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RESONANCE VIBRATIONS OF AIRCRAFT PROPELLERS

By Fritz Liebers

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I. NOTATION

t (second),	time.
x (cm),	length coordinate in direction of propeller radius.
l (cm),	blade length.
$\xi = x/l$,	nondimensional length coordinate in direction of propeller radius.
b (cm),	width of propeller blade.
d (cm),	thickness
ϵ (cm),	radius of propeller hub.
α,	angle of attack of a propeller-blade element.
$\Delta\alpha$ (x),	angle of torsional vibration.
$\Delta\beta$,	angle of bending vibration of rigid propeller blade.
y (x), y (ξ),	line of torsional or bending vibrations.
m,	variable parameter in group of functions for assumed line of bending vibrations (equation 18).

*"Resonanzschwingungen von Luftschrauben." Luftfahrtforschung, May 16, 1930, pp. 137-152. See also "Contribution to the Theory of Propeller Vibrations," by the same author. Zeitschrift für technische Physik, Vol. X, 1929, pp. 361-369. (T.M. No. 568, N.A.C.A., 1930.)

$F(x)=F_0 f(x)$ (cm^2),	section of propeller blade at point x.
F_0 (cm^2),	root section of propeller blade.
$J(x)=J_0 i(x)$ (cm^4),	sectional inertia moment against torsion or bending at point x.
J_0 (cm^4),	sectional inertia moment at blade root.
κ, ϑ,	exponents indicating reduction of cross section and cross-sectional moment of propeller blade (equation 17).
G $(kg\ cm^{-1})$,	modulus of shear.
E $(kg\ cm^{-2})$,	modulus of elasticity.
ρ_m $(kg\ cm^{-4}\ s^2)$,	density of material.
ρ $(kg\ cm^{-4}\ s^2)$;	density of air.
J_m $(kg\ cm\ s^2)$,	inertia moment of propeller-blade element of unit length about torsional axis of rotation (elastic axis).
θ $(kg\ cm\ s^2)$,	inertia moment of propeller blade with reference to fixed axis.
S $(kg\ s^2)$,	static moment of propeller blade with reference to fixed axis.
ω (s^{-1}),	angular velocity of propeller.
n (r.p.m.),	revolution speed of propeller.
λ (s^{-1}),	cyclic frequency of vibrations.
λ_0 (s^{-1}),	cyclic frequency of vibrations on stand.
$\nu = 60\,\lambda/2\pi$ (min^{-1}),	vibrations per minute.
c (kg cm),	elasticity constant (equation 7).
c_m,	moment coefficient of aerodynamic force with reference to elastic axis.

c_a, lift coefficient.

v (cm s^{-1}), flight speed.

II. MANNER OF PRESENTATION

The field opened up by the investigation of the vibrations of aircraft propellers is so large that a division of the general problem and great simplifications are absolutely necessary if intelligible results are to be obtained. It is therefore important to state clearly the assumptions and line of reasoning employed for simplifying the problem and for its still very troublesome solution. It is usually not very satisfactory, when the problem is first tackled mathematically and the preliminary assumptions and associated physical results are interpolated in the calculation. Hence the attempt here is first made to describe the way the calculation is to follow and then to carry out the actual calculation, which finally leads to numerical results.

III. SURVEY OF PROBLEM, METHOD OF SOLUTION AND GENERAL RESULTS

1. Possibility of Different Kinds of Propeller Vibrations

Propeller vibrations have come to be of practical importance, since propeller breaks have frequently occurred with the well-known characteristics of fatigue fractures. Light-metal propellers have suffered various kinds of breaks due to vibration, so that their increasing use in aviation calls for a more thorough investigation.

Propeller vibrations may be produced in essentially different ways:

a) Unstable vibrations.- First (the same as for airplane wings) there may be, at a certain critical speed, unstable vibrations, which are increased by absorption of energy from the air until amplitudes are reached which result in fractures. If, however, we consider the great torsional rigidity of a propeller blade with respect to the aerodynamic forces acting upon it, we can estimate mathematically (Section IV, 1, c), that the critical speed,

at which such vibrations can be produced, is so great that this kind of vibrations is very improbable.

Independently of any mathematical confirmation, it is obvious that the unstable vibrations, which are often fatal for wings, can hardly be dangerous for propeller blades. To a certain degree, a propeller blade resembles a wing consisting entirely of a single spar. The elastic stabilizing forces must here stand in a very different relation to the aerodynamic forces than in the case of an airplane wing. In the latter, the elastic forces are mainly concentrated in the spars, while the wing itself merely serves to absorb the aerodynamic forces and transmit them to the spars.

b) Resonance vibrations due to discharge of vortices.- Another cause of propeller vibrations may be sought in a resonance between the natural frequencies of propeller blades and the frequency of the periodical discharge of vortices from the boundary layer in what is ordinarily treated as a stationary condition. This refers, e.g., to the phenomenon which leads to the familiar "singing" of wires in the wind. It should not be confounded with the formation of free vortices due to variations in the thrust or in the circulation about the blade. These vortices result from every propeller vibration, however produced.

The nature of this vortex discharge is so complex, however, and its calculation hitherto so difficult, that it may be considered impossible to explain theoretically any vibration phenomena produced by it. It should be noted that the most important parts, the blade tips, revolve, on rapid propellers, at nearly or quite the velocity of sound. It is very problematic as to whether the vortex discharge contains sufficient energy to produce the vibrations. This can be determined only by experiment.

c) Resonance vibrations due to unequal impacts of the propeller blades.- Still another cause of propeller vibrations, and probably the most important one, is their production by unequal impact of the blades, due to an uneven field of flow, such as always exists for a propeller revolving near the airplane.

The propeller blade, e.g., sweeps close by a strut or wing. In particular a pusher propeller behind the wing is disturbed by the vortex trail from the wing. On multiengine airplanes it sometimes happens that the propeller

disks overlap one another. These and similar phenomena affect the impact velocity of the air and the angle of attack of the propeller blade as it enters the disturbed region. Consequently, the propeller stresses vary during each revolution and may produce resonance vibrations of the blades. Ordinarily these vibrations occur once, though sometimes twice, during each revolution.

Since all aircraft propellers work more or less in zones of disturbed air flow, and since propeller vibrations and breaks have occurred on airplanes where the arrangement of the propellers as mentioned above was manifestly unfavorable, resonance vibrations produced by external disturbances represent one of the practically most important classes of all conceivable kinds of propeller vibrations. It is therefore important for this class of vibrations to be investigated among the first.

2. Natural Frequencies of Propeller Blades with Reference to Torsional and Bending Vibrations

Knowledge of the natural frequencies of the propeller blades is of fundamental importance in every case for explaining the particular vibration phenomenon, of whatever nature it may be. The natural frequencies depend on the material, on its elastic properties and on the shape of the propeller blade. They are also affected by the aerodynamic and centrifugal forces acting on the propeller and consequently by the velocity and revolution speed. By "aerodynamic forces" is meant the variations in the propeller thrust and in the torsional moment of the aerodynamic force, due to the variations in the angle of attack caused by the vibrations.

