buckling are divided into the period of actual shock and the period of free oscillations following the actual shock.

The investigation leads to functions which are proportional to the maximum stresses in time and space due to the shock stresses in buckling. These functions are then compared for the case of shock load lower than Eulerian load with the maximum stresses in static load. It is found

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*"Stossartige Knickbeanspruchung schlanker Stäbe im elastischen Bereich bei beiderseits gelenkiger Lagerung." Luftfahrtforschung, July 6, 1933, pp. 55-64.

that the former are smaller for short shock periods and vice versa; that is, in the extreme case, twice as high as the latter.

From a comparison of the functions decisive for the maximum stresses, it appears that the Eulerian load may be safely exceeded in shock-like buckling stresses, provided the shock period is sufficiently short; further, that, whereas the stresses under shock load above the Euler load show an unrestricted increase with the shock period, the stresses in shock loads below the Euler load reach an upper limit which is not exceeded during any shock period.

The report closes with an analysis of the interdependence between the shock stress in buckling and the shock impulse $\int P\,dt$. It is found that, contrary to common belief, the stress with equal shock impulse is sensibly affected by the shock period. For that reason the determination of the stress stipulates not only the time integral $\int P\,dt$, but also the shock force and the shock period - a fact which is of essential importance from the experimental point of view.

I. INTRODUCTION

The analysis of static buckling stresses affords, as is known, a problem in stability. It poses and answers the question up to what limit the compression may be increased for given bar dimensions without exceeding the range within which an unequivocally definable condition of equilibrium exists. Several equilibrium conditions are possible after this boundary has been exceeded. On approaching the stability limit the rise of the deformation is such that the bar usually loses its carrying capacity before reaching the equilibrium condition. For this reason, the determination of the stability limit is of decisive importance.

Contrariwise, stressing a bar suddenly in buckling, the suddenness being the short-time interval between load change and loading period, as shown in this report, the stability limit is no longer as significant as in the static case, and may be safely exceeded, provided the shock period is so short as to leave the bar no time to deform as would correspond to the static equilibrium condition.

From this it follows that the calculation of the deformations and stresses with respect to time is the primary issue rather than the determination of the stability limit when analyzing the shock load in buckling.

II. NOTATION

L	kg,	force component parallel to bar axis $(\bar{x})$.
D	kg,	force component at right angles to bar axis $(\bar{y})$.
$X = -P$	kg,	force component parallel to axis x.
Y	kg,	force component at right angles to axis x.
M	kg m,	bending moment.
q_d	kg m^{-1},	outside force at right angles to bar axis
q_l	kg m^{-1},	outside force parallel to bar axis.
m	kg,	moment loading of bar elements.
r	m,	radius of curvature of the elastic line.
α,		slope of the elastic line.
$\underline{y}$	m,	deviation of bar axis from straight line in unloaded condition.
ϵ	m,	"amplitude" of bar axis in unloaded condition.
η	m,	deflection. (See fig. 2.)
ξ	m,	shifting in direction of x.
$S = \int P\,dt$	kg s,	shock impulse. (See footnote, page 26.)
E	kg m^{-2},	elasticity modulus.
l	m,	length of bar.
i	m,	radius of gyration of bar section.

F	m^2,	area of bar section.
J	m^4,	inertia moment of bar section.
ρ	kg s^2 m^{-4},	density.
σ_x	kg m^{-2},	normal stress in x direction.
ϵ_x,		elongation in x direction.
t	s,	time interval.
T	s,	period of oscillation of the free fundamental oscillation.
τ	s,	shock period.
p	s^{-1},	frequency of oscillation.
v	m s^{-1},	velocity of sound in bar material.
φ,		phase lag.
a,		ratio of shock load to Eulerian load.
b,		ratio of shock period to oscillation period of the free, transverse fundamental oscillation.
c,		ratio of the maximum moments (taken absolute) in the static and dynamic case.
λ,		proper values.
A, B, C, D, k_1, k_2,		constants.

Indices:

$n = 1, 2, 3$,	the natural numerals.
0	refers to quantities appearing with tension = 0.
E,	quantities representative of the Eulerian buckling load.

$\dot{}$, $\ddot{}$, etc.,	dots over a symbol denote its 1st, 2d, etc., derivation in time rato.
$'$, $''$, etc.,	dashes over a symbol denote its 1st, 2d, etc. derivation with respect to a length (x).

III. THE DIFFERENTIAL EQUATIONS OF THE SYSTEM

Within the curvilinear system of coordinates $\bar{x}\ \bar{y}$ the equilibrium equations for an element of the bent bar of length ds are as follows (fig. 1):

$$\left(L + \frac{\partial L}{\partial s} ds\right) \cos d\varphi - \left(D + \frac{\partial D}{\partial s} ds\right) \sin d\varphi - L + q_l\ ds = 0 \tag{1}$$

$$\left(L + \frac{\partial L}{\partial s} ds\right) \sin d\varphi + \left(D + \frac{\partial D}{\partial s} ds\right) \cos d\varphi - D + q_d\ ds = 0 \tag{2}$$

$$\left(M + \frac{\partial M}{\partial s} ds\right) + \left(L + \frac{\partial L}{\partial s} ds\right) ds \sin \frac{d\varphi}{2} +$$

$$+ \left(D + \frac{\partial D}{\partial s} ds\right) ds \cos \frac{d\varphi}{2} - M - \frac{1}{2} q_l\ ds^2 \sin \frac{d\varphi}{2} +$$

$$+ \frac{1}{2} q_d\ ds^2 \cos \frac{d\varphi}{2} + m\ ds = 0 \tag{3}$$

With $r\ d\ \varphi = ds$, whereby r = curvature radius, these equations, upon $d\varphi \rightarrow 0$ and disregarding the infinitely small quantities of the 2d and 3d order and with

$$\sin d\varphi \sim d\varphi$$

$$\cos d\varphi \sim 1$$

reduce to

$$\frac{\partial L}{\partial s} - \frac{1}{r} D + q_l = 0 \tag{1a}$$

$$\frac{1}{r} L + \frac{\partial B}{\partial s} + q_d = 0 \tag{2a}$$

$$\frac{\partial M}{\partial s} + D + m = 0 \tag{3a}$$

The deviations of the bar axis from a straight line in unstressed condition as well as the deflections in the processes analyzed hereinafter, are assumed small compared to the length of the bar, and the choice is a rectangular system of coordinates x y such that in first approximation axis x coincides with the axis of the bar in unstressed attitude. Then the relations between the force components in the curvilinear system $\bar{x}\ \bar{y}$ and the force components in the rectangular system x, y are

$$L = X \cos \alpha + Y \sin \alpha$$

$$D = - X \sin \alpha + Y \cos \alpha$$

or, since $\alpha = \sin \alpha = \tan \alpha = y'$,

$$\frac{d\alpha}{dx} = \frac{1}{r} = y''$$

$$\frac{dx}{ds} = \cos \alpha = 1$$

$$L = X + Y\, y'$$

$$D = - X\, y' + Y.$$

As a result:

$$\frac{\partial L}{\partial s} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x}\, y' + Y\, y''$$

$$\frac{\partial D}{\partial s} = - \frac{\partial X}{\partial x}\, y' + \frac{\partial Y}{\partial x} - X\, y''$$

$$\frac{\partial M}{\partial s} = \frac{\partial M}{\partial x}.$$

These terms are written into (1a) to (3a), whereby, omitting the small quantities of the 2d order, the equilibrium equations for the slightly bent bar become:

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x}\, y' + q_l = 0 \qquad (1b)$$

$$- \frac{\partial X}{\partial x}\, y' + \frac{\partial Y}{\partial x} + q_d = 0 \qquad (2b)$$

$$\frac{\partial M}{\partial x} - X\, y' + Y + m = 0 \qquad (3b)$$

By eliminating Y they reduce to

$$\frac{\partial X}{\partial x} - q_d\ y' + q_l = 0 \tag{4}$$

$$\frac{\partial^2 M}{\partial x^2} - X\ y'' - q_d + \frac{\partial m}{\partial x} = 0 \tag{5}$$

Assuming zero outside load at the bar element, i.e., q_l and q_d to be mass forces of the bar element and m the mass moment of the bar element $q_d\ y'$ are negligible relative to q_l, because, first, in sufficiently thin bars the oscillation frequency and through it the mass acceleration in transverse direction is small relative to the corresponding quantities in the longitudinal direction; second, y' is a small quantity according to the premises.

