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BEHAVIOR OF VORTEX SYSTEMS

By A. Betz

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BEHAVIOR OF VORTEX SYSTEMS*

By A. Betz

Progressive application of the Kutta-Joukowsky theorem to the relationship between airfoil lift and circulation affords a number of formulas concerning the conduct of vortex systems. The application of this line of reasoning to several problems of airfoil theory yields an insight into many hitherto little observed relations.

The report is confined to plane flow, hence all vortex filaments are straight and mutually parallel (perpendicular to the plane of flow).

I. GENERAL THEOREMS

1. Kutta-Joukowsky theorem.- When a body, about which the line integral of the flow is other than zero, i.e., with a circulation Γ, is in motion relative to this fluid with speed v, it is impressed by a force perpendicular to the direction of motion, which per unit length is

$$P = \rho\, v\, \Gamma \qquad (1)$$

(Kutta-Joukowsky theorem, fig. 1). If there is no motion in the fluid other than the circulatory flow, then v is the speed at infinity relative to the fluid. But, if the fluid executes still other motions aside from the circulation, say, when several vortices, or sources and sinks are existent, it is not forthwith clear which is to be considered as the relative speed. On the other hand, we do know that v should be the speed of the body relative to that flow which would prevail in the place of the body in its absence. The body is thereby assumed as infinitely small, otherwise different speeds could prevail at different places of the body. (This case can be worked up by inte-

*"Verhalten von Wirbelsystemen." Z.f.a.M.M., vol. XII, no. 3, June 1932, pp. 164-174.

gration from infinitely small bodies.) This finer distinction of the Kutta-Joukowsky theorem is readily understood when bearing in mind that a free vortex, upon which no force can act, moves at the same speed as the flow in the place of the vortex if the latter were nonexistent. Consequently, the Kutta-Joukowsky theorem must afford the force zero for the motion at this speed, that is, the speed in the Kutta-Joukowsky theorem must be measured relative to this motion. But this may also be shown direct by appropriate derivation of the Kutta-Joukowsky theorem. The general rule for this derivation consists of computing those pressures in a coordinate system, in which the body rests (steady motion), which as result of superposition of circulation and translation act upon a control area enveloping the body and the momentums which enter and leave through it. Thus when we choose as control area a cylinder enveloping the body so closely that the speed of translation in the whole region of the control area can be considered as constant, this selfsame translatory speed contiguous to the body becomes the speed v in the Kutta-Joukowsky theorem, although it is the speed which would prevail at this point if the body were nonexistent.

2. The center of gravity of finite vortex zones.- If there are a number of vortices in a fluid, each individual one is within a flow which as field of all other vortices is determined by their magnitude and arrangement, and each vortex moves with this flow. Visualizing these vortices replaced by individual solid bodies with the same circulation as the vortices (say, rotating cylinders), the flow also is the same. Preventing these bodies from moving with the flow without effecting a change in their circulation, each cylinder is impressed according to Kutta-Joukowsky by a force and we must, in order to hold it, exert an opposite force upon it. For a body with circulation Γ_n, existing at a point with speed v_n, this force is

$$P_n = \rho\, v_n\, \Gamma_n \qquad (2)$$

and is at right angles to v_n. The resultant of the forces exerted on the cylinders must be taken up by the walls at the boundaries of the fluid, i.e., it must be equal to the resultant from the pressures of the fluid onto the boundary wall.*

Now, if the fluid is very much extended so that its

*See footnote, page 3.

assumedly rigid and quiescent boundaries are everywhere far removed from the vortices (or bodies with circulation), the resultant of the pressures onto the boundary walls approaches zero when the vortices are restrained. For the flow velocity v produced by the vortices decreases inversely proportional to the first power of the distance, while the relevant pressure differences (Bernoulli's equation $p - p_0 = - \frac{\rho}{2} v^2$) drop inversely proportional to the square of the distance; the surface of the boundary increases linearly with the distance, so that the force produced as sum of pressure difference and surface is a decrease inversely proportional to the distance.

But when this force on the boundary walls vanishes, the resultant force on our body must disappear also, or in other words,

$$\sum_n P_n = \rho \sum_n v_n \Gamma_n = 0 \tag{3}$$

As the forces P_n or the speeds v_n may assume any direction, Σ is considered a vectorial addition of the forces or speeds, respectively. Instead of that the components in the X and Y direction may be added separately, in which case

*By restraining the vortices the flow becomes steady (provided that there are no singularities other than those vortices, and that the boundary walls are rigid and quiescent). For which reason the pressures can be computed by the simple Bernoulli equation $p + \frac{\rho}{2} v^2 =$ constant. For free vortices the type of flow within a stated time interval is the same as for restrained vortices, but it is usually no longer steady, for the vortices travel, that is, change their arrangement in space. Therefore the pressures change also, because for nonsteady flow the generalized Bernoulli equation $p + \frac{\rho}{2} v^2 + \rho \frac{\partial \Phi}{\partial t} =$ constant is applicable (Φ = flow potential, for steady flow $\frac{\partial \Phi}{\partial t} = 0$). With free vortices there is no force as is in the restrained vortices, so that the resultant force on the boundary walls must disappear. This is precisely obtained by the accelerating forces $\rho \frac{\partial \Phi}{\partial t}$. Consequently, the forces on the fluid boundaries used here and in the following are those forces which would occur if the vortices were restrained, i.e., by steady flow.