The calculation of the frequencies in terms of the above quantities forms the essential part of the present report, which deals especially with resonance vibrations of aircraft propellers caused by load variations.

a) Limitation to one kind of vibration.- The propeller vibrations may be represented by both the coordinates, torsional and bending. On account of the distortion exhibited by the separate cross sections of the propeller and the corresponding main cross-sectional axes with respect to one another, torsional and bending vibrations must always occur simultaneously. If, therefore, for mathematical reasons, the natural frequencies for torsion and

bending are calculated separately, each with temporary disregard of the other, values are obtained, one of which is somewhat greater than the smaller and the other somewhat smaller than the greater of the natural frequencies of the propellor, i.e., of the frequencies of the naturally occurring combined vibrations of the propeller. (Cf. Courant-Hilbert, (Methoden d. math. Phys., Vol. I, Chap. 5, 3.) The difference between the separately calculated frequencies for torsion and for bending and the frequencies of the combined vibrations cannot be very great, because (as found by calculation and experiment) the torsional and bending frequencies of propeller blades differ greatly from each other.

As regards the frequencies of a nonrevolving propeller blade, they can be calculated more or less accurately, but can be most accurately and quickly determined by experiment. The frequencies of a revolving propeller, exposed to aerodynamic and centrifugal forces are, however, of most interest here. Hence the question to be answered is: "How do the aerodynamic and centrifugal forces affect the torsional and bending frequencies of a given revolving propeller with given frequencies on the stand?" In order to determine first only the order of magnitude of these effects in comparison with the other forces, it is only necessary to replace the tapering, twisted and spatially bent propeller blade by the simplest possible substitute, provided it absorbs practically the same aerodynamic and centrifugal forces as the real propeller.

b) Torsional vibrations in terms of the aerodynamic and centrifugal forces.- In the sense of this simplification and for the preliminary determination of the torsional frequencies of a revolving propeller blade, the latter is replaced by a blade of uniform section. By solving the differential equation for the torsional vibrations with allowance for the aerodynamic forces, it is found that, through the influence of the aerodynamic forces, even with torsionally weak propellers and very high revolution and flight speeds, the torsional frequency is reduced by such small amounts (1 to 2%), that it can be entirely disregarded. This result is surprising, at first, because the aerodynamic forces represent no damping of the torsional vibrations, but an increasing moment proportional to the amplitude. (See equation (1), Section IV, 1, a.) For the centrifugal force, a small component of which falls in the direction of the cross section of the blade, it is only necessary to demonstrate that its participation increases the torsional frequency.

If it is now demonstrated by calculation or experiment that the torsional frequencies of resting propellers are of the order of magnitude of 10,000 vibrations per minute, it can be inferred, on the basis of the conclusion that the aerodynamic forces of a revolving propeller are unimportant and that the centrifugal forces can only increase their torsional frequency, that resonances between vibrations occurring with the frequency of the simple or twofold revolution speed of normally 1000 to 2000 r.p.m. and that the torsional vibrations of the propeller do not come into consideration. This is the first result which simultaneously restricts the problem under discussion.

c) Rectilinear bending vibrations of a rigid propeller under the action of the aerodynamic and centrifugal forces.- If the bending vibrations are treated in the same manner (by replacing them provisionally by rectilinear vibrations of the assumedly rigid propeller blade, flexibly mounted on the hub perpendicular to the plane of rotation), there is again found, in very simple manner, a negligibly small dependence of the bending frequency on the aerodynamic forces. This result is to be expected in bending vibrations, because in this case the aerodynamic forces only signify a damping proportional to the bending velocity. (Cf. equation (10) in Section IV, 2,a.) It can therefore affect the frequency only in the second order. On the contrary, the centrifugal forces, which exert a strong return moment, contribute greatly toward increasing the bending frequencies. Moreover, since the true bending frequencies of actual propellers at rest are 1000 to 2000 per minute, the possibility of resonance between bending vibrations and disturbances with the period of the simple or twofold revolution speed is directly indicated.

This is the second important result. From this point on, no theory of propeller vibrations can be satisfied with estimates of the hitherto customary nature. The gist of our task is rather the most accurate possible calculation of the bending frequencies of propeller blades in terms of the revolution speed, in which the effect of the aerodynamic forces can be disregarded.

d) Restriction of the bending frequency of elastic propellers by an upper and a lower limit.- Even the restricted task meets many difficulties. The unknown bending-vibration line varies, like the frequencies, with the

angular velocity of the propeller. The solution of the general integro-differential equation is practically impossible. Another possibility of determining the vibration-elastic line is offered by the Rayleigh principle of the minimum natural frequency of a vibration-elastic system which, for the case under consideration, may be briefly expressed as follows:* Of all possible bending lines compatible with the marginal and continuity conditions of the problem, the true bending line is the one which yields the minimum frequency.

From the mathematical viewpoint, the Rayleigh principle presents a variation problem. If it is solved by the methods of variation calculation (in the present case practically by the so-called direct methods of variation calculation), the Rayleigh theory determines the elastic deformations during the vibration, and consequently the bending frequencies of the revolving propeller, with perfect accuracy. (See Courant-Hilbert, Meth. d. math. Phys., Vol. I, Chap. 2, par. 2.) The above-mentioned methods, however, are generally troublesome and, for convergence reasons, do not always yield satisfactory results. (Courant-Hilbert, Vol. I, Chap. 4, par. 1,4 and 2,4.)

In any case, however, the following statement, based on the Rayleigh principle, is correct: If, instead of the actual deformations at the various angular velocities, other bending lines, which satisfy the marginal and continuity conditions, but which are otherwise arbitrary, are adopted as the basis for the calculation of the bending frequencies of the propeller blade, the calculated frequencies will surely be too great. The hypothetically calculated bending frequencies represent the upper limit for the still unknown true bending frequencies. The dif-

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*Rayleigh's theory ("Theory of Sound," Vol. I, par. 88 and 89) says nothing about the marginal and continuity conditions of the hypothetical deformations adopted for comparison. It says only that, when the comparison functions differ from the true "natural" deformations by small variations, the corresponding frequencies differ by small quantities of the second order, from which it may be further deduced that the frequency of the vibrations corresponding to any hypothetical form of vibration lies between the minimum and maximum values of the natural frequencies of the system. The above restricted conception might be better for practical application.

ference will be less in proportion to the suitability of the hypothetical bending lines, i.e., according to the excellence of the choice of the determining factors. These are the distribution of the elastic and inertia forces dependent on the shape (taper) of the propeller and the centrifugal forces, likewise dependent on the shape and also on the revolution speed of the propeller.

e) The upper limit.- This was determined as follows: An infinite number of bending lines was assumed, which, in the first place, are continuously deformable into one another, secondly, satisfy the continuity and marginal conditions of the problem and, lastly, were so selected that they represent approximations of the bending frequencies of uniform and tapered blades. Moreover, the group contains curves, which may be considered as derived from the above-mentioned curves by continuous deformation due to the increasing effect of the centrifugal forces with increasing revolution speed. In the simplest case, one uses a group of curves, selected according to these viewpoints, in the form of a function, $y = y(\xi, m)$, having only one arbitrarily variable parameter. In this function y is the amplitude of the deflection, ξ the coordinate in the longitudinal direction of the propeller blade, and m the variable parameter. (The actually chosen group of simple infinitely numerous bending lines is represented in Section IV, 2,c by equation (18) and Figure 4.)