Thus the differential equations read:

$$\frac{\partial X}{\partial x} + q_l = 0 \tag{4a}$$

$$\frac{\partial^2 M}{\partial x^2} - X\ y'' - q_d + \frac{\partial m}{\partial x} = 0 \tag{5a}$$

Now $\underline{y}$ is the deviation of the bar axis from straight line in unstressed condition and η is the deflection, so that $\underline{y} + \eta$ must be substituted for y in (5a) (fig. 2).

X and M are expressed in terms of deformation:

$$X = \sigma_x\ F = E\ F\ \epsilon_x = E\ F\ \frac{\partial \xi}{\partial x}$$

$$M = E\ J\ \frac{\partial^2 \eta}{\partial x^2}$$

and the mass forces q_l and q_d, and the mass moment m as

$$q_l = -\ \rho\ F\ \frac{\partial^2 \xi}{\partial t^2}$$

$$q_d = -\ \rho\ F\ \frac{\partial^2 \eta}{\partial t^2}$$

$$m = -\rho\ J\ \frac{\partial}{\partial t^2}\left(\frac{\partial \eta}{\partial x}\right).$$

Herewith (4a) and (5a) become:

$$E \frac{\partial^2 \xi}{\partial x^2} - \rho \frac{\partial^2 \xi}{\partial t^2} = 0 \tag{4b}$$

$$EJ \frac{\partial^4 \eta}{\partial x^4} - X \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 y}{\partial x^2} \right) + \rho F \frac{\partial^2 \eta}{\partial t^2} - \rho J \frac{\partial^4 \eta}{\partial x^2 \partial t^2} = 0 \tag{5b}$$

These two equations (4b) and (5b) constitute the differential equations of the buckling stress due to shock of a bar with constant cross-sectional dimensions.

IV. SOLUTION FOR A BAR HINGED AT EITHER END

1. The Free Longitudinal Oscillations of the Bar

We repeat the well-known formulas for the free longitudinal oscillations of a straight bar. They are obtained by resolving (4b) conformably to the generalized equation:

$$\xi = (A \sin \lambda x + B \cos \lambda x)\,(k_1 \sin p_0 t + k_2 \cos p_0 t)$$

The bar is assumed to be clamped or fixed at one end but left free to move longitudinally at the other. Then the boundary equations are:

$$\xi = 0 \qquad \text{for} \quad x = 0$$

$$\frac{\partial \xi}{\partial x} = \frac{X}{EF} = 0 \quad '' \quad x = l.$$

$B = 0$ because of the first boundary condition, thus reducing the equation to

$$\xi = \sin \lambda x \,(k_1 \sin p_0 t + k_2 \cos p_0 t).$$

Double differentiation according to x and t affords:

$$\frac{\partial^2 \xi}{\partial x^2} = - \lambda^2 \xi$$

$$\frac{\partial^2 \xi}{\partial t^2} = - p^2 \xi,$$

which, written in (4b) gives

$$-\lambda^2 E + p_o^2 \rho = 0 \qquad p_o = \lambda \sqrt{\frac{E}{\rho}} = v \lambda,$$

where $v = \sqrt{\frac{E}{\rho}}$ is the velocity of sound in the material.

Furthermore, because of

$$\frac{\partial \xi}{\partial x} = \lambda \cos \lambda x (k_1 \sin p t + k_2 \cos p t)$$

together with the second boundary equation we have:

$$\lambda_n = \frac{\pi}{2l}, \frac{3\pi}{2l}, \ldots\ldots \frac{2n - 1}{2l} \pi.$$

Consequently, the frequencies are:

$$p_{on} = \frac{\pi}{2l} v, \frac{3\pi}{2l} v, \ldots\ldots \frac{2n - 1}{2l} \pi v$$

and the period is

$$T_{on} = \frac{2\pi}{p} = 4 \frac{l}{v}, \frac{4}{3} \frac{l}{v}, \ldots\ldots \frac{4}{2n - 1} \frac{l}{v} .$$

2. The Free Transverse Oscillations of the Bar

Disregarding the rotatory inertia for the case of $X = 0$, equation (5b) becomes:

$$EJ \frac{\partial^4 \eta}{\partial x^4} + \rho F \frac{\partial^2 \eta}{\partial t^2} = 0 \qquad (6)$$

The generalized solution is:

$$\eta = \sum_n \sin (p_{on} t + \varphi_n) (A_n \sin \lambda_n x + B_n \cos \lambda_n x +$$
$$+ C_n \operatorname{sh} \lambda_n x + D_n \operatorname{ch} \lambda_n x).$$

The insertion of

$$\frac{\partial^4 \eta_n}{\partial x^4} = \lambda_n^4 \eta_n \quad \text{and} \quad \frac{\partial^2 \eta_n}{\partial t^2} = - p_{on}^2 \eta_n$$

in (6) gives

$$p_{on} = \lambda_n^2 \sqrt{\frac{EJ}{\rho F}} \qquad (7)$$

The boundary conditions for the bar hinged at both ends read:

$$\left.\begin{aligned} \eta &= 0 \\ \frac{\partial^2 \eta}{\partial x^2} &= 0 \end{aligned}\right\} \text{for} \quad x = 0 \quad \text{and} \quad x = l,$$

thus modifying (6) to

$$\eta = \sum_n \eta_n = \sum_n A_n \sin (p_n t + \varphi_n) \sin \lambda_n x \tag{7a}$$

whereby

$$\lambda_n = \frac{\pi}{l}, \frac{2\pi}{l}, \ldots\ldots \frac{n\pi}{l}$$

By virtue of (7) the frequencies are:

$$p_{on} = \frac{\pi^2}{l^2}\sqrt{\frac{EJ}{\rho F}}, \frac{4\pi^2}{l^2}\sqrt{\frac{EJ}{\rho F}}, \ldots \frac{n^2\pi^2}{l^2}\sqrt{\frac{EJ}{\rho F}} = \frac{n^2\pi^2}{l^2} v\ i,$$

where i = radius of inertia of the cross-sectional area. As a result the oscillation period of the (n-1)th higher oscillation is:

$$T_{on} = \frac{2\pi}{p_{on}} = \frac{2\,l^2}{n^2\,\pi\,v\,i}.$$

The ratio of oscillation period of the transverse and longitudinal oscillations for the fundamental harmonic $(n = 1)$ in the bar hinged at both ends to that of the bar left free to shift longitudinally at one end, is:

$$\frac{2\,l^2}{\pi\,v\,i}\,\frac{v}{4\,l} = \frac{1}{2\pi}\,\frac{l}{i}.$$

For slenderness ratios $\frac{l}{i} > 2\pi$ the transverse oscillations are consequently of lower frequency than the longitudinal oscillations.