$$\sum_n P_{nx} = \rho \sum_n v_{ny} \Gamma_n = 0 \qquad (3a)$$

and

$$\sum_n P_{ny} = \rho \sum_n v_{nx} \Gamma_n = 0 \qquad (3b)$$

with x and y as components along X and Y of the respective vectors. If we release the bodies, whereby they can be replaced again by common vortices, they move at the speed v_n, and our preceding equations constitute a general prediction as to the displacement of the vortices within a fluid without extraneous forces, especially in a fluid extended to infinity. To illustrate: visualize the vortices replaced by mass points (material system) whose mass is proportional to the vortex intensities. Admittedly, we must also include negative masses, in which the vortices with one sense of rotation correspond to positive masses and those with opposite sense of rotation correspond to negative masses. Then we may speak of a center of gravity of a vortex system, while meaning the center of gravity of the corresponding mass system. Applying this interpretation, the vortex motion can be expressed as follows:

Theorem 1.- The motion of vortices in a fluid upon which no extraneous forces can act (fluid extended to infinity), is such that their center of gravity relative to the rigid fluid boundaries or relative to the fluid at rest at infinity remains unchanged. This theorem has already been developed, although in a different way, by Helmholtz, in his well-known work (reference 1). The premise is, of course, the absence of further singular points in the fluid other than the stipulated vortices.

If the fluid is bounded by rigid walls and it is possible to make some prediction as to the resultant force on the boundary walls by restrained vortices (steady flow), then equation (2) gives an account of motion of the center of gravity of the vortex.

If the resultant force on the walls is P, then

$$\rho \Sigma v_n \Gamma_n = P \qquad (4)$$

With v_o = velocity in center of gravity relative to the rigid walls, we have

$$\rho v_o \Sigma \Gamma_n = \rho \Sigma v_n \Gamma_n = P$$

hence,

$$v_0 = \frac{P}{\rho \Sigma \Gamma_n} \tag{5}$$

which is at right angles to force P. Thus,

Theorem 2.- When, by restrained vortices (steady flow), the pressures exerted by the fluid onto the boundary walls produce a resultant force, then the movement of the center of gravity of the free vortices is such as an airfoil whose circulation equals the sum of the circulations of the vortices would need to have in quiescent, infinitely extended fluid to make its lift equal to this resultant force.

As a rule the pressures on the boundary walls and thus their resultant force are not summarily known, although it is possible to make at least certain predictions in many cases. For example, if the fluid is bounded on one side by a flat wall or enclosed between two parallel walls, the resultant force can only be perpendicular to those boundary walls. Since the center of gravity of the vortices moves perpendicularly to this force,

Theorem 3 reads as follows: If there are vortices between two flat, parallel walls or on one side of a flat boundary wall, the distance of the center of gravity of the vortex from these walls remains unchanged (it moves parallel to the walls). This result has already been obtained for a small number of vortices by numerical calculation of the vortex paths.*

3. Inertia moment of finite vortex zones.- Again visualize the vortices as being held fast in a fluid and decompose the speeds on each vortex into a component radially toward or away from the center of gravity and one at right angles thereto. If r is the distance of a vortex with circulation Γ away from the center of gravity, and v_r the radial (outwardly directed) speed component, this vortex is impressed with a force

$$T = \rho \Gamma v_r$$

which is perpendicular to r and therefore forms a moment $T r$ with respect to the center of gravity. The tangential component v_t (perpendicular to r) produces a force along r which does not set up a moment about the center of gravity. The sum of the forces impressed upon the vortices can be divided into a resultant passing through the center

*W. Muller's report before the meeting of the members of the Ges. f. angew. Math. u. Mech., at Göttingen, 1929; and of physicists, at Prague, 1929.

of gravity (radial force component due to v_t) and a moment

$$M = \rho \Sigma \Gamma v_r r.$$

They must be equal and opposite to the forces and moments acting on the fluid boundaries. Releasing the vortices, the center of gravity moves conformably to the laws of the individual force. Moreover, the vortices move also in radial direction at speed v_r. Since

$$v_r = \frac{\partial r}{\partial t}$$

we obtain

$$\rho \Sigma \Gamma r \frac{\partial r}{\partial t} = \frac{\rho}{2} \frac{\partial}{\partial t} \Sigma \Gamma r^2 = M \quad (6)$$

where M = moment of extraneous forces, by restrained vortices, with respect to the center of gravity. If this is zero,* we have

$$\Sigma \Gamma r^2 = \text{constant} \quad (7)$$

$\Sigma \Gamma r^2$ is a quantity which corresponds to the polar mass moment of inertia $\Sigma m r^2$ relative to the center of gravity. Consequently, it may be designated as inertia moment of the vortex system and we obtain

<u>Theorem 4</u>.- When, by restrained vortices, the extraneous forces acting on a fluid have no moment with respect to the center of gravity of the vortex system within this fluid, the inertia moment of this system of vortices remains constant.

If the moment of the extraneous forces is in the same sense as the chosen positive vortex rotation, this inertia moment increases according to equation (6) and vice versa.

<u>4. Vortex systems whose total circulation is zero</u>.- The kinetic energy of a potential vortex in infinitely extended flow in a circular ring between r and dr and thickness layer 1 is

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*Whether or not there is an extraneous moment in a given case requires a more careful analysis than the problem of extraneous forces, since, for example, the forces decrease toward zero with $1/r$ when the boundary surfaces are enlarged, whereas the moments may remain finite because of the added factor r as lever arm.

$$\frac{\rho}{2}\left(\frac{\Gamma}{2r\pi}\right)^2 2\,r\,\pi\,dr = \frac{\rho}{2}\,\frac{\Gamma^2}{2\pi}\,\frac{dr}{r}\;.$$

Integration over the whole fluid (from $r = 0$ to $r = \infty$) yields by approximation to $r = 0$ as well as to $r = \infty$, the energy as ∞. For which reason it is physically impossible to realize such vortices. The difficulty with $r = 0$ is obviated because the physical vortices always have a nucleus of finite diameter, in which the speed no longer rises with $1/r$ toward ∞, but remains finite. But by $r = \infty$ the difficulty remains (apart from the energy the rotary momentum likewise $= \infty$). As a result, the production of vortices in an infinitely extended fluid can only be effected by pairs, so that the sum of the circulations is zero. The velocity field of such a doublet drops at great distancesinversely as the square of the distance so that the fluid energy remains finite for any extension. Hence,

<u>Theorem 5</u>.- The total circulation of all votices in an infinitely extended fluid is zero. No vortex system with finite total circulation can occur unless the fluid is finitely limited. And of course, a part of the vortices in an infinitely extended fluid can also be at such a remote distance as to be of no account for the flow at that particular point. There may then be vortex systems with one-sided total circulation, in which the very vortices which supplement the total circulation to zero are very remote from it. Since, however, energy and momentum of two opposite vortices increase with the distance, very great distances are encountered only in cases of very great energy input. The case of a vortex system with zero total circulation is consequently relatively frequent and deserves special consideration, since the center of gravity of such a system lies, as we know, at infinity, so that the preceding theorems are not summarily applicable in part.

Combining one part of the vortices into one group and the others into another group, we can analyze each group by itself, as, for instance, the clockwise rotating vortices in one, and the anticlockwise vortices in another, although this is not necessary. The only condition is that the total circulation of the one group be equal and opposite to that of the other group and other than zero.

In the absence of forces and moments on the fluid,* as, say, by infinitely extended fluid, the forces and moments on the restrained vortices must be zero or, in other words, the resultant force on one group must be equal and opposite to the force on the other and be on the same line with it. But these forces need not necessarily pass through the center of gravity of each of the two groups. When the vortices are released the center of gravity of each of the two groups moves perpendicularly to this one-sided force and at the same speed. This is expressed in

Theorem 6, as follows: The motion of the centers of gravity of two groups of vortices with equal and opposite total circulation is mutually parallel and has the same speed, hence constant distance.

Knowing at first absolutely nothing about the direction of the opposite force, we can make no prediction as to the direction of motion. When this opposite force passes through the center of gravity of a group, this group is without extraneous moments and its inertia moment is then constant (theorem 4).** As a rule this force does not exactly go through the center of gravity of the two groups. But when they are separate to a certain extent and closed in themselves, the force almost always passes very close to the center of gravity, in which case we can then consider the inertia moments at least approximately as constant.

If the force does not pass through the centers of gravity of the groups, their inertia moment changes. But if the force is parallel to the line connecting the two

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*In such vortex systems the moments also are forthwith small when pushing beyond the rigid boundary walls. (Compare footnote** below.
**It was always assumed that no singularity other than the vortex system existed. But with the two groups and each considered by itself, the assumption ceases to hold. However, the previous considerations can be generalized so that the forces needed to restrain the vortices of the momentarily disregarded group, become the extraneous forces on the fluid. It is readily seen that theorem 4 is equally applicable in this sense to a group of vortices in the presence of further vortices.

centers of pressure (S_1, S_2, fig. 2), which is manifested by their perpendicular motion to the connecting line, the moment of the force relative to the two centers of gravity is equal and opposite. The result is that the inertia moment of one group increases at the same rate as that of the other group decreases. (One inertia moment is usually positive, the other negative; their absolute values thus increase or decrease to the same extent.)