If the frequency λ is now calculated by means of the energy theorem on the basis of the bending line $y = y(\xi, m)$, at first left undetermined by the arbitrary parameter m, λ then appears as a function of m. Furthermore, the expression for λ also contains the constructional characteristics of the given propeller, namely, the thickness of the material ρ_m, the modulus of elasticity E, the length l, the course of the cross section $F(\xi)$ throughout the length, and the angular velocity ω. The frequency may therefore be expressed by the formula

$$\lambda = \lambda\ (m,\ \rho_m,\ E,\ l,\ F,\ J,\ \omega)$$

For the given constructional characteristics, ρ_m, E, l, F, J and the given angular velocity ω, λ is a function of the parameter m alone, and there remains only the solution of the most elementary problem, to determine that value of m for which λ becomes a minimum. This quite definite value $m = m'$ characterizes, however (again with

respect to the Rayleigh theory), within the curve $y_1(\xi,m)$, the bending line which most closely approaches the true line. Likewise, $\lambda(m') = \lambda_{min}$ is the value of the frequency, which, with the adoption of the vibration-elastic lines $y(\xi,m)$ for agreement with the Rayleigh minimum principle, most closely approaches the true frequency. It is surely too great, however.

f) The lower limit.- This is obtained by means of the following generally applicable theorem set forth in Section IV, 2,f (q.v.):

The square of the frequency of an elastic system, subjected to several forces, is always greater than the sum of the frequency squares, which the system would have if, at the given time, only one of the forces were acting on the system. This theorem can be expressed by the formula:

$$\lambda^2 > \Sigma \lambda_i^2 \quad (i = 1, 2, \ldots)$$

The propeller blade is acted on simultaneously by the elastic and centrifugal forces. The theorem then maintains that, if the centrifugal force is at first entirely disregarded, and only the elastic forces are considered, the propeller has a definite bending frequency λ_o. This is the same, however, as the frequency of the nonrevolving propeller. If, on the other hand, the elastic forces are disregarded, and only the centrifugal forces are regarded as acting, the propeller has another frequency λ_ω. The latter, however, is the frequency of the propeller revolving at infinite speed, or, as it may also be expressed, of a perfectly flexible rope having the same mass and revolving about an axis like a sling. Its vibration frequency, as may be easily calculated (Section IV, 2,a), is now exactly the same as the angular velocity ω of the sling. Hence, the above-mentioned theorem yields the extremely simple formula

$$\lambda^2 > \lambda_o^2 + \omega^2;$$

valid for propellers of any shape, indicating that the square of the frequency of the revolving propeller is greater than the sum of the square of the frequency of the propeller at rest and the square of the angular velocity.

It should be remembered that the relations are really not quite so simple, since the propeller blade is mounted

on a hub which must be regarded as rigid. See further details in Section IV, 2,g regarding a necessary correction of the above formula, which takes into account the effect of the hub. In any case, we have now found a lower limit for the bending frequency of the propeller (approximate limit, since the value $\lambda_0^2 + \omega^2$ can never be actually assumed).

It can be demonstrated that the above inequality is not only a lower limit, but also an approximation for the bending frequency of the propeller.

g) Difference between the upper and lower limits.- On the basis of the last remark and the fact that even the upper limit, on account of the physically based assumptions for the bending lines, can show no great deviation from the true value of the bending frequencies, it must be concluded that the difference between the upper and the lower limit is small enough to satisfy the practical requirements for accuracy. In fact, the actual calculation (Section IV) by the here-described method shows that the difference between the upper and lower limits is about 5 per cent in the most unfavorable case, and normally much less. For very small and very great centrifugal forces the error tends toward zero. (See Figure 8 in Section IV, 2,e.)

After the amount of the greatest error had been determined, the problem of calculating the bending frequencies of a propeller in terms of the revolution speed was practically solved.

The practical numerical applications of the above-mentioned principles with respect to propellers endangered by resonance vibrations are introduced near the end of the report. (Sections V and VI.) The calculations will first be actually made, as already explained in a general way.

IV. MATHEMATICAL APPLICATION OF THE PRINCIPLES EXPLAINED IN SECTION III

Corresponding to the statements in Section III, the effect of the aerodynamic and centrifugal forces on the frequencies of a propeller will first be calculated. Simultaneously the question will be answered as to whether

the torsional and bending frequencies can approximate the revolution speed.

1. Torsional Vibrations

a) Preliminary assumptions and theorems.- Since no theory of the torsional vibrations of aircraft propellers is involved, but only the solution of the above-mentioned problems, the calculations may be based on the simplest assumptions.

For the simple determination of the torsional frequencies, the actual propeller blade, which is tapered and warped from root to tip, is replaced by a blade of uniform section and torsional inertia moment. For tapered blades the frequencies will have still higher values than those here calculated.

Moreover, the small components of the centrifugal force, which lie in the direction of the blade section, are entirely disregarded. The torsional frequencies could only be further increased by its consideration. These assumptions validate the following equation

$$J_m \frac{\partial^2 \Delta\alpha}{\partial t^2} = G J \frac{\partial^2 \Delta\alpha}{\partial x^2}$$

where $\Delta\alpha$ is the torsional angle; x, the length coordinate in the direction of the propeller radius; J_m, the inertia moment of a wing element of unit length about the axis of revolution; J, the cross-sectional moment against torsion; G, the modulus of shear; and t, the time.

To this equation there is added still another term which accounts for the aerodynamic forces.

$$J_m \frac{\partial^2 \Delta\alpha}{\partial t^2} = G J \frac{\partial^2 \Delta\alpha}{\partial x^2} - \frac{\rho}{2} (\omega^2 x^2 + v^2) b^2 \frac{\partial c_m}{\partial \alpha} \Delta\alpha \qquad (1)$$

The second term represents the additional moment about the axis of revolution, which is exerted by the aerodynamic forces, when the angle of attack α of the wing element is changed by the torsional vibrations. c_m is the non-dimensional moment coefficient of the aerodynamic force with respect to the axis of revolution. ρ is the air density; $\sqrt{\omega^2 x^2 + v^2}$, the resultant velocity, with the component ωx equaling the tangential velocity and v

the parallel flow in the propeller disk, approximating the flight speed. The width of the blade is represented by b.