3. Transverse Oscillations Due to Constant Shock Load

Here we analyze the specific case of a bar hinged at both ends being stressed under constant shock load X during a shock period τ. We idealize the case by disregarding the longitudinal oscillation, i.e., assume the tension (normal force) to be identical at every point of the bar during the period of shock. With the neglected rotatory inertia the resolvable partial differential equation is, according to (5b):

$$EJ \frac{\partial^4 \eta}{\partial x^4} - X \frac{\partial^2 \eta}{\partial x^2} + \rho F \frac{\partial^2 \eta}{\partial t^2} - X \frac{\partial^2 \underline{y}}{\partial x^2} = 0 \tag{8}$$

The deviations of the bar axis from a straight line, i.e., the original bar form, are expanded in Fourier series conformably to the local function $\sin \lambda n x$ of (7a):

$$\underline{y} = \sum_n \epsilon_n \sin \lambda_n x = \epsilon_1 \sin \lambda x + \epsilon_2 \sin 2 \lambda x + \ldots + \epsilon_n \sin n \lambda x + \tag{9}$$

where $\lambda = \frac{\pi}{l}$ is the first proper value of the free transverse oscillation (see section IV,2) and ϵ_n = constant. Equation (8) resolves to

$$\eta = \sum_n \eta_{tn} \eta_{xn} = \eta_{t1} \sin \lambda_1 x + \eta_{t2} \sin \lambda_2 x + \ldots + \eta_{tn} \sin \lambda_n x + \ldots$$

$$= \eta_{t1} \sin \lambda x + \eta_{t2} \sin 2 \lambda x + \ldots + \eta_{tn} \sin n \lambda x + \ldots,$$

where $\eta_t = f(t)$

$\eta_x = \varphi(x)$.

Putting these values of $\underline{y}$ and η in (8) gives

$$E J \lambda^4 \sum_n n^4 \eta_{tn} \sin n \lambda x + X \lambda^2 \sum_n n^2 \eta_{tn} \sin n \lambda x + \rho F \sum_n \frac{d^2 \eta_{tn}}{dt^2} \sin n \lambda x + X \lambda^2 \sum_n n^2 \epsilon_n \sin n \lambda x = 0.$$

The equation for the fundamental equation is:

$$E J \lambda^4 \eta_{t1} + X \lambda^2 \eta_{t1} + \rho F \frac{d^2 \eta_{t1}}{dt^2} + X \lambda^2 \epsilon_1 = 0 \tag{10}$$

and for the (n-1)th higher oscillation:

$$n^4 \; EJ \; \lambda^4 \; \eta_{tn} + n^2 \; X \; \lambda^2 \; \eta_{tn} + \rho F \; \frac{d^2 \eta_{tn}}{dt^2} + n^2 \; X \; \lambda^2 \; \epsilon_n = 0 \tag{11}$$

These equations are valid during the actual shock period. After the shock the bar executes free oscillations whose initial conditions are contingent upon the deflection and rate at the termination of the shock.

a) The fundamental oscillation

The Eulerian buckling load for the bar hinged at both ends and subjected to static load (see also V) is:

$$P_E = E \; J \; \frac{\pi^2}{l^2} = E \; J \; \lambda^2 \tag{12}$$

By denoting the ratio of shock force to Eulerian load with a, or in other words, presume

$$a = - \frac{X}{P_E}, \tag{13}$$

$X = - a \; E \; J \; \lambda^2$ and (10) becomes

$$\rho \; F \; \frac{d^2 \eta_t}{dt^2} + E \; J \; \lambda^4 \; (1 - a) \; \eta_t = a \; \epsilon \; E \; J \; \lambda^4 \tag{14}$$

index 1 being omitted for simplicity.

The resolution of this equation must differentiate between three cases:

1. $X > - P_E$, that is, $a < 1$

2. $X = - P_E$, " " , $a = 1$

3. $X < - P_E$, " " , $a > 1$

α) Shock load lower than the Eulerian load*

$(a < 1)$.

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*This also includes all cases with negative a, that is, the cases of shock stresses in tension.

Putting $\eta_t = \bar{\eta}_t + \bar{\bar{\eta}}_t$ in (14), where $\bar{\eta}_t$ denotes the resolution of the homogeneous equation and $\bar{\bar{\eta}}_t$ the effect of the disturbing function, the resolution of the homogeneous equation

$$\rho\ F\ \frac{d^2 \bar{\eta}_t}{dt^2} + E\ J\ \lambda^4\ (1 - a)\ \bar{\eta}_t = 0$$

gives $\bar{\eta}_t = k_1 \sin p\ t + k_2 \cos p\ t$,

because $\frac{EJ}{\rho F}\ \lambda^4\ (1 - a) > 0$. Consequently,

$$\frac{d^2 \bar{\eta}_t}{dt^2} = -\ p^2\ \bar{\eta}_t$$

The insertion of these values in the homogeneous equation gives the frequency:

$$p = \lambda^2 \sqrt{\frac{EJ}{\rho F}} \sqrt{1 - a} = p_0 \sqrt{1 - a} \tag{15}$$

where p_0 = frequency of the free fundamental oscillation (see IV,2).

The effect of the disturbing function on the right-hand side of (14) is found from

$$E\ J\ \lambda^4\ (1 - a)\ \bar{\bar{\eta}}_t = a\ \epsilon\ E\ J\ \lambda^4$$

at

$$\bar{\bar{\eta}}_t = \frac{a}{1 - a}\ \epsilon.$$

Consequently,

$$\eta_t = k_1 \sin p\ t + k_2 \cos p\ t + \frac{a}{1 - a}\ \epsilon \tag{16}$$

whence,

$$\dot{\eta}_t = p\ (k_1 \cos p\ t - k_2 \sin p\ t) \tag{17}$$

Assuming $\eta_t = 0$ and $\dot{\eta}_t = 0$ for the start of the shock, $t = 0$, (16) and (17) give

$$k_2 + \frac{a}{1 - a}\ \epsilon = 0; \quad k_2 = -\ \frac{a}{1 - a}\ \epsilon; \quad k_1 = 0.$$

The solution is:

$$\eta_t = \frac{a}{1 - a} \epsilon (1 - \cos p t) \tag{18}$$

and consequently,

$$\dot{\eta}_t = \frac{a}{1 - a} \epsilon p \sin p t \tag{19}$$

Let τ = shock period. Then at the end of the shock:

$$\eta_\tau = \frac{a}{1 - a} \epsilon (1 - \cos p \tau) \tag{20}$$

$$\dot{\eta}_\tau = \frac{a}{1 - a} \epsilon p \sin p \tau \tag{21}$$

The incipient free oscillations following the end of the shock are, according to (6):

$$\eta = \eta_t \eta_x = C \sin (p_o t + \varphi) \sin \lambda x \tag{22}$$

with φ = phase shifting. For this period of the processes, (20) and (21) are the initial equations. Thus,

$$C \sin (p_o \tau + \varphi) = \frac{a}{1 - a} \epsilon (1 - \cos p \tau)$$

$$C p_o \cos (p_o \tau + \varphi) = \frac{a}{1 - a} \epsilon p \sin p \tau.$$

The addition of the squares of these equations gives:

$$C^2 = \frac{a^2}{(1 - a)^2} \epsilon^2 \left[(1 - \cos p \tau)^2 + \frac{p^2}{{p_o}^2} \sin^2 p \tau\right]$$

or, with due regard to (15),

$$C = \frac{a}{1 - a} \epsilon \sqrt{2 - 2 \cos p \tau - a \sin^2 p \tau}.$$

We use the oscillation period of the free oscillation $T = \frac{2\pi}{p_o}$ as time scale and introduce

$$\tau = b T = \frac{2\pi}{p_o} b,$$

so that $p\,\tau = 2\,\pi\,b\sqrt{1 - a}$, and

$$C = \frac{a}{1 - a}\,\epsilon\sqrt{2-2\,\cos(2\,\pi\,b\sqrt{1-a}) - a\,\sin^2(2\,\pi\,b\sqrt{1-a})}.$$