<u>Theorem 7</u>.- If the motion of the centers of gravity of two groups of vortices of equal and opposite total circulation within an infinitely extended quiescent fluid is perpendicular to the line connecting the gravity centers, the algebraic sum of the inertia moments of the groups remains unchanged.

When the force forms an angle with this connecting line (S_1 S_2, fig. 3), that is, when one velocity component v_x is along its connecting line, the inertia moment of one group increases more than that of the other decreases or vice versa. In any case, the sum of the inertia moments of the two groups is changed. It amounts, in fact, to

$$\frac{\partial}{\partial t} (\Sigma\, r_1^2\, \Gamma_1 + \Sigma\, r_2^2\, \Gamma_2) = 2\, v_x\, a\, \Sigma\, \Gamma \tag{8}$$

according to equation (6) and figure 3. ($\Sigma\, r_1^2\, \Gamma_1$ is the inertia moment of one group, $\Sigma\, r_2^2\, \Gamma_2$ that of the other, a is the distance of the two centers of gravity, and $\Sigma\, \Gamma$ the total circulation of one group.)

The sum of the inertia moments increases when the motion of the centers of gravity in direction of the group with positive circulation is toward the group with negative circulation. It decreases for opposite direction. Thus $\Sigma\Gamma$ in equation (8) denotes the total circulation of that group, which moves toward the other.

II. APPLICATION

In the practical application of these theorems, it frequently is not so much a case of a number of individual vortices, but rather of continuously distributed vortices. But that presents no difficulty; it merely means substituting $\int$- terms for Σ terms. It is, however, something else when the vortex systems extend to infinity and at the same time have infinitely large circulation. But with some care, they also are amenable to solution by these theorems.

1. Vortices back of an airplane wing.- According to airfoil theory (see Handb. d. Phys., vol. VII, p. 239 ff), an area of discontinuity is formed behind a wing by optimum lift distribution (minimum by given lift), which has the same speed of downwash at every point. Thus the flow behind an airfoil may be visualized as if a rigid plate, the area of discontinuity, were downwardly displaced at constant speed and thereby sets the fluid in motion (fig. 4). This, however, is applicable only in first approximation when the interference velocities (foremost of which is the speed of displacement w) are small compared to the flight speed. For this motion would only be possible for any length of time if the area of discontinuity actually were rigid. By flowing around the edges, laterally directed suction forces P occur, which only could be taken up by a rigid plate. These forces are absent when the area of discontinuity is other than rigid, as a result of which the suction P effects other motions; starting at the edges, it unrolls and gradually forms two distinct vortices (fig. 5).

With l = wing span, the circulation per unit length of $\frac{d\Gamma}{dx}$ for such an area of discontinuity is distributed across the span conformably to the following equation

$$\frac{d\Gamma}{dx} = \Gamma_0 \left(\frac{2}{l}\right)^2 \frac{x}{\sqrt{1 - \left(2\,\frac{x}{l}\right)^2}} \tag{9}$$

with Γ_0 = circulation about the wing in its median plane. The downward velocity of the area of discontinuity prior to development is

$$w = \frac{\Gamma_0}{l}. \tag{10}$$

The area of discontinuity may be regarded as a continuously distributed system of vortices with zero total circulation. The distribution of the vortices is given in equation (9). Combining the two symmetrical halves ($-\frac{l}{2} \leqq x \leqq 0$ and $0 \leqq x \leqq \frac{l}{2}$) into one group each, the distance of the center of gravity of the two groups must remain the same, according to theorem 6. The center of gravity of a system of vortices conformable to equation (9) from 0 to $l/2$ lies, as is readily computable, at a distance

$$x_0 = \frac{\pi}{4} \frac{l}{2} \tag{11}$$

from the center, so that the distance of the centers of gravity of the two groups becomes

$$a = 2 x_0 = \frac{\pi}{4} l \tag{12}$$

This,. then, is accordingly also the distance of the centers of gravity of the two formative individual vortices (fig. 5). The steady symmetry of the process in the present case is indicative of the consistently parallel displacement of the centers of gravity and consequently, that the individual vortices are also symmetrical to the original plane of symmetry.

The process of convolution or development with respect to time can also be followed by similar considerations, although this calls for considerable mathematical work. Up to the present the course of the process has been explored very accurately in its first stages, during which the developed part was still small compared to the whole area of discontinuity (reference 3).

In the present report an attempt is made to gain approximate information regarding the magnitude of the tip vortices and the circulatory distribution within them. The vortices of the area of discontinuity are divided at some point x and those lying to the left are grouped into one; those to the right of it (full line in fig. 6) into another. Then it is assumed that the opposite forces on the two groups - the vortices being restrained - pass through the center of gravity of both groups, which actually proves fairly correct, because of the comparatively strong concentration of the vortices toward the tips and the ensuing distinct separation of both groups. Now the inertia moment of one vortex group must remain approximately constant during development. The total circulation of one group of the undeveloped area of discontinuity from x to $l/2$ is

$$\Gamma_x = \int_x^{l/2} \frac{\partial\Gamma}{\partial x} dx = \Gamma_0 \sqrt{1 - \left(\frac{2x}{l}\right)^2} \tag{13}$$

For the ensuing calculation the angle φ is used in place of the variable x, which is bound up with x through

$$\cos\varphi = \frac{2x}{l} \quad \text{and} \quad \sin\varphi = \sqrt{1 - \left(\frac{2x}{l}\right)^2} \tag{14}$$

Thus the vortex distribution (equation 9) becomes:

$$\frac{\partial\Gamma}{\partial x} = \Gamma_0 \left(\frac{2}{l}\right)^2 \frac{x}{\sqrt{1 - \left(\frac{2x}{l}\right)^2}} = \Gamma_0 \frac{2}{l} \cot\varphi \tag{9a}$$

$$\Gamma_x = \Gamma_0 \sqrt{1 - \left(\frac{2x}{l}\right)^2} = \Gamma_0 \sin\varphi \tag{13a}$$

The distance of the center of gravity of this group is:

$$x_1 = \frac{1}{\Gamma_x} \int_x^{l/2} \frac{\partial\Gamma}{\partial x} x dx = \frac{1}{\sin\varphi} \frac{l}{2} \int_0^{\varphi} \cos^2\varphi d\varphi = \frac{l}{4 \sin\varphi} [\varphi + \tfrac{1}{2}\sin 2\varphi] \tag{15}$$

The inertia moment of the group with respect to the center of the area of discontinuity $(x = 0)$ is:

$$J_0 = \int_x^{l/2} \frac{\partial\Gamma}{\partial x} x^2 dx = \Gamma_0 \left(\frac{l}{2}\right)^2 \int_0^{\varphi} \cos^3\varphi\, d\varphi = \Gamma_0 \left(\frac{l}{2}\right)^2 \sin\varphi\left(1 - \frac{1}{3}\sin^2\varphi\right) \tag{16}$$

The inertia moment of the group with respect to its center of gravity is:

$$J_x = J_0 - \Gamma_x x_1^2 = \Gamma_0 \left\{\left(\frac{l}{2}\right)^2 \sin\varphi\left(1 - \frac{1}{3}\sin^2\varphi\right) - \left(\frac{l}{4}\right)^2 \frac{1}{\sin\varphi} [\varphi + \tfrac{1}{2}\sin 2\varphi]^2\right\} \tag{17}$$

This inertia moment must be present again after the convolution.

Now the coiled-up group is assumed to be circular; that is, the asymmetry stipulated by the mutual interference of the two coiled-up vortices is disregarded, so that the circulation may be presented as a pure function of radius ($\Gamma = f(r)$). The vortex group from x to $l/2$ is coiled up into a spiral which fills the circle with radius r. Then the circulation Γ_r must be equal to the circulation of the original vortex group

$$\Gamma_r = \Gamma_x \tag{18}$$

and likewise, the inertia moment of the vortices coiled up in this circle must be equal to the original inertia moment of the vortex group

$$J_r = \int_0^r \frac{\partial \Gamma}{\partial r} r^2 \, dr = J_x. \tag{19}$$

Permitting r to increase by dr, then decreases x by dx and increases φ by $d\varphi$ under these premises. The result is an increase of

$$\frac{\partial \Gamma_r}{\partial r} dr = \frac{\partial \Gamma_x}{\partial \varphi} d\varphi = \Gamma_o \cos \varphi \, d\varphi \tag{20}$$

in circulation, and of

$$\frac{\partial \Gamma_r}{\partial r} r^2 \, dr = \frac{\partial J_x}{\partial \varphi} d\varphi \tag{21}$$

in inertia moment.

Then the differentiation of (17) yields

$$\frac{\partial J_x}{\partial \varphi} = \Gamma_o \left(\frac{l}{2}\right)^2 \left[\cos^3 \varphi + \frac{\cos\varphi}{4 \sin^2 \varphi} \left(\varphi + \frac{1}{2} \sin 2\varphi\right)^2 - \frac{1}{2 \sin \varphi} \left(\varphi + \frac{1}{2} \sin 2\varphi\right) (1 + \cos 2\varphi)\right]$$

which, written into (21) and with regard to (20) gives

$$\left(\frac{2r}{l}\right)^2 = \cos^2\varphi + \frac{1}{4\sin^2\varphi}\left(\varphi + \frac{1}{2}\sin 2\varphi\right)^2 - \frac{\cos\varphi}{\sin\varphi}\left(\varphi + \frac{1}{2}\sin 2\varphi\right) \tag{22}$$

Since $\sin\varphi = \sqrt{1 - \left(\frac{2x}{l}\right)^2}$, (equation 22) connotes the relationship between r and x, that is, it gives the size of the circle into which a piece of the original area of discontinuity has changed. And, knowing the circulation $\Gamma = \Gamma_0 \sin\varphi$, the equation also discloses the distribution of the circulation in the coiled-up tip vortex. Figure 7 shows the respective values of Γ and x versus r, and also the distribution of the vortex density $\frac{d\Gamma}{d(r^2\pi)}\frac{l^2}{20\,\Gamma_0}$. When forming the pertinent boundary transitions, equation (22) yields

$$\left(\frac{2r}{l}\right)^2 = \frac{1}{9}\sin^4\varphi \tag{23}$$

for very small values of φ, so that

$$\frac{r}{\frac{l}{2} - x} = \frac{2}{3} \tag{24}$$

In other words, a small boundary piece of the area of discontinuity coils up into a circle, whose radius is 2/3 of the length of the original piece.

For $\varphi = \frac{\pi}{2}$ we have $\frac{2r}{l} = \frac{\pi}{4}$, which means that the radius of the tip vortices is $\frac{\pi}{4}\frac{l}{2}$. Since the center of the tip vortices is $\frac{\pi}{4}\frac{l}{2}$ distant from the plane of symmetry, it would indicate that the two tip vortices precisely touch each other. But for such close proximity, our assumption that the individual tip vortices shall be symmetrical circles, ceases to hold: the speed between the two vortices is substantially greater than it is outside, with the result that the individual streamlines are outwardly displaced. So in reality the vortices should not touch each other. The established approximate result however, may, because of its simplicity, give a ready picture of the order of magnitude of the vortices. According to figure 7, the relationship between r and x is fairly linear. Hence $r = \frac{2}{3}\left(\frac{l}{2} - x\right)$ in the greater part of the vortex conformable to (24), and it is only in the outer edge of the vortex that the factor 2/3 changes to $\pi/4$. The curve for the distribution of the vortex density shows the main part of the vortices to be very much concentrated around the center despite their comparatively great extent.

2. Phenomena behind cascades of airfoils.- Cascades of airfoils also form areas of discontinuity aft of the airfoils (fig. 8), and whose motion relative to the undisturbed flow would, by optimum lift distribution, be as for rigid surfaces, if the edges could absorb the suction. But in reality they develop with respect to time. (See Handb. d. Phys., vol. VII, p. 272 ff.)

Let us analyze the practically always existing case wherein the distance a' of the surfaces is small compared to their span. Assuming the areas of discontinuity to be actually rigid, the flow around the rigid surfaces far behind the airfoils would, near the edge, be as shown in figure 9, when choosing a system of coordinates within which these surfaces rest. The motion in this system of coordinates being steady, Bernoulli's equation can be employed. Inasmuch as the interference velocity between the surfaces far removed from the edge is evanescently small relative to the surfaces, whereas outside in the undisturbed flow the relative velocity is equal and opposite to the velocity of displacement w, Bernoulli's equation yields

$$p = \rho \frac{w^2}{2} \tag{25}$$

positive pressure between the surfaces with respect to the pressure in the undisturbed flow on the side of the surfaces.* This positive pressure balances the suction at

*Directly behind the cascades the pressures and velocities are different. By contraction or expansion of the lateral edges of the areas of discontinuity (positive or negative contraction) equilibrium is, however, established with the pressure of the lateral undisturbed flow, resulting in a correspondingly different speed. (See Handb. d. Phys., vol. VII, p. 259 ff.) Here and in the following the conditions subsequent to this balance are considered only. For many purposes it should be noted that owing to the width changes of the hypothetical rigid area of discontinuity the suction at the edges has a component along the direction of flow.

the plate edges. For an analysis of the horizontal forces acting upon a fluid strip of the height of the surface spacing a', reveals on one side a force $a' \rho \frac{w^2}{2}$, as result of the pressure difference within and without, and on the other the suction at one plate edge. No momentums are transmitted by the boundary surfaces, therefore the suction must be

$$P = a' \rho \frac{w^2}{2} \tag{26}$$

In such a system of surfaces the vortices are very much concentrated at the boundaries, at great distance from the edge, that is, in the entire middle part of the surfaces the relative velocities are practically zero and with it, of course, the velocity differences on both sides of the surfaces, i.e., the vortices. As a result, the effects of the developed and of the undeveloped areas of discontinuity are equal at distances which are great compared to spacing a', since the spatial transformation of the vortices during development is subordinate as against the great distance. Nevertheless, there is a fundamental difference as far as the flow is concerned between the theoretical process with undeveloped rigid surfaces and the actual process with developed individual vortices, a fact which up to now has never been pointed out, to my knowledge.

The vortex group at one side is in the velocity field of the vortices of the other. Owing to its remoteness, this field does not change appreciably during the development. Thus assuming the vortices as restrained before and after development, the mutual force exerted by the vortices, remains the same, and with it the velocity in the center of gravity of the developed and the undeveloped vortices. But when visualizing the areas of discontinuity as rigid, they are then no longer exempt from forces because of the suction P, and in that case the velocity is greater by an amount

$$\Delta w = \frac{P}{\rho \Gamma} \tag{27}$$

than with the free vortices. Γ is herein the circulation about the part of the area of discontinuity lying on one side of the plane of symmetry, respectively, about the single vortex developed therefrom (for the rest equal to the circulation about the airfoil in its median part). Following the line integral in figure 9, it is readily seen that

$$\Gamma = a' w \tag{28}$$

hence, with due regard to (26):

$$\Delta w = \frac{w}{2} \tag{29}$$

Then the velocity of the free vortices is:

$$w' = w - \Delta w = \frac{w}{2} \tag{30}$$

As a matter of fact, the process of development is such that the center of gravity of the vortices clustered around each edge, lags behind the velocity of the central main part of the surfaces. Whereas the latter moves at velocity w, the center of gravity of the vortices moves at a speed $w/2$ and it maintains this speed in the final attitude after development.

However, this speed $w/2$ can also be deduced direct from the field of the opposite vortices. At great distance it is identical with that of a vertical row of concentric vortices (fig. 10). But at medium distance the field of such a vortex row is a constant speed $\pm w' = \pm \frac{\Gamma}{2a'}$ downward on one side and upward on the other. Between the two rows the fields of the two rows add up to speed w, so that

$$w = 2 w' \tag{31}$$

The signs for the fields outside of the rows are contrary, hence the speed is zero. Each vortex row itself moves under the effect of the momentary other row, that is, its speed is

$$w' = \frac{w}{2} \tag{32}$$

But there is yet another result which is not as readily conceived as the change in vortex velocity. For the rigid surfaces we had within the deflected flow a positive pressure $q = \rho \frac{w^2}{2}$ which balanced the suction at the edges. After development the suction is absent, so that there is also no more positive pressure within between the vortex rows, as can be proved from the Bernoulli equation. In the chosen system of coordinates of figure 10, w is the speed of the inside flow, 0 that of the outside flow, and $w/2$ that of the vortices. To insure steady conditions, we must select a coordinate system in which the vortices rest. Then the speed of the inside flow is $w/2$ and that of the outside flow $-w/2$ (fig. 11). Both are of equal absolute magnitude, hence of equal pressure within and without the vortex rows.

Now this change of pressure during development is not without influence on the flow inside. Analyzing a cut through the airfoil cascades (figs. 8 and 12) while applying Bernoulli's equation to the speeds in front of and behind the cascades, reveals by pressure balance (developed vortices, fig. 12),

$$c_3 = c_1 \tag{33}$$

and by $p = \rho \frac{w^2}{2}$ (undeveloped vortices)

$$c_2{}^2 + w^2 = c_1{}^2 \tag{34}$$

Therefore the speed is greater after development than before $(c_3 > c_2)$.

This result, while at first sight perhaps somewhat peculiar, can also be elucidated in a different fashion. Looking at the cascades from the side, once with undeveloped vortex surfaces (fig. 8), and then with developed vortices (fig. 12), the direction of the detached vortices is manifestly different because their own speed relative to the undisturbed flow is different (w and $w/2$). The interference velocity w, which may be considered as vortex field, is perpendicular to the vortices, and has therefore a somewhat different position in both cases. For nondeveloped vortices, the vortices lie in the direction of c_2, w is perpendicular to c_2 (fig. 8), and since c_2 is composed of undisturbed velocity c_1 and interference velocity w, we have

$$c_2{}^2 = c_1{}^2 - w^2.$$

In the developed state the vortices move with natural speed $w/2$, that is, they are between c_1 and c_3. Since w and therefore $w/2$ in turn are perpendicular to the vortices, the velocity vectors c_1, c_3, and w form a triangle in which the vortex line is the median line (fig. 12). But this implies that $c_3 = c_1$.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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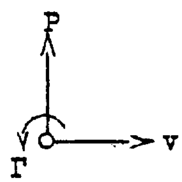

Figure 1.-Kutta-Joukowsky theorem.

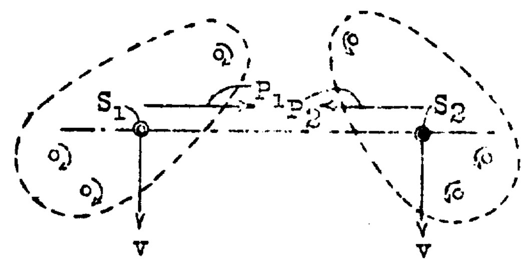

Figure 2.-Two groups of vortices whose centers of gravity move perpendicular to their connecting line.

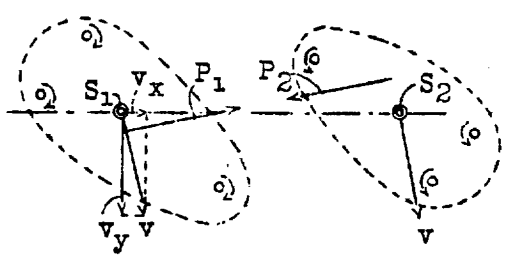

Figure 3.-Two groups of vortices whose centers of gravity move obliquely to their connecting line.

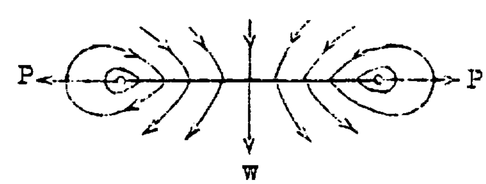

Figure 4.-Displacement of a rigid plate in a fluid.

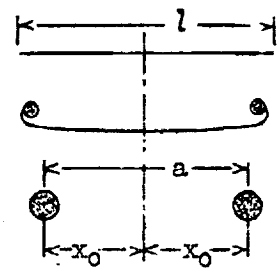

Figure 5.-Development of area of discontinuity behind an airplane wing.

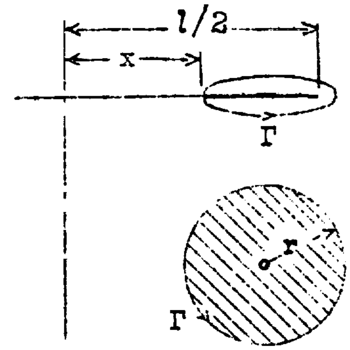

Figure 6.-Part of area of discontinuity and circle over which it is distributed after development.

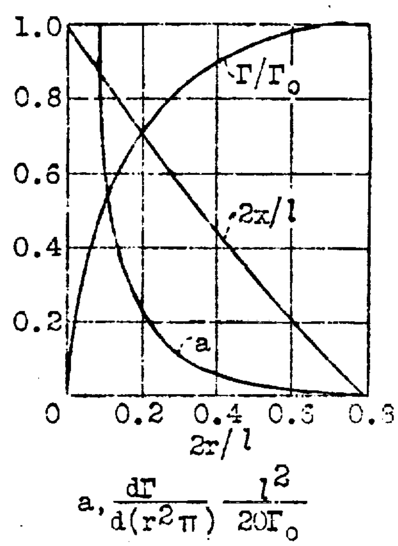

Figure 7.-Relationship between developed and non-developed area of discontinuity.

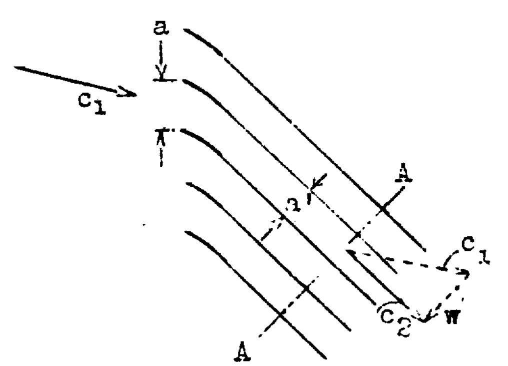

Figure 8.-Flow past cascades of airfoils with hypothetical rigid areas of discontinuity.

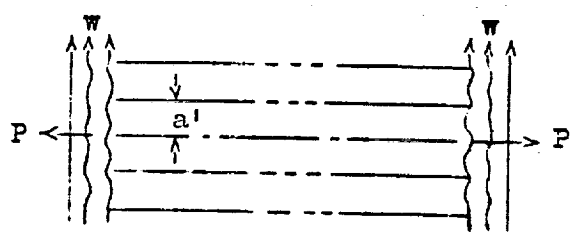

Figure 9.-Flow about the theoretical non-developed areas of discontinuity in cut A-A.

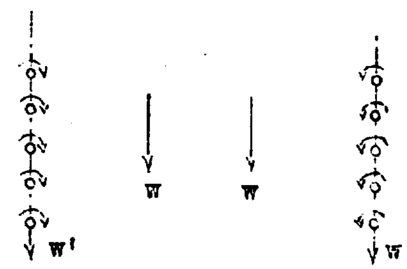

Figure 10.-Velocities relative to undisturbed flow.

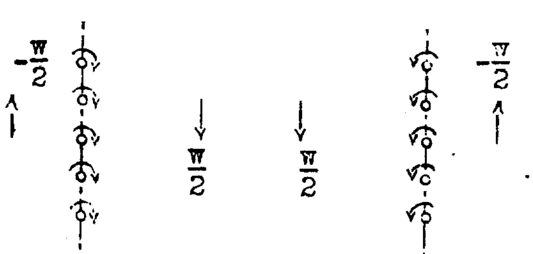

Figure 11.-Velocities relative to vortices after development (steady flow).

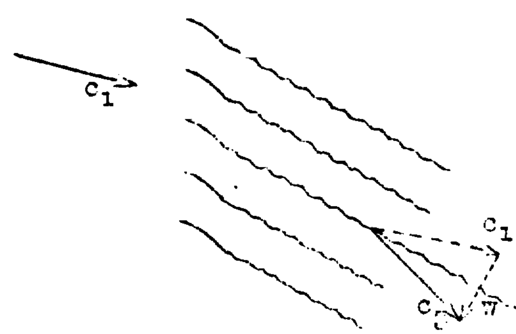

Figure 12.-Composition of velocity after development.