Equation (1) assumes the vibration to be "slow." Both the torsional velocity $(\partial\Delta\alpha/\partial t)$ and the involved variation in the effective angle of attack of a blade element were disregarded, as also the effect of the vortices periodically released from the vibrating blade. That the resulting error is unimportant, is demonstrated by a subsequent consideration of the reduced frequency,* which is the criterion for the slowness of a blade vibration. The reduced frequency of a propeller blade is 10 to 100 times smaller than that of a normal wing.

b) Solution of the torsional equation.- The lowest natural frequency of the vibration represented by equation (1) is obtained by the formula

$$\Delta\alpha\,(x,t) = y\,(x)\,\sin\lambda\,t.$$

Equation (1) then becomes

$$G\,J\,\frac{d^2y}{dx^2} + \left[J_m\,\lambda^2 - \frac{\rho}{2}\,(\omega^2\,x^2 + v^2)\,b^2\,\frac{\partial\,c_m}{\partial\alpha}\right] y = 0 \qquad (2)$$

If

$$\frac{1}{GJ}\left(J_m\,\lambda^2 - \frac{\rho}{2}\,v^2\,b^2\,\frac{\partial\,c_m}{\partial\alpha}\right) = C_1$$

and

$$\frac{1}{GJ}\left(\frac{\rho}{2}\,\omega^2\,b^2\,\frac{\partial\,c_m}{\partial\alpha}\right) = C_2$$

then equation (2) will read:

$$\frac{d^2y}{dx^2} + (C_1 - C_2\,x^2)\,y = 0 \qquad (3)$$

If the solution of equation (3) is put in the form of an infinite potential series and their coefficients are determined by the introduction of the formula into the differential equation, a simple calculation yields

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*Birnbaum, "Das ebene Problem des schlagenden Flügels." Zeitschr. f. angew. Math. u. Mech., 1924, p. 277.

$$y = \sin \sqrt{C_1}\, x - C_2 \sqrt{C_1} \left(\frac{6}{5!} x^5 - \frac{26}{7!} C_1 x^7 + \frac{68}{9!} C_1^2 x^9 - \right.$$

$$\left. - \frac{252}{9!} C^2 x^9 + \ldots \right) \qquad (4)$$

This series and likewise the series obtained from it by differentiation converge as alternating series with terms whose absolute values continuously decrease. Hence, equation (4) represents a solution of equation (3).

The marginal condition at the point of fixation ($y = 0$ for $x = 0$) is directly fulfilled. The condition for the free end ($x = l$) requires that the torsional moment shall there disappear:

$$\left(\frac{d\,y}{d\,x} \right)_{x=l} = 0.$$

This condition yields an equation for determining the frequency λ.

If, instead of $x = l$, the differentiated equation (4) is solved by using only the first terms according to λ, by putting λ as the sum of the known torsional frequency in the absence of aerodynamic forces and of a correction factor Δ, we obtain

$$\lambda = \frac{\pi}{2\,l} \sqrt{\frac{G\,J}{J_m}} - \Delta.$$

If further solved according to Δ, it becomes

$$\lambda = \sqrt{\frac{G\,J}{J_m\,l^2} \left(\frac{\pi}{2} - \Delta \right)^2 - \frac{\frac{\rho}{2} v^2 b^2 \frac{\partial c_m}{\partial \alpha}}{J_m}}$$

$$= \sqrt{\lambda_o^2 \frac{4}{\pi^2} \left(\frac{\pi}{2} - \Delta \right)^2 - \frac{\frac{\rho}{2} v^2 b^2 \frac{\partial c_m}{\partial \alpha}}{J_m}} \qquad (5)$$

Here λ_o denotes the frequency of the propeller at rest. Moreover, the following formula is very accurate.

$$\Delta = \frac{-1 + 0.086\ C_2 l^4 + \sqrt{1 - 0.172\ C_2 l^4 + 0.0148\ C_2^2 l^8}}{0.0216\ C_2 l^4} \qquad (5a)$$

Δ and the second expression under the radical in equation (5) disappear for the propeller at rest: $\omega = 0$, $v = 0$. In this case equation (5) becomes

$$\lambda = \lambda_o = \frac{\pi}{2 l} \sqrt{\frac{G J}{J_m}} \qquad (6)$$

c) Results.- The numerical evaluation of formulas (5) and (6) with respect to the resonance vibrations of the propeller blades is here the most important. If we select, as a sample of thin-bladed metal propellers, a rectangular bar of the following dimensions and material constants:

$l = 150$ cm, $\quad G = 3 \times 10^5$ kg cm^{-2} and

$b = 20$ " $\quad \rho_m = 3 \times 10^{-6}$ kg cm^{-4} s^2

$d = 2$ "

we will surely obtain unfavorable results, since a real metal propeller is torsionally about twice as rigid. If calculated according to formula (6), the stand frequency of the chosen example is $\lambda_o = 660$ s^{-1}.

If we also adopt the following characteristic numbers for the operating conditions of the propeller,

$v = 6000$ cm s^{-1}, $\quad \rho = 1.25 \times 10^{-9}$ kg cm^{-4} s^2

and

$$\frac{\partial c_m}{\partial \alpha} = 0.4^\circ \quad \text{(circular measure)}$$

and calculate with these numbers the torsional frequencies according to equations (5) and (5a) for all possible angular velocities ω of the propeller, we obtain the values plotted in Figure 1.

It is known that, in the practical range of angular velocities of $0 \leqq \omega \leqq 200$, the aerodynamic forces have almost no effect on the torsional frequencies of the propeller.

Remembering that the adopted form of calculation represents, through its assumptions in that respect, an underestimate of the torsional frequencies (see also Section III, 2,a) and that the chosen example was especially unfavorable, we obtain the further result that the natural torsional frequencies of propellers differ so greatly from the frequency of the impulses corresponding to the revolution speed, that resonance in the form of torsional vibrations does not enter into the question.

Figure 1 represents a second result. The torsional vibrations become unstable when the expression under the radical in equation (5) disappears. This happens for our example, when $\omega \approx 1300\ s^{-1}$. Obviously equation (5) becomes meaningless for $\omega > 200\ s^{-1}$, since the peripheral velocities of the propeller are so great in this region that the aerodynamic assumptions of the calculation lose their validity. Figure 1 shows, however, that torsional vibrations, such as can be calculated for wings on the basis of the simple assumptions (static instability) here made, do not enter into the problem for propeller blades, so long as their peripheral velocities remain below the velocity of sound.* It is true that the critical angular velocity for torsional vibrations may be considerably reduced by combination with bending vibrations, though it is improbable that it would drop into the region of $\omega < 200\ s^{-1}$. It is found* that the critical velocity is most reduced, when the torsional and bending vibrations most nearly agree. For propellers, however, the ratio of bending to torsional frequency differs much from unity.

Somewhat different is the possibility of vibrations being occasioned by the special aerodynamic relations in the vicinity of the velocity of sound. Here, however, no mathematical determination is yet possible.

In limiting the present report to resonance vibrations produced by external disturbances, the torsional vibrations on the basis of the above-mentioned results can be disregarded in the further investigation.

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*Blenk and Liebers, "Gekoppelte Torsions-, Biegungs- und Querruderschwingungen von freitragenden und halbfreitragenden Flugeln," Luftfahrtforschungen, Vol. IV (1929), p. 69.

2. Bending Vibrations

a) Estimation of effect of aerodynamic and centrifugal forces on bending vibrations.- We will again attempt to simplify the general problem by estimating, from an idealized example, the magnitude of the aerodynamic and centrifugal forces in comparison with the elastic forces.

For this purpose the flexible propeller blade is replaced by a rigid blade flexibly attached to the hub by a spring, so that it can vibrate perpendicularly to the plane of revolution. (Fig. 2.) If this blade is bent through the small angle $\Delta\beta$, the reacting elastic moment will be

$$M = - c \Delta \beta \tag{7}$$

The centrifugal force, acting on a blade element of length dx and cross section F(x) at the distance x from the center of revolution, exerts on the lever arm $x \sin \Delta \beta$ a moment:

$$d M_z = - \rho_m F(x) \, d x \, \omega^2 \, (x \cos \Delta \beta + \epsilon) \, x \sin \Delta \beta$$

Here ϵ is the radius of the nonvibrating propeller hub. For the whole blade (on the assumption of a small angle $\Delta\beta$) the reacting centrifugal force is

$$M_z = - \rho_m \omega^2 \Delta \beta \left(\int_0^l F(x) \, x^2 \, d x + \epsilon \int_0^l F(x) \, x \, d x \right)$$

$$= - \omega^2 \Delta \beta \, (\theta + \epsilon S) \tag{8}$$

where θ denotes the inertia moment and S the static moment of the blade, both with reference to the axis of fixation.

Due to the velocity of the bending motion $d\Delta\beta/dt$, the effective angle of attack of a blade element varies, such variation being proportional to the distance x from the axis of fixation. Consideration of the velocity triangle of the relative motion (fig. 3) yields a variation of

$$\delta \alpha = \frac{x}{\sqrt{\omega^2 x^2 + v^2}} \frac{d \Delta \beta}{d t} \tag{9}$$

in the angle of attack. Hence there is produced on the blade element of length dx and width b(x) an additional air damping, exerting a moment of $x \cos \Delta \beta$ on

the lever arm (fig. 2), which, according to the customary aerodynamic method of notation, is written

$$d\,M_L = -\frac{\partial c_a}{\partial_a}\,\delta\,\alpha\,\frac{\rho}{2}\,(\omega^2\,x^2 + v^2)\,b(x)\,dx\,x\cos\Delta\beta$$

The aerodynamic force is overestimated only when the variable blade width $b(x)$ is exchanged with the maximum width b. If this is integrated after introducing the value (equation 9) for $\delta\alpha$, the equation for the whole blade reads

$$M_L = -\frac{\rho}{2}\,b\,\frac{\partial c_a}{\partial\alpha}\,\frac{d\,\Delta\beta}{dt}\int_0^l \sqrt{\omega^2\,x^2 + v^2\,x^2}\,dx.$$

The elementary integral in M_L is very difficult to write. If its value is expressed by a series, we obtain with sufficient accuracy the aerodynamic moment

$$M_L = -\frac{\rho}{2}\,b\,\frac{\partial c_a}{\partial\alpha}\,\frac{d\,\Delta\beta}{dt}\,\omega\,\frac{l^4}{4}\left[1 - \left(\frac{v}{l\omega}\right)^2\right] \qquad (10)$$

After combining all of the forces acting on the vibrating blade, the descriptive equation of motion reads:

$$\theta\,\frac{d^2\,\Delta\beta}{d\,t^2} = M_E + M_Z + M_L$$

or, with the aid of equations (7), (8), and (10),

$$\theta\,\frac{d^2\,\Delta\beta}{d\,t^2} + \frac{\rho}{2}\,b\,\frac{\partial c_a}{\partial\alpha}\,\omega\,\frac{l^4}{4}\left[1 + \left(\frac{v}{l\omega}\right)^2\right]\frac{d\,\Delta\beta}{d\,t} + \left[\omega^2\,(\theta + \epsilon\,S) + c\right]\Delta\beta = 0 \qquad (11)$$

If the solution of equation (11) is put in the form

$$\Delta\beta = e^{(\kappa + i\lambda)t}$$

the frequency λ becomes

$$\lambda = \sqrt{\frac{c}{\theta} + \omega^2\left(1 + \epsilon\,\frac{S}{\theta}\right) - \omega^2\left(\frac{\rho}{2}\right)\frac{b^2\,l^8}{64\,\theta^2}\left(\frac{\partial c_a}{\partial\alpha}\right)^2\left[1 + \left(\frac{v}{l\omega}\right)^2\right]^2}$$

$$= \sqrt{\lambda_o^2 + \lambda_\omega^2 - \lambda_L^2} \qquad (12)$$

where λ_o is the frequency when only the elastic force acts.

λ_ω is the frequency when only the centrifugal force acts.

λ_L is the decrease in the frequency due to the aerodynamic force.

If we now choose the following values, corresponding to a thin-bladed tapering metal propeller at 1500 r.p.m.

$l = 150$ cm $\qquad \rho_m = 3 \times 10^{-6}$ kg cm^{-4} s^2

$b = 20$ " $\qquad \theta = 1/12 \ \rho_m F_o l^3 = 33.75$ kg cm s^2

$d = 2$ " $\qquad S/\theta = 2/l = 0.0133$ cm^{-1}

$\epsilon = 15$ " $\qquad c/\theta = \lambda_o^2 = 2.25 \times 10^4$ s^{-2}

$\rho = 1.25 \times 10^{-9}$ kg cm^{-4} s^2

$v = 6000$ cm s^{-1}

$\omega = 150$ cm s^{-1}

$$\frac{\partial c_a}{\partial \alpha} = 4.5$$

we obtain, according to equation (12)

$\lambda_o^2 = 22500$ s^{-2}; $\lambda_\omega^2 = 27000$ s^{-2}; $\lambda_L^2 = 300$ s^{-2};

and $\lambda^2 = 49200$ sec^{-2};

or the corresponding revolution speeds

$\nu_o = 1430$ min^{-1}; $\nu_\omega = 1570$ min^{-1}; $\nu_L = 170$ min^{-1}

and $\nu = 2120$ min^{-1}.

For the provisionally assumed ideal propeller blade, the above numbers give a good idea of the probable order of magnitude of the bending frequencies of actual aircraft propellers and of the importance of the individual components of the frequency λ and also of ν. The aerodynamic forces, which are noticeable only as damping forces, are

negligible as regards the magnitude of the bending frequencies. In their order of magnitude, the bending frequencies approximate the revolution speed, thus making resonance vibrations possible.

Before proceeding to a more accurate investigation of the actual bending frequencies of propellers, we will mention several special cases of formula (12), which will subsequently be used.

1. Omitting the unimportant term $\frac{\partial c_a}{\partial \alpha} = 0$, derived from the aerodynamic forces, equation (12) becomes

$$\lambda = \sqrt{\lambda_0^2 + \lambda\omega^2} = \sqrt{\lambda_0^2 + \omega^2 \left(1 + \epsilon \frac{S}{\theta}\right)} \qquad (13)$$

In case of the linear reduction of the cross section $F = F_0 \left(1 - \frac{x}{l}\right)$, as approximately happens for nearly all propeller types, equation (13) reads:

$$\lambda = \sqrt{\lambda_0^2 + \omega^2 \left(1 + 2 \frac{\epsilon}{l}\right)} \qquad (13a)$$

For comparison, we give the corresponding formula,

$$\lambda = \sqrt{\lambda_0^2 + \omega^2 \left(1 + \frac{3}{2} \frac{\epsilon}{l}\right)} \qquad (13b)$$

for a propeller blade of uniform cross section.

