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ACCURATE CALCULATION OF MULTISPAR CANTILEVER AND

SEMICANTILEVER WINGS WITH PARALLEL WEBS UNDER DIRECT

AND INDIRECT LOADING

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Minimum structural weight of a wing is ensured when
within bounds of the selected support system, the struc-
tural material having the highest efficiency (σ_b/γ for
tension, E/γ for buckling) is so disposed that its cross-
sectional areas are everywhere utilized up to the strength
limit. In particular, uneconomic bending stresses (because
of the only locally restricted maximum stress of the mate-
rial) should be avoided wherever possible. This stipula-
tion of resolving all stresses into pure tension and com-
pression intimates the utility of truss beams, and thus
explains their ever-increasing use in the design of the
modern large airplanes. The outer shape of a cantilever
wing and with it the height of the beam is determined by
the aerodynamically defined airfoil. The problem then is
to decide on the type of construction, a matter which de-
pends on the nature of the externally imposed loads.
Their resultant can be readily resolved into components
parallel and normal to the wing chord. The parallel com-
ponent is invariably, as a glance at Figure 1 shows, a
small fraction of the normal component and, in addition,
encounters the wing chord as beam height; hence, will not
become decisive for beam shape or calculation. In direct
contrast, the normal component is predominant and, because
of the small beam height, decisive; aside from that the
points of application of their resultants are extremely
variable. (See fig. 1.) In view of this variability,
the lightest weight is contingent upon the whole struc-
ture helping to carry the load, i.e., a very high combined
effect between all supporting parts. By combined effect
we imply, for example, the subsequently examined multispar

wing, in which a spar not only carries this load but at the
same time is helped by adjacent spars through the agency
of the ribs. Combined effect is impossible in a two-spar
wing unless the bond between rib and spar affords the
transmission of a torsion moment to the spar and the latter
itself is resistant to torsion. A two-spar lattice wing
has no combined effect. In three-spar or multispar wings
torsional stiffness of the individual spar is no longer
necessary; resistance to bending suffices.

In our case, i.e., multispar cantilever wing, the par-
allel webs are interconnected by transverse ribs.

Consistently with our preceding stipulation, we now
visualize spars and ribs as truss beam or grillage. The
normal loads are taken up and transmitted by the normal
lift-truss of the spar. The equally normal rib-truss to-
gether with the spar lift-truss form a beam grillage and
effect the requisite bond. The individual lift-trusses
of this grillage are not rigid in torsion nor is the gril-
lage stable against loads acting in its plane. To take
up the small parallel loads the shear strength of a corru-
gated metal skin is used exclusively, which also takes up
the air loads and their transmission to the lift-truss
flanges of the spar in form of an equivalent load. The
unloaded transverse structure (ribs) merely serves to pro-
duce the combined effect and prevents overstress in the
metal skin. These ribs make the complete structure stat-
ically indeterminate. Such designs should ordinarily be
avoided for wing structures because the statically inde-
terminate structure is usually heavier than the equivalent
statically determinate structure, due to the incomplete
usability of the sections of the members. But this par-
ticular case is an exception. The great variability of
the location of the applied load produces, in view of the
combined effect through the statically indeterminate lat-
tices, such a far-reaching unloading of the respective
member under load as to make the complete statically in-
determinate structure lighter in spite of it. This com-
bined effect, moreover, becomes vitally important when,
due to failure of one part, the remaining members must
take over the load of the other.

For our purposes we select the spar spacing at 1/10
wing chord, and specifically, we arrange seven spar lift-
trusses.

Rib spacing and thickness are defined in keeping with the fact that the combined effect is considerable even with few small ribs and shows but little increase when increased in number or size. Maximum combined effect even with comparatively small ribs is then obtained when the latter are in the vicinity of maximum spar deflection, i.e., at the wing tips.

In accordance with calculation, we presume equal stiffness in bending in all six ribs. (See fig. 2.)

After thus explaining the most essential points for obtaining the requisite "lightest" wing, there remains the statical anlysis of the complete structure, so as to be able to decide upon the dimensions of the individual parts. This is done by an unusual method, the principal advantages of which are:

1. Freedom of static examination from special load case by introduction of influence lines, consequently one investigation covering all load cases (pull-up, glide, steep dive, inverted flight, pull-up in inverted flight, etc.).

2. Simple inclusion of direct loads in the static analysis.

3. Provision for actual support conditions on fuselage and eventual bracing.

4. Regard as to limit of rib stiffness.

5. Calculability of covering effect.

6. Examination of each spar inertia moment occurrence.

7. Utmost possibility of approach to actual load attitude.

8. Special qualification for calculating extremely large and "all-wing" airplanes.

In conclusion, it may be observed that the study is equally applicable, with suitable modifications, of course, to wings with varying chord, and that the consideration of a load variable along the spars (tapering at the tip, for example), presents no obstacles.

Here also it is essential to differentiate between external loads parallel and normal to the plane of the grillage. The parallel loads are balanced by the shear strength of the skin; the normal loads are treated as follows (see fig. 2): With the stipulated four-point support the complete spars and two ribs, let us say, $x = 6$, and $x = 7$, suffice to maintain stability, with the result that 12 ribs may be altogether removed. Since each rib as beam on seven supports is five times statically indeterminate, the total number of the statically indeterminates is $12 \times 5 = 60$. Substituting the word "rib" for "spar," and vice versa, yields the same final result.

General rule: A grillage, extraneously supported by four conditions (limitations of displacement and distortion) with members unstable in torsion is externally statically determinate and internally as many times statically indeterminate as it has internal joints (from which four members emanate). Every external additional condition stipulates an external static indeterminateness. Albeit this total number of statically indeterminate quantities is now reduced to 30, because of the symmetry of the loads, an exact treatment by conventional methods of statics is still out of the question, so recourse is had to the method of differential equations as broached by Professor Ernst Melan, Vienna. Instead of the truss beams we substitute solid beams with inertia moment $J = F \cdot n^2 /2$. In addition, we discard the hitherto customary, but very arbitrary assumption of cantilever wing rigidly restrained at the fuselage. It is felt that this "free body" diagram, as illustrated in Figure 2, is more nearly correct. It is the same as considering the two semi-wings, connected directly or by means of a central frame, as static unit on which the load, now symmetrical to the airplane axis, is imposed and wherein the load encounters its opposing forces in the attachment forces of a fuselage or in direct wing loads as encountered when approaching the "all-wing" type of airplane, where engines, tanks, cabins, controls, etc., are mounted on the wing. Since these are conditions very largely dependent upon the respective type of construction, we confine ourselves to representing the method with a statically determinate arrangement of the opposing forces, as indicated in Figure 2, and explaining any occurring changes under other support conditions during the ensuing calculation. That this method is particularly suited for the calculation of giant airplane types, such as the Junkers, Rumpler, and Dornier is obvious. An-

other feature from the structural point of view is that the perfectly free truss is unstable against asymmetrical normal loads, such as are imposed by aileron effect, in spite of the covering, to the extent that, due to the lack of any torsional stiffness of the grillage members, the entire grillage can, without resistance, deform into a hyperboloidal area, because all grillage members remain straight. This instability can be remedied by restraining at least four joints of the grillage, say, four fuselage attachment points relative to the grillage or else by resorting to spatial stiffening. In the numerical execution, we confine the accuracy to that obtained by slide rule.

To comprehend the static behavior of the complete wing structure and in particular of the combined effect between the individual spar walls, as well as for providing the requisite bases of the proportions, we rely on the influence lines. Our influence lines, somewhat at variance with their orthodox definition, have as ordinates the influence of an equivalent load of 1 per unit length of spar, extending over one whole spar (from $x = 0$ to $x = 13$ in our example), when this load constant by itself, travels from spar to spar in rib direction — no direct rib loading being presumed herein — to the requisite magnitude (spar moment, rib moment, displacement of joint) at the respective point. Altogether there are seven different load positions. When one spar $y = $ constant sets up a uniform load along its whole length, all other spars are unloaded. Hence the computation of all requisite quantities for each of these seven load positions makes it possible to plot each influence line of these quantities. By virtue of the symmetry of the wing structure to the center spar, it is necessary to investigate only four instead of seven attitudes. To assure inclusion of direct wing loads the procedure is as follows: arbitrary, direct wing loads are always known and constant in magnitude and place, hence are counted among the other known external loads which merely have to comply with the general conditions of external loads on the free grillage. In order to be able to retain our method of defining the influence lines we select one of the "unit loads" of the whole wing (all spars loaded with 1 per meter length), bearing in mind the relation between actual weight of the direct load (engines, cabin, fuel tanks, etc.) and indirect actual wing loading, a corresponding part of the direct loading apply, during loading of one of the m spars, the mth part of the direct loading at the requisite

points, define the reactions necessary hereto (occurring
either as body attachment forces or as direct control
forces on the "all-wing" airplane), after which we can
proceed as in the ordinary case.

To illustrate: Let an average, equable spar loading
of 150 kg/m² be synonymous to the mean actual wing load-
ing; an engine of 800 kg actual weight is assumedly mount-
ed on the top of the wing in form of a single load. Our
influence line calculation was carried out with a stipu-
lated equivalent spar load of 1000 kg/m². Consequently,
we must, in order to maintain similar loading conditions,
determine a corresponding part of the direct loading
$(800 \times \frac{1000}{150} = 5330 \text{ kg})$ in view of the relationship be-
tween actual weight of direct loading (800 kg) and mean
surface loading (respectively, of the equivalent spar load
150 kg/m² dependent thereon). So when loading the wing
with the originally assumed equivalent load (1000 kg/m²)
over all spars and using up the computed direct loading
proportion (5330 kg), this system of forces must be brought
into equilibrium by normal forces applied at those points
"where, by nature, fuselage attachment forces or control
forces must occur." Our method of defining the influence
lines stipulates loading of only one of the m spars at
one time, so that obviously only the mth part of the
direct load portion must be applied, because all m spars
together are to correspond to the complete portion again.
After defining the reactions for this loading attitude,
the calculation can be carried through for this case.
The method is applicable to any airplane attitude, even
for case C where, as is known, no component of the direct
loading must be incurred in the normal direction of the
wing.

Before beginning the actual calculation, the inertia
moment of spars and ribs and some notations must be de-
fined. The spars are assumed as truss-cantilever of equal
resistance in bending, tapering linearly in height from
the root toward the wing tip. Counting x beginning at
the wing tip and stipulating an equivalent load q per
meter of length, we have

$$M_x = q \, x^2/2; \quad \sigma = M_x/W_x; \quad W_x = M_x/\sigma = q \, x^2/2 \, \sigma$$

and therefrom,
$$J = W \, h/2 = q \, h/4 \, \sigma \, x^2;$$

which, with $h = \alpha x$ reveals moment of inertia

$$J_x = q\, \alpha/4\sigma\, x^3 = k\, x^3,$$

wherein k is the spar inertia moment at the first joint
$x = 1$. Besides, we stipulate equal stiffness in bending
in all spars, and equal and constant inertia moment of
magnitude k in all ribs, so that $J_y = k$.

Notation: spar spacing $h = 1$; rib spacing $l = 4$.

$$\varphi_x = \frac{l}{E\,J_x} = \frac{4}{E\,k\,x^3}; \qquad \psi_y = \frac{h}{E\,J_y} = \frac{1}{Ek};$$

$$\varphi_1 = \varphi_{13} = \frac{4.000}{Ek}; \qquad \varphi_2 = \varphi_{12} = \frac{0.500}{Ek};$$

$$\varphi_3 = \varphi_{11} = \frac{0.148}{Ek}; \qquad \varphi_4 = \varphi_{10} = \frac{0.063}{Ek};$$

$$\varphi_5 = \varphi_9 = \frac{0.032}{Ek}; \qquad \varphi_6 = \varphi_8 = \frac{0.019}{Ek};$$

$$\varphi_7 = \frac{0.019}{Ek}.$$

As cantilever beam of equal resistance in bending, the
spar demands, consistently with the foregoing, an inertia
moment conformably to a cubic parabola. A truss beam with
constant flange profile and linearly changing height would
yield a square parabola. For that reason the flange pro-
files must be graded. These conditions, which are alto-
gether subject to the particular case in point, are ac-
counted for here by using in each spar panel the maximum
instead of the mean inertia moment for defining φ . For
designating φ_x the number of the joint following the
panel was used.

φ is independent of y , ψ is independent of x
and y . The displacement at joints x , y normal to plane
of grillage is z_{xy} .

The moments at the joints in direction x : $L_{x-1,y}$;
L_{xy} ; $L_{x+1,y}$

The moments at the joints in direction y : $M_{x,y-1}$;
$M_{x,y}$; $M_{x,y+1}$.

The support pressure of the freely borne beam at joints x,y set up by external loads, is $T_{xy} = 4$. The support pressure of the moment loading of the freely supported beam at joint x,y induced by this member is $P_{xy} = 8/3$. (Only the spars $y = $ constant are loaded, that is, uniformly with unit loading 1 per unit length, hence $P_{xy} = \frac{1}{2} \times \frac{2}{3} f l = \frac{1}{2} \times \frac{2}{3} \times 2 \times 4 = \frac{8}{3}$, because $l = 4$, $f = \frac{q l^2}{8} = 2$.) Force and displacement are counted positive in the same direction. The bending moments are positive when the center of curvature of the deformed member lies on the side of negative z_{xy}.

The development of the differential equations begins with Clapeyron's three-moment equation:

$$\varphi_x L_{x-1} + 2(\varphi_x + \varphi_{x+1}) L_x + \varphi_{x+1} L_{x+1} +$$

$$+ 6 \left(\frac{z_{x+1} - z_x}{l_{x+1}} - \frac{z_x - z_{x-1}}{l_x} \right) = - 6 \left(\frac{P_x \varphi_x}{l_x} + \frac{Q_x \varphi_{x+1}}{l_{x+1}} \right)$$

modified for our purposes to:

$$\varphi_x L_{x-1} + 2(\varphi_x + \varphi_{x+1}) L_x + \varphi_{x+1} L_{x+1} +$$

$$+ 6/l \ (z_{x+1} - 2 z_x + z_{x-1}) = - 6 P/l \ (\varphi_x + \varphi_{x+1}).$$

For abbreviation, we write:

$$6 P_{xy}/l \ (\varphi_x + \varphi_{x+1}) = U_{xy}$$

$$\varphi_x L_{x-1} + 2(\varphi_x + \varphi_{x+1} (L_x + \varphi_{x+1} L_{x+1} = \vartheta(\varphi L)$$

$$\psi_x M_{x-1} + 2(\psi_x + \psi_{x+1}) M_x + \psi_{x+1} M_{x+1} = \vartheta(\psi M)$$

$$1/l \ (z_{x+1} - 2 z_x + z_{x-1}) = D_x(z)$$

$$1/h \ (z_{y+1} - 2 z_y + z_{y-1}) = D_y(z)$$

which afford the partial differential equations

$$\vartheta_x(\varphi L) + 6 D_x(z) = - U_{xy} \qquad (1)$$

$$\vartheta_y(\psi M) + 6 D_y(z) = 0 \qquad (2)$$

Each joint having three coordinated unknown quantities (L, M, z) a third equation is needed and obtained from the equilibrium of the joints perpendicular to the plane of the grillage. The proportion of moments L at the joints in direction of the external load is:

$$1/l \ (L_{x-1,y} - 2 \ L_{xy} + L_{x+1,y}) = D \ x(L).$$

Moments M at the joints contribute:

$$1/h \ (M_{x,y+1} - 2 \ M_{xy} + M_{x,y-1}) = D_y(M).$$

Contribution of external load: T_{xy}.

Thus follows the third equation of the simultaneous system:

$$D \ x(L) + D \ y(M) = T_{xy} \tag{3}$$

Since our lift structure has free boundaries only, the conditions stipulate that the moment in direction normal to the boundary must disappear: $M_{xy} = 0$; $L_{xy} = 0$. The cross stress in the first panel outside of the area must therefore become zero. This is proportional to the difference in moments at the ends of the members:

$y = 0$; $M_{x,y-1} - M_{xy} = 0$; consequently, $M_{x,y-1} = 0$

$y = m$; $M_{x,y+1} - M_{xy} = 0$;   " $M_{x,y+1} = 0$

$x = 0$; $L_{x-1,y} - L_{xy} = 0$;   " $L_{x-1,y} = 0$

$x = n$; $L_{x+1,y} - L_{xy} = 0$;   " $L_{x+1,y} = 0$.

If the wing structure has more than four fuselage attachments, or if outlying joints are prevented from shifting by strutting, as in the semicantilever wing, for instance, the attachment pressures imposed on these redundant attachment points are statically indeterminate in their normal component.

These pressures are temporarily introduced as unknown external loads and the procedure is as follows:

1. Elastically restrained joint: $z_{xy} = k\, T_{xy}$, when T_{xy} represents the unknown attachment pressure at joint x,y. As shown elsewhere, we have

$$z_{xy} = \sum_{i=0}^{m} \xi_i\, \eta_i$$

and

$$\bar{\xi}_i = [1/l\ (\bar{X}_{x-1} - 2\,\bar{X}_x + \bar{X}_{x+1}) + t^i\,(x)]\,(-\psi/\omega_i),$$

wherewith

$$k\, T_{xy} = \sum_{i=0}^{m} \eta_i\, 4/\omega_i\, [1/l\ (\bar{X}_{x+1} - 2\,\bar{X}_x + \bar{X}_{x+1}) + t_{i\,(x)}]$$

becomes the general increment equation for defining the statically indeterminable attachment pressure at each redundant elastically restrained joint.

For redundant fuselage attachment the premise of rigidly restrained joints ordinarily should hold.

2. Rigidly restrained joints: $z_{xy} = 0$, consequently, $\bar{\xi}_{xy} = 0$, and the simple equation for defining the external statically indeterminates with rigidly restrained joint is:

$$1/l\ (\bar{X}_{x+1} - 2\,\bar{X}_x + \bar{X}_{x-1}) + t^i\,(x) = 0,$$

wherein the unknown T_{xy} is contained within $t^i\,(x)$ as external load.

Now we proceed to Professor Melan's method of resolving the differential equations:

1. The general resolutions L_{xy}, M_{xy}, z_{xy} of the unhomogeneousness, simultaneous system of partial differential equations,

$$\partial x\,(\varphi L) + 6\, Dx\,(z) = -\, U_{xy}, \quad \partial y\,(\psi M) + 6\, Dy\,(z) = 0,$$

$$Dx\,(L) + Dy\,(M) = -\, T_{xy}$$

can be developed from the proper resolutions Y_i, η_i of the homogeneous simultaneous system of the same partial differential equations,

$$\vartheta x(\varphi L) + 6 \, Dx(z) = 0, \quad \vartheta y(\psi M) + 6 \, Dy(z) = 0,$$

$$Dx(L) + Dy(M) = \lambda \, z,$$

and from the resolutions $\overline{X}_i$ and $\overline{\xi}_i$ of an unhomogeneous system in the following form:

$$L_{xy} = \sum_{i=0}^{m} \overline{X}_i \, \eta_i, \quad M_{xy} = \sum_{i=0}^{m} Y_i \, \overline{\xi}_i, \quad z_{xy} = \sum_{i=0}^{m} \overline{\xi}_i \, \eta_i$$

2. For defining the proper resolutions of the homogeneous system of partial differential equations the simultaneous system resolves itself with

$$L_{xy} = X_{lx} \, Y_{ly}, \quad M_{xy} = X_{mx} \, Y_{my}, \quad z_{xy} = \xi_x \, \eta_y$$

into two groups of two ordinary differential equations each:

$$\vartheta x(\varphi X) + 6 \, Dx(\xi) = 0, \quad Dx(X) - \lambda' \xi = 0 \quad \text{for} \quad X_i \text{ and } \xi_i$$

and

$$\vartheta y(\psi Y) + 6 \, Dy(\eta) = 0, \quad Dy(Y) - \lambda'' \eta = 0 \quad \text{for} \quad Y_i \text{ and } \eta_i.$$

They reveal λ' and λ'' as roots of the frequency equation produced by making the coefficient determinants of that system of equation $= 0$ which takes the place of the differential equation.

3. For defining resolutions $\overline{X}_i$ and $\overline{\xi}_i$ of the unhomogeneous system the load terms U_{xy} and T_{xy} are developed according to the proper resolutions η^i determined above.

$$U_{xy} = \sum_{i=0}^{m} u_i(x) \, \eta_i, \quad T_{xy} = \sum_{i=0}^{m} t^i(x) \, \eta_i,$$

whereby, if the η_i are regulated,

$$u_i(x) = \sum_{y=0}^{m} U_{xy} \, \eta^i y \quad \text{and} \quad t^i(x) = \sum_{y=0}^{m} T_{xy} \, \eta^i y.$$

The result is the simultaneous system of ordinary differential equations

$$\vartheta x(\varphi \, \overline{X}_i) + 6 \, Dx(\overline{\xi}_i) = - u_i(x), \quad Dx(\overline{X}_i) + \lambda_i'' \, \overline{\xi}_i = t_i(x)$$

$$\text{for} \quad \overline{X}_i \text{ and } \overline{\xi}_i.$$

4. Now the general resolutions L_{xy}, M_{xy}, $z_{\overline{xy}}$ are computed conformably to the equations

$$L_{xy} = \sum_{i=0}^{m} \overline{X}_i \, \eta_i, \qquad M_{xy} = \sum_{i=0}^{m} Y_i \, \overline{\xi}_i, \qquad z_{xy} = \sum_{i=0}^{m} \overline{\xi}_i \, \eta_i$$

by matrix multiplications.

This is a brief outline of the general method of resolution and we proceed to the concrete calculation of the first loading attitude (spar 0 loaded).

In this case the unhomogeneous, simultaneous partial differential equations read

$$\vartheta x (\varphi L) + 6 \, Dx(z) = - U_{xy}, \quad \vartheta y (\psi M) + 6 \, Dy(z) = 0$$

$$Dx(L) + Dy(M) = - T_{xy},$$

wherein

$U_{xy} = 0$ for all $y = 1, 2, 3, 4, 5, 6$

$U_{xy} = 6/l \times 8/3 \, (\varphi_x + \varphi_{x+1}) = 4 \, (\varphi_x + \varphi_{x+1})$ for $y = 0$

and

$T_{xy} = 0$ for all $y = 1, 2, 3, 4, 5, 6$

$T_{xy} = 4$ for $y = 0$.

The proper resolutions of the homogeneous equations

$$\vartheta x (\varphi L) + 6 \, Dx(z) = 0, \quad \vartheta y (\psi M) + 6 \, Dy(z) = 0,$$

$$Dx(L) + Dy(M) = \lambda \, z$$

are developed in direction y (of the unloaded members).

By the agency of

$$L_{xy} = X_{lx} \, Y_{ly}, \quad M_{xy} = X_{mx} \, Y_{my}, \quad z_{xy} = \xi_x \, \eta_y,$$

the homogeneous, simultaneous system of partial differential equations can be resolved into two groups of two ordinary differential equations with constant coefficients each

$$\vartheta x(\varphi X) + 6\, Dx(\xi) = 0, \quad Dx(X) = \lambda'' \xi \quad \text{for} \quad X \text{ and } \xi$$

and

$$\vartheta y(\psi Y) + 6\, Dy(\eta) = 0, \quad Dy(Y) = \lambda'' \eta \quad \text{for} \quad Y \text{ and } \eta,$$

wherein the second group for Y and η is of primary interest.

Bearing in mind the interconnections

$$\vartheta\, (\psi_y\, Y^i) = \psi_y (\Delta^2 + 6)\, Y^i$$

$$D(\eta) = \Delta^2\, \eta/h$$

(Δ^2 ... difference of second order, e.g.:

$$\Delta^2 y = y_{x+1} - 2\, y_x + y_{x-1}),$$

wherein, $\qquad\qquad \psi_y = \psi = \text{constant}$

$$h_y = h = \qquad "$$

the equations

$$\vartheta x(\psi Y) + 6\, Dy(\eta) = 0, \quad Dy(Y) = \lambda' \eta$$

can be written in the form:

$$h\, \psi\, (\Delta^2 + 6)\, Y_i + 6\, \Delta^2\, \eta_i = 0$$

$$\Delta^2(Y_i) = h\, \lambda_i'' \, \eta_i.$$

With

$$H_i = \psi\, h\, Y^i \quad \text{and} \quad \omega_i = h^2\, \lambda_i'' \, \psi$$

we obtain the normal form

$$(\Delta^2 + 6)\, H_i + 6\, \Delta^2\, \eta_i = 0$$

$$\Delta^2\, H_i = \omega_i\, \eta_i$$

with the boundary conditions for

$$y = 0 \ldots H^i{}_{y-1} = H^i{}_y = 0$$

$$y = 6 \ldots H^i{}_y = H^i{}_{y+1} = 0.$$

The normalized solutions for $H^i{}_y$ and η_i of this system of equations with such boundary conditions may be found in the previously mentioned volume of Bleich-Melan. The desired values are

$$Y_i = H_i/\psi h; \quad \lambda_i = \omega_i/h^2 \psi.$$

To determine the solutions for $\overline{X}_i$ and $\overline{\zeta}_i$ of the un-homogeneous equations, we first develop the load terms U_{xy} and T_{xy} according to the proper resolutions η_i:

$$U_{xy} = \sum_{i=0}^{m} u_i(x) \, \eta_i; \quad u_i(x) = \sum_{i=0}^{m} U_{xy} \, \eta^i(x);$$

$$T_{xy} = \sum_{i=0}^{m} t^i(x) \, \eta_i; \quad t^i(x) = \sum_{i=0}^{m} T_{xy} \, \eta^i{}_y.$$

In the construction of the $u_i(x)$ it is to be observed that $U_{xy} = 0$ for all y other than zero, hence the sum itself reduces to one single term; likewise in the construction of $t^i(x) \ldots T_{xy} = 0$ for all y other than zero. In addition, the reactions at the joints $x = 6.7$ are here also to be treated as external loads.

Now follows the calculation of $\overline{\zeta}_i$ and $\overline{X}_i$ from

$$\vartheta x(\varphi_x \overline{X}_i) + 6 \, Dx(\overline{\zeta}_i) = - u_i(x)$$

$$Dx(\overline{X}_i) + \lambda_i{}'' \, \overline{\zeta}_i = - t_i(x).$$

It being impossible to give a general solution of this unhomogeneous simultaneous system of ordinary differential equations of the second order with variable coefficient in closed form, numerical resolution is resorted to. Resolved from the second equation and written into the first $\overline{\zeta}_i$ is eliminated, and $\overline{X}_i$ yields

$$\lambda'' \vartheta (\varphi \overline{X}_i) - 6 \, D \, D(\overline{X}_i) = - u_{(x)} \lambda'' + 6 \, D(t_i(x))$$

(a differential equation of the fourth order for $\overline{X}_i$).

Specifically at point x:

$$\vartheta_x \, \bar{X}_{x-2} + \epsilon_x \, \bar{X}_{x-1} + \gamma_x \, \bar{X}_x + \epsilon_{x+1} \, \bar{X}_{x+1} + \vartheta_{x+2} \, \bar{X}_{x+2} =$$

$$= - \omega_i / \psi \; h^2 \; u_{(x)} + 6/l \cdot (t_{x+1} - 2t_x + t_{x-1}),$$

wherein

$$\vartheta_x = - 6/l^2 \qquad \epsilon_x = 24/l^2 + \omega_i / \psi \; h^3 \; \varphi_x$$

$$\gamma_x = - 2 \, [18/l^2 - (\varphi_x + \varphi_{x+1}) \, \omega_i / \psi \; h^2].$$

This equation is to be posed for x = 1 in the n-panel system the first time, for x = n - 1 the last time, and for x for all in between. Agreeably to the redundant unknowns $\bar{X}_{-1}$, $\bar{X}_0$, $\bar{X}_n$, $\bar{X}_{n+1}$, there are four boundary conditions for the free boundaries:

x = 0...$\bar{X}_{-1}$ = $\bar{X}_0$ = 0   x = 13...X_{13} = X_{14} = 0.

Because of the symmetry of the supporting structure in question, from the remaining twelve unknowns, it is paramount that

$$\bar{X}_1 = \bar{X}_{12}, \; \bar{X}_2 = \bar{X}_{11}, \; \bar{X}_3 = \bar{X}_{10} \text{ etc.},$$

so that only the equations from x = 1 to x = 6 have to be worked out.

This group of six equations, with six unknowns, is now developed for i = 0, i = 1, i = 2 to i = 6 and resolved, yielding the numerical values for X_i.

As to $\bar{\xi}_i$, recourse is had to the second equation of the unhomogeneous simultaneous system and results in

$$\bar{\xi}_i = - \psi / \omega_i \, [1/l \, (\bar{X}_{x+1} - 2 \, \bar{X}_x + \bar{X}_{x-1}) + t^i_{(x)}];$$

wherefrom $\bar{\xi}_i$ can be readily computed. For i = 0 and i = 1; this formula, however, leads to the unknown value per cent and in this range $\bar{\xi}_i$ is defined by resolving the first equation of the unhomogeneous simultaneous system. In this manner the numerical values for $\bar{\xi}_i$ and E k $\bar{\xi}$ are computed and all moments at the spar and the joints can be expressed according to the formulas

$$L_{xy} = \sum_{i=0}^{m} \bar{X}_i \, \eta_i \quad \text{and} \quad M_{xy} = \sum_{i=0}^{m} \bar{Y}_i \, \bar{\zeta}_i.$$

The displacement at the joints, developed from

$$z'_{xy} = \sum_{i=0}^{m} \bar{\zeta}_i \, \eta_i,$$

contain, in view of assuming z_{60}, z_{70}, z_{66}, $z_{76} = 0$
a displacement (or torsion) which the grillage executes
as rigid body and which is readily eliminated by linear in-
terpolation. (This displacement (or torsion) should not
be confused with the previously cited hyperboloidal curva-
ture of the complete structure.)

For the second load attitude (spar I loaded) the un-
homogeneous simultaneous equations read:

$$\vartheta x(\varphi L) + 6\, Dx(z) = -U_{xy} \qquad \vartheta y(\psi M) + 6\, Dy(z) = 0$$

$$DX(L) + Dy(M) = -T_{xy},$$

wherein

$U_{xy} = 0$ for all $y = 0, 2, 3, 4, 5, 6$

$U_{xy} = 6/l \times 8/3 \,(\varphi_x + \varphi_{x+1}) = 4(\varphi_x + \varphi_{x+1})$ for $y = 1$

and

$T_{xy} = 0$ for all $y = 0, 2, 3, 4, 5, 6$

$T_{xy} = 4$ for $y = 1$.

The solutions of the homogeneous system of equations
are identical with those for loading condition I. The so-
lutions $\bar{X}_i$ and $\bar{\zeta}_i$ of the unhomogeneous equations are de-
fined as before and so also the desired unknowns. The
other load attitudes (third, fourth attitude) now can be
computed as the others, so that, because of the symmetry
of the structure to the median spar, the moments at the
joints, at the ribs and joint displacements are known for
any possible load attitude and the influence lines can be
plotted. The evaluation of these lines with the perti-
nent load attitude then yields the moments at the joints
and therefrom all stresses in the members.

The numerical data of the discussed four load atti-
tudes are tabulated and appended in Tables I to IV.

If no combined effect were introduced by suitable ribs between the individual spars, each spar would have to carry its full proportion of the load, and under these conditions the moments at the spar joints are readily computed from $L_x = \frac{qx^2}{2}$ at

$$L_0 = 0, \ L_1 = -8, \ L_2 = -32, \ L_3 = -72, \ L_4 = -128,$$

$$L_5 = -200, \ L_6 = -288.$$

In order to represent clearly the results of the static calculation and for ready reference in a particular case the influence lines for all moments at spar joints, rib joints and displacements of joints must be drawn. Because of the symmetry of the beam to spar III, plotting of the influence lines is confined to those for spars 0 to III. In Figure 3 is given a sample of the actually drawn influence lines, those remaining being readily apparent from the data in the tables.

Aside from the exact determination of the internal loads of the grillage, the exact amount of its deflections is not only of interest from the general point of view, but it is rather precisely these which enable us to draw conclusions with an accuracy unattainable heretofore, regarding the statical combined effect between grillage and a metal skin, be it of smooth or corrugated strip.

SUMMARY

In a great majority of modern large cantilever or semicantilever airplanes the wing forms a lattice beam whose members consist of spars (from 2 to 10, according to special design) and ribs. Spars and ribs may be solid or lattice, the latter being more economical in most cases.

Loads in form of attached bodies, engines, propellers, controls, useful loads, wing bracing, etc., are impressed at arbitrary joints of the lattice beam (comprising both wings) in accordance with the particular case of a cantilever, semicantilever, indirectly, or more or less directly loaded wing. Such highly statically indeterminate lattice beams yield readily to accurate determination by means of differential equations, regardless of type of

load, change of moments of inertia of spars and ribs, spar spacing, body attachment conditions, etc. Because the loads vary, the use of influence lines is recommended.

In the present report the computation is actually carried through for the case of parallel spars of equal resistance in bending without direct loading, including plotting of the influence lines; for other cases the method of calculation is explained. The development of large size airplanes can be speeded up by accurate methods of calculation, such as this.

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Table I — Results of attitude "spar 0 loaded"

Displacements of joints $z_{xy} \cdot Ek$

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	+250,97	+177,75	+116,95	+70,94	+37,03	+15,31	0
1	+191,40	+132,51	+86,84	+53,30	+29,09	+16,48	+9,32
2	+134,62	+89,39	+58,34	+37,15	+24,00	+16,02	+14,00
3	+78,21	+49,49	+32,60	+22,12	+16,32	+12,56	+13,35
4	+21,83	+12,51	+8,98	+7,61	+7,27	+6,80	+8,30
5	−24,82	−22,57	−13,50	−6,84	−2,75	+0,39	+0,34
6	−88,69	−56,07	−35,19	−20,95	−10,95	−8,23	0

Moments on rib joints M_{xy}

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	0	0	0	0	0	0	0
1	−0,94	−2,35	−2,03	−1,30	+0,24	+2,18	+4,86
2	−1,40	−3,32	−2,73	−1,18	+1,25	+3,05	+5,59
3	−1,43	−3,06	−2,27	−0,48	+0,31	+2,42	+4,57
4	−1,01	−2,04	+0,55	+0,01	+0,95	+1,42	+2,85
5	−0,40	−0,74	+0,37	+0,20	+0,44	+0,36	+0,88
6	0	0	0	0	0	0	0

Moments on spar joints L_{xy}

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	0	−4,02	−14,10	−31,74	−59,07	−103,61	−172,80
1	0	−2,17	−10,89	−25,77	−47,42	−73,41	−93,10
2	0	−2,00	−8,60	−19,61	−32,07	−42,29	−46,24
3	0	−1,47	−5,59	−11,55	−16,22	−17,96	−18,33
4	0	−0,61	−1,94	−2,79	−2,28	−0,66	+0,42
5	0	+0,50	+2,35	+5,68	+9,55	+12,67	+13,87
6	0	+1,74	+6,83	+13,77	+20,39	+25,19	+28,08

Table II — Results of attitude "spar I loaded"

Displacements of joints $z_{xy} \cdot Ek$

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	+196,21	+137,77	+90,38	+54,41	+27,98	+10,18	0
1	+159,25	+112,41	+75,44	+48,85	+30,77	+19,31	+13,33
2	+121,94	+86,29	+59,62	+41,72	+30,32	+24,02	+20,78
3	+84,02	+60,33	+43,37	+32,80	+26,68	+23,41	+21,81
4	+48,18	+35,29	+27,18	+22,40	+19,72	+18,35	+17,61
5	+11,19	+11,33	+11,10	+10,83	+10,52	+10,00	+9,88
6	−25,03	−12,63	−5,08	−1,22	+0,18	+0,26	0

Moments on rib joints M_{xy}

	0,13	1,14	2,11	3,10	4,9	5,8	6,7
0	0	0	0	0	0	0	0
1	+0,52	+1,22	+1,28	+1,97	+3,11	+4,91	+6,54
2	−0,27	−0,39	+0,29	+1,71	+3,62	+5,48	+6,57
3	+0,42	−1,04	−0,11	+1,58	+3,33	+4,52	+5,16
4	−0,27	−0,96	−0,13	+1,15	+2,33	+3,25	+3,64
5	−0,42	−0,68	−0,26	+0,31	+0,80	+1,15	+1,47
6	0	0	0	0	0	0	0

Moments on spar joints L_{xy}

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	0	−1,93	−9,69	−23,27	−45,59	−81,14	−138,32
1	0	−3,57	−9,99	−21,85	−39,26	−60,25	−76,69
2	0	−1,45	−7,52	−16,63	−27,09	−35,23	−39,36
3	0	−1,18	−5,52	−11,39	−15,95	−17,92	−18,34
4	0	−0,07	−2,88	−3,61	−7,21	−7,56	−6,79
5	0	−0,02	−0,03	+1,05	+0,02	+1,73	+3,46
6	0	+0,78	+3,37	+3,19	+7,08	+3,78	+5,78

Table III — Results of attitude "spar II loaded"

Displacements of joints $z_{xy} \cdot Ek$

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	+188,88	+85,00	+57,22	+35,07	+18,45	+6,73	0
1	+109,70	+80,36	+56,60	+39,13	+27,03	+19,70	+15,95
2	+100,03	+74,53	+54,63	+41,17	+32,59	+28,24	+20,32
3	+89,50	+66,99	+50,07	+39,35	+32,07	+30,15	+28,59
4	+78,41	+58,21	+43,55	+33,99	+28,61	+25,38	+23,06
5	+67,10	+49,20	+35,98	+26,37	+19,65	+16,66	+13,37
6	+55,72	+43,98	+28,03	+17,81	+9,75	+3,59	0

Moments on rib joints M_{xy}

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	0	0	0	0	0	0	0
1	+0,39	+0,83	+1,13	+1,77	+2,71	+4,11	+5,61
2	+1,10	+2,62	+3,16	+4,52	+6,37	+7,98	+9,67
3	+0,52	+0,92	+1,91	+3,57	+5,31	+6,66	+8,40
4	+0,18	+0,12	+0,98	+2,27	+3,64	+4,73	+4,68
5	+0,04	+0,11	+0,35	+0,93	+1,57	+2,26	+2,80
6	0	0	0	0	0	0	0

Moments on spar joints L_{xy}

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	0	−1,02	−4,77	−12,57	−26,70	−51,49	−93,16
1	0	−0,89	−5,49	−13,09	−24,76	−39,35	−50,17
2	0	−2,75	−6,80	−14,13	−21,70	−27,09	−27,82
3	0	−0,80	−5,52	−11,39	−16,11	−18,27	−18,77
4	0	−0,92	−4,51	−9,39	−14,17	−17,10	−19,36
5	0	−0,84	−3,12	−6,99	−13,14	−21,57	−28,98
6	0	−0,69	−1,61	−4,43	−11,42	−25,41	−49,76

Table IV — Results of attitude "spar III loaded"

Displacements of joints $z_{xy} \cdot Ek$

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	+86,91	+61,30	+42,31	+24,94	+12,78	+4,37	0
1	+87,47	+64,64	+46,28	+32,66	+23,43	+17,66	+14,67
2	+89,76	+67,49	+49,59	+38,82	+31,75	+27,66	+25,31
3	+90,32	+68,80	+51,04	+41,37	+35,09	+31,51	+29,20
4	+89,76	+67,49	+49,59	+38,25	+31,75	+27,66	+25,31
5	+97,47	+64,64	+46,28	+32,66	+23,43	+17,66	+14,67
6	+86,91	+61,30	+42,31	+24,94	+12,78	+4,37	0

Moments on rib joints M_{xy}

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	0	0	0	0	0	0	0
1	+0,32	+0,60	+0,78	+1,63	+2,38	+3,32	+4,00
2	+0,73	+1,47	+1,79	+3,66	+5,11	+6,29	+6,96
3	+1,47	+3,36	+3,62	+5,97	+7,61	+8,57	+9,39
4	+0,73	+1,47	+1,79	+3,66	+5,11	+6,29	+6,96
5	+0,32	+0,60	+0,78	+1,63	+2,38	+3,32	+4,00
6	0	0	0	0	0	0	0

Moments on spar joints L_{xy}

	0,13	1,12	2,11	3,10	4,9	5,8	6,7
0	0	−0,95	−3,36	−9,18	−20,70	−41,25	−75,35
1	0	−0,87	−4,28	−10,13	−19,10	−30,69	−39,93
2	0	−0,82	−5,10	−10,65	−16,24	−19,99	−21,16
3	0	−2,72	−6,39	−12,12	−16,01	−17,04	−16,31
4	0	−0,82	−5,10	−10,65	−16,24	−19,99	−21,16
5	0	−0,87	−4,28	−10,13	−19,10	−30,69	−39,93
6	0	−0,95	−3,36	−9,18	−20,70	−41,25	−75,35

Influence lines of moments at
spar joints $L_{x,2}$; moments at
rib joints $M_{x,2}$; displacement
of joints Ek $z_{x,2}$

Fig. 3   Influence lines of
         wing structure.

Fig. 2   Moments of inertia
         of a multi-spar,
cantilever wing structure
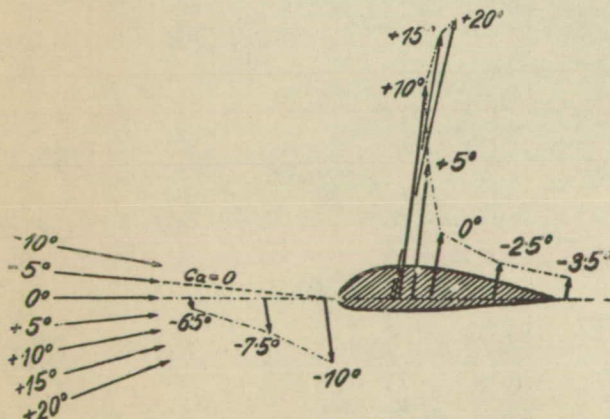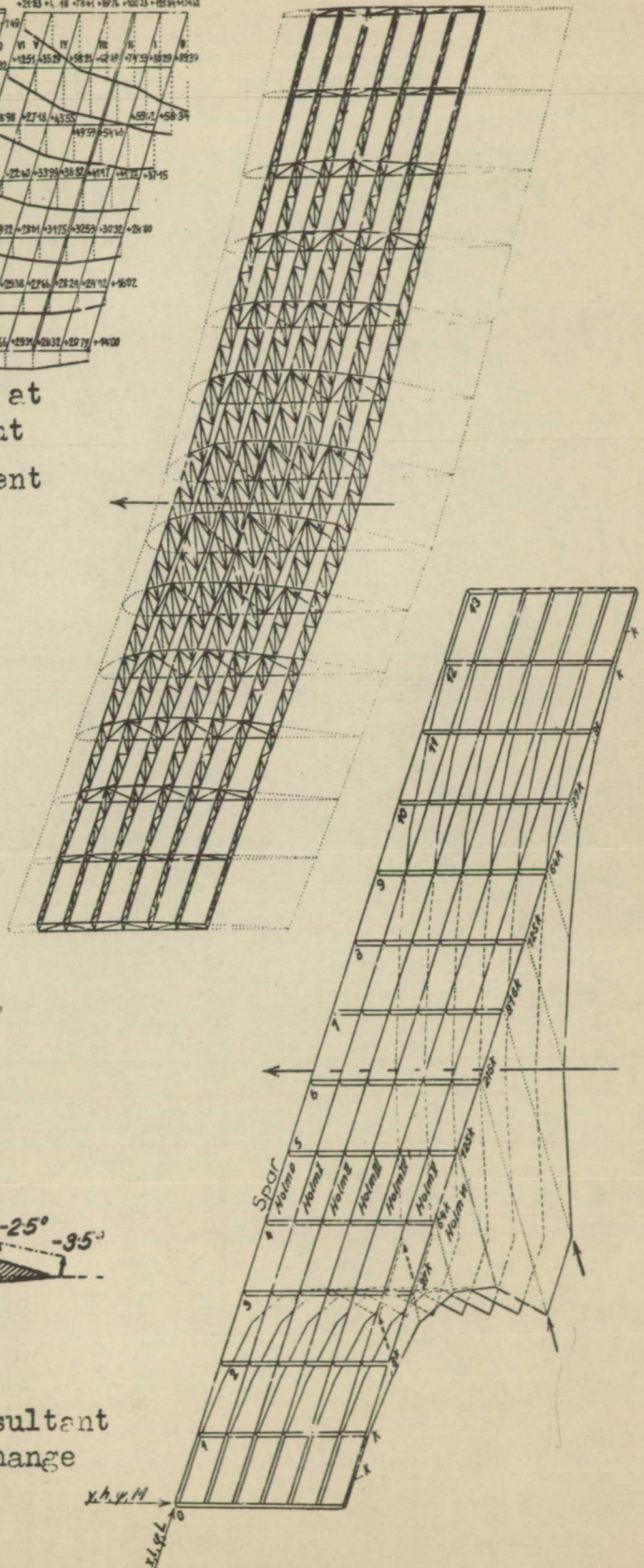with parallel webs. ———>

Fig. 1   Displacement of resultant
         of air load with change
in angle of attack.