Our interest centers about the maximum bending moment which is proportional to the maximum curvature $(\partial^2\eta/\partial x^2)$. Formula (22) concedes

$$\frac{\partial^2\eta}{\partial x^2} = -\,C\,\lambda^2\,\sin\,(p_0\,t + \varphi)\,\sin\,\lambda\,x.$$

The curvature is maximum in the center of the bar $(x = \frac{l}{2})$ and amounts to

$$\max\frac{\partial^2\eta}{\partial x^2} = C\,\lambda^2 = C\,\frac{\pi^2}{l^2} = \epsilon\,\frac{\pi^2}{l^2}\,f\,(a,b)$$

where

$$f(a,b) = \frac{a}{1-a}\sqrt{2-2\,\cos(2\pi b\sqrt{1-a})-a\,\sin^2(2\pi b\sqrt{1-a})} \qquad (23)$$

This value is decisive for the maximum moment <u>after</u> the actual shock period. However, it may happen that a still greater moment occurs <u>during</u> the actual shock period. Such is the case when the shock lasts at least long enough - until the highest possible deflection has been reached once. According to (18), the greatest possible deflection during the shock occurs once when $\cos pt = -1$. Consequently, if

$$\tau \gtreqless \frac{\pi}{p} = \frac{\pi}{p_0}\,\frac{1}{\sqrt{1 - a}}$$

or

$$b \gtreqless \frac{\pi}{pT}\,\frac{1}{\sqrt{1 - a}} = \frac{1}{2\sqrt{1 - a}}$$

the maximum moment is already reached <u>during</u> the shock.

On the other hand, according to (18),

$$\eta = \frac{a}{1 - a}\,\epsilon\,(1 - \cos\,p\,t)\,\sin\,\lambda\,x$$

during the shock, hence

$$\max \frac{\partial^2 \eta}{\partial x^2} = 2 \frac{a}{1 - a} \epsilon \lambda^2 = \epsilon \frac{\pi^2}{l^2} f(a,b),$$

is approximately proportional to the maximum moment, whereby

$$f(a,b) = 2 \frac{a}{1 - a} \tag{24}$$

To sum up: For shock periods $b < \frac{1}{2\sqrt{1 - a}}$, formula (23) is valid; for shock periods $b \gtreqless \frac{1}{2\sqrt{1 - a}}$, (24) is the decisive quantity for the maximum moment. For $b = \frac{1}{2\sqrt{1 - a}}$, (23) steadily resolves to (24). The latter represents an absolute maximum value of $f(a,b)$, which may not be exceeded with any shock period.

β) The shock load equals the Eulerian load ($a = 1$).

In this case (14) reduces to

$$\frac{d^2 \eta_t}{dt^2} = \frac{EJ}{\rho F} \lambda^4 \epsilon.$$

Integrating twice gives:

$$\eta_t = \frac{EJ}{2\rho F} \lambda^4 \epsilon t^2 + k_1 t + k_2.$$

The initial conditions:

$$\eta_t = 0 \quad \text{and} \quad \dot{\eta}_t = 0 \quad \text{for} \quad t = 0$$

concede $k_1 = 0$ and $k_2 = 0$. Therefore,

$$\eta_t = \frac{EJ}{2\rho F} \lambda^4 \epsilon t^2 \tag{25}$$

and for the end of the shock,

$$\eta_t = \frac{EJ}{2\rho F} \lambda^4 \epsilon \tau^2 \tag{26}$$

and

$$\dot{\eta}_t = \frac{EJ}{\rho F} \lambda^4 \epsilon \tau \tag{27}$$

The time interval after the actual shock is again computed with (22) and the initial conditions for this shock period are posed in (26) and (27). Thus,

$$C \sin (p_o \tau + \varphi) = \frac{EJ}{2\rho F} \lambda^4 \epsilon \tau^2$$

$$C p_o \cos (p_o \tau + \varphi) = \frac{EJ}{\rho F} \lambda^4 \epsilon \tau$$

hence,

$$C = \frac{EJ}{\rho F} \lambda^4 \epsilon \tau \sqrt{\frac{\tau^2}{4} + \frac{1}{p_o{}^2}}$$

or, when taking

$$\tau = \frac{2\pi}{p_o} b \quad \text{and} \quad \frac{EJ}{\rho F} \lambda^4 = p_o{}^2$$

into consideration,

$$C = 2 \pi b \epsilon \sqrt{\pi^2 b^2 + 1.}$$

Equation (22) again yields

$$\max \frac{\partial^2 \eta}{\partial x^2} = C \lambda^2 = \epsilon \frac{\pi^2}{l^2} f(a,b),$$

where

$$f(a,b) = 2 \pi b \sqrt{\pi^2 b^2 + 1} \tag{28}$$

Contrary to the case of $a < 1$, equation (25), applicable during the shock, is now aperiodic, hence the deflection during the shock may not exceed that at the end of the shock. But the latter is, conformable to (28), the start of the free oscillation.

γ) The shock force exceeds the Eulerian load ($a > 1$)

In this case,

$$\frac{EJ}{\rho F} \lambda^4 (1 - a) < 0,$$

and the solution of the homogeneous equation corresponding to (14), manifests:

$$\bar{\eta}_t = k_1 \operatorname{sh} p\, t + k_2 \operatorname{ch} p\, t,$$

whence,

$$\frac{d^2 \bar{\eta}_t}{dt^2} = p^2\, \bar{\eta}_t,$$

which, written in the homogeneous equation, gives

$$p^2 = \frac{EJ}{\rho F} \lambda^4 \;(a - 1) = p_o^2 \;(a - 1) \tag{29}$$

p_o = frequency of the free fundamental oscillation.

$$E\, J\, \lambda^4 \;(1 - a)\, \bar{\eta}_t = a\, \epsilon\, E\, J\, \lambda^4$$

concedes the particular integral $\bar{\bar{\eta}}_t$ at

$$\bar{\bar{\eta}}_t = - \frac{a}{a - 1}\, \epsilon.$$

Consequently,

$$\eta_t = k_1 \operatorname{sh} p\, t + k_2 \operatorname{ch} p\, t - \frac{a}{a - 1}\, \epsilon \tag{30}$$

$$\dot{\eta}_t = p\, (k_1 \operatorname{ch} p\, t + k_2 \operatorname{sh} p\, t) \tag{31}$$

Assume $\eta_t = 0$ and $\dot{\eta}_t = 0$ for $t = 0$. Then (30) and (31) give

$$k_2 = \frac{a}{a - 1}\, \epsilon \quad \text{and} \quad k_1 = 0$$

whence,

$$\eta_t = \frac{a}{a - 1}\, \epsilon\; (\operatorname{ch} p\, t - 1) \tag{32}$$

$$\dot{\eta}_t = \frac{a}{a - 1}\, \epsilon\, p \operatorname{sh} p\, t.,$$

and for the end of the shock $t = \tau$:

$$\eta_\tau = \frac{a}{a - 1}\, \epsilon\; (\operatorname{ch} p\, \tau - 1) \tag{33}$$

$$\dot{\eta}_\tau = \frac{a}{a - 1}\, \epsilon\, p \operatorname{sh} p\, \tau \tag{34}$$

For the free oscillations *after* the shock (22) and the initial conditions (33) and (34) are again applicable. Then,

$$C \sin (p_0 \tau + \varphi) = \frac{a}{a - 1} \epsilon (\text{ch } p \tau - 1)$$

$$C p_0 \cos (p_0 \tau + \varphi) = \frac{a}{a - 1} \epsilon p \text{ sh } p \tau$$

and, with due allowance for (29)

$$C = \frac{a}{a - 1} \epsilon \sqrt{2 - 2 \text{ ch } p \tau + a \text{ sh}^2 p\tau}$$

or with

$$p \tau = 2 \pi b \frac{p}{p_0} = 2 \pi b \sqrt{a - 1},$$

$$C = \frac{a}{a - 1} \epsilon \sqrt{2-2 \text{ ch}(2\pi b \sqrt{a-1}) + a \text{ sh}^2(2\pi b \sqrt{a - 1})}.$$

According to (22) the maximum curvature is again

$$\max \frac{\partial^2 \eta}{\partial x^2} = C \lambda^2 = \epsilon \frac{\pi^2}{l^2} f(a,b)$$

where

$$f(a,b) = \frac{a}{a-1} \sqrt{2-2 \text{ ch}(2\pi b \sqrt{a-1}) + a \text{ sh}^2(2\pi b \sqrt{a-1})} \quad (35)$$

Since (32), applicable during the shock, is aperiodic for η_t and increases with t, no greater deflection can occur *during* the shock than the maximum deflection reached *after* the shock; but that is comprised in (35).

b) The (n-1)th higher oscillation

Substituting $X = - a E J \lambda^2$, equation (11) becomes:

$$\rho F \frac{d^2 \eta_{tn}}{dt^2} + n^2 E J \lambda^4 (n^2 - a) \eta_{tn} = n^2 a E J \lambda^4 \epsilon_n \quad (36)$$

Again we differentiate between:

1. $a < n^2$
2. $a = n^2$
3. $a > n^2$

of which the first is of primary interest. Even with the first higher oscillation $(n = 2)$, cases 2 and 3 refer to shock loads at least $2^2 = 4$ times as high as the Eulerian load. Such excesses of the Eulerian load may be disregarded and the analysis confined to $a < n^2$.

The resolution of (36) for $a < n^2$ is similar to that of (14) for $a < 1$. Let the solution of the homogeneous equation be

$$\bar{\eta}_{tn} = k_1 \sin p_n t + k_2 \cos p_n t$$

so that

$$\frac{d^2 \bar{\eta}_{tn}}{dt^2} = - p_n^2 \bar{\eta}_{tn}$$

which gives the frequency

$$p_n = n \lambda^2 \sqrt{\frac{EJ}{\rho F}} \sqrt{n^2 - a} = n p_{01} \sqrt{n^2 - a} \qquad (37)$$

p_{01} = frequency of free fundamental oscillation. (See section IV,2.)

The effect of the disturbing function follows from

$$n^2 E J \lambda^4 (n^2 - a) \bar{\bar{\eta}}_{tn} = n^2 a E J \lambda^4 \epsilon_n$$

at

$$\bar{\bar{\eta}}_{tn} = \frac{a}{n^2 - a} \epsilon$$

whence the solution of (8) at

$$\eta_{tn} = k_1 \sin p t + k_2 \cos p t + \frac{a}{n^2 - a} \epsilon_n.$$

k_1 and k_2 are again defined from the initial conditions: $\eta_{tn} = 0$ and $\dot{\eta}_{tn} = 0$ for $t = 0$. The result is

$$\eta_{tn} = \frac{a}{n^2 - a} \epsilon_n (1 - \cos p_n t) \qquad (38)$$

$$\dot{\eta}_{tn} = \frac{a}{n^2 - a} p_n \epsilon_n \sin p_n t = \frac{n a}{\sqrt{n^2 - a}} p_{01} \epsilon_n \sin p_n t$$

For $t = \tau$, we have:

$$\eta_{\tau n} = \frac{a}{n^2 - a} \epsilon_n (1 - \cos p_n \tau) \tag{39}$$

$$\dot{\eta}_{\tau n} = \frac{n\ a}{\sqrt{n^2 - a}} p_{01} \epsilon_n \sin p_n \tau \tag{40}$$

After the shock the bar executes the (n-1)th free higher oscillation, which is governed by

$$\eta_n = C_n \sin (n^2 p_{01} t + \varphi_n) \sin n \lambda x \tag{41}$$

With C and φ defined from (39) and (40) as initial conditions:

$$C_n \sin (n^2 p_{01} \tau + \varphi_n) = \frac{a}{n^2 - a} \epsilon_n (1 - \cos p_n \tau)$$

$$C_n n^2 p_{01} \cos(n^2 p_{01} \tau + \varphi_n) = \frac{n\ a}{\sqrt{n^2 - a}} p_{01} \epsilon_n \sin p_n \tau$$

gives C at

$$C_n = \frac{a}{n^2 - a} \epsilon_n \sqrt{2 - 2 \cos p_n \tau - \frac{a}{n^2} \sin p_n \tau}.$$

Because of (37) it is

$$p_n \tau = 2 n \pi b \sqrt{n^2 - a}$$

when $\tau = \frac{2 \pi}{p_{01}} b$. Therefore,

$$C_n = \frac{a}{n^2 - a} \epsilon_n \sqrt{2 - 2 \cos(2n\pi b\sqrt{n^2 - a}) - \frac{a}{n^2} \sin^2(2n\pi b\sqrt{n^2 - a})}.$$

According to (41) the curvature is

$$\frac{\partial^2 \eta_n}{\partial x^2} = - C_n n^2 \lambda^2 \sin (n^2 p_{01} t + \varphi_n) \sin n \lambda x$$

and the maximum curvature,

$$\max \frac{\partial^2 \eta_n}{\partial x^2} = C_n n^2 \lambda^2 = \frac{\pi^2}{l^2} \epsilon_n f(n,a,b),$$

where $f(n,a,b) =$

$$= \frac{n^2\ a}{n^2 - a} \sqrt{2 - 2\ \cos(2n\pi b\sqrt{n^2 - a}) - \frac{a}{n^2}\ \sin^2(2n\pi b\sqrt{n^2 - a})} \qquad (42)$$

The validity of (42) for the maximum moment is decisive only when it occurs _after_ the shock. If the actual shock lasts at least long enough to permit once the occurrence of the highest possible deflection, then the maximum moment occurs during the actual shock. This is the case according to (38) when $\cos p_n\ t = -1$, that is, when the duration of the shock is

$$\tau \geqq \frac{\pi}{p_n} = \frac{\pi}{n\ p_{01}}\ \frac{1}{\sqrt{n^2 - a}}$$

or

$$b \geqq \frac{1}{2\ n\sqrt{n^2 - a}}$$

In this case the curvature is

$$\frac{\partial^2 \eta_n}{\partial\ x^2} = \frac{a}{n^2 - a}\ \epsilon_n\ n^2\ \lambda^2\ (1 - \cos p_n\ t)\ \sin n\ \lambda\ x$$

according to

$$\eta_n = \frac{a}{n^2 - a}\ \epsilon(1 - \cos p_n\ t)\ \sin n\ \lambda\ x$$

and the maximum curvature is

$$\max\ \frac{\partial^2 \eta_n}{\partial\ x^2} = \frac{2a}{n^2 - a}\ \epsilon_n\ n^2\ \lambda^2 = \epsilon_n\ \frac{\pi^2}{l^2}\ f(n,a,b),$$

with

$$f(n,a,b) = \frac{2\ a\ n^2}{n^2 - a} \qquad (43)$$

Summed up: for shock period

$$b < \frac{1}{2\ n\sqrt{n^2 - a}}$$

equation (42) is applicable;

for

$$b \geqq \frac{1}{2\,n\sqrt{n^2 - a}}$$

(43) is valid as the quantity deciding the maximum moment.

When $b = \frac{1}{2\,n\sqrt{n^2 - a}}$, (42) becomes (43). The latter represents the maximum value of $f(n,a,b)$, which is not exceeded in any shock period.

V. NUMERICAL INTERPRETATION AND DEDUCTIONS

The behavior of the bar during shock load is best evaluated by comparing it with its behavior under static load.

The differential equation of the static load is:

$$E\,J\,\frac{d^2\eta}{dx^2} - X\,\eta = X\,\underline{y} \tag{44}$$

with $\underline{y}$ given from (9). Limited to the first term of (9), the resolution gives:

$$\eta = C_1 \sin \lambda x + C_2 \cos \lambda x$$

or

$$\eta = C_1 \sin \lambda x \tag{45}$$

since $\eta = 0$ when $x = 0$.

Putting (45) in (44) gives $\lambda = \frac{\pi}{l}$ and

$$C_1 = \frac{\epsilon_1}{-\frac{E\,J\,\pi^2}{X\,l^2} - 1},$$

consequently,

$$\eta = \frac{\epsilon_1 \sin \lambda x}{-\frac{E\,J\,\pi^2}{X\,l^2} - 1} \tag{46}$$

for

$$-X = P_E = E\,J\,\frac{\pi^2}{l^2},$$

that is, the Eulerian load, $\eta = \infty$. Therefore, and with consideration of (13),

$$\eta = \frac{a}{1 - a} \epsilon_1 \sin \lambda x$$

whence

$$\max \frac{d^2\eta}{dx^2} = \frac{\pi^2}{l^2} \epsilon_1 f(a) \tag{47}$$

with $f(a) = \frac{a}{1 - a}$.

The ratio c of the moments due to shock load and static load is:

$$c = \sqrt{2 - 2 \cos(2\pi b \sqrt{1 - a}) - a \sin^2(2\pi b \sqrt{1 - a})} \tag{48}$$

according to (23) and (47) when $a < 1$ and the duration of the shock is

$$b \leqq \frac{1}{2\sqrt{1 - a}}$$

and

$$c = 2 \tag{49}$$

according to (24) and (47) when the shock period is

$$b \geqq \frac{1}{2\sqrt{1 - a}}$$

It is readily seen that the c terms are dependent on the magnitude of the eccentricity ϵ. (See fig. 3.)

It will be noted that the ratio c of the dynamic and static stress for short shock periods is smaller, at longer periods greater than 1 and its maximum value 2. Moreover, for equal shock durations, ratio c is smaller as the shock load is higher.

When plotting, as in figure 4, that shock load b against shock load a for which the dynamic equals the static stress, it is seen that comparatively long shocks are necessary vicinal to the Eulerian load to raise the dynamic stress on a level with the static stress. With a shock equivalent to 0.97 times (approximately) the Eulerian load, this shock period equals the natural oscillation period of the free bar.

If the form of the bar is such as to exactly produce the (n-1)th higher harmonic, by the same argument the nth stability limit is

$$- X = EJ \frac{n^2 \pi}{l^2}$$

when the bar produces only the fundamental oscillation and the decisive function for the maximum moment is

$$f(a,n) = \frac{n^2 a}{n^2 - a} \tag{50}$$

When

$$b \leqq \frac{1}{2\, n \sqrt{n^2 - a}}$$

the comparison with (42) and (43) gives

$$c = \sqrt{2-2 \cos(2n\pi b \sqrt{n^2-a}) - \frac{a}{n^2} \sin^2(2n\pi b \sqrt{n^2-a})} \tag{51}$$

and

$$c = 2$$

when

$$b \geqq \frac{1}{2\, n \sqrt{n^2 - a}} \text{.}$$

The range of validity of (51) is limited to very short shock periods. Even for $n = 2$ and $a = 1$, the upper limit of b is 0.144 only. Since for very short shock periods the premises of the calculations are in any case hardly met (reference 1), the evaluation of (51) may be foregone, especially since the case where exactly only the (n-1)th higher oscillation occurs, is practically without significance.

By contrast, the case where the bar shape is such as to incur several oscillations concurrently, is much more important. But obviously this case does not lend itself to general treatment, because the results are substantially affected by the relative magnitude and the sign of ϵ_n in (9).

The resolution of (44) is applicable only to the cases for the evaluation of the data of the dynamic investigation in which the load lies below the stability limit, i.e., for $n = 1$ in the $a < 1$ range; for $n = 2$ in the $a < 4$ range, etc. For loads above the stability limit (44) should be disregarded in favor of the more exact equation of the elastic line, the resolution of which is, however, quite complicated. For that reason the comparison of the data is limited to $n = 1$ for $a < 1$, $a = 1$, and $a > 1$, respectively. The results are illustrated in figure 5, in logarithmic scale. The functions $f(a,b)$ from (23), (24), (28), and (35), proportional to the maximum moments are plotted for divers a against b.

It is readily seen that the Eulerian load may be exceeded in buckling stresses due to shock, provided the shock period itself is short enough. Moreover, the maximum dynamic stresses are fostered by increasing shock load a and period b. But, while attaining a limit value for shock loads below the Eulerian load $(a < 1)$ for a given duration of shock, which cannot be exceeded in any shock period, they increase arbitrarily at shock loads above the Eulerian load.

VI. EFFECT OF SHOCK IMPULSE $\int P\,dt$ ON THE STRESS

Frequently it is assumed that the stress due to shock load P is dependent only on the shock impulse $\int P\,dt$, that is, individually unaffected by the magnitude of the shock load and the duration.*

The results of the present paper disclose the error of this assumption, for otherwise only the product $a\,b$ would appear as sole variable of the terms for $f(a,b)$. Again, it may be asked whether or not it would be approximately correct. For that reason, we compute the functions $f(a,b)$ versus b for several values of $S = \int P\,dt = ab$.

*It is common practice to designate the time integral $\int P\,dt$ as "shock load," whereas the quantity P is not specifically expressed. This practice is probably due to the concept that only the time integral $\int P\,dt$ is decisive for the shock process, whereas no special importance attaches to quantity P. But the authors of this paper have, on the strength of their investigations, drawn different conclusions, and believe it, in fact, to be more logical to express P, which has the dimension of a force, as "shock load" and time integral $\int P\,dt$ with the dimension of a force times time interval as "shock impulse."

Concentrating on the fundamental oscillation, there are three ranges of b for a stated value of $S = ab$:

1. range $0 < b < b_1$ for $a > 1$.
2. " $b_1 < b < b_2$ for $a < 1$ to the extent that the maximum stress occurs <u>after</u> the actual shock period.
3. range $b_2 < b$ when the stress occurs <u>during</u> the actual shock period.

The values for b_1 and b_2 may be defined as follows:

$$a = 1 \quad \text{for} \quad b_1, \quad \text{consequently,} \quad b_1 = S;$$

$$b_2 = \frac{1}{2\sqrt{1 - a}} \quad \text{for} \quad b_2,$$

or

$$b_2 = \frac{1}{2}\left(S + \sqrt{S^2 + 1}\right)$$

because

$$a = \frac{S}{b_2}.$$

For $f(a,b)$:

equation	(35)	is valid in the range of	$0 < b < b_1$
"	(23)	" " " " " "	$b_1 < b < b_2$
"	(24)	" " " " " "	$b_2 < b$
"	(28)	" " " " " "	$b = b_1$

The value of $f(a,b)$ for $b = 0$ is obtained by putting $a = \frac{S}{b}$ in (35) and permitting b to approach zero. Then,

$$\lim_{b=0} f(a,b) = \sqrt{\frac{S}{b}\,\mathrm{sh}^2\left(2\pi\sqrt{S b - b^2}\right)} =$$

$$= \pi S \frac{(S - 2b)\,\mathrm{sh}\left(4\pi\sqrt{S b - b^2}\right)}{\sqrt{S b - b^2}} = 2\pi S.$$

With these formulas we computed $f(a,b)$ for $S = 0.25$; 0.50, 0.75, 1.00, and 1.25, as well as for various b values. The results are plotted in figure 6. The ordinates of the individual curves are noticeably not approximately constant with the parameter S or, in other words, the stress due to a shock load cannot even be approximately given in function of the shock impulse. On the contrary, shock load and shock period must be individually known if the shock stress is to be determined.

This result is of great importance for shock tests. Shock load and shock period must be included in such experiments, although this will be more difficult to accomplish than recording the shock impulse.

APPENDIX

Effect of Minor Changes of the Original Bar Shape on the Results

The premise of the interpretation of the results was the selection of the original bar shape such as to precisely insure the occurrence of the fundamental oscillation due to shock load. In the example hereinafter, we attempt to show the effect of a minor change in the original shape of the bar on the results. For simplicity the range is restricted to $a < 1$, that is, to the range within which the ratio c of the dynamic and static stress is readily obtainable.

We assume the shape of the bar such as to develop aside from the fundamental oscillation, yet the second higher oscillation. Then (9) reads:

$$y = \epsilon_1 \sin \frac{\pi}{l} x + \epsilon_3 \sin 3 \frac{\pi}{l} x$$

with $\frac{\pi}{l} = \lambda$.

For shock periods,

$$b \geqq \frac{1}{2\sqrt{1 - a}};$$

the equation

$$b \geqq \frac{1}{2n\sqrt{n^2 - a}} = \frac{1}{6\sqrt{9 - a}}$$

is particularly applicable. According to IV,3a and IV,3b the maximum deflections occur during the shock. The curvature of the elastic line is expressed by

$$\left|\frac{\partial^2 \eta}{\partial x^2}\right| = \left|a\ \lambda^2 \left[\frac{\epsilon_1}{1-a}(1 - \cos p\ t) \sin \lambda\ x + \right.\right.$$

$$\left.\left. + \frac{9\ \epsilon_3}{9-a}(1 - \cos p_3\ t) \sin 3\ \lambda\ x\right]\right| \qquad (52)$$

The points of the maximum and minimum values of this function result from the resolution of

$$\left|\frac{\partial^3 \eta}{\partial x^3}\right| = \left|a\ \lambda^3 \left[\frac{\epsilon_1}{1-a}(1 - \cos p\ t) \cos \lambda\ x + \right.\right.$$

$$\left.\left. + \frac{27\ \epsilon_3}{9-a}(1 - \cos p_3\ t) \cos 3\ \lambda\ x\right]\right| = 0 \qquad (53)$$

With

$$\cos 3\ \lambda\ x = 4 \cos^3 \lambda\ x - 3 \cos \lambda\ x$$

(53) resolves to $\cos \lambda\ x = 0$ (54a)

and

$$\frac{\epsilon_1}{1-a}(1-\cos pt) + \frac{27\ \epsilon_3}{9-a}(1-\cos p_3 t)(4 \cos^2 \lambda\ x - 3) = 0 \qquad (54b)$$

First we consider the maximum for $x = \frac{l}{2}$ of (54a). Here (52) gives:

$$\left|\frac{\partial^2 \eta}{\partial x^2}\right| = \left|a\ \lambda^2 \left[\frac{\epsilon_1}{1-a}(1-\cos pt) - \frac{9\ \epsilon_3}{9-a}(1-\cos p_3 t)\right]\right| \qquad (55)$$

To the extent that $|\epsilon_3| \leqq |\epsilon_1,|$ hence $\left|\frac{9\ \epsilon_3}{9-a}\right| \leqq \left|\frac{\epsilon_1}{1-a}\right|$, the limits of the maximum curvature are

$$\left|2a\ \lambda^2\left(\frac{\epsilon}{1-a} - \frac{9\ \epsilon_3}{9-a}\right)\right| \leqq \left|\frac{\partial^2 \eta}{\partial x^2}\right|_{\substack{max \\ x=\frac{l}{2}}} \leqq \left|2a\ \lambda^2 \frac{\epsilon_1}{1-a}\right| \qquad (56a)$$

when ϵ_1 and ϵ_3 have the same sign, and

$$\left| 2a\lambda^2 \; \frac{\epsilon_1}{1-a} \right| \leqq \left| \frac{\partial^2 \eta}{\partial x^2} \right|_{\substack{max \\ x=\frac{l}{2}}} \leqq \left| 2a\lambda^2 \left(\frac{\epsilon_1}{1-a} - \frac{9\,\epsilon_3}{9-a} \right) \right| \qquad (56b)$$

when ϵ_1 and ϵ_3 have different signs.

Comparing these results with the value of the curvature at $x = l/2$ in the static case

$$\left| \frac{d^2\eta}{dx^2} \right|_{x=\frac{l}{2}} = \left| a \, \lambda^2 \left(\frac{\epsilon_1}{1 - a} - \frac{9\,\epsilon_3}{9 - a} \right) \right|,$$

the ratio c of the dynamic and static stress is found to lie between

$$2 \leqq c \leqq 2 \left| \frac{\epsilon_1 \; (9 - a)}{\epsilon_1 \, (9 - a) - 9\,\epsilon_3 \, (1 - a)} \right| \qquad (57a)$$

when ϵ_1 and ϵ_3 have the same sign, and

$$2 \left| \frac{\epsilon_1 (9 - a)}{\epsilon_1 (9 - a) - 9 \, \epsilon_3 (1 - a)} \right| \leqq c \leqq 2 \qquad (57b)$$

when they have a different sign.

This leaves the question, whether or not at some point other than $x = \frac{l}{2}$ an upper limit of $\left| \frac{\partial^2 \eta}{\partial x^2} \right|_{max}$ may occur which exceeds the values given in (57a) and (57b).

Superior limit values of $\left| \frac{\partial^2 \eta}{\partial x^2} \right|_{max}$ may occur at points other than $x = \frac{l}{2}$ only when x meets equation (54b). Let x_1 be a real root of this equation. Then

$$\cos^2 \lambda \, x_1 = \frac{3}{4} - \frac{1}{108} \, \frac{9 - a}{1 - a} \, \frac{\epsilon_1}{\epsilon_3} \, \frac{1 - \cos p \, t}{1 - \cos p_3 t}$$

hence,

$$\sin 3 \lambda \, x_1 = \sin \lambda \, x_1(-1+4 \cos^2 \lambda \, x_1) =$$

$$= \sin \lambda \, x_1\left(2 - \frac{1}{27} \, \frac{9-a}{1-a} \, \frac{\epsilon_1}{\epsilon_3} \, \frac{1-\cos p \; t}{1-\cos p \; t}\right)$$

Then, according to (52) the curvature at $x = x_1$ is:

$$\left|\frac{\partial^2 \eta}{\partial x^2}\right|_{x=x_1} = \left| a\,\lambda^2 \sin \lambda\, x_1 \left[\frac{2}{3}\,\frac{\epsilon_1}{1-a}\,(1 - \cos p\, t) + \right.\right.$$

$$\left.\left. + \frac{18\,\epsilon_3}{9-a}\,(1 - \cos p_3\, t)\right]\right| \qquad (58)$$

The upper limits of (58) are:

$$\left|\frac{\partial^2 \eta}{\partial x^2}\right|_{x=x_1} \leqq \left| 2a\,\lambda^2 \sin \lambda\, x_1 \left(\frac{2}{3}\,\frac{\epsilon_1}{1-a} + \frac{18\,\epsilon_3}{9-a}\right)\right| \qquad (59a)$$

when ϵ_1 and ϵ_3 show the same sign, and

$$\left|\frac{\partial^2 \eta}{\partial x^2}\right|_{x=x_1} \leqq \left| \frac{4}{3}\,\lambda^2\,\frac{a\,\epsilon_1}{1-a} \sin \lambda\, x_1\right| \qquad (59b)$$

when otherwise.

The requirement that the curvature at $x = x_1$ for equal signs ϵ_1 and ϵ_3 shall at the most be the same as the maximum value of (56a) results in

$$\left| 2a\,\lambda^2 \sin \lambda\, x_1 \left(\frac{2}{3}\,\frac{\epsilon_1}{1-a} + \frac{18\,\epsilon_3}{9-a}\right)\right| \leqq \left| 2a\,\lambda^2\,\frac{\epsilon_1}{1-a}\right| .$$

This condition is always met:

$$\left|\frac{2}{3}\,\frac{\epsilon_1}{1-a} + \frac{18\,\epsilon_3}{9-a}\right| \leqq \left|\frac{\epsilon_1}{1-a}\right| ,$$

i.e., for

$$\left|\frac{\epsilon_3}{\epsilon_1}\right| \leqq \frac{1}{6} \qquad (60)$$

because

$$\frac{1-a}{9-a} \leqq \frac{1}{9} .$$

For different signs of ϵ_1 and ϵ_3 the result is similar, according to (56b) and (59b):

$$\left| \frac{4}{3}\,\lambda^2\,\frac{a\,\epsilon_1}{1-a} \sin \lambda\, x_1\right| \leqq \left| 2a\,\lambda^2 \left(\frac{\epsilon_1}{1-a} - \frac{9\,\epsilon_3}{9-a}\right)\right| .$$

This condition is met for every value of ϵ_3 as can readily be seen.

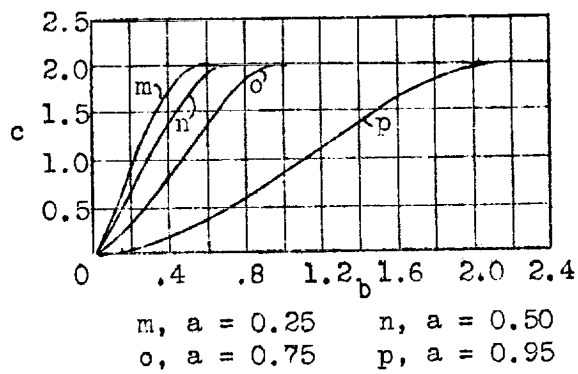

Figure 3.-Ratio c of maximum moments under dynamic and static load versus ratio b of shock period to period of free oscillations for various ratios a of longitudinal force to Eulerian load.

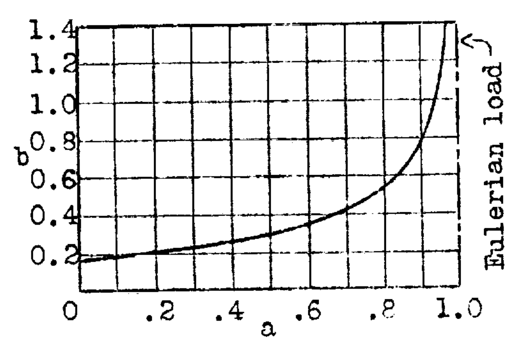

Figure 4.-"Shock period" b for divers "shock loads" a, for which the maximum dynamic and static stress are equal, i.e. c = 1 .

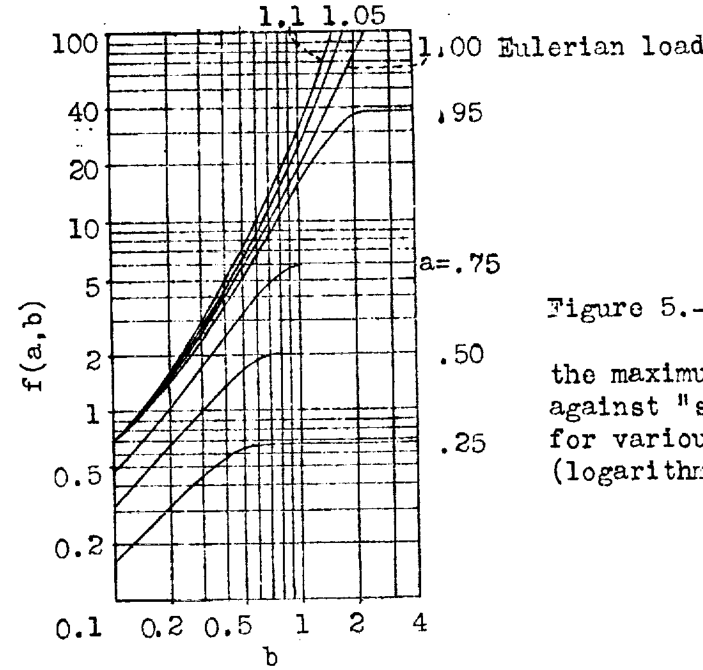

Figure 5.- Quantity f(a,b) proportional to the maximum moments plotted against "shock period" b, for various "shock loads" a. (logarithmic scale)

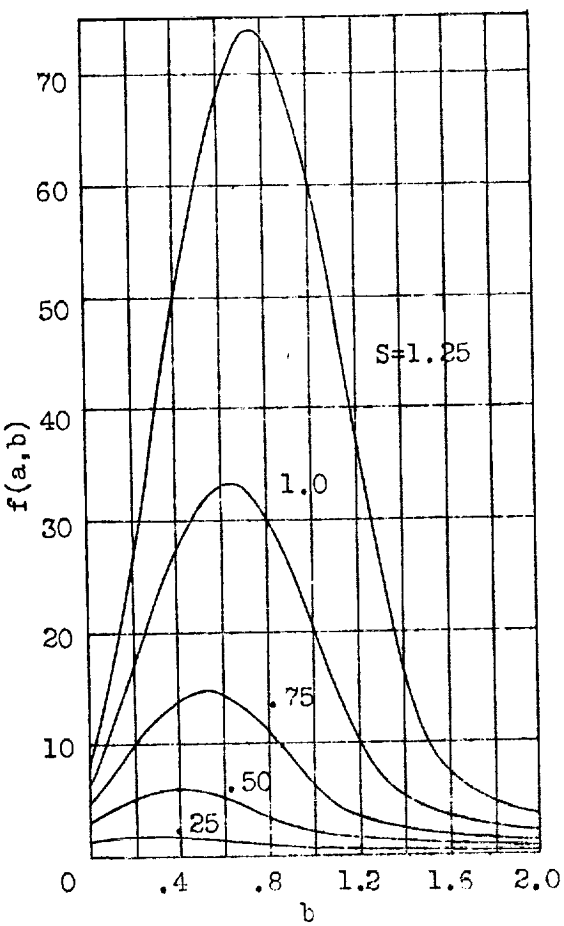

Figure 6.- Stress f(a,b) versus "shock load" b, for various shock impulses S = f(a,b)

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