2. With a vanishingly small hub ($\epsilon = 0$), we have

$$\lambda = \sqrt{\lambda_0^2 + \omega^2} \qquad (14)$$

for any desired cross-sectional taper.

3. With vanishing elasticity of the propeller blade ($\lambda_0 = 0$) or at very great angular velocities ($\omega = \infty$), we have

$$\lambda = \lambda_\omega = \omega \sqrt{1 + \epsilon \frac{S}{\theta}} \qquad (15)$$

If simultaneously $\epsilon = 0$, equation (15) becomes

$$\lambda = \omega \qquad (16)$$

The assumption of vanishing elasticity corresponds to the case of a propeller blade flexibly attached to its hub, or to the case of a rope which vibrates perpendicularly to its plane of revolution (which, due to the rectilinear form of vibration, obviously amounts to the same thing).

b) Determination of the bending frequencies of elastic propeller blades (preliminary assumptions).- In order to obtain reliable numerical data for practical application, the bending frequencies (only the order of magnitude of which has thus far been estimated) must now be calculated by a more accurate method, which takes into consideration the elastic properties of the blades.

After we decided (as explained in Section III, 2,a) to reckon with simple bending vibrations, not combined with torsion, we considered the propeller blade as an unwarped blade, in which the main axes of all the cross sections are parallel to one another. Hereby the direction of the main axis corresponding to the smallest inertia moment in each cross section can be inclined to the propeller disk, and the vibrations are not necessarily perpendicular to the plane of revolution. The position of equilibrium, about which the blade vibrates, may be a deflection from the plane of the propeller disk, either produced by the thrust or included in the original design, provided only that it remain in the region in which the acting forces can be treated with sufficient accuracy as linear functions of the coordinates. Moreover, we have made all necessary assumptions for the application of the elementary elasticity theory.

It only remains to characterize the shape of the propeller blade by designating the course of the cross-sectional area and inertia moment. Both are assumed to vary with the propeller radius according to the simple potential law:

$$\left.\begin{array}{l}\text{Cross section, } F=F_0\, f(\xi)=F_0(1-\xi)^{\kappa}\\ \text{Inertia moment, } J=J_0\, i(\xi)=J_0(1-\xi)^{\vartheta}\end{array}\right\}\quad \left.\begin{array}{l}\xi=\dfrac{x}{l}\\ 0\leqq\xi\leqq 1\end{array}\right\} \qquad (17)$$

x is the distance of a cross section from the center and l is the blade length. The exponents κ and ϑ are any desired parameters to be adapted to the given propeller blade. According to the dimensions of ordinary propeller

blades, we can nearly always write $\kappa = 1$. ϑ usually lies between 2 and 3.5*.

c) Calculation of the bending frequencies as a minimum problem.- In explanation of why the general differential equation for the bending vibrations of revolving propeller blades is not used for solving the problem, attention is called to the difficulties and the inconvenience of the results of the mathematically exact treatment of the bending vibrations of tapering nonrevolving bars**, so that graphs or other approximation methods*** must generally be employed.

Section III,2,d explains the simple method of solution which can be used, when the Rayleigh principle of variation leads from the minimum of the natural frequencies approximately to an ordinary minimum problem with only one variable.

For this purpose, on the basis of a suitable but fundamentally arbitrary selection from the numerous forms of vibration $y(\xi)$, we select the following group of simple infinitely numerous bending lines:

$$\frac{y(\xi,m)}{y_{max}} = \frac{1}{m+2}\left[(1-\xi)^{m+3}+(m+3)\,\xi-1\right] \qquad (18)$$

and adopt only this as the comparative function for the Rayleigh minimum requirement. Here m is a continuously varying parameter, to every value of which there corresponds a different vibration line. This group of curves (eq.18) is plotted in Figure 4.

For any value of m, equation 18 may be considered as an infinite potential series, whose coefficients, from the outset, are so established as to satisfy the marginal conditions. For whole numbers m, there is derived from it an

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* F. Seewald, "Beitrag zur Ermittlung der Beanspruchungen und der Formänderungen von Luftschrauben." Jahrb. d. W.G.L. 1926. See Figures 4 and 7, in which the results of testing an ordinary metal propeller are plotted.

** Schwerin, Verh. II. Internat. Kongr. f. techn. Mech., Zurich, 1926, p. 138.

*** W. Hort, "Berechnung der Eigentöne nicht gleichförmiger, insbesondere verjüngter Stäbe." Z.f.techn, Phys. 1925, p. 181. See also Hütte, Vol. I, edition 25, p. 403.

infinite potential series with $m + 3$ terms.

The four marginal conditions, disappearance of amplitude ($y = 0$) and inclination ($y' = 0$) at the point of fixation ($\xi = 0$) and of the bending moment ($y'' = 0$) and shearing force ($y''' = 0$) at the free end ($\xi = 1$), are satisfied by equation (18) for $m > 0$. All the marginal conditions are no longer fulfilled, however, for $m \leq 0$, a case which is here of relatively little importance. According to Rayleigh's theory, the curves (18) may even then be adopted as hypothetical bending lines. The resulting frequencies can likewise be too large.

For $m = 1$, equation (18) passes into the known bending line of the blade of uniform cross section fixed at one end, under the load of its own weight. It should be here noted that, for the blade of uniform cross section in the absence of centrifugal forces, the static bending line and the true vibration line so nearly coincide, that the difference in the frequency of the two lines is less than $\frac{1}{2}$ per cent. The curves $m < 1$ and $m > 1$ are derived from the line $m = 1$ by continuous transformation in the direction of an increase or decrease in the mean curvature. For the line $m = \infty$, equation (18) becomes a straight line.

The Rayleigh minimum requirement will now select the strongly bent curves as bending-elastic lines from the group of curves placed at its disposal in the case of small revolution speeds and centrifugal forces. The flatter bending lines will better satisfy the minimum requirement as the revolution speed increases until, at infinitely great revolution speeds where the elastic forces must be disregarded, the straight line ($m = \infty$) best satisfies the Rayleigh requirement.

d) General solution of the minimum problem.- For the actual solution of the minimum problem, the group of bending lines (18) must now be introduced in some expression for the frequency λ. The energy equation is suitable for this purpose. If the propeller blade is in the position of equilibrium, only kinetic energy is present. In the position of the maximum vibration amplitudes, however, the kinetic energy disappears and all the energy is stored up in the form of potential energy. This is the sum of the absorbed energy of transformation and of the work performed against the centrifugal forces, calculated from the position of equilibrium to the attainment of the maximum vibration amplitude. According to the energy theorem, both

energies (the kinetic energy T and the sum of the transformation work U_1 and the centrifugal work U_2) must be equal.

For the fundamental vibration $Y = y(\xi) \sin \lambda t$ the kinetic energy in the position of equilibrium is

$$T = \int_0^l \frac{d\,\mu\, v^2_{max}}{2} = \frac{1}{2} \rho_m\, l\, \lambda^2\, F_0 \int_0^1 f(\xi)\, y^2(\xi)\, d\,\xi \qquad (19)$$

($d\,\mu$ = mass of blade element, which vibrates with the velocity v_{max} through the zero position.)

The transformation work U_1 is

$$U_1 = \int_0^l \frac{M^2\, d\,x}{2\, E\, J} = \frac{E\, J_0}{2\, l^3} \int_0^1 i(\xi)\, y''^2(\xi)\, d\,\xi \qquad (20)$$

($M = E\, J\, y''$ = bending moment.)

Lastly the centrifugal work is the product of the centrifugal force and the radial displacement ζ of its point of application during a vibration. (Fig. 5.) It is therefore,

$$U_2 = \int_0^l d\,\mu\, x\, \omega^2\, \zeta = \rho_m\, l^2\, \omega^2\, F_0 \int_0^1 f(\xi)\, \xi\, \zeta\,(\xi)\, d\,\xi$$

The radial displacement ζ at the distance x from the blade center, according to a simple geometric consideration (see Stodola "Dampf- und Gasturbinen," par. 195), is

$$\zeta = \frac{1}{2} \int_0^x \left(\frac{dy}{dx}\right)^2 d\,x =$$

$$= \frac{1}{2\,l} \int_0^\xi y'^2(\xi)\, d\,\xi.$$

The centrifugal force is therefore finally

$$U_2 = \rho_m\, l^2\, \omega^2\, F_0 \int_0^1 f(\xi)\, \xi\, \frac{1}{2l} \int_0^\xi y'^2(\xi)\, d\,\xi\, d\,\xi \qquad (21)$$

If we put $T = \lambda^2\, T'$ and solve according to λ^2, the energy equation $T = U_1 + U_2$ then reads

$$\lambda^2 = \frac{U_1 + U_2}{T'}$$

$$= \frac{E J_o}{\rho_m F_o l^4} \frac{\int_0^1 i\, y''^2\, d\xi}{\int_0^1 f\, y^2\, d\xi} + \omega^2 \frac{\int_0^1 f\, \xi \int_0^\xi y'^2\, d\xi\, d\xi}{\int_0^1 f\, y^2\, d\xi} \qquad (22)$$

If we here introduce the values $f(\xi,\kappa)$ and $i(\xi,\vartheta)$ for the course of the cross section and inertia moment over the blade length (equation 17) and the indeterminate expression $y(\xi,m)$ for the vibration elastic line according to equation (18), the determinate integrals will then be functions of m,κ and ϑ. Equation (22) then assumes the form

$$\lambda^2 = \frac{E J_o}{\rho_m F_o l^4} X_1 (m,\kappa,\vartheta) + \omega^2 X_2 (m,\kappa) \qquad (23)$$

where the function X_1 and X_2 are identical with the two quotients of the determinate integrals in equation (22).

Hence we have an equation, which, for given characteristics of the propeller (l, F, J, E, ρ_m, κ, ϑ) and given angular velocity ω, yields the frequency λ in terms of the parameter m. The minimum value of the frequency, $\lambda = \lambda_{min}(m)$ (which is contained in equation (23) and which, according to Rayleigh, is the one most closely approximating the true value) is then expressed by

$$\frac{\partial \lambda}{\partial m} (l, F, J, E, \rho_m, \kappa, \vartheta, \omega, m) = 0 \qquad (24)$$

The value m (derived from equation 24) = m (l, F, J, E, ρ_m, κ, ϑ, ω) introduced into equations (23) and (22), yields the desired frequency.

The problem is thus practically solved. Before passing, however, to the actual mathematical determination of the minima of equations (22) and (23), the latter equation is transformed as follows. For the case of rest ($\omega = 0$), the second term drops out and we have for the bending frequency on the stand

$$\lambda_e^2 \equiv \lambda^2_{\omega=0} = \frac{E J_o}{\rho_m F_o l^4} X_{1\,min} (m',\kappa,\vartheta) \qquad (25)$$

By m' is expressed the fact that the minimum function

X_1 is already found. $X_{1\,min}$ is a constant for determinate pairs of values κ, ϑ.

If the value (25) is introduced into equation (23), the latter then becomes

$$\lambda^2 = \frac{X_1\ (m,\kappa,\vartheta)}{X_{1\,min}\ (m',\kappa,\vartheta)}\ \lambda_0^2 + X_2\ (m,\kappa)\,\omega^2 \tag{26}$$

or

$$\left(\frac{\lambda}{\lambda_0}\right)^2 = \bar{X}_1\ (m,\kappa,\vartheta) + X_2\ (m,\kappa)\ \left(\frac{\omega}{\lambda_0}\right)^2 \tag{27}$$

whereby the new abbreviation $\bar{X}_1 = \frac{X_1}{X_{1\,min}}$ is introduced. In the form of equation (27) there are only two important variables, λ/λ_0 and ω/λ_0. This greatly simplifies the calculation, for we now have, through equation (27), equal κ and ϑ values, i.e., for all propellers of the same taper, regardless of other structural differences, a single ratio between the bending frequency and the angular velocity of the propeller.

e) Bending frequencies of resting and revolving propellers.- In order to solve numerically the minimum problem, stated in general form in the last paragraph, the functions X_1 and X_2 in equation (27) must now be written in explicit form. $\bar{X}_1$ (the same as X_1) and X_2 are, according to equation (23), abbreviations for the quotients of the determinate integrals in equation (22). These must therefore be evaluated by substituting for f, i and y their values from equations (17) and (18). If the integrals in equation (22), freed from their coefficients, are designated by $\bar{U}_1$, $\bar{U}_2$ and $\bar{T}'$, we then have

$$\bar{U}_1 = \int_0^1 i\ y''^2\, d\,\xi = \int_0^1 (1-\xi)^{\vartheta}\ (m+3)^2\ (1-\xi)^{2(m+1)}\ d\,\xi =$$

$$= \frac{(m+3)^2}{2\,m+3+\vartheta} \tag{28}$$

and likewise,

$$\bar{U}_2 = \int_0^1 f\ \xi \int_0^{\xi} y'\, d\,\xi\, d\,\xi =$$

$$= \left(\frac{m+3}{m+2}\right)^2 \left[\frac{1}{(\kappa+1)\ (\kappa+2)}\left(\frac{1}{2m+5} + \frac{2}{\kappa+3} - \frac{2}{m+3}\right) - \right.$$

$$- \frac{1}{(2m+5)\ (2m+\kappa+6)\ (2m+\kappa+7)} +$$

$$\left. + \frac{2}{(m+3)\ (m+\kappa+4)\ (m+\kappa+5)}\right. \tag{29}$$

$$\overline{T}' = \int_0^1 f\, y^2\, d\,\xi = \frac{1}{(m+2)^2}\left[\frac{2\,(m+3)}{(\kappa+1)\,(\kappa+2)}\left(\frac{m+3}{\kappa+3}-1\right)+\right.$$

$$\left.+\frac{1}{\kappa+1}+\frac{1}{2m+\kappa+7}+\frac{2}{m+\kappa+4}\left(\frac{m+3}{m+\kappa+5}-1\right)\right] \tag{30}$$

If we derive from this, by division, $\overline{U}_1/\overline{T}' = X_1(m, \kappa, \vartheta)$ and $\overline{U}_2/\overline{T}' = X_2(m, \kappa)$, and introduce the quotients into equation (22), we obtain equation (23). If we now determine the minimum of $\overline{U}_1/\overline{T}' = X_{1\,min}$ and introduce this into equation (23), we obtain equation (27).

The course of the functions X_1 and X_2 is shown graphically in Figure 6, in which X_1 and X_2 are plotted against the parameter m and indeed, for the most important cases of linear cross-sectional taper of the propeller blades ($\kappa = 1$) with simultaneous reduction of the inertia moment between $0 \leq \vartheta \leq 3$. From Figure 6, it may be seen that X_2 with increasing m tends toward the limit 1, corresponding to equation (18), which changes to a straight line for $m = \infty$. (See also fig. 4.) This limit is the minimum of X_2 for all values of κ, as may be verified by calculation. The various functions X_1 for the corresponding ζ values have their minima in the vicinity of $m = 0$. These minima are again represented in Figure 7.

α) The frequency of a propeller at rest.- $X_{1\,min}$ accords with equation (25) of the numerical factor, dependent on the tapering factors κ and ϑ, in the expression for the frequency λ_0^2 of the propeller at rest. Figure 7, therefore, already contains a numerical table for the nondimensional stand frequency

$$\lambda_0^2 = \frac{\rho_m F_0 l^4}{E J_0}$$

of all propeller blades with linear cross-sectional taper and any decrease in the inertia moment between $\vartheta = 0$ and $\vartheta = 3$. This class comprises practically all conventional propellers. In designing a new propeller, it would be well to make use of Figure 7. If, however, it concerns the stand frequency of a finished propeller, its experi-

mental determination is always simplest and most reliable.

For verifying the calculation, the stand frequencies are indicated by crosses in Figure 7, as obtained from Hort's formulas for slightly tapered blades. These values all lie above the ones here determined. Since, however, the latter, on account of their derivation, can certainly not be too small, they come nearer the true frequencies than the values of Hort.

For the case of simultaneously linear reduction of cross section and inertia moment ($\kappa = 1$, $\vartheta = 1$), we have for comparison the exact solution of Ono:*

$$\lambda_0^2 \frac{\rho_m F_0 l^4}{E J_0} = 51.2$$

in perfect agreement with the value here obtained (as indicated in Figure 7 by a small square). Likewise, for the case of the blade with uniform cross section ($\kappa = \vartheta = 0$), our calculation must be carried to the third decimal place, in order to show a discrepancy in comparison with the exact value 12.3596.

For stronger tapers, where the Hort method yields too high frequencies, the more accurate integral-equation method of Schwerin is used for comparison. It yields, e.g., in the case corresponding to $\kappa = 1$, $\vartheta = 2$, the value of λ_0^2 indicated in Figure 7 by a small triangle. In λ_0 our calculation differs from this by only about 2 per cent.

β) The frequencies of a propeller in motion.- For the calculation of the bending frequencies of a revolving propeller, we now have to determine the minima of λ/λ_0 by equation (27) for all pertinent tapers κ and ϑ. It was not advisable to determine this in the usual manner by solving the derivative $\partial(\lambda/\lambda_0)/\partial m$ put equal to zero according to m, on account of the long and troublesome expressions for $\bar{X}_1$ and $\bar{X}_2$ for differentiating. (See formulas 28, 29, and 30.) (This was not done even in the determination of λ_0.) It is more practical to dispense with any closed expression for the frequencies and to determine them directly from equation (27), by observing for what values of the parameter m with given κ and ϑ

*A. Ono, Journ. Soc. Mech. Eng., Tokyo, 1925.

values, the expression $(\lambda/\lambda_0)^2$ increases its minima for the different ratios $(\omega/\lambda_0)^2$. Since Figure 6 places all the pertinent values of the functions X_1 and X_2 (as likewise the function $\bar{X}_1$, which differs from X_1 only by the constant factor $1/X_{1\,min}$) at our disposal, the essential part of this work is already done. The results are shown in Figure 8.

Figure 8, moreover, contains a nomogram from which the bending frequency at any angular velocity can be read directly for every propeller of any strength, dimensions and taper (i.e., values of λ_0). In this connection there is assumed a knowledge of the frequency on the stand, as based on equation (25) and Figure 7, or (still better), on experimental determination.

The mutual position of the curves in Figure 8 is in the right sense. For constant λ_0, the ratio κ/ϑ, i.e., the less the cross section tapers in proportion to the inertia moment, the greater will be the effect of the centrifugal forces and just so much higher will be the frequencies. For the various cases of tapering in the region of κ, the ϑ values represent the differences in the growth of the frequencies, as shown in Figure 8, though only to a slight degree. The cases of quadratic cross-sectional tapering ($\kappa = 2$) with simultaneous reduction of the inertia moment ($0 \leqq \vartheta \leqq 3$), which were likewise calculated, yielded curves so closely coinciding with those already plotted, that they had to be left out of Figure 8. Greater differences occur with stronger tapers, for which, however, only the region up to the line $\lambda = \lambda_0 + \omega$ is available, which is valid for a blade of any cross section with vanishing inertia moment ($\vartheta = \infty$).*

We conclude therefore that two propellers, which may otherwise differ in all their dimensions, material constants and (within wide limits) even in their tapers, but which have the frequency λ_0 on the stand, also have practically coincident bending frequencies while revolving.

This fact justifies the simplification effected by replacing the curves of Figure 8, which embrace all prac-

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* $\vartheta = \infty$ indicates the vanishing of the bending stiffness of the blade. Equation (16) of Section IV, 2,a is therefore valid ($\lambda = \omega$ and $\lambda = \lambda_0 + \omega$ with $\lambda_0 = 0$).

tically important cases, by a single curve. This curve is closely approximated by the interpolation formula

$$\frac{\lambda}{\lambda_\theta} = 1 + \frac{7\left(\frac{\omega}{\lambda_o}\right)^2}{6 + 7\left(\frac{\omega}{\lambda_o}\right)} \qquad (31)$$

The curve corresponding to equation (31) takes its course in the region $0 \leqq \omega/\lambda_o \leqq 3$ in the middle of the group of curves plotted in Figure 8. Hence equation (31) represents, in abbreviated form, the net result of this theorem.

γ) Frequencies of a revolving propeller with respect to the effect of the hub.- Up to this point, we have entirely disregarded the fact that the conventional propeller blade is attached to an assumedly rigid hub, which is not affected by the bending vibrations of the blade. As a result of this arrangement, the blade does not turn about an axis passing through its root, but about an axis outside the actual blade. This results in augmenting the centrifugal forces acting on a blade element and increases the bending frequencies.

There would have been no serious difficulty in making allowance for this propeller effect in our formulas and calculations. Still our formulas would have been rendered more complex by the addition of another parameter, the ratio of the hub radius to the blade length. Hence it was at first disregarded, in order to avoid purely mathematical difficulties. In consideration of the small hub radius of a propeller as compