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TORSIONAL STRESSES IN BOX BEAMS WITH CROSS SECTIONS PARTIALLY RESTRAINED AGAINST WARPING

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TORSIONAL STRESSES IN BOX BEAMS WITH CROSS SECTIONS PARTIALLY RESTRAINED AGAINST WARPING*

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SUMMARY

The present report gives a method for computing the torsion of boxes with thin shear-resistant or simply tension-resistant walls under any torsional load, support and dimension. The final stress condition is developed from that of a principal system with unconstrained sectional buckling corresponding to Bredt's formula and an additional stress condition due to constrained cross-sectional buckling. This is computed by means of the deflection condition of the principal system from a statically indeterminate calculation. Conformably, the torsional rigidity of the final system is derived from that of the principal system with unconstrained sectional buckling. It is shown that the additional stresses due to constrained sectional buckling become so much more decisive as the dimensions of the individual side walls vary from each other. In general, it suffices to determine the additional stresses from a single or, at the most, double, statically indeterminate partial system at the points of strong buckling constraint by assuming in their plane rigid bulkheads or using the formulas derived for regular systems. The discrepancy in torsional rigidity from that computed by Bredt's formula, is insignificant except for short boxes.

I. INTRODUCTION

Boxes (fig. 1) consisting of four side walls with reinforcing bulkheads are extensively used in airplane and

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*"Die Beanspruchung dünnwandiger Kastenträger auf Drillung bei behinderter Querschnittswölbung." Z.F.M., December 14, 1933, pp. 645-655; and December 28, 1933, pp. 684-692.

ship construction. They usually form the center boxes of metal- and ply-covered wings, airplane bodies, ship hulls, and double bottoms of ships. In this connection, the box is predominately stressed in bending (which is not, however, within the scope of this paper (reference 1)) and in torsion.

The analysis of thin-walled boxes under torsion is very simple, provided unconstrained cross-sectional buckling is ensured, and in which case Bredt's theory for thin-walled tubes is applicable (reference 2).

The stress of a box with shear-resistant walls under torsion with constrained cross-sectional buckling has already been investigated for specific cases of loading, support, and dimensions, namely: by Eggenschwyler (reference 3) for the freely supported, unconstrained box with sinusoidally distributed torsional loading, and by Reissner (reference 4) for the one-sidedly buckling resistant box with evenly distributed torsional load. Whereas Eggenschwyler assumes a box with constant cross section, Reissner, ~~moreover~~ however, treats the box with linearly changing height but otherwise constant dimensions. The premise of both authors is the stiffening of the box by infinitely closely spaced, rigid bulkheads.

These assumptions as well as the limitations relative to load and size are not made in the present report because in practice we encounter at times, boxes of different load and size with less rigid bulkheads at wider spacing. As a result the problem then reverts to the resolution of an equation of differences, i.e., of a system of equations rather than of a differential equation, as in the cited reports.

From the point of view of statics, the box with shear-resistant walls requires no corner stiffeners when stressed in torsion, although for structural reasons and because of the concurrent stress in bending, it is nevertheless so reinforced as a rule and therefore falls within the scope of this study.

The plate girders used in airplane design are very thin metal webs with no regard to shear stiffness. The webs act between the edge stiffeners and the upright stiffeners (spaced suitably close, 1/2 to 1/5 beam height) as "tension diagonal fields." As demonstrated by Wagner (reference 5) for the flat-plate girder; the analysis of such beams is very simple when assuming edge flanges stiff in bending, and can be readily extended to include box beams with such very thin plate walls.

The case of box beams with longitudinal walls and bulkheads consisting of trusses, has been treated elsewhere (reference 6).

II. PRINCIPLES

1. Notation (see figs. 1 and 2)

The bulkeads are denoted by

$$0, 1, 2 \ldots\ldots r - 1, r, r + 1 \ldots\ldots n - 1, n;$$

positive longitudinally when from left to right. The divisions between consecutive bulkheads or "cells" carry the figure of their right-hand bulkhead. The bulkhead corners, positive longitudinally when clockwise are indicated by I, II, III, and IV. The coordinate systems for the individual longitudinal walls of the cells are illustrated in figure 2. In addition:

a, spacing of bulkheads

b, c, sides of bulkheads

$d_b, d_c, \bar{d}$, wall thickness of horizontal and upright longitudinal walls and bulkheads.

l, length of system

F, cross section of corner stiffener

M_r, individual torque about the longitudinal axis at the rth bulkhead

m_r, distributed torsional load in rth cell

$\bar{M}_r$, torsion moment in rth cell

B, C, couples of horizontal and vertical forces

t, tangential edge loading (kg/m)

Individual torque M and couples B and C are positive when clockwise as seen in positive fore-and-aft direction, and twisting moment $\overline{M}$ is positive when the resultant of the outside torque at the left-hand part of the system is counterclockwise as seen in positive fore-and-aft direction. Folding the longitudinal walls in the plane of the forward longitudinal wall, we figure as positive: outside loads when downward, transverse loads and bending moments when the resultant of the outside loads at the left wall portion is upward or turns clockwise. Tensile stresses are positive.

2. Outside Loads

The outside torsional load of the box may appear as individual torque M (m kg) at the bulkheads, or as torsional load m (m kg/m) distributed across the length of the cell. The effect of a distributed torsional load is duly allowed for by dividing it over two supports across the bulkheads, conformably to the support pressure of a beam and computing the box for the thus produced individual torque. The effect of the intermediate loading is accounted for by analyzing the equilibrium condition from the inverse support torque and the intermediate torsional load (fig. 2). The individual torque M and the distributed torsional load m can be applied as couple of horizontal and vertical forces M_b and M_c or m_b and m_c. The torque M_i at the bulkheads then gives the effective twisting moment in a cell r:

$$\overline{M}_r = - \sum_{i=o}^{r-1} M_i = + \sum_{i=r}^{n} M_i$$

3. Principal System and Elasticity Equations

Division of the box loaded under individual torque M (fig. 1) at the bulkheads into a number of separate boxes or "cells," affords a "principal system," wherein the cells themselves are in equilibrium when the twisting moments $\overline{M}_r$ to be transmitted between each pair of cells are applied as outside torque (fig. 3). The stress condition of this system resulting from the twisting moment applied at the cells is readily defined. To illustrate: the box with shear-resisting walls gives an attitude of pure stress in shear, which can be computed with Bredt's formulas (section III,2 - equation 15).

The distortion of the individual cells induces sectional buckling, i.e., the intersections of an originally flat cross section are no longer on a plane after loading (fig. 4). If two consecutive cells are unlike as regards load, form, or size, the cross-sectional buckling at the intersecting bulkheads is generally not the same. To reestablish the connection of the cells in the box, it is necessary to apply normal stresses between the cells so as to balance the buckling differences. Such normal stresses are equally necessary when one end of the bulkhead is to be joined to a rigid plane.

For reasons of equilibrium the normal stresses transmitted between the cells must be such that their resultant effect can be represented in groups of four equal asymmetrical normal stresses of amount X in the corners of the bulkheads (fig. 3). Then a rational assumption for the distribution of the normal stresses narrows the problem down to a statically indeterminate group of forces X at each bulkhead, thus reducing the analysis of the box to a statically indeterminate system whose static indetermination with n cells and free-to-buckle connection of the end bulkheads is $(n - 1)$ times; for one-sided and two-sided buckling resistant fixation is n and $(n + 1)$ times.

The effect of each group $X_r = 1$ on the principal system (fig. 3) is confined to the two contiguous cells r and $(r + 1)$. Thus the determination of quantities X_r with elastic bulkheads is effected by means of a system of elasticity equations with five terms, the rth equation being

$$\delta_{r,r-2} X_{r-2} \pm \delta_{r,r-1} X_{r-1} + \delta_{r,r} X_r$$

$$+ \delta_{r,r+1} X_{r+1} + \delta_{r,r+2} X_{r+2} = -\delta_{r,o}$$

By assuming the bulkheads as being rigid in plane, the equations contain three terms.

The absolute terms $\delta_{r,o}$ and the factors $\delta_{r,i}$ denote the mutual buckling of the right-hand bulkhead of cell r relative to the left-hand bulkhead of cell $(r + 1)$ as a result of the outside load and condition $X_i = 1$. The buckling is denoted by the group of the four corner deflections in the sense of the "buckling forces X." Then the first problem is to define $\delta_{r,o}$ and $\delta_{r,i}$, which stipulates knowing the stress attitude on the cells in the principal system due to twisting moment M_r and buckling force X_r.

4. Distribution of Stresses

The elasticity equations merely state that the mutual deflection of the corners of two adjacent cells must disappear at the common bulkhead. So, strictly speaking, a distribution of the normal stresses must be presumed such that no individual buckling of the side wall sections occurs between the corners (fig. 4).

By virtue of the stipulated thin-walled box the cell walls are plates within which occurs a plane stress condition with normal stresses σ_x and σ_y and shear stresses τ. Under the most elementary assumption the normal stresses σ_x are linearly distributed over the longitudinal walls X (Navier formula), which according to the rigorous elasticity theory, stipulates in the whole plate a parabolic distribution of shear stress τ as well as a vanishing normal stress σ_y in transverse direction. However, owing to the parabolic shear distribution and with finite transverse elongation, the plate sections (fig. 4) experience an S-shaped buckling as a result of the normal stresses, and which becomes so much more pronounced as the plate is higher relative to its width. In consequence, the Navier stress attitude is at variance with the previously established boundary condition: no individual buckling of longitudinal walls between the bulkhead corners.

For this reason, Airy's stress function was used to determine a stress condition for $X = 1$, which accounts for the boundary condition of vanishing individual buckling of longitudinal walls at the bulkheads. The results are given in section VI in comparison with Navier's stress formula for a relatively wide box (figs. 23 and 24). The local stress discrepancies, particularly at the corners of the loaded bulkhead, are not inconsiderable, although they are minimized when the corners are reinforced. The more precise investigation reveals the shear stress to be more evenly distributed toward the bulkheads, although the total stresses, according to the rigorous elasticity theory are not very much different from those conformable to the Navier method, and since these are decisive for the deflection factors, they may be computed, without appreciable error, under an assumedly linear distribution of normal stresses. Then the shear stress distribution is parabolic, although it may also be assumed as uniformly distributed with sufficient accuracy. Following the calculation of the statically indeterminate X, the final stress condition can then be defined with the formulas given in the appendix.

III. CONDITION OF STRESS AND DEFLECTION OF PRINCIPAL SYSTEM

Rectangular

1. Square Cell with Shear-Resistant Walls

and warping

When stressing a cell in torque $\overline{M}$ or buckling X (fig. 3), the bulkheads transmit couples B and C to the longitudinal walls. The couples can be defined from two equilibrium equations (fig. 5). Owing to $\overline{M}$, couples B and C must first set up the moment $\overline{M}$ at the bulkheads, then produce longitudinal forces A_b and A_c at the individual longitudinal walls which disappear when the latter are joined together. The forces B and C due to X must first have a vanishing moment at the bulkheads, and second, produce forces A_b and A_c which, upon assembly of the longitudinal walls, form the there applied forces X. For an arbitrary rth cell of the prismatic box, we have, owing to $\overline{M}$,

$$B_o = -\frac{\overline{M}}{2\,c}, \quad C_o = -\frac{\overline{M}}{2\,b} \tag{1}$$

owing to $X_r = 1$ at right (rth) bulkhead,

$$B_r = -\frac{b}{2\,a}, \quad C_r = \frac{c}{2\,a} \tag{2}$$

When loading the individual shear-resistant walls then with the defined B and C forces, their boundary elongations ϵ usually differ at the common edges I, II, III, IV. To afford equal elongation, stipulates mutually tangential effective edge loads t (kg/m) between the edges of the individual longitudinal walls (fig. 5). These loads t_I to t_{IV} are determined with four equations, one of which, say at edge I, reads

$$\epsilon_{Ib} = \epsilon_{Ic} \tag{3}$$

These equations yield in the generalized case a three-term equation system for t, which is readily resolved.

If the box cell is symmetrical to the two median planes as regards form and size, the edge loads t under twist ($\overline{M}$) and buckling (X) must for reasons of symmetry be the same at all four edges, so that one equation will then suffice to define t.

With M as bending moment and W as section modulus, the assumption of linear distribution of the normal stresses across the section, equation (3) becomes

$$+\frac{M_b}{E\,W_b} = -\frac{M_c}{E\,W_c}$$

A constant section modulus across the length of the cell and differentiation according to x (distance from left-end bulkhead) gives:

$$\frac{1}{W_b}\frac{d\,M_b}{dx} + \frac{1}{W_c}\frac{d\,M_c}{dx} = 0 \tag{4}$$

whence

$$\frac{d\,M_b}{dx} = Q_b + t\ b$$

$$\frac{d\,M_c}{dx} = Q_c + t\ c$$

which, written in (4) gives

$$t = -\frac{Q_b/W_b + Q_c/W_c}{b/W_b + c/W_c} \tag{5}$$

2. Stress Attitude of Shear-Resistant Longitudinal Walls

By virtue of a group of longitudinal forces $X_r = 1$ at the rth cell of the principal system (fig. 3), it is

$$t_r = -\frac{1}{2a}\,\frac{b/W_b - c/W_c}{b/W_b + c/W_c} \tag{6}$$

after inserting $Q_b = - B_r$ in (5) and $Q_c = - C_r$ from (2).

If the longitudinal walls are plates of constant thickness d_b and d_c, equation (6) becomes with $W_b = \frac{b^2\ d_b}{6}$ and $W_c = \frac{c^2\ d_c}{6}$:

$$t_r = + \frac{1}{2\ a} \frac{b\ d_b - c\ d_c}{b\ d_b + c\ d_c} \tag{7}$$

With (2) and (7) the longitudinal stresses at the edges due to $X_r = 1$ (fig. 6) are:

$$\sigma_r = \pm \frac{M_b}{W_b} = \mp \frac{M_c}{W_c} = \pm \frac{-B_r\ x + \int_0^x t_r\ b\ dx}{b^2\ d_b/6} = \pm \frac{b\ x/a}{b\ d_b + c\ d_c} \tag{8}$$

The upper prefix applies to edges I and III, the other to II and IV. As a result of $X_r = 1$, the shear stresses are first evenly distributed (fig. 7), due to the edge loading t_r:

$$d_b\ \tau_{b,r}^{(t)} = d_c\ \tau_{c,r}^{(t)} = - t_r = - \frac{1}{2\ a} \frac{b\ d_b - c\ d_c}{b\ d_b + c\ d_c} \tag{9}$$

In addition, however, it results in shear stresses pertinent to the normal stresses, which remain constant over the cell length and distribute themselves parabolically across the longitudinal wall section (fig. 7):

$$\left.\begin{aligned} \tau_{b,r}^{(\sigma)} &= \frac{3}{2\ a} \frac{b}{b\ d_b + c\ d_c} \left[1 - \left(\frac{y}{b/2}\right)^2\right] \\ \tau_{c,r}^{(\sigma)} &= - \frac{3}{2\ a} \frac{c}{b\ d_b + c\ d_c} \left[1 - \left(\frac{z}{c/2}\right)^2\right] \end{aligned}\right\} \tag{10}$$

With $\tau_{b,r}$ and $\tau_{c,r}$ as the maximum shear stress in the center of the horizontal and the vertical walls, we have, because of $X_r = 1$:

$$\left.\begin{aligned} \tau_{b,r} &= + \frac{1}{2\ a\ d_b} \frac{2\ b\ d_b + c\ d_c}{b\ d_b + c\ d_c} \\ \tau_{c,r} &= - \frac{1}{2\ a\ d_c} \frac{b\ d_b + 2\ c\ d_c}{b\ d_b + c\ d_c} \end{aligned}\right\} \tag{11}$$

τ IN MID. OF WALL 9+10

With shear stresses $\tau^{(\sigma)}$ evenly distributed across the wall sections, the total shear due to $X_r = 1$, is

$$\tau_{b,r} = \frac{1}{2\ a\ d_b},\quad \tau_{c,r} = - \frac{1}{2\ a\ d_c} \tag{12}$$

As a result of condition $X_{r-1} = 1$ at the left bulkhead, it is

$$\sigma_{r-1} = \pm \frac{6\ \left(1 - \frac{x}{a}\right)}{b\ d_b + c\ d_c} \tag{13}$$

The shear τ_{r-1} due to $X_{r-1} = 1$ is of the same magnitude but inverse sign as that due to $X_r = 1$.

The same stress attitude due to $X = 1$ is afforded when computing the individual longitudinal walls for the pertinent end load B and C separately and allowing for the adjacent walls with a "supporting or effective width", $b_m = b/6$ and $c_m = c/6$. The result is the same as previously, with

$$W_b = \frac{b}{6}\ (b\ d_b + c\ d_c) \quad \text{and} \quad W_c = \frac{c}{6}\ (b\ d_b + c\ d_c),$$

$$M_b = \pm \frac{b}{2\ a}\ x \qquad \text{and} \quad M_c = \mp \frac{c}{2\ a}\ x$$

the edge stress

$$\sigma_r = \frac{M_b}{W_b} - \frac{M_c}{W_c} = \pm \frac{6}{b\ d_b + c\ d_c} \frac{x}{a}\ .$$

In the more precise stress condition (see Appendix) the effective width decreases particularly in the vicinity of the point of application of X. The value $b_m = b/6$ and $c_m = c/6$ is the limit value for a relatively long box.

Owing to the torsion moment $\bar{M}$ on the principal system (fig. 3), it is

$$t_o = -\frac{\bar{M}}{2\,b\,c}, \tag{14}$$

independent of the wall thickness of the longitudinal walls when writing $Q_b = -B_o$ and $Q_c = -C_o$ of (1) into (5).

The longitudinal stresses at the edges due to $\bar{M}$ are with (1) and (14):

$$\sigma_o = \frac{-B_o\,x + \int_0^x t_o\,b\,dx}{W_b} = 0$$

Thus the distortion of the free cell due to $\bar{M}$ causes a pure stress attitude in shear

$$d_b\,\tau_{b,o} = d_c\,\tau_{c,o} = -t_o = \frac{\bar{M}}{2\,b\,c} \tag{15}$$

Consequently (15) is identical to Bredt's formula for thin-walled tubes of enclosed section F_i,

$$\tau = \frac{\bar{M}}{2\,F_i\,d}.$$

The torque at the bulkhead supports due to distributed torsional load m is (fig. 2):

$$M_i = \frac{m_i\,a_i}{2} + \frac{m_{i+1}\,a_{i+1}}{2}.$$

The twisting moments $\bar{M}$ thus formed produce the same stress attitude as before. The intermediate loading sets up an additional stress attitude whose effect grows with the bulkhead spacing. If m_b is the horizontal and m_c the vertical torsional load initiated in the longitudinal walls, (5) gives with

$$Q_b = \frac{m_b}{c}\left(\frac{a}{2} - x\right), \quad Q_c = \frac{m_c}{b}\left(\frac{a}{2} - x\right)$$

the tangential edge loading:

$$t_z = -\frac{1}{b\ c}\ \frac{c\ d_c\ m_b + b\ d_b\ m_c}{b\ d_b + c\ d_c}\left(\frac{a}{2} - x\right) \qquad (16)$$

Whence the stress attitude due to the intermediate loading is

$$\left.\begin{aligned}
\sigma_z &= \pm\frac{6}{b\ d_b+c\ d_c}\ \frac{m_b - m_c}{2\ b\ c}\ x\ (a - x),\\
d_b\ \tau^{(t)}_{b,z} &= d_c\ \tau^{(t)}_{c,z} = -\ t_z,\\
\tau^{(\sigma)}_{b,z} &= \frac{3}{2}\ \frac{b}{b\ d_b+c\ d_c}\ \frac{m_b - m_c}{b\ c}\left(\frac{a}{2} - x\right)\left[1 - \left(\frac{y}{b/2}\right)^2\right]\\
\tau^{(\sigma)}_{c,z} &= -\frac{3}{2}\ \frac{c}{b\ d_b+c\ d_c}\ \frac{m_b - m_c}{b\ c}\left(\frac{a}{2} - x\right)\left[1 - \left(\frac{z}{c/2}\right)^2\right]
\end{aligned}\right\} \qquad (17)$$

However, the same stress condition is obtained again when the individual longitudinal walls are computed separately for their respective torsional load m_b and m_c and an effective width, $b_m = b/6$ and $c_m = c/6$ is assumed. The effective width for broad or high boxes decreases according to the precise stress attitude, conformably to the equations given in the appendix.

With equal horizontal and vertical torsional load $m_b = m_c = m/2$, a pure shear condition

$$d_b\ \tau_{b,z} = d_c\ \tau_{c,z} = \frac{m}{2\ b\ c}\left(\frac{a}{2} - x\right)$$

is reached again. For shear stresses $\tau^{(\sigma)}$ evenly distributed across the wall it reduces

$$\tau_{b,z} = \tau^{(t)}_{b,z} + \tau^{(\sigma)}_{b,z} = \frac{m_b}{b\ c\ d_b}\left(\frac{a}{2} - x\right)$$

and

$$\tau_{c,z} = \tau^{(t)}_{c,z} + \tau^{(\sigma)}_{c,z} = \frac{m_c}{b\ c\ d_c}\left(\frac{a}{2} - x\right)$$

3. Stress Attitude of Bulkheads

If the bulkheads themselves consist of elastic shear resisting plates of $\bar{d}$ thickness, the attitude due to twisting moment $\bar{M}$ and buckling force X on the principal system (fig. 3) is a pure stress in shear when assuming rigid pin-ended edge members.

Due to $X_r = 1$ the tangential edge load $\bar{t}_r$ (fig. 8a) sets up in the left and right bulkheads of the rth cell the shear:

$$\bar{\tau}_r = -\frac{\bar{t}_r}{\bar{d}} = -\frac{1}{2\,a\,\bar{d}} \quad \text{and} \quad = +\frac{1}{2\,a\,\bar{d}} \tag{18}$$

The amount of shear in the bulkheads due to $\bar{M}$ depends on the applied torque M at the individual bulkheads. If they are, to begin with, equally divided by the square cell over the horizontal and vertical longitudinal walls, no shear stress occurs. Otherwise the shear stress attitude is obtained from the equilibrium condition at the bulkhead between the applied torque and the inversely acting, evenly divided torque.

With M_b as the horizontally, and M_c as the vertically, applied proportion of the torque, the shear due to the tangential edge load $\bar{t}_o$ (fig. 8b and c) is:

$$\bar{\tau}_o = -\frac{\bar{t}_o}{\bar{d}} = -\frac{M_b - M_c}{2\,b\,c\,\bar{d}} \tag{19}$$

4. Attitude of Deflection with Shear-Resistant Walls

With the stresses of the principal system (fig. 3) as defined in the preceding sections, the deflection factors of the elasticity equations can now be established. In the plane stress attitude presumed for the individual walls, the deflections (to be visualized as potential energy) - with d = wall thickness - follow from:

$$\delta_{i,k} = d\Big[\int\int \sigma_{x,i}\,\epsilon_{x,k}\,dx\,dy + \int\int \sigma_{y,i}\,\epsilon_{y,k}\,dx\,dy + \int\int \tau_i\,\gamma_k\,dx\,dy\Big]$$

$$= d\Big[\int\int \frac{\sigma_{x,i}\,\sigma_{x,k}}{E}\,dx\,dy - \frac{1}{m}\int\int \frac{\sigma_{x,i}\,\sigma_{y,k}}{E}\,dx\,dy + \int\int \frac{\sigma_{y,i}\,\sigma_{y,k}}{E}\,dx\,dy - \frac{1}{m}\int\int \frac{\sigma_{y,i}\,\sigma_{x,k}}{E}\,dx\,dy + \int\int \frac{\tau_i\,\tau_k}{G}\,dx\,dy\Big].$$

With linear stress distribution for σ_x,

$$\delta_{i,k} = d\Big[\int\int \frac{\sigma_{x,i}\,\sigma_{x,k}}{E}\,dx\,dy + \int\int \frac{\tau_i\,\tau_k}{G}\,dx\,dy\Big] \tag{20}$$

is simplified, because $\sigma_y = 0$.

With the premised symmetry of the box relative to the median planes, the integration for the stress attitudes defined in the preceding section gives the deflections due to condition $X_r = 1$ at

$$\left.\begin{aligned}
E\,\delta_{r,r}^{(\sigma)} &= 8\left[\frac{a_r}{(b\,d_b+c\,d_c)_r} + \frac{a_{r+1}}{(b\,d_b+c\,d_c)_{r+1}}\right]\\
G\,\delta_{r,r}^{(\tau)} &= \frac{1}{2a_r}\left[\frac{b}{d_b} + \frac{c}{d_c} + \frac{4}{5}\,\frac{b^3\,d_b+c^3\,d_c}{(b\,d_b+c\,d_c)^2}\right]_r +\\
&\quad \frac{1}{2a_{r+1}}\left[\frac{b}{d_b} + \frac{c}{d_c} + \frac{4}{5}\,\frac{b^3\,d_b+c^3\,d_c}{(b\,d_b+c\,d_c)^2}\right]_{r+1}\\
E\,\delta_{r,r-1}^{(\sigma)} &= 4\,\frac{a_r}{(b\,d_b+c\,d_c)_r}\\
G\,\delta_{r,r-1}^{(\tau)} &= -\frac{1}{2a_r}\left[\frac{b}{d_b} + \frac{c}{d_c} + \frac{4}{5}\,\frac{b^3\,d_b+c^3\,d_c}{(b\,d_b+c\,d_c)^2}\right]_r\\
&\qquad (\delta_{r,r+1}\ \text{correspondingly}).
\end{aligned}\right\} \tag{21}$$

And in addition:

$$
\left.
\begin{aligned}
G\,\bar{\delta}_{r,r} &= \frac{1}{4a_r^2}\,\frac{bc}{\bar{d}_{r-1}} + \left(\frac{1}{2a_r} + \frac{1}{2a_{r+1}}\right)^2 \frac{b\,c}{\bar{d}_r} \\
&\qquad + \frac{1}{4a_{r+1}^2}\,\frac{bc}{\bar{d}_{r+1}} \\
G\,\bar{\delta}_{r,r-1} &= -\frac{1}{2a_r}\left(\frac{1}{2a_{r-1}} + \frac{1}{2a_r}\right)\frac{bc}{\bar{d}_{r-1}} \\
&\qquad - \frac{1}{2a_r}\left(\frac{1}{2a_r} + \frac{1}{2a_{r+1}}\right)\frac{bc}{\bar{d}_r} \\
G\,\bar{\delta}_{r,r-2} &= +\frac{1}{4a_{r-1}\,a_r}\,\frac{bc}{\bar{d}_{r-1}}\,. \\
&(\bar{\delta}_{r,r+1} \text{ and } \bar{\delta}_{r,r+2} \text{ correspondingly}),
\end{aligned}
\right\}
\tag{22}
$$

when the bulkheads are elastic and rigid in shear (fig. 9a).

With bulkheads rigid in their plane, the terms $\delta_{i,k}$ disappear, leaving three terms. If the bulkheads are built up of members of length s and of longitudinal stiffness $E\,F$ (fig. 9b) it is necessary to put $\Sigma\, s^3/EF$ instead of $bc/G\bar{d}$ in the $\bar{\delta}_{i,k}$ values.

If the bulkheads represent bending resistant frames in their plane, with stiffness in bending $E\,J_b$ or $E\,J_c$ of the bars or uprights (fig. 9c), $bc/G\bar{d}$ must be replaced by $\frac{b^2\,c^2}{24\,E}\left(\frac{b}{J_b} + \frac{c}{J_c}\right)$. These figures stipulate longitudinally stiff edge members of the bulkheads. The last term $\frac{4}{5}\,\frac{b^3\,d_b + c^3\,d_c}{b\,d_b + c\,d_c}$ appearing in the $\delta_{i,k}^{(\tau)}$ values results from the parabolic distribution of the shear $\tau^{(\sigma)}$ set up by the longitudinal stresses σ. With a uniform distribution of the shear this term disappears.* Then the special

*Reissner's report contains the assumption of uniform maximum shear distribution across the section for one condition τ_i or τ_k. This gives for the last term in $\delta_{i,k}^{(\tau)}$:

$$\frac{b^2 + c^2}{b\,d_b + c\,d_c} \quad \text{instead of} \quad \frac{4}{5}\,\frac{b^3\,d_b + c^3\,d_c}{(b\,d_b + c\,d_c)^2}$$

case of equal cells and rigid bulkheads affords:

$$\left.\begin{aligned} G\ \delta_{r,r} &= \frac{G}{E}\,\frac{16\ a}{b\ d_b + c\ d_c} + \frac{1}{a}\left(\frac{b}{d_b} + \frac{c}{d_c}\right)_r \\ G\ \delta_{r,r-1} &= \frac{G}{E}\,\frac{4\ a}{b\ d_b + c\ d_c} - \frac{1}{2a}\left(\frac{b}{d_b} + \frac{c}{d_c}\right) \end{aligned}\right\} \qquad (21a)$$

The effect of the different assumptions for the shear distribution on the statically indeterminate X is illustrated in figure 20 (numerical example, section V). The absolute terms $\delta_{r,o}$ due to twisting moment M are obtained when applying the outside torque M_1 equally in the horizontal and vertical longitudinal walls

$$G\ \delta_{r,o} = +\left(\frac{b}{d_b} - \frac{c}{d_c}\right)_r \frac{M_r}{2bc} - \left(\frac{b}{d_b} - \frac{c}{d_c}\right)_{r+1} \frac{M_{r+1}}{2b\ c} \qquad (23)$$

If the horizontal quota $M_{b,r}$ is unlike the vertical quota $M_{c,r}$ of the torque and the bulkheads are elastic, the additive absolute terms are:

$$\left.\begin{aligned} G\ \bar{\delta}_{r-1,o} &= \frac{M_{b,r} - M_{c,r}}{4\ a_r\ \bar{d}_r} \\ G\ \bar{\delta}_{r,o} &= -\left(\frac{1}{a_r} + \frac{1}{a_{r+1}}\right)\frac{M_{b,r} - M_{c,r}}{4\ \bar{d}_r} \\ G\ \bar{\delta}_{r+1,o} &= \frac{M_{b,r} - M_{c,r}}{4\ a_{r+1}\ \bar{d}_r} \end{aligned}\right\} \qquad (24)$$

If the bulkheads are rigid the $\bar{\delta}_{i,o}$ terms disappear. Moreover, it will be noted that with similar application of equal torque at all bulkheads and equal bulkhead design, no $\bar{\delta}_{i,o}$ terms occur except as edge values.

In the special case of equal cells and equal application of torque or rigid bulkheads, it is

$$G\ \delta_{r,o} = \left(\frac{b}{d_b} - \frac{c}{d_c}\right)\frac{M_r}{2\ b\ c} \qquad (23a)$$

For torsional load m_b or m_c evenly distributed between the bulkheads, the additive absolute terms due to the intermediate loading are:

$$E\ \delta_{r,z} = + \frac{a_r^3}{b\ c}\left[\frac{m_b - m_c}{b\ d_b + c\ d_c}\right]_r + \frac{a_{r+1}^3}{b\ c}\left[\frac{m_b - m_c}{b\ d_b + c\ d_c}\right]_{r+1} \tag{25}$$

which reduces to

$$E\ \delta_{r,z} = \frac{2\ a^3}{bc}\ \frac{m_b - m_c}{b\ d_b + c\ d_c} \tag{25a}$$

for equal size and loading of consecutive cells.

If the box is rigidly joined to one end bulkhead (0 or n) (buckling resistant constraint), the edge terms are found from (21) to (25) by assuming a rigid cell (d or $\overline{d} = \infty$) joining the edge cell (1 or n).

5. Square Cells with Reinforced Corners

If the corners of a box with shear-resistant walls are reinforced with stiffeners of section F, the stress condition can be determined with the edge loading t from (5) in the same manner as described in section III,2. Because of $X_r = 1$ this condition is:

$$\left.\begin{aligned}
\sigma_r &= \pm \frac{6\ x/a}{b\ d_b + c\ d_c + 6\ F} \\
\tau_{b,r} &= \frac{1}{2a\ d_b}\left[1 - \frac{2b\ d_b - 3b\ d_b\left[1 - \left(\frac{y}{b/2}\right)^2\right]}{b\ d_b + c\ d_c + 6\ F}\right] \\
\tau_{c,r} &= -\frac{1}{2a\ d_c}\left[1 - \frac{2c\ d_c - 3c\ d_c\left[1 - \left(\frac{z}{c/2}\right)^2\right]}{b\ d_b + c\ d_c + 6\ F}\right]
\end{aligned}\right\} \tag{26}$$

With shear uniformly distributed across the section, it is:

$$\tau_{b,r} = \frac{1}{2a\ d_b}, \quad \tau_{c,r} = -\frac{1}{2a\ d_c} \tag{26a}$$

Because of outside twisting moments $\overline{M}$, the stress attitude is one of pure shear:

$$\tau_{b,o} = \frac{H}{2b\ c\ d_b} \quad \text{and} \quad \tau_{c,o} = \frac{H}{2b\ c\ d_c} .$$

The result of evenly distributed torsional loading is the additional normal stress (compare (17)):

$$\sigma_z = \pm \frac{6}{b\ d_b + c\ d_c + 6\ F}\ \frac{m_b - m_c}{2\ b\ c}\ x\ (a - x)$$

and the evenly distributed shearing stress:

$$\tau_{b,z} = \frac{m_b}{b\ c\ d_b}\left(\frac{a}{2} - x\right) \quad \text{and} \quad \tau_{c,z} = \frac{m_c}{b\ c\ d_b}\left(\frac{a}{2} - x\right).$$

The deflection factors due to $X = 1$ for a box with shear-resistant walls and corner stiffeners are obtained by writing

$$\frac{1}{6}\ (b\ d_b + c\ d_c) + F \quad \text{instead of} \quad \frac{1}{6}\ (b\ d_b + c\ d_c)$$

in formulas (21) for the box without corner stiffeners. The terms $\delta_{i,k}$ due to the elastic bulkheads as well as the absolute terms $\delta_{r,o}$ from (23) and (24) due to outside torque, remain. In the additive absolute terms $\delta_{r,z}$ due to evenly graded torsional load $\frac{1}{6}\ (b\ d_b + c\ d_c)$ must also be replaced by $[\frac{1}{6}\ (b\ d_b + c\ d_c) + F]$. The assumption of uniform shear distribution here is the more appropriate as the corner reinforcement is greater relative to the wall sections (fig. 20).

6. Square Cells with Walls Resistant in Tension Only

The rigidity in shear in the walls of a box stiffened at the corners can be disregarded, and the walls built of very thin webs. Then the web walls are only stressed in tension; they form "tension diagonal fields" (reference 8).

The oblique tension stresses σ in the web (fig. 10a) set up normal stresses and bending moments in the longitudinal flanges. With upright stiffeners spaced suitably closely e ($\frac{1}{2}$ to $\frac{1}{8}$ of beam height h), the bending stress is ineffective. The uprights balance the vertical components of the tension σ applied at the top and bottom flange. The produced stress attitude is particularly simple when assuming bending resistant flanges together with longitudinally and bending resistant pin-ended uprights. The permanent tension stresses σ produced in each panel between two uprights form an angle $\alpha = 45^{\circ}$, independent of the transverse force Q acting in the panel

$$\sigma = -\frac{2\ Q}{h\ d}\ (Q_i = -\sum_{1}^{i-1} P)$$

With finite longitudinal stiffness of the uprights, angle α becomes less, averaging about 40° for practical cases. The stress attitude is also somewhat modified with finite bending stiffness of the uprights. Instead of the sudden change in tension σ at the uprights, there is a rise in tension from the tension toward the compression flange even in the panel. However, the assumption of longitudinally and bending resistant uprights suffices for our purposes.

The stress attitude in the longitudinal flanges and uprights due to equal panel-point loads P, is illustrated in figure 10b. The flanges and uprights are first subjected to linearly increasing tension L' and V' as a result of the directly applied components of the tension stresses in the same direction; second, to constant tension L'' and V'' as a result of the support reaction transmitted by the flanges and uprights of the vertically acting components of the tension stresses. In figure 10 the load L' in the top flange is a tension; in the bottom flange, compression; the load L'' in both flanges is compressive.

The argument for the flat plate beam with very thin web is readily applicable to the box with four such longitudinal walls wherein, however, the longitudinal flanges now refer to two walls. The loading of a square cell (fig. 3) under twist $\overline{M}$ produces in the horizontal and vertical side walls oblique tension stresses corresponding in amount and direction to the outside couples B_o and C_o (equation (1)):

$$\sigma_{b,o} = -2\ \frac{Q_b}{b\ d_b} = -\ \frac{\overline{M}}{bc\ d_b},\ = \sigma_{c,o} - 2\ \frac{Q_c}{c\ d_c} = -\ \frac{\overline{M}}{bc\ d_c} \qquad (27)$$

The tension L'_o (fig. 10b) of the horizontal and vertical longitudinal walls cancels, whereas, the superposed

L''_o amounts to:

$$L''_o = L_o = + \left(\frac{Q_b}{2} + \frac{Q_c}{2}\right) = + \frac{b + c}{4\ b\ c} M \qquad (28)$$

For condition $X = 1$ (fig. 3) the direction of the tensile stresses formed as a result of the outside torsional load is used as basis, because this attitude governs the direction of the tension stresses in the final system. Owing to $X = 1$, there also will be oblique compressive stresses.

The result of $X_r = 1$ at the right bulkhead of cell r becomes with $Q_b = - B_r$ and $Q_c = - C_r$ according to (2)

$$\sigma_{b,r} = - \frac{1}{a\ d_b}, \quad \sigma_{c,r} = + \frac{1}{a\ d_c} \qquad (29)$$

Besides, owing to $X_r = 1$, there are the longitudinal flange stresses:

$$\left.\begin{aligned} L'_{I,r} &= - L'_{II,r} = L'_{III,r} = - L'_{IV,r} = + \frac{x}{a} \\ L''_{I,r} &= L''_{II,r} = L''_{III,r} = L''_{IV,r} = + \frac{b - c}{4\ a} \end{aligned}\right\} \qquad (30)$$

and, owing to $X_{r-1} = 1$ at the left bulkhead:

$$\left.\begin{aligned} L'_{I,r-1} &= - L'_{II,r-1} = L'_{III,r-1} = - L'_{IV,r-1} = 1 - \frac{x}{a} \\ L''_{I,r-1} &= L''_{II,r-1} = L''_{III,r-1} = L''_{IV,r-1} = - \frac{b - c}{4\ a} \end{aligned}\right\} \qquad (30a)$$

The appearance of tension diagonal fields in the bulkheads is followed by oblique tension stresses of double the amount of the shear stresses in section III,3. Then the deformation factors are according to (20):

$$\delta_{i,k}^{(D)} = d \int \int \frac{\sigma_i\ \sigma_k}{E}\, dx\ dy$$

or

$$\delta_{i,k}^{(D)} = \frac{\sigma_{b,i}\ \sigma_{b,k}}{E}\, a\ b\ d_b \quad \text{and} \quad \frac{\sigma_{c,i}\ \sigma_{c,k}}{E}\, a\ c\ d_c$$

for one field, when assuming constant tension stresses in the horizontal and vertical fields of the cell.

The proportion of one longitudinal flange of section F to the deformation factors is

$$\delta_{i,k}^{(L)} = \int L_i\, L_k \frac{dx}{E\, F}$$

Being presumed rigid, the uprights do not contribute

$$\delta_{i,k}^{(V)} = 0$$

Thus, with symmetrical size, the factors due to $X_r = 1$, are:

$$\left.\begin{aligned}
E\,\delta_{r,r}^{(D)} &= 2\left[\left(\frac{b}{d_b} + \frac{c}{d_c}\right)_r \frac{1}{a_r} + \left(\frac{b}{d_b} + \frac{c}{d_c}\right)_{r+1} \frac{1}{a_{r+1}}\right] \\
E\,\delta_{r,r}^{(L)} &= 4\left[\frac{1}{3}\left(\frac{a_r}{F_r} + \frac{a_{r+1}}{F_{r+1}}\right) + \frac{(b-c)^2}{16}\right. \\
&\qquad\qquad \left.\left(\frac{1}{a_r\, F_r} + \frac{1}{a_{r+1}\, F_{r+1}}\right)\right] \\
E\,\delta_{r,r-1}^{(D)} &= -\,2\left(\frac{b}{d_b} + \frac{c}{d_c}\right)_r \frac{1}{a_r} \\
E\,\delta_{r,r-1}^{(L)} &= 4\left[\frac{a_r}{6\,F_r} - \frac{(b-c)^2}{16\,a_r\,F_r}\right] \\
&\qquad (\delta_{r,r+1} \text{ correspondingly})
\end{aligned}\right\} \quad (31)$$

Similarly, the absolute terms due to $\overline{M}$ are:

$$\left.\begin{aligned}
E\,\delta_{r,o}^{(D)} &= \left(\frac{b}{d_b} - \frac{c}{d_c}\right)_r \frac{2\,\overline{M}_r}{bc} - \left(\frac{b}{d_b} - \frac{c}{d_c}\right)_{r+1} \frac{2\,\overline{M}_{r+1}}{bc} \\
E\,\delta_{r,o}^{(L)} &= \frac{b^2 - c^2}{4\,bc}\left(\frac{\overline{M}_r}{F_r} - \frac{\overline{M}_{r+1}}{F_{r+1}}\right)
\end{aligned}\right\} \quad (32)$$

For the special case of equal cells and rigid bulkheads, we have:

$$\left.\begin{aligned} E\,\delta_{r,r} &= \frac{4}{a}\left(\frac{b}{d_b}+\frac{c}{d_c}\right)+\frac{8}{3}\frac{a}{F}+\frac{(b-c)^2}{2\,a\,F} \\ E\,\delta_{r,r-1} &= -\frac{2}{a}\left(\frac{b}{d_b}+\frac{c}{d_c}\right)+\frac{2}{3}\frac{a}{F}-\frac{(b-c)^2}{4\,a\,F} \\ E\,\delta_{r,o} &= \left[\left(\frac{b}{d_b}-\frac{c}{d_c}\right)+\frac{b^2-c^2}{8\,F}\right]\frac{2\,M_r}{bc} \end{aligned}\right\} \qquad (33)$$

A comparison of $\delta_{i,k}^{(D)}$ of (31) and (32) with the $\delta_{i,k}^{(T)}$ terms of (21) and (23) reveals that the only difference between the deformation quotas of the tension-resistant walls and the shear-resistant walls with evenly distributed shear is in the factor $4\,G/E$. For elastic bulkheads the $\delta_{i,k}$ terms given in (22) become additive to the above deformation figures $\delta_{i,k}^{(D)}$ and $\delta_{i,k}^{(L)}$. If the bulkheads permit tensional diagonal fields, then G is to be replaced by $E/4$.

7. Blunt-Wedge Cells

The attitude of stress and deformation established by assuming an evenly distributed shear stress for the square cell of shear-resistant walls with and without corner stiffeners can be likened to that of a box with flanges of

$$F_l = F + \frac{1}{6}\,(b\;d_b + c\;d_c)$$

section concentrated in the corners. The transverse force (shear) Q_b and Q_c, respectively, transmitted by the bulkheads is, in square cells, taken up by the walls of section $F_b = b\;d_b$ and $F_c = c\;d_c$, respectively, and the bending moments by the flanges of section F_l. The transverse forces of the wall T and the forces of the longitudinal flange L, are

$$T_b = \tau_b\,F_b = Q_b \quad \text{and} \quad T_c = \tau_c\,F_c = Q_c$$

$$L_I = -\,L_{II} = +\,L_{III} = -\,L_{IV} = \left(\frac{Q_b}{b} - \frac{Q_c}{c}\right)x$$

The deformation quota of walls and flanges are found from:

$$\delta_{i,k}^{(T)} = \int_0^a T_{b,i}\, T_{b,k} \frac{dx}{G\, F_b} \text{ and } \int_0^a T_{c,i}\, T_{c,k} \frac{dx}{G\, F_c}$$

$$\delta_{i,k}^{(L)} = \int_0^a L_i\, L_k \frac{dx}{E\, F_l}.$$

By assuming evenly distributed shear stresses the attitude of stress and deformation can be developed in similar fashion for the blunt-wedge cell (fig. 11a) with the proviso, however, that the longitudinal flanges also aid in taking up the shear from the bulkheads. With the dimensions of the rth cell (fig. 11a), the ~~cross~~ shear stresses of the walls are:

$$T_b = Q_b - Q_b \frac{b_r - b_{r-1}}{b_x} \frac{x}{a} = Q_b \frac{b_{r-1}}{b_x}$$

and

$$T_c = Q_c - Q_c \frac{c_r - c_{r-1}}{c_x} \frac{x}{a} = Q_c \frac{c_{r-1}}{c_x}$$

which, when putting

$$\xi = \frac{x}{a}, \quad \beta = \frac{b_r - b_{r-1}}{b_{r-1}}, \quad \gamma = \frac{c_r - c_{r-1}}{c_{r-1}}$$

$$b_x = b_{r-1} (1 + \beta\, \xi) \quad \text{and} \quad c_x = c_{r-1} (1 + \gamma\, \xi),$$

becomes

$$T_b = Q_b \frac{1}{1 + \beta\, \xi} \quad \text{and} \quad T_c = Q_c \frac{1}{1 + \gamma\, \xi}$$

The load on the longitudinal flanges of length l is:

$$L = \pm \left(\frac{Q_b}{b_x} - \frac{Q_c}{c_x}\right) \frac{x}{a}\, l$$

$$= \pm \left[\frac{Q_b}{b_{r-1}} \frac{\xi}{1 + \beta\, \xi} - \frac{Q_c}{c_{r-1}} \frac{\xi}{1 + \gamma\, \xi}\right] l$$

Then the integration of

$$\delta_{i,k}^{(T)} = \int_0^{a_b} T_{b,i}\, T_{b,k} \frac{dx_b}{G\, F_b} \quad \text{and} \quad \int_0^{a_c} T_{c,i}\, T_{c,k} \frac{dx_c}{G\, F_c}$$

$$\delta_{i,k}^{(L)} = \int_0^{l} L_i\, L_k \frac{d\,l}{E\, F_l}$$

over the true lengths a_b and a_c and l (fig. 11b) give the deformation of walls and flanges, wherein

$$F_l = F + \frac{1}{6}\,(b_x\, d_b + c_x\, d_c)$$

$$F_b = b_x\, d_b \quad \text{and} \quad F_c = c_x\, d_c,$$

must be introduced.

With constant corner reinforcement F the changing flange section F_l may be replaced by a constant mean section.

With $dx_b = a_b\, d\,\xi$ and $dx_c = a_c\, d\,\xi$ and $d\,l = l\, d\,\xi$

together with the insertion of the previously defined T and L values, we have:

$$\delta_{i,k}^{(T)} = \frac{a_b}{G}\,\frac{Q_{b,i}}{b_{r-1}}\,\frac{Q_{b,k}}{d_b}\int_0^1 \frac{d\,\xi}{(1+\beta\,\xi)^3} \quad \text{and} \quad \frac{a_c}{G}\,\frac{Q_{c,i}}{c_{r-1}}\,\frac{Q_{c,k}}{d_c}\int_0^1 \frac{d\,\xi}{(1+\gamma\,\xi)^3}$$

$$\delta_{i,k}^{(L)} = \frac{l^3}{E\, F_l}\left[\frac{Q_{b,i}\, Q_{b,k}}{b_{r-1}^2}\int_0^1 \frac{\xi^2\, d\,\xi}{(1+\beta\,\xi)^2} + \frac{Q_{c,i}\, Q_{c,k}}{c_{r-1}^2}\int_0^1 \frac{\xi^2\, d\,\xi}{(1+\gamma\,\xi)^2}\right.$$

$$\left. - \frac{Q_{b,i}\, Q_{c,k} + Q_{c,i}\, Q_{b,k}}{b_{r-1}\, c_{r-1}}\int_0^1 \frac{\xi^2\, d\,\xi}{(1+\beta\,\xi)\,(1+\gamma\,\xi)}\right]$$

The integrals therein are:

$$\int_0^1 \frac{d\,\xi}{(1+\beta\,\xi)^3} = \frac{1+\beta/2}{(1+\beta)^2}$$

$$\int_0^1 \frac{\xi^2\, d\,\xi}{(1+\beta\,\xi)^2} = \frac{1}{\beta^3}\left[\beta\,\frac{\beta+2}{\beta+1} - 2\,\ln\,(\beta+1)\right]$$

$$\int_0^1 \frac{\xi^2 \, d\xi}{(1 + \beta \xi)(1 + \gamma \xi)} = \frac{1}{\beta \gamma} + \frac{\ln(\beta + 1)}{\beta^2 (\beta - \gamma)} + \frac{\ln(\gamma + 1)}{\gamma^2 (\gamma - \beta)}$$

$\left(\int_0^1 \frac{d\xi}{(1 + \gamma \xi)^2} \right.$ and $\int_0^1 \frac{\xi^2 \, d\xi}{(1 + \gamma \xi)^2}$ correspondingly)

From these deformation quotas $\delta_{i,k}^{(T)}$ and $\delta_{i,k}^{(L)}$ the coefficients and the absolute terms of the elasticity equations can be worked up.

The cross stresses Q_b and Q_c must be included conformably to the existing load cases. They are obtained from two elasticity equations in the same manner as for the square cell. With the abbreviation

$$K_r = b_r \, c_{r-1} + c_r \, b_{r-1}$$

the cross stresses at the bulkhead $(r - 1)$ as a result of $X_r = 1$, is

$$Q_{b,r} = \frac{b_{r-1} \, b_r \, c_r}{a_r \, K_r}, \quad Q_{c,r} = - \frac{c_{r-1} \, b_r \, c_r}{a_r \, K_r}.$$

The cross stresses $Q_{b,r-1}$ and $Q_{c,r-1}$ at bulkhead r are as a result of $X_{r-1} = 1$ of opposite sign and are obtained by exchanging sign r and $(r - 1)$ at b and c.

Owing to torsion moment $\overline{M}$, it is:

$$Q_{b,o} = \frac{b_r}{K_r} \overline{M}, \quad Q_{c,o} = \frac{c_r}{K_r} \overline{M}$$

For the blunt pyramid cell $(\beta = \gamma)$, we have, due to $X_r = 1$,

$$Q_{b,r} = \frac{b_r}{2 \, a_r}, \quad Q_{c,r} = - \frac{c_r}{2 \, a_r}$$

and, owing to $\overline{M}$,

$$Q_{b,o} = \frac{\overline{M}}{2 \, c_{r-1}}, \quad Q_{c,o} = \frac{\overline{M}}{2 \, b_{r-1}}.$$

In such pyramid cells the stress attitude of the principal system due to $\overline{M}$ is particularly simple. It is

$$d_b \tau_{b,o} = d_c \tau_{c,o} = \frac{T_{b,o}}{b_x} = \frac{T_{c,o}}{c_x} = \frac{\overline{M}}{2 b_x c_x} = \frac{\overline{M}}{2 F_x}; \; L_o = 0,$$

that is, a pure shearing stress, according to Bredt's formula (15). The proportions of the absolute terms $\delta_{r,o}^{(L)}$ also disappear in this particular case. By contrast the shear due to $\overline{M}$ in blunt-wedge cells, varies from Bredt's formula:

$$d_b \tau_{b,o} = \frac{\overline{M}}{K_r} \frac{b_r b_{r-1}}{b_x^2} \quad \text{and} \quad d_c \tau_{c,o} = \frac{\overline{M}}{K_r} \frac{c_r c_{r-1}}{c_x^2}$$

Besides, $L_o \neq 0$.

The attitude of the stress and deformation of bulkheads of blunt-wedge cells is obtained as for the square cell. If the bulkheads are rigid in shear the stress attitude is one of pure shear again.

The stress and deformation for blunt-wedge cells with tension-resistant walls can be defined in the same manner as for blunt-wedge cells having shear-resistant walls, when starting with the assumption that the oblique tensile stresses in wall center correspond to the evenly distributed shearing stresses. (See Z.F.M., vol. 20, no. 9, 1929, p. 232.)

IV. FINAL STRESS OF BOX

1. Resolution of Elasticity Equations

With the $\delta_{i,k}$ terms as defined in the preceding section, the elasticity equations can now be written and resolved. Those containing three or five terms can be disregarded, since a number of methods have already been developed for resolving them (reference 8).

In most cases an approximate solution is sufficient. As a rule the premise of bulkheads rigid in their plane, is admissible. Unequal application of torque at the horizontal and vertical longitudinal walls ($M_b \neq M_c$) with this assumption results in discrepancies of the redundant members from the exact values (fig. 19), which become so much greater as the successive bulkheads differ in size and loading. Besides, the redundancy effect at the more distant bulkheads can ordinarily be disregarded. The system is divided into partial systems of two, or at the most, three cells each, and the redundancies determined from one equation each with one unknown or from two equations each with two unknown factors (fig. 16). Thereby only the partial systems in the vicinity of greater constraint of sectional buckling need to be taken into consideration, i.e., adjacent to the fixation and at the points of sudden change of load or size of the cells. Lastly, the solutions for the box with equal cells, given hereafter, can be applied to systems having unequal cells when proceeding on the basis of a box whose cells have throughout the size of the cells at the point of constrained sectional buckling, as, say, at the fixation (fig. 16).

In the particular case of equal cells and bulkheads rigid in their plane, the equation represents a difference equation of the second order with constant values:

$$X_{r-1} - 2\,\alpha\,X_r + X_{r+1} = \nu_r$$

With regularly changing load the solution can be given in closed form (reference 9). For a box with equal cells and rigid bulkheads constrained against buckling on one side, this solution becomes for a torque M_o at the free end with $\alpha > 1$ or $\alpha < -1$ (upper or lower sign):

$$X_r = \pm \frac{\operatorname{Sinh} r\,\varphi}{\operatorname{Cosh} n\,\varphi}\,(\pm 1)^{r+n}\,\frac{\nu}{\sqrt{\alpha^2 - 1}}\,\frac{a\,M_o}{bc} \tag{34}$$

and, as a result of evenly distributed torsional loading m with $\alpha > 1$ or $\alpha < -1$:

$$X_r = \left[\pm \frac{\operatorname{Sinh} r\,\varphi}{\operatorname{Cosh} n\,\varphi}\,(\pm 1)^{r+n}\,\frac{\nu}{\sqrt{\alpha^2 - 1}} - \frac{1}{n}\left(1 - \frac{\operatorname{Cosh}(n-r)\varphi}{\operatorname{Cosh} n\,\varphi}\,(\pm 1)^r\right)\frac{\nu}{2(\alpha-1)}\right] n\,\frac{a^2\,m}{bc} \tag{35}$$

On account of the intermediate loading, we also have:

$$X_r^{(Z)} = -\left[1 - \frac{\operatorname{Cosh}(n-r)\varphi}{\operatorname{Cosh} n\,\varphi}\,(\pm 1)^r\right]\frac{\nu^{(Z)}}{2\,(\alpha-1)}\,\frac{a^2\,m}{bc}\,.$$

For the box with shear-resistant, longitudinal walls, we write:

$$\left.\begin{aligned} \alpha &= \frac{1+\rho}{1-\frac{\rho}{2}}, \quad \varphi = \operatorname{Cosh}^{-1} |\alpha|, \\ \nu &= \frac{\eta}{1-\frac{\rho}{2}}, \quad \nu^{(Z)} = \frac{\rho/4}{1-\frac{\rho}{2}} \frac{m_b - m_c}{m} \end{aligned}\right\} \tag{36}$$

whereby*

$$\rho = \frac{16\, a^2 \frac{G}{E}}{\left(\frac{b}{d_b} + \frac{c}{d_c}\right)\left(\frac{b}{d_b} + \frac{c}{d_c}\right)}, \quad \eta = \frac{\frac{b}{d_b} - \frac{c}{d_c}}{\frac{b}{d_b} + \frac{c}{d_c}} \tag{37}$$

For the box with simple tension-resistant walls and corner stiffeners F, we have:

$$\alpha = \frac{1 + \rho_1 + \rho_2}{1 - \frac{1}{2}\rho_1 + \rho_2}, \quad \varphi = \operatorname{Cosh}^{-1} |\alpha|, \quad \nu = \frac{1 + \frac{b+c}{b-c}\rho_2}{1 - \frac{1}{2}\rho_1 + \rho_2} \tag{36a}$$

where

$$\rho_1 = \frac{\frac{2}{3} a^2}{\left(\frac{b}{d_b} + \frac{c}{d_c}\right) F}, \quad \rho_2 = \frac{\frac{1}{8}(b-c)^2}{\left(\frac{b}{d_b} + \frac{c}{d_c}\right) F}, \quad (\eta \text{ as above}) \tag{37a}$$

With

$$\operatorname{Sinh} n\,\varphi \sim \operatorname{Cosh} n\,\varphi \quad \text{and} \quad \frac{1}{\operatorname{Cosh} n} \sim 0$$

for $\alpha > 1$ or $\alpha < -1$ due to torque, the longitudinal forces produced at bulkhead $r = n$ are:

$$X_n = \pm \frac{\nu}{\sqrt{\alpha^2 - 1}} \frac{a\, M_o}{bc} \tag{38}$$

---

*The ρ and η values are for uniform shear distribution. For parabolic distribution the denominator has, aside from $\left(\frac{b}{d_b} + \frac{c}{d_c}\right)$ the term $\frac{4}{5}\frac{b^3 d_b + c^3 d_c}{(bd_b + cd_c)^2}$, and for evenly distributed maximum shear stress the term $\frac{b^2 + c^2}{bd_b + cd_c}$. With corner stiffeners F, the term $(bd_b + cd_c + 6\,F)$ replaces $bd_b + cd_c$.

and, as a result of evenly distributed torsional loading:

$$X_n = \left[\pm \frac{\nu}{\sqrt{\alpha^2 - 1}} - \frac{1}{n} \frac{\nu}{2\ (\alpha - 1)}\right] n \frac{a^2\ m}{bc} \qquad (39)$$

With the values from (36) inserted, we have:

$$X_n = \frac{\eta}{\sqrt{3\ \rho\ (1+\rho/4)}} \frac{a}{b\ c} M_o \qquad (38a)$$

and

$$X_n = \left[\frac{1}{\sqrt{3\ \rho\ (1 + \rho/4)}} - \frac{1}{n} \frac{1}{3\ \rho}\right] \frac{\eta\ n\ a^2\ m}{b\ c} \qquad (39a)$$

The values $\frac{1}{\sqrt{3\ \rho\ (1 + \rho/4)}}$ and $\frac{1}{n} \frac{1}{3\ \rho}$ are plotted in figure 12, for various ρ and n. Formulas (38) and (39) ensure a rapid determination of the redundancy at the point of fixation of the one-sidedly restrained box.

In the special case of $\alpha = 1$, that is, $\rho = 0$, the solution of the difference equation for a torque at the end or for uniform torsional load distribution is:

$$X_r = r\ \eta\ \frac{a\ M_o}{bc} \qquad (34a)$$

$$X_r = \frac{r^2}{2}\ \eta\ \frac{a^2\ m}{bc} \qquad (35a)$$

The case $\rho = 0$ occurs when the horizontal or vertical walls become infinitely thin. In such case the box becomes a two-walled grillage (truss); then d_b and $d_c = 0$ $\eta = \pm 1$, and (34a) and (35a) represent the $1/c$ and $1/b$ time curve of bending moment of the truss.

But $\rho = 0$ can also occur when the bulkhead spacing a approaches 0. In that case a special limiting transition is necessary because n must be made infinite.

Disregarding the small terms of the higher order when ρ and φ are small, (36) can be written as

$$\operatorname{Cosh} \varphi - 1 = \alpha - 1 = \frac{\varphi^2}{2},\ 2\ (\alpha - 1) = 3\ \rho = \varphi^2;$$

$$\sqrt{\alpha^2 - 1} = \sqrt{3\ \rho} = \varphi,\ \nu = \eta$$

Putting $x = r\,a, \quad l = n\,a,$

$$\psi^2 = (n\ \varphi)^2 = \frac{48\ l^2\ \frac{G}{E}}{\left(\frac{b}{d_b} + \frac{c}{d_c}\right)\left(\frac{b}{d_b} + \frac{c}{d_c}\right)} \tag{40}$$

becomes

$$X_r = \frac{\operatorname{Sinh} \psi\ \frac{x}{l}}{\operatorname{Cosh} \psi}\ \frac{\eta}{\psi}\ \frac{l\ M_o}{bc} \tag{34b}$$

and

$$\left.\begin{aligned} X_r &= \left[\frac{\operatorname{Sinh} \psi\ \frac{x}{l}}{\operatorname{Cosh} \psi}\ \frac{\eta}{\psi} - \left(1 - \frac{\operatorname{Cosh} \psi\left(1 - \frac{x}{l}\right)}{\operatorname{Cosh} \psi}\right)\frac{\eta}{\psi^2}\right]\frac{l^2\ m}{b\ c} \\ &= -\left[1 - \frac{\operatorname{Cosh} \psi\left(1 - \frac{x}{l}\right) + \psi\ \operatorname{Sinh} \psi\ \frac{x}{l}}{\operatorname{Cosh} \psi}\right]\frac{\eta}{\psi^2}\ \frac{l^2\ m}{b\ c} \end{aligned}\right\} \tag{35b}$$

The last formula is identical with Reissner's formula* for the edge stress of the one-sidedly restrained box with infinitely close, rigid bulkheads and the same size with uniform torsional load distribution.

The longitudinal forces at the restrained bulkhead with infinitely close bulkhead spacing are:

$$X_n = \frac{\eta}{\psi}\ \frac{l\ M_o}{bc} \quad \text{and} \quad X_n = (\psi - 1)\ \frac{\eta}{\psi^2}\ \frac{l^2\ m}{bc}$$

The course of the redundant members in dependence of the bulkhead spacing is analyzed in a subsequent section.

2. Final Stresses and Loads

In the box with shear-resistant walls the final shearing stresses of the rth cell as a result of torque M at the bulkheads, are:

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*Compare (8a) (reference 3). Conformable to his assumption on the shear distribution the term $\frac{b^2+c^2}{bd_b+cd_c}$ must be added in the denominator of ψ^2 and η.

$$\tau_r = \tau_{r,o} + \tau_{r,r-1}\, X_{r-1} + \tau_{r,r}\, X_r \tag{41}$$

For square cells, they are:

$$\tau_r = \tau_{r,o} + \tau_{r,r}\,(X_r - X_{r-1}) \tag{41a}$$

owing to $\tau_{r,r-1} = -\tau_{r,r}$. Here the shear $\tau_{r,o}$ of (15) and $\tau_{r,r}$ of (9) and (10) or (11) and, if corner stiffeners are used, of (26) must be inserted.*

The final longitudinal stresses in the rth cell are:

$$\sigma_r = \sigma_{r,r-1}\, X_{r-1} + \sigma_{r,r}\, X_r \tag{42}$$

For square cells the longitudinal stresses of (8) and (13) are inserted.

If the intermediate loading is taken into account for uniform torsional load distribution m, stresses $\tau_{r,z}$ and $\sigma_{r,z}$ (17) must be added.

The oblique tensile stresses in the box with only tension-resistant longitudinal walls, are:

$$\sigma_r = \sigma_{r,o} + \sigma_{r,r-1}\, X_{r-1} + \sigma_{r,r}\, X_r \tag{43}$$

For square cells, it becomes:

$$\sigma_r = \sigma_{r,o} + \sigma_{r,r}\,(X_r - X_{r-1}) \tag{43a}$$

with $\sigma_{r,o}$ (27) and $\sigma_{r,r}$ (29). The final loads in the longitudinal flanges are:

$$\left.\begin{aligned} L_r &= L_{r,o} + L_{r,r-1}\, X_{r-1} + L_{r,r}\, X_r \\ &= (L' + L'')_{r,o} + (L' + L'')_{r,r-1}\, X_{r-1} \\ &\qquad + L' + L'')_{r,r}\, X_r \end{aligned}\right\} \tag{44}$$

At the left and right bulkhead of the rth cell of a square cell, it is:

*The sign r for the cells in III,2,5,6, was omitted, since it pertains to the rth cell throughout.

$$\left.\begin{aligned} L_r &= L''_{r,o} + X_{r-1} + L''_{r,r}\,(X_r - X_{r-1}) \\ \text{and}\qquad L_r &= L''_{r,o} + X_r + L''_{r,r}\,(X_r - X_{r-1}) \end{aligned}\right\} \tag{44a}$$

($L_{r,o}$ (28), $L_{r,r}$ (30).)

The final stress attitude in shear-resistant bulkheads, is:

$$\bar{\tau}_r = \bar{\tau}_{r,o} + \bar{\tau}_{r,r-1}\, X_{r-1} + \bar{\tau}_{r,r}\, X_r + \bar{\tau}_{r,r+1}\, X_{r+1} \tag{45}$$

and for square cells,

$$\bar{\tau}_r = \bar{\tau}_{r,o} + \bar{\tau}_{r,r-1}\,(X_{r-1} - 2\,X_r + X_{r+1}) \tag{45a}$$

($\bar{\tau}_{r,o}$ (16), $\bar{\tau}_{r,r-1}$ (19).)

For simple tension-resistant design and rigid-edge members of bulkhead, oblique tension stresses of double the amount occur.

3. Angle of Twist - Twisting Rigidity (reference 10)

They both play an important role in proportioning airplane wings. The final angle of twist $\Delta\,\varphi_r$ of a cell of a box beam may be composed of the angle of twist $\Delta\,\varphi_{r,o}$ of the cell in the principal system due to $\bar{M}_r$ and angles of twist $\Delta\,\varphi_{r,r-1}$ and $\Delta\,\varphi_{r,r}$ due to the tension groups $X_{r-1} = 1$ and $X_r = 1$ applied at the cell:

$$\Delta\,\varphi_r = \Delta\,\varphi_{r,o} + \Delta\,\varphi_{r,r-1}\, X_{r-1} + \Delta\,\varphi_{r,r}\, X_r$$

For square cells with shear-resisting walls, we have:

$$G\,\Delta\,\varphi_{r,o} = d_r \int\!\!\int \tau^2_{r,o}\,\bar{M}_r\, dx\, dy = \frac{a_r\,\bar{M}_r}{2\,b^2\,c^2}\left(\frac{b}{d_b} + \frac{c}{d_c}\right)$$

$$G\,\Delta\,\varphi_{r,r-1} = -\,G\,\Delta\,\varphi_{r,r} = d_r \int\!\!\int \tau_{r,o}\,\tau_{r,r-1}\, dx\, dy$$

$$= -\,\frac{1}{2\,b\,c}\left(\frac{b}{d_b} - \frac{c}{d_c}\right)$$

and consequently:

$$G\,\Delta\,\varphi_r = \frac{a_r\,\bar{M}_r}{2\,b^2\,c^2}\left(\frac{b}{d_b} + \frac{c}{d_c}\right)_r - \frac{1}{2bc}\left(\frac{b}{d_b} - \frac{c}{d_c}\right)_r (X_{r-1} - X_r)$$

Then the angle of twist of the box between any two bulkheads i and k is:

$$G\ \varphi_{i,k} = \sum_{r=i+1}^{k} G\ \Delta\ \varphi_r = \frac{1}{2b^2c^2}\sum_{r=i+1}^{k} \overline{M}_r\ a_r\left(\frac{b}{d_b}+\frac{c}{d_c}\right)_r$$

$$-\frac{1}{2bc}\Bigg[\left(\frac{b}{d_b}-\frac{c}{d_c}\right)_{i+1} X_i + \left\{\left(\frac{b}{d_b}-\frac{c}{d_c}\right)_{i+2}\right.$$

$$\left.-\left(\frac{b}{d_b}-\frac{c}{d_c}\right)_{i+1}\right\}X_{i+1} + \ldots\ldots - \left(\frac{b}{d_b}-\frac{c}{d_c}\right)_k X_k\Bigg]$$

With constant dimensions of box, it is:

$$G\ \varphi_{i,k} = \frac{\frac{b}{d_b}+\frac{c}{d_c}}{2\ b^2\ c^2}\sum_{r=i+1}^{k} \overline{M}_r\ a_r - \frac{\frac{b}{d_b}-\frac{c}{d_c}}{2\ b\ c}\ (X_i - X_k)$$

Here $\sum_{i+1}^{k} \overline{M}_r\ a_r$ represents the volume of the twist-moment area between the bulkheads I and k. Accordingly, the angle of twist of the whole box between its end bulkheads is:

$$G\ \varphi_n = \frac{\frac{b}{d_b}+\frac{c}{d_c}}{2\ b^2\ c^2}\sum_{1}^{n} \overline{M}_r\ a_r - \frac{\frac{b}{d_b}-\frac{c}{d_c}}{2\ b\ c}\ (X_o - X_n)$$

$$= G\ \varphi_{n,o}\ (1 - \zeta) \qquad (46)$$

whereby:

$$\zeta = \frac{\frac{b}{d_b}-\frac{c}{d_c}}{\frac{b}{d_b}+\frac{c}{d_c}}\ \frac{(X_o - X_n)\ b\ c}{\sum_1^n \overline{M}_r\ a_r} = \eta_o\ \frac{(X_o - X_n)\ b\ c}{\sum_1^n \overline{M}_r\ a_r} \qquad (47)$$

Consequently, if the buckling of the box with constant dimensions at its end bulkheads is not prevented $(X_o = X_n = 0)$, its angle of twist is as that of the principal system:

$$\varphi_n = \varphi_{n,o} = \frac{\sum_1^n \overline{M}_r\ a_r}{G\ J_{d,o}} \qquad (48)$$

and its rigidity to twisting:

$$G\,J_d = G\,J_{d,o} = \frac{2\,b^2\,c^2\,G}{\frac{b}{d_b} + \frac{c}{d_c}} \tag{49}$$

is identical with that of thin-walled tubes of closed section F_1 computed according to Bredt's formula:

$$G\,J_{d,o} = \frac{4\,F_1^2\,G}{\oint \frac{ds}{d}}$$

The decrease in angle of twist of the box constrained against buckling compared to the angle of twist of the main system computed by Bredt's formula, is expressed by the factor ζ from (47), which is affected by the dimension as well as the type of loading and support. For the one-sidedly fixed box with equal cells the X_n values of (38a) and (39a) afford due to torque at the end:

$$\left.\begin{aligned} \sum_1^n \overline{M}_r\,a_r &= -\,n\,a\,M_o \\ \zeta &= \frac{1}{n}\,\frac{\eta_o\,\eta}{\sqrt{3\,\rho\left(1 + \frac{\rho}{4}\right)}} \end{aligned}\right\} \tag{50}$$

and, because of uniform torsional loading m:

$$\left.\begin{aligned} \sum_1^n \overline{M}_r\,a_r &= \sum_1^n \left(r - \frac{1}{2}\right) a^2\,m = \frac{n^2}{2}\,a^2\,m \\ \zeta &= \left[\frac{1}{\sqrt{3\,\rho\left(1 + \frac{\rho}{4}\right)}} - \frac{1}{n}\,\frac{1}{3\,\rho}\right]\frac{2\eta\,\eta_o}{n} \end{aligned}\right\} \tag{51}$$

Here ρ and η are as defined in (37) (reference 10). With uniform distribution of shearing stress:

$$\eta = \eta_o = \frac{\frac{b}{d_b} - \frac{c}{d_c}}{\frac{b}{d_b} + \frac{c}{d_c}}$$

For narrow bulkhead spacing $(a \rightarrow 0,\ n \rightarrow \infty)$ and with the abbreviation ψ according to (40) for ζ due to torque at the end or uniform torsional loading, we can write

$$\zeta = \frac{\eta\,\eta_o}{\psi}$$

and

$$\zeta = \frac{2\,\eta\,\eta_o}{\psi}\left(1 - \frac{1}{\psi}\right)$$

With equal wall thickness $d_b = d_c$ and

$$\psi^2 = \frac{48\ l^2\ \frac{G}{E}}{(b+c)^2},\quad \eta = \eta_o = \frac{b-c}{b+c},\quad \frac{G}{E} = 0.4:$$

$$\zeta = 0.23\ \frac{(b-c)^2}{l\,(b+c)}$$

$$\zeta = 0.46\ \frac{(b-c)^2}{l\,(b+c)}\left[1 - 0.23\ \frac{b+c}{l}\right]$$

According to the preceding formulas for ζ the fixity effect on the angle of twist and the mean twisting rigidity of the whole box need not be allowed for except with short systems and such of markedly other than square section, respectively, with markedly different wall thicknesses d_b and d_c. (See table I, section IV,4.)

Similarly, for the box with only tension-resistant, longitudinal walls and corner stiffener F, the mutual angle of twist of the end bulkheads is:

$$E\ \varphi_n = \frac{\frac{b}{d_b} + \frac{c}{d_c} + \frac{(b+c)^2}{8\ F}}{\frac{1}{2}\ b^2\ c^2}\ \sum_1^n\ \bar{M}_r\ a_r$$

$$- \frac{\frac{b}{d_b} - \frac{c}{d_c} + \frac{b^2 - c^2}{8\ F}}{\frac{1}{2}\ b\ c}\ (X_o - X_n)$$

$$= E\ \varphi_{n,o}\ (1 - \zeta).$$

The angle of twist of the box with unrestrained end buckling is:

$$\varphi_{n,o} = \frac{\sum_1^n\ \bar{M}_r\ a_r}{G\ J_{d,o}}$$

and the twisting rigidity is:

$$G\,J_{d,o} = \frac{\frac{1}{2}\,b^2\,c^2\,E}{\frac{b}{d_b} + \frac{c}{d_c} + \frac{(b+c)^2}{8\,F}}$$

TABLE I

Ratio of final stresses τ_n and σ_n and twisting rigidity $G\,J_d$ to $\tau_{n,o}$ and twisting rigidity $G\,J_{d,o}$ for various side and wall thickness ratios b/c and d_b/d_c.

d_b/d_c	b/c	$\frac{\sigma_n}{\tau_{cn,o}}$	At the edges $\frac{\tau_{bn}}{\tau_{bn,o}} = \frac{\tau_{cn}}{\tau_{cn,o}}$	In center of wall $\frac{\tau_{bn}}{\tau_{bn,o}}$	In center of wall $\frac{\tau_{cn}}{\tau_{cn,o}}$	$\frac{GJ_d}{GJ_{d,o}} = \frac{\varphi_{n,o}}{\varphi_n}$
1	1	0	1	1	1	1
	2	0.701	1.031	0.843	1.125	1.006
	4	1.360	1.124	0.627	1.248	1.028
	6	1.670	1.199	0.484	1.318	1.047
	8	1.860	1.257	0.377	1.367	1.063
	16	2.125	1.411	0.100	1.493	1.114
2	1	-0.496	0.969	1.157	0.875	1.006
	2	0	1	1	1	1
	4	0.437	1.091	0.780	1.130	1.008
	6	0.649	1.167	0.666	1.209	1.022
	8	0.780	1.229	0.498	1.275	1.038
	16	1.095	1.438	0.083	1.480	1.091
1	1	0	1	1	1	1
1/2		0.994	0.969	0.875	1.157	1.006
1/4		2.72	0.876	0.752	1.373	1.028
1/8		5.24	0.743	0.633	1.623	1.063
1/32		13.25	0.441	0.387	2.171	1.200
0		76.7	0	0	3	∞ ($GJ_{d,o}=0$)

See tables II and III at end of report.

The diminution factor ζ due to fixation is:

$$\zeta = \frac{\frac{b}{d_b} - \frac{c}{d_c} + \frac{b^2 - c^2}{8\ F}}{\frac{b}{d_b} + \frac{c}{d_c} + \frac{(b + c)^2}{8\ F}} \ \frac{b\ c\ (X_o - X_n)}{\sum_1^n \bar{M}_r\ a_r}$$

For the one-sidedly restrained box with equal cells the corresponding values due to a torque at the end or uniform torsional loading of (38) and (39) may be introduced.

4. Fixation Effect with Different Dimensional Systems

and Transverse Stiffening

The formulas for the redundancies (section IV,1) manifest how the dimensions of the system and the transverse stiffening with buckling constraint and variable twisting moment affect the discrepancy of the final stresses and twisting rigidity from the values of the principal system. It is seen that this discrepancy depends on the value η (37) (section IV,1). It increases as the side lengths b/d_b and c/d_c referred to the corresponding wall thickness vary from each other. Table I gives the discrepancy of the final system from the principal system in the nth cell with evenly distributed torsional loading at different aspect ratios b/c and wall thicknesses d_b/d_c for a restrained box of given length l, given section $b\ c$ and given bulkhead spacing $a = l/8$.

Because of the readily damped-out redundancy effect, the length l and the number of cells n with fixed cell length a is of no particular influence on the decisive change of the stress condition due to buckling constraint. The effect of l and n with fixed length of system, i.e., of the transverse stiffening on the stress condition is more complicated. The data in tables II and III represent the final stress condition due to evenly distributed torsional loading m_c for various numbers of cells $n = l/a$ in contrast to the stress condition of the principal system (σ_z and $\tau_o + \tau_z$) as applied to a box with shear resistant sides of given length l and given section. The effect of the intermediate loading applied at the vertical side walls was taken into consideration. The original dimensions of the box are $b/c = 4$, $d_b/d_c = 2$, $l/c = 24$,

$d_b = c/100$. The stresses are given at $x/c = 2, 4, 6 \ldots 24$, with x denoting the distance from the free end.

The redundancies with different bulkhead spacing a, were computed according to (35), and are shown as ordinates against the bulkhead figure in figure 13. The graph also shows the curve resulting from infinitely close bulkheads (35b) when the boundary is exceeded. The curve for $n = 24$ already is practically the same as that of the boundary curve for $n = \infty$, so that in this case of $a < c$ bulkhead spacing, the assumption of infinitely close bulkheads is admissible.

After a certain bulkhead spacing $(a \sim l/6)$ has been exceeded, there is no appreciable change in the final stress condition, according to tables II and III; likewise, the additional stress condition due to intermediate loading is no longer of great significance. From this follows that the purpose of transverse stiffening, to ensure the best uniform stress distribution, and consequently, greater twisting rigidity, is already obtained by comparatively wide bulkhead spacing. On top of that, the buckling constraint causes a load in the narrow upright and a release in the wide horizontal side walls, whose effect, however, is, with the usual bulkhead spacings, restricted to the vicinity of the fixation. Accordingly, it suffices as a rule, to simply allow for the change in stress condition through the tensions X_n, and for the rest, consider the stresses of the principal system as the final ones.

V. NUMERICAL EXAMPLES

The following examples apply to a one-sidedly restrained box with transverse stiffeners spaced at $n = 5$, as, for instance, used in the center box of an airplane wing (fig. 14). The loading consists of a torque applied at the end (of the wing-tip portion, for example): $M_o = 1200$ m kg and an evenly distributed torsion at the upright side walls: $m_c = 500$ m kg/m. The result is the twisting moment area shown in figure 15.

1. Box with Shear-Resistant Walls

<u>a) Rigid bulkheads</u>.- The shear-resistant side walls are assumed to have stepped wall thickness and rigid bulk-

heads. Dimensions: a = 80 cm, b = 120 cm, c = 40 cm.

Wall thickness in mm

r =	1	2	3	4	5	F = 0
$d_b = d_c$	1.5	1.5	1.8	1.8	2.0	$\frac{G}{E} = \frac{5}{13}$

Coefficients of deformation: For the special case of equal cell length and equal thickness in vertical and horizontal walls, equation (21) gives the coefficients of the redundancies:

$$G\ \delta_{r,r} = 8\ \frac{G}{E}\ \frac{a}{b + c}\left[\frac{1}{d_r} + \frac{1}{d_{r+1}}\right] +$$

$$\frac{1}{2\ a}\left[\left(b + c + \frac{4}{5}\ \frac{b^3 + c^3}{(b + c)^2}\right)\left(\frac{1}{d_r} + \frac{1}{d_{r+1}}\right)\right],$$

$$G\ \delta_{r,r-1} = 4\ \frac{G}{E}\ \frac{a}{b + c}\ \frac{1}{d_r} - \frac{1}{2\ a}\left[b + c + \frac{4}{5}\ \frac{b^3 + c^3}{(b + c)^2}\right]\frac{1}{d_r}\ .$$

The load figures, according to (23), are:

$$G\ \delta_{r,o} = \frac{b - c}{2\ b\ c}\left[\frac{1}{d_r}\ \bar{M}_r - \frac{1}{d_{r+1}}\ \bar{M}_{r+1}\right].$$

The evaluation of the deformation coefficients yields the system of elasticity equations of table IV.

TABLE IV. Elasticity Equations with Rigid Bulkheads

k \ i	1	2	3	4	5	Load figures due to M_0=1200 mkg	m_c=500 mkg/m
1	38.5	-3.9	-	-	-	0	-2220
2	-3.9	35.3	-3.2	-	-	1111	-1295
3	-	-3.2	32.1	-3.2	-	0	-1850
4	-	-	-3.2	30.5	-2.9	555	-1020
5	-	-	-	-2.9	14.5	5000	+7500

mkg X 7.23298 = ft.-lb.

Redundancies: The resolution gives

M_0 = 1200 mkg	m_c = 500 mkg/m
X_1 = 3	X_1 = - 62
X_2 = 33	X_2 = - 49
X_3 = 9	X_3 = - 62
X_4 = 53	X_4 = 10
X_5 = 357	X_5 = 521

shown as a in figure 16, which likewise indicates the redundancies under the assumption that all cells are equal; that is, as the cell at the restraint (lines b). In the latter case, the redundancies may be computed according to (34) and (35). With the dimensions of the nth cell, we have:

$\rho=1.14$, $\eta=0.37$, $\alpha=4.99$, $\varphi=\cosh^{-1}|\alpha|=2.29$, $\nu=0.86$

for M_0 = 1200 mkg

$$\frac{\nu}{\sqrt{\alpha^2-1}}\frac{a}{bc}M_0 = +\ 353$$

for m_c = 500 mkg/m

$$\frac{\nu}{2(\alpha-1)}\frac{a^2 m_c}{bc} = +\ 72$$

$$\frac{\nu}{\sqrt{\alpha^2-1}}\frac{n\ a^2 m}{bc} = +\ 587$$

Comparing a and b, it is seen that, even with unequal cell dimensions, the redundancies computed with the dimensions of the constrained cell on the assumption of equal cells, yields close approximative values. For rough calculation it suffices to compute the redundancies independently from a one-term equation ($X_r = -\delta_{r,o}/\delta_{r,r}$) each. The ensuing redundancies are shown in figure 16, denoted by c.

Final stresses.- Formulas (41a) and (42) give the maximum shear and tension stresses (at wall center and at edge I) of tables V to VII. The effect of the intermediate loading due to m_c is disregarded. (See section IV,4.)

Figure 17 shows the shear stresses $\tau_{br,o}$ and $\tau_{cr,o}$

at the principal system due to external load M_0 and m_c and the additional shearing stresses $\tau_{br,x}$ and $\tau_{cr,x}$ due to redundancies X_r. Accordingly, the restraint in the horizontal side wall produces in the nth cell, as a result of constrained cross-sectional buckling, a decrease in stress of the order of 30 percent and in the vertical side wall, an increase of approximately 20 percent. In the other cells the discrepancies of the final stresses from those of the principal system are minor.

The course of the tensions σ_r is shown in figure 18. The abrupt change at the bulkheads is due to the stepped dimension.

b) Elastic bulkheads.- Here we include the redundancies for the case of elastic, shear-resistant walls ($\bar{d}$ = 1 mm). The end bulkheads are assumed rigid as before. Then the special case of equal cell length and equal bulkhead thickness $\bar{d}$ gives, according to (22) (section III,4) the additional coefficients,

$$G\,\bar{\delta}_{r,r} = \frac{3}{2}\,\frac{b\,c}{a^2\,\bar{d}},$$

$$G\,\bar{\delta}_{r,r-1} = -\frac{b\,c}{a^2\,\bar{d}},\quad G\,\bar{\delta}_{r,r-2} = \frac{b\,c}{4\,a^2\,\bar{d}}$$

TABLE V

Shear at wall center and tension at edge I (kg/cm²) due to condition X_r = 1 kg at principal system.

r	$\tau^{(t)}_{br,r} = \tau^{(t)}_{cr,r}$	$\tau^{(\sigma)}_{br,r}$	$\tau^{(\sigma)}_{cr,r}$	$\tau_{br,r}$	$\tau_{cr,r}$	$\sigma_{r,r}$ and $\tau_{r,r-1}$
1	-0.0208	0.0937	-0.0312	0.0729	-0.0520	0.2500
2	-0.0208	0.0937	-0.0312	0.0729	-0.0520	0.2500
3	-0.0174	0.0781	-0.0260	0.0607	-0.0434	0.2083
4	-0.0174	0.0781	-0.0260	0.0607	-0.0434	0.2083
5	-0.0156	0.0703	-0.0234	0.0547	-0.0390	0.1872

TABLE VI

Final shear and tension (kg/cm²) due to M_o = 1200 m kg

r	$\bar{M}_r$ (mkg)	$\tau_{br,o} = \tau_{cr,o}$	$(X_r - X_{r-1})$	$\tau_{br,r}(X_r - X_{r-1})$	$\tau_{cr,r}(X_r - X_{r-1})$	τ_{br}	τ_{cr}	σ_r
1	-1200	-83.3	3	0.2	- 0.2	-83.1	-83.5	0.8
2	-1200	-83.3	30	2.2	- 1.6	-81.1	-84.9	8.3 / 6.9
3	-1200	-69.5	-24	-1.5	+ 1.0	-71.0	-68.5	1.9
4	-1200	-69.5	44	2.7	- 1.9	-66.8	-71.4	11.1 / 9.2
5	-1200	-62.5	304	16.6	-11.9	-45.9	-74.3	67.0

TABLE VII

Final shear and tension (kg/cm²) due to m_c = 500 mkg/m

r	$\bar{M}_r$ (mkg)	$\tau_{br,o} = \tau_{cr,o}$	$(X_r - X_{r-1})$	$\tau_{br,r}(X_r - X_{r-1})$	$\tau_{cr,r}(X_r - X_{r-1})$	τ_{br}	τ_{cr}	σ_r
1	- 200	-13.9	- 62	-4.5	+ 3.2	-18.4	- 10.7	-15.5
2	- 600	-41.7	+ 13	+0.9	- 0.7	-40.8	- 42.4	-12.3 / -10.2
3	-1000	-57.9	- 13	-0.8	+ 0.6	-58.7	- 57.3	-12.9
4	-1400	-81.1	+ 72	4.4	- 3.1	-76.7	- 84.2	2.1 / 1.9
5	-1800	-93.8	+511	28.0	-19.9	-65.8	-113.7	97.6

TABLE VIII

Elasticity equations for elastic bulkheads
(bulkheads o and n are rigid)

i / k	1	2	3	4	5	Load figures due to M_o=1200 mkg	m_c=500 mkg/m
1	(L) 38.5 (Q) 9.4 Σ 47.9	- 3.9 - 7.5 -11.4	0 1.9 1.9	-	-	0 0 0	-2220 -1250 -3470
2	- 3.9 - 7.5 -11.4	35.3 11.3 46.6	- 3.2 - 7.5 -10.7	0 1.9 1.9	-	1111 0 1111	-1295 0 -1295
3	0 1.9 1.9	- 3.2 - 7.5 -10.7	32.1 11.3 43.4	- 3.2 - 7.5 -10.7	0 1.9 1.9	0 0 0	-1850 0 -1850
4	-	0 1.9 1.9	- 3.2 - 7.5 -10.7	30.5 9.4 39.9	- 2.9 - 3.8 - 6.7	555 0 555	-1020 -1250 -2270
5	-	-	0 1.9 1.9	- 2.9 - 3.8 - 6.7	14.4 1.9 16.3	5000 0 5000	+7500 +1250 +8750

In like manner, equation (24) gives the additional load figures for the torque M_c applied equally at all bulkheads as vertical moments:

$$G\,\overline{\delta}_{1,o} = G\,\overline{\delta}_{n-1,o} = -\,G\,\overline{\delta}_{n,o} = -\,\frac{M_c}{4\bar{a}\,\bar{d}}$$

There are no additional load values for a torque at the end when the end bulkheads are rigid.

The above additional coefficients give the system of elasticity equations of table VIII. The side (L) and bulkhead (Q) quotas to the deformation coefficients are shown separately. If X_r is the solution of the equations with five terms for the original load values from the side walls and $\overline{X}_r$ the solution for the additional load values from the bulkheads, we have:

because of $M_0 = 1200$ mkg		because of $m_c = 500$ mkg/m	
$X_1 = 6$	$\overline{X}_r = 0$	$X_1 = -\ 57$	$\overline{X}_1 = -\ 28$
$X_2 = 24$		$X_2 = -\ 59$	$\overline{X}_2 = -\ 8$
$X_3 = 8$		$X_3 = -\ 66$	$\overline{X}_3 = -\ 9$
$X_4 = 74$		$X_4 = 41$	$\overline{X}_4 = -\ 22$
$X_5 = 334$		$X_5 = 484$	$\overline{X}_5 = +\ 69$

The course of the redundancies X_r and $\overline{X}_r$ is shown in figure 19 by b and c, in comparison with those (a) for rigid bulkheads of figure 16. It is seen that $(M_{b,r} = M_{c,r} = M_r/2)$ the difference between assumedly elastic and rigid bulkheads is small, so long as the torque is uniformly applied. The application of torque as horizontal couple $(M_{b,r} = M_r)$ produces equivalent additional values $\overline{X}_r$ (lines c) with contrary sign. From these two limit cases of application $(M_{c,r} = M_r$ and $M_{b,r} = M_r)$ all intermediary cases may be deduced.

2. Box with Corner Stiffeners

The restrained box with shear-resistant walls is now assumed to be fitted with corner stiffeners $F = 3$ cm². The outside dimensions are the same as before, the wall thickness in all cells the same $(d_b = d_c = 2$ mm). The assumption of rigid bulkheads gives the redundancies according to (34) and (35).

The graph (fig. 20) shows the redundancies X_r (lines 2a), the redundancies for an assumed uniformly, rather than parabolically distributed shearing stress and maximum shear for loading M_o (lines 2b and 2c), and finally the redundancies for an assumed shear distribution of a box without corner stiffeners (lines 1a to 1c). It is seen that the effect of the shear distribution decreases as the corner stiffening increases.

The final stress curve is not shown separately. The additional shearing stresses due to redundancy X_r are somewhat higher than in V,1, but the additional tensile stresses are slightly lower.

3. Box with Simple Walls Resistant to Tension

The restrained box with the same dimensions as before, is now assumed to have only tension-resistant side walls of thickness $d_b = 0.6$ mm, $d_c = 0.4$ mm, and corner stiffeners $F = 6$ cm^2.

For the particular case of equal cells and rigid bulkheads the redundancies X_r, according to (34) and (35), give:

$$\rho_1 = 0.237, \quad \rho_2 = 0.044, \quad \eta = 0.333, \quad \alpha = 1.382,$$

$$\varphi = 0.848, \text{ and } \nu = 0.455$$

because of $M_0 = 1200$ mkg	because of $m_c = 500$ mkg/m
$X_1 = 26$	$X_1 = -183$
$X_2 = 72$	$X_2 = -203$
$X_3 = 173$	$X_3 = -76$
$X_4 = 417$	$X_4 = 298$
$X_5 = 955$	$X_5 = 1205$

The redundancy is shown as 3, in figure 20.

The calculation of the oblique tensile stresses σ_r as well as of the flange stresses L_r is effected by (43a) and (44a). Figure 21 gives $d_b\ \sigma_{br,o} = d_c\ \sigma_{cr,o}$ due to outside loading at the principal system, together with the additional values $-d_b\ \sigma_{br,x} = d_c\ \sigma_{cr,x}$ due to X_r. The stress decrease in the horizontal side wall and the increase in the vertical side wall is the same in this system and amounts to about 30 percent in the nth cell. Cells adjacent to the restraint likewise manifest a greater influence of X_r when the box has only tension-resistant walls.

Figure 22 shows the flange stresses $L''_{r,o}$ due to outside load at the principal system separate from the additional flange stresses $L_{r,x}$ due to X_r. The great effect of X_r on the flange stresses is thus manifest.

VI. APPENDIX

Stress Condition According to the Rigorous Theory of Elasticity*

These equations have a consistent error check with original

The result of the tension group $X = 1$ for the boundary condition of vanishing buckling of side-wall sections is the following stress condition.

In the horizontal wall:

$$\sigma_{x,b} = -\Sigma \frac{1}{m\pi} \frac{b/a}{b_m d_b + c_m d_c} \operatorname{Csch} k \frac{b}{2} \left[\frac{\operatorname{Sinh} k y}{\operatorname{Sinh} k \frac{b}{2}} - \frac{y}{b/2} \frac{\operatorname{Cosh} k y}{\operatorname{Cosh} k \frac{b}{2}} - \frac{4}{kb} \frac{\operatorname{Sinh} k y}{\operatorname{Cosh} k \frac{b}{2}} \right] \cos k x,$$

$$\sigma_{y,b} = +\Sigma \frac{1}{m\pi} \frac{b/a}{b_m d_b + c_m d_c} \operatorname{Csch} k \frac{b}{2} \left[\frac{\operatorname{Sinh} k y}{\operatorname{Sinh} k \frac{b}{2}} - \frac{y}{b/2} \frac{\operatorname{Cosh} k y}{\operatorname{Cosh} k \frac{b}{2}} \right] \cos k x,$$

$$\tau_b = \frac{1}{2a\ d_b} + \Sigma \frac{1}{m\pi} \frac{b/a}{b_m d_b + c_m d_c} \operatorname{Csch} k \frac{b}{2} \left[\frac{\operatorname{Cosh} k y}{\operatorname{Sinh} k \frac{b}{2}} - \frac{y}{b/2} \frac{\operatorname{Sinh} k y}{\operatorname{Cosh} k \frac{b}{2}} - \frac{2}{kb} \frac{\operatorname{Cosh} k y}{\operatorname{Cosh} k \frac{b}{2}} \right] \sin k x,$$

In the vertical wall:

$$\sigma_{x,c} = +\Sigma \frac{1}{m\pi} \frac{c/a}{b_m d_b + c_m d_c} \operatorname{Csch} k \frac{c}{2} \left[\frac{\operatorname{Sinh} k z}{\operatorname{Sinh} k \frac{c}{2}} - \frac{z}{c/2} \frac{\operatorname{Cosh} k z}{\operatorname{Cosh} k \frac{c}{2}} - \frac{4}{kc} \frac{\operatorname{Sinh} k z}{\operatorname{Cosh} k \frac{c}{2}} \right] \cos k x,$$

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*A detailed development of the following formulas will be given in a report soon to be published in the Ingenieur-Archiv.

$$\sigma_{y,c} = -\Sigma \frac{1}{m\pi} \frac{c/a}{b_m d_b + c_m d_c} \operatorname{Csch} k \frac{c}{2} \left[\frac{\operatorname{Sinh} k z}{\operatorname{Sinh} k \frac{c}{2}} - \frac{z}{c/2} \frac{\operatorname{Cosh} k z}{\operatorname{Cosh} k \frac{c}{2}} \right] \cos k x,$$

$$\tau_c = \frac{1}{2a \; d_c} - \Sigma \frac{1}{m\pi} \frac{c/a}{b_m d_b + c_m d_c} \operatorname{Csch} k \frac{c}{2} \left[\frac{\operatorname{Cosh} k z}{\operatorname{Sinh} k \frac{c}{2}} - \frac{z}{c/2} \frac{\operatorname{Sinh} k z}{\operatorname{Cosh} k \frac{c}{2}} - \frac{z}{kc} \frac{\operatorname{Cosh} k z}{\operatorname{Cosh} k \frac{c}{2}} \right] \sin k x.$$

The effective width b_m is:

$$b_m = \frac{b}{2} \frac{\frac{1}{kb} \operatorname{Sinh} k b - 1}{\operatorname{Cosh} k b - 1}$$

$$k = \frac{m\pi}{2a}$$

(c_m correspondingly).

The normal stresses σ_x and the shearing stresses for a box of aspect ratio $a : b : c = 2 : 4 : 1$ has been computed according to the preceding formulas and plotted in figures 23 and 24. The dash lines show the stress for the assumption of linear normal stress distribution σ_x and parabolic shear distribution τ.

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Table II

Survey of the normal edge stresses of a restrained box with shear-resistant walls due to $\frac{m_r}{c^2} = 1$ for different bulkhead spacing $a = l/n$.

σ_x = tension due to intermediate loading.
$\sigma_r^{(M)}$ = " " " $X_r^{(M)}$
$\sigma_r^{(Z)}$ = " " " $X_r^{(Z)}$

$n = l/a$	σ	$x/c = 2$	4	6	8	10	12	14	16	18	20	22	24
1	σ_x	−735	−1333	−1800	−2130	−2330	−2400	−2330	−2130	−1800	−1333	−735	0
	$\sigma_r^{(M)}$	2	4	6	8	10	12	14	16	18	20	22	24
	$\sigma_r^{(Z)}$	198	396	595	793	991	1190	1389	1587	1785	1984	2182	2380
	Σ	−535	−933	−1199	−1329	−1329	−1198	−927	−527	+3	+671	1469	2404
2	σ_x	−333	−533	−600	−533	−333	0	−333	−533	−600	−533	−333	0
	$\sigma_r^{(M)}$	−7	−15	−21	−29	−36	−43	−21	1	23	45	68	90
	$\sigma_r^{(Z)}$	83	167	250	332	414	496	472	448	424	400	376	353
	Σ	−257	−381	−371	−230	+45	+453	+118	−84	−153	−88	+111	+443
4	σ_x	−133	−133	0	−133	−133	0	−133	−133	0	−133	−133	0
	$\sigma_r^{(M)}$	−7	−13	−19	−17	−15	−13	−21	−31	−39	25	91	157
	$\sigma_r^{(Z)}$	37	76	114	109	104	99	100	100	101	101	100	100
	Σ	−103	−70	+95	−41	−44	+86	−54	−64	+62	−7	+58	+257
6	σ_x	−67	0	−67	0	−67	0	−67	0	−67	0	−67	0
	$\sigma_r^{(M)}$	−9	−17	−16	−16	−16	−16	−16	−16	−20	−25	87	195
	$\sigma_r^{(Z)}$	23	46	45	43	44	44	44	44	44	44	44	44
	Σ	−53	+29	−38	+27	−39	+28	−39	+28	−43	+19	+64	+239
8	σ_x	−33	−33	0	−33	−33	0	−33	−33	0	−33	−33	0
	$\sigma_r^{(M)}$	−10	−15	−16	−16	−16	−16	−16	−16	−15	−7	69	216
	$\sigma_r^{(Z)}$	15	24	24	24	24	24	24	24	24	24	24	24
	Σ	−28	−24	+8	−25	−25	+8	−25	−25	+9	−16	+60	+240
24	σ_x	0	0	0	0	0	0	0	0	0	0	0	0
	$\sigma_r^{(M)}$	−13	−15	−16	−16	−16	−16	−16	−15	−12	−1	48	+249
	$\sigma_r^{(Z)}$	1	2	3	3	3	3	3	3	3	3	3	3
	Σ	−12	−13	−13	−13	−13	−13	−13	−12	−9	+2	+51	+252

Table III

Survey of maximum shear (in wall center) of restrained box with shear-resistant walls due to $\frac{m_r}{c^2} = 1$ for different bulkhead spacing $a = l/n$.

τ_0 = shear due to $M_r = -(r - 1/2)\, a\, m_r$
τ_x = " " " intermediate loading
$\tau_r^{(M)}$ = " " " $X_r^{(M)}$
$\tau_r^{(Z)}$ = " " " $X_r^{(Z)}$

a) Horizontal side wall

$n = l/a$	τ	$\frac{x}{c} = 2$	4	6	8	10	12	14	16	18	20	22	24
1	τ_0	−150	−150	−150	−150	−150	−150	−150	−150	−150	−150	−150	−150
	τ_x	−111	−89	−67	−45	−22	0	+22	+45	+67	+89	+111	+133
	$\tau_r^{(M)}$	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1
	$\tau_r^{(Z)}$	+70	+70	+70	+70	+70	+70	+70	+70	+70	+70	+70	+70
	Σ	−190	−168	−146	−124	−101	−79	−57	−34	−12	+10	+32	+54
2	τ_0	−75	−75	−75	−75	−75	−75	−225	−225	−225	−225	−225	−225
	τ_x	−44	−22	0	+22	+44	+67	−44	−22	0	+22	+44	+67
	$\tau_r^{(M)}$	−3	−3	−3	−3	−3	−3	+8	+8	+8	+8	+8	+8
	$\tau_r^{(Z)}$	+29	+29	+29	+29	+29	+29	−8	−8	−8	−8	−8	−8
	Σ	−93	−71	−49	−27	−5	+18	−269	−247	−225	−203	−181	−158
4	τ_0	−38	−38	−38	−113	−113	−113	−188	−188	−188	−263	−263	−263
	τ_x	−11	+11	+33	−11	+11	+33	−11	+11	+33	−11	+11	+33
	$\tau_r^{(M)}$	−2	−2	−2	+1	+1	+1	−3	−3	−3	+23	+23	+23
	$\tau_r^{(Z)}$	+14	+14	+14	−2	−2	−2	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
	Σ	−37	−15	+7	−125	−103	−81	−203	−180	−158	−251	−229	−207

Table III continued

$n = l/a$	τ	$\frac{x}{c} = 2$	4	6	8	10	12	14	16	18	20	22	24
6	τ_0	−25	−25	−75	−75	−125	−125	−175	−175	−225	−225	−275	−275
	τ_x	0	+22	0	+22	0	+22	0	+22	0	+22	0	+22
	$\tau_r^{(M)}$	−3	−3	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	−2	−2	+40	+40
	$\tau_r^{(Z)}$	+8	+8	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
	Σ	−20	+2	−75	−53	−125	−103	−175	−153	−227	−205	−235	−213
8	τ_0	−19	−56	−56	−94	−131	−131	−169	−206	−206	−244	−281	−281
	τ_x	+6	−6	+17	+6	−6	+17	+6	−6	+17	+6	−6	+17
	$\tau_r^{(M)}$	−4	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	+3	+52	+52
	$\tau_r^{(Z)}$	+5	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
	Σ	−12	−62	−39	−88	−137	−114	−163	−212	−189	−235	−235	−212
24	τ_0	−19	−44	−69	−94	−119	−144	−169	−194	−219	−244	−269	−294
	τ_x	+6	+6	+6	+6	+6	+6	+6	+6	+6	+6	+6	+6
	$\tau_r^{(M)}$	−3	−1	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	+1	+2	+6	+22	+95
	$\tau_r^{(Z)}$	+1	0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
	Σ	−15	−39	−63	−88	−113	−138	−163	−187	−211	−232	−241	−193

b) Vertical side wall

$n = l/a$	τ	$\frac{x}{c} = 2$	4	6	8	10	12	14	16	18	20	22	24
1	τ_0	−300	−300	−300	−300	−300	−300	−300	−300	−300	−300	−300	−300
	τ_x	+527	+421	+315	+210	+105	0	−105	−210	−315	−421	−527	−633
	$\tau_r^{(M)}$	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
	$\tau_r^{(Z)}$	−83	−83	−83	−83	−83	−83	−83	−83	−83	−83	−83	−83
	Σ	+143	+37	−69	−174	−279	−384	−489	−594	−699	−805	−911	−1017
2	τ_0	−150	−150	−150	−150	−150	−150	−450	−450	−450	−450	−450	−450
	τ_x	+212	+106	0	−106	−212	−317	+212	+106	0	−106	−212	−317
	$\tau_r^{(M)}$	+3	+3	+3	+3	+3	+3	−9	−9	−9	−9	−9	−9
	$\tau_r^{(Z)}$	−35	−35	−35	−35	−35	−35	+10	+10	+10	+10	+10	+10
	Σ	+30	−76	−182	−288	−394	−499	−237	−343	−449	−555	−661	−766
4	τ_0	−75	−75	−75	−225	−225	−225	−375	−375	−375	−525	−525	−525
	τ_x	+53	−53	−158	+53	−53	−158	+53	−53	−158	+53	−53	−158
	$\tau_r^{(M)}$	+3	+3	+3	−1	−1	−1	+4	+4	+4	−27	−27	−27
	$\tau_r^{(Z)}$	−16	−16	−16	+2	+2	+2	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
	Σ	−35	−141	−246	−171	−277	−382	−318	−424	−529	−499	−605	−710
6	τ_0	−50	−50	−150	−150	−250	−250	−350	−350	−450	−450	−550	−550
	τ_x	0	−106	0	−106	0	−106	0	−106	0	−106	0	−106
	$\tau_r^{(M)}$	+4	+4	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	+2	+2	−46	−46
	$\tau_r^{(Z)}$	−10	−10	+1	+1	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
	Σ	−56	−162	−149	−255	−250	−356	−350	−456	−448	−554	−596	−702
8	τ_0	−38	−113	−113	−188	−263	−263	−338	−413	−413	−488	−563	−563
	τ_x	−26	+26	−79	−26	+26	−79	−26	+26	−79	−26	+26	−79
	$\tau_r^{(M)}$	+4	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	−4	−61	−61
	$\tau_r^{(Z)}$	−6	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
	Σ	−66	−87	−192	−214	−237	−342	−364	−387	−492	−518	−598	−703
24	τ_0	−38	−88	−138	−188	−238	−288	−338	−388	−438	−488	−538	−588
	τ_x	−26	−26	−26	−26	−26	−26	−26	−26	−26	−26	−26	−26
	$\tau_r^{(M)}$	+4	+1	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	−1	−2	−7	−26	−112
	$\tau_r^{(Z)}$	−1	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0	≈ 0
	Σ	−61	−113	−164	−214	−264	−314	−364	−415	−466	−521	−590	−726

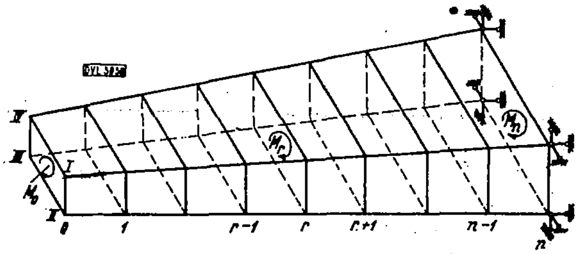


Figure 1.- Box with torque at bulkheads.

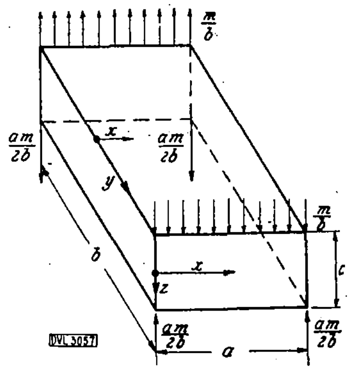


Figure 2.- Square cell with intermediate loading.

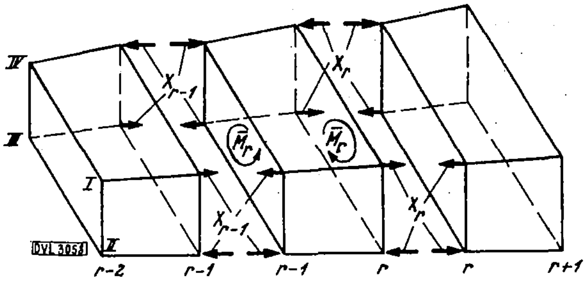


Figure 3.- Principal system with twisting moments $\bar{M}$ and buckling forces X.

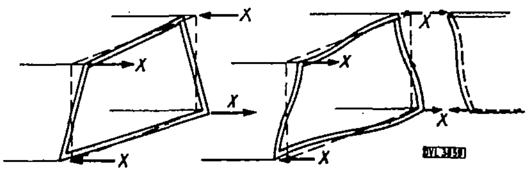


Figure 4.- Corner deformation with and without individual buckling of longitudinal walls of cell.

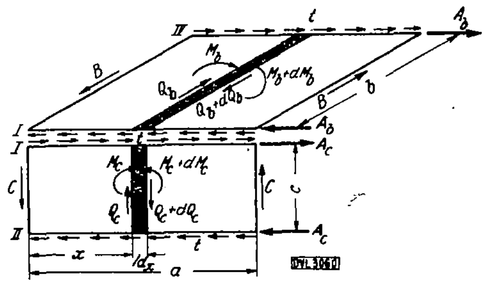


Figure 5.- Edge loading of longitudinal walls of cell.

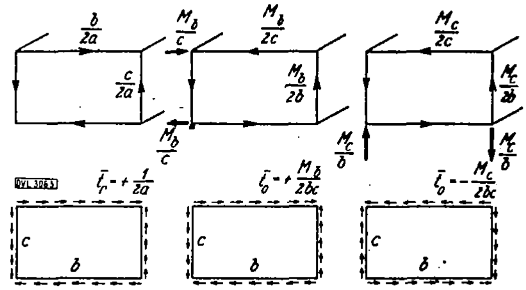


Figure 8.- Edge loading of bulkheads, (a) due to $X_r = 1$, (b) due to M_b, (c) due to M_c .

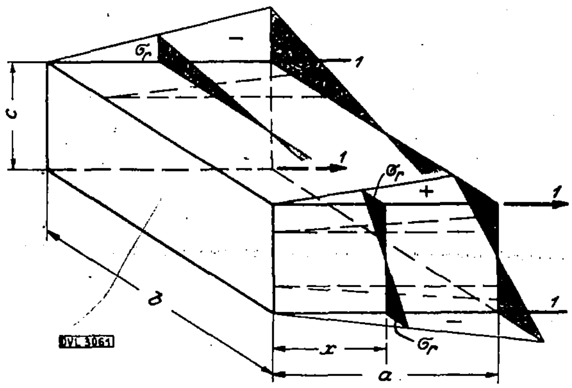


Figure 6.- Normal stresses σ_r due to $X_r = 1$.

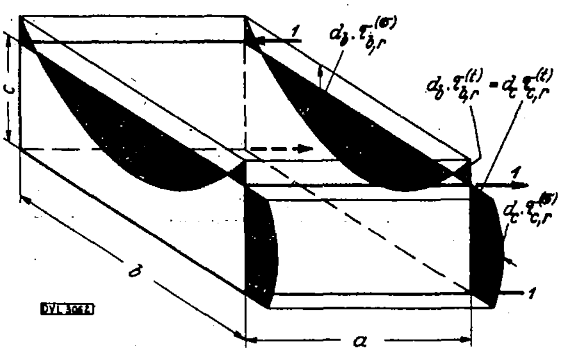


Figure 7.- Shear stresses τ_r due to $X_r = 1$.

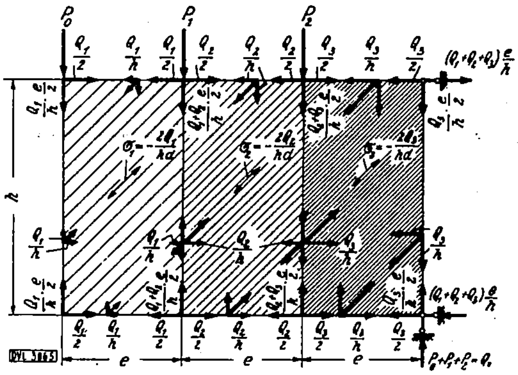
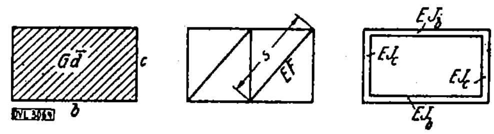


Fig. 10 a



Plate, Truss, Frame or box.

Figure 9.- Bulkhead designs.

Fig. 11 a

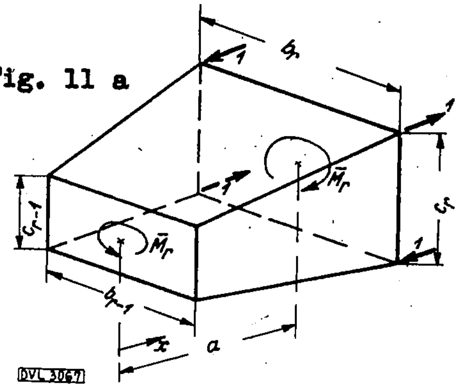


Fig. 10 b

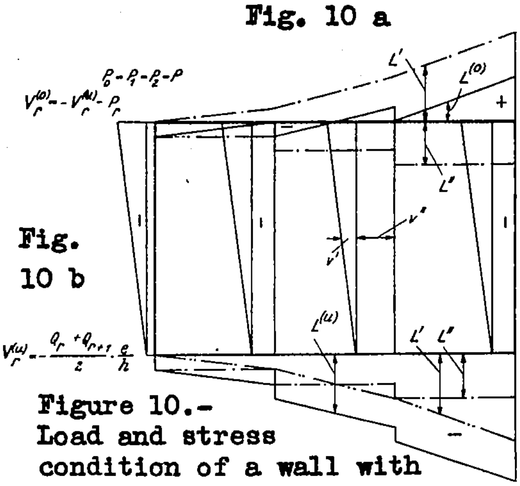


Figure 10.- Load and stress condition of a wall with very thin web and bending resistant flanges and uprights.

Fig. 11 b

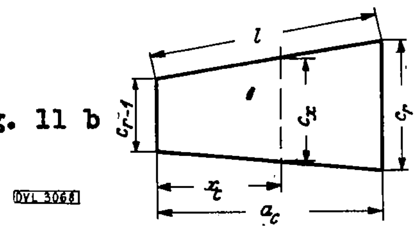


Figure 11.-(a) blunt-wedge cell in torsional and buckling load, (b) vertical longitudinal wall.

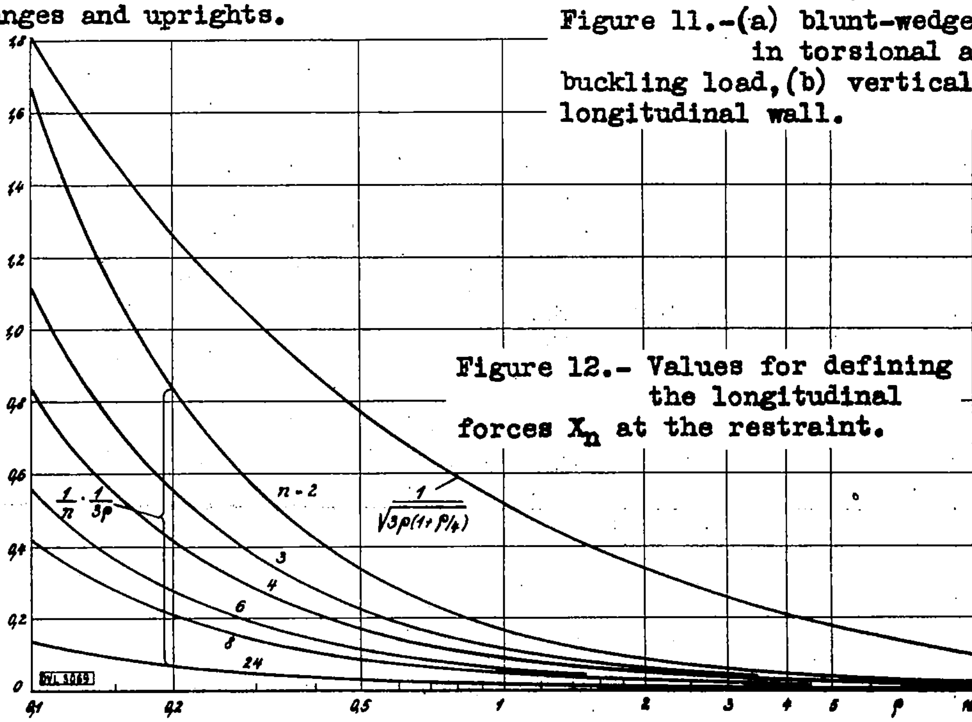


Figure 12.- Values for defining the longitudinal forces X_n at the restraint.

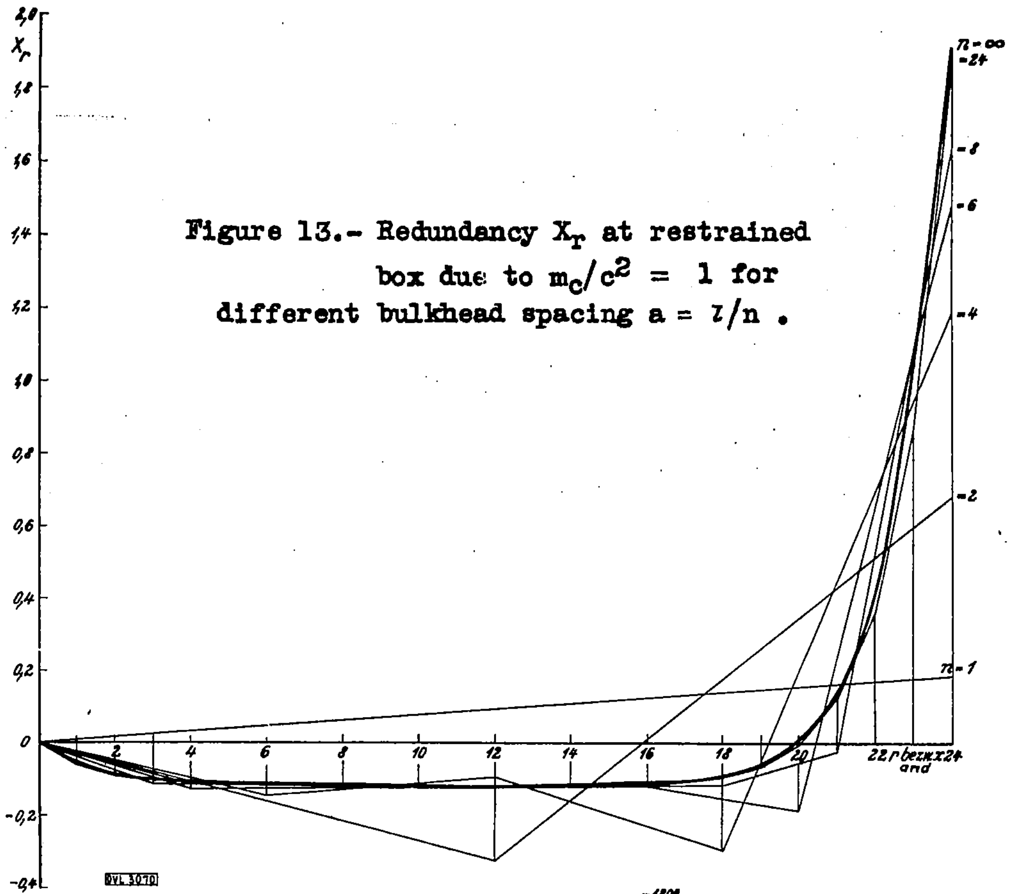


Figure 13.- Redundancy X_r at restrained box due to $m_c/c^2 = 1$ for different bulkhead spacing $a = l/n$.

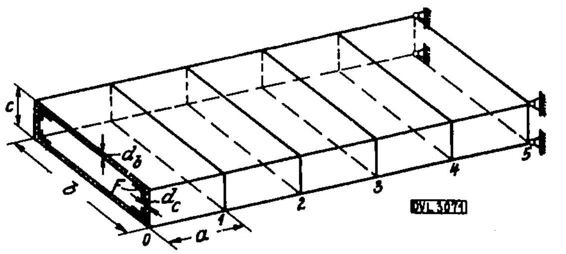


Figure 14.- One-sidedly restrained box with equally spaced transverse stiffeners.

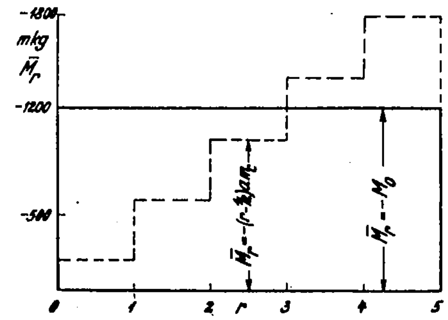


Figure 15.- Twisting moment areas for sole torque M_o = 1200 mkg and uniformly distributed torsional loading m_c = 500 mkg/m .

(mkg × 7.23298 = ft.-lb.)

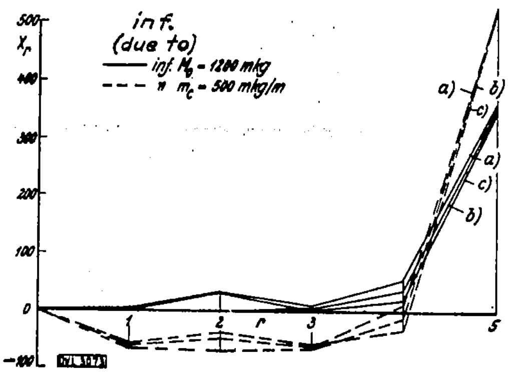


Figure 16.- Redundancy X_r of restrained box with rigid bulkheads.
(a) unequal size of cells,
(b) equal size of cells,
(c) unequal size and approximation ($X_r = -\frac{\delta r,o}{\delta r,r}$

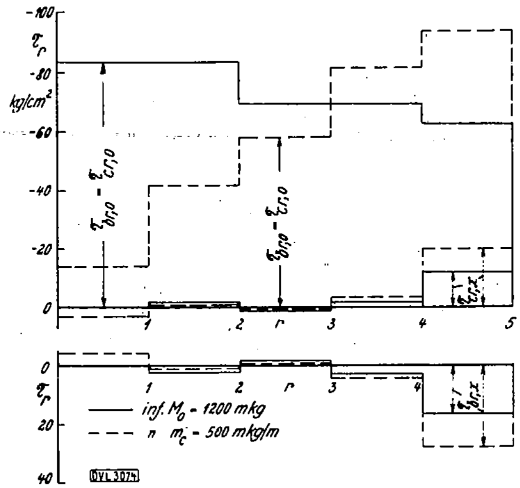


Figure 17.- Final shear stresses $(\tau_{r,o} + \tau_{r,x})$ in mid-wall.

inf.
(due to)

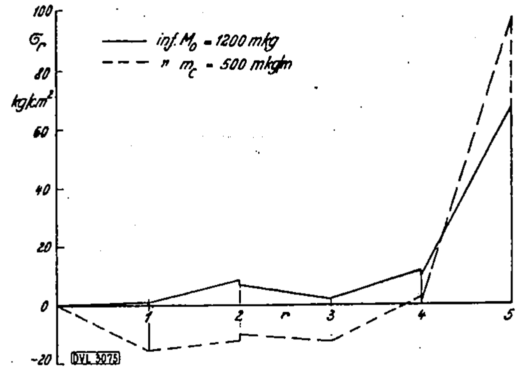


Figure 18.- Final normal stresses σ_r at edge I .

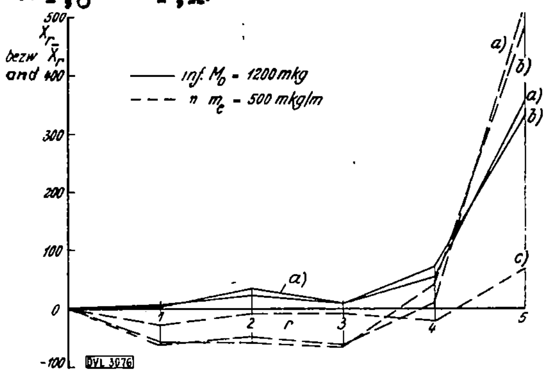


Figure 19.- X_r of restrained box with stepped dimensions
(a) X_r with rigid bulkheads,
(b) X_r with elastic bulkheads (and bulkhead o and n rigid) and equal application of torque ($M_{b,r} = M_{c,r} = M_r/2$),
(c) additive $\pm \overline{X}_r$ for uneven application of torque ($M_{b,r} = 0$, $M_{c,r} = M_r$ and $M_{b,r} = M_r$, $M_{c,r} = 0$).

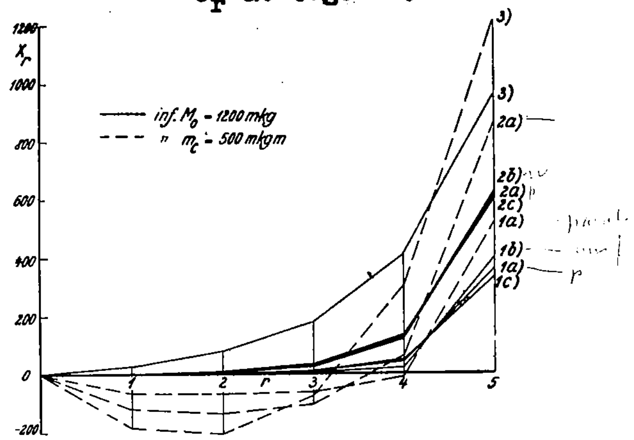


Figure 20.- X_r of restrained box with equal cells and rigid bulkheads.
(1) box with shear resistant walls,
(2) " " " " " and corner stiffeners,
(a) under parabolic shear distribution,
(b) under uniform shear distribution,
(c) under uniform maximum shear distribution,
(3) box with only tension resistant walls.

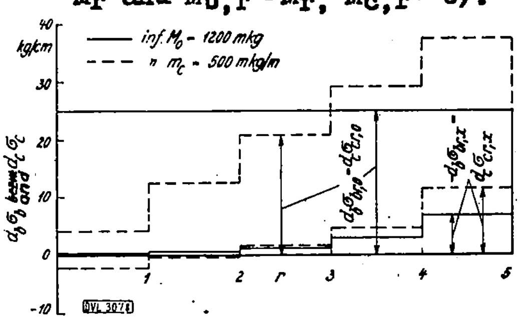


Figure 21.- Final tensile stresses $(d \cdot \sigma_{r,o} + d \cdot \sigma_{r,x})$.

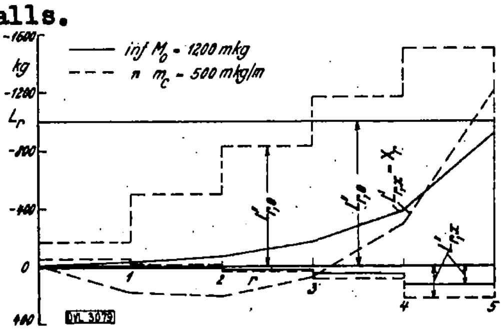


Figure 22.- Final flange forces $(L_{r,o} + L'_{r,x} + L''_{r,x})$.

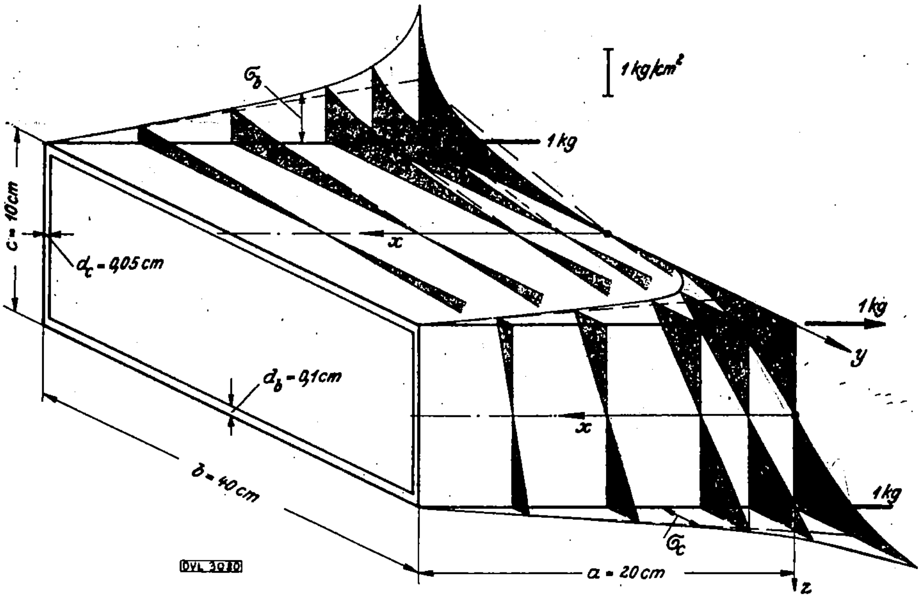


Figure 23.- Normal stresses σ_x due to X = 1 kg according to rigorous elasticity theory.

kg/cm^2 x 14.2235 = lb./sq.in.

kg x 2.20462 = lb.

cm x .3937 = in.

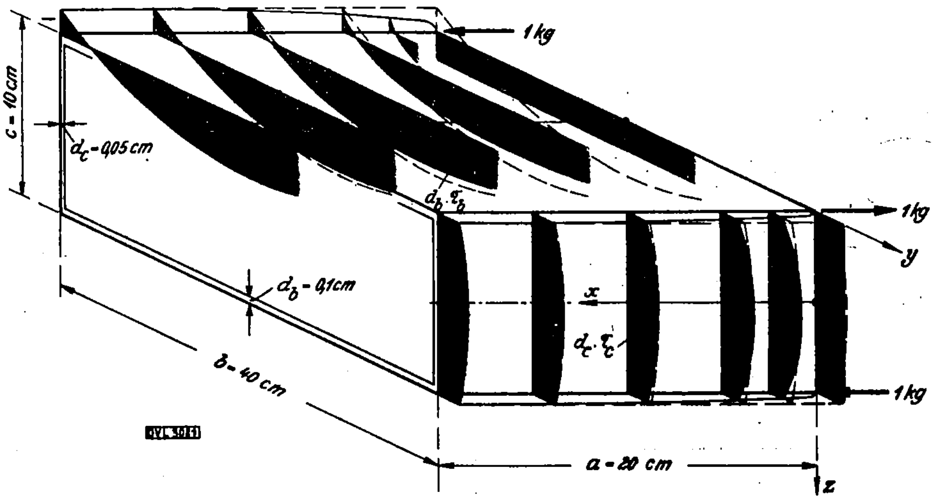


Figure 24.- Shear stresses τ due to X = 1 kg according to rigorous elasticity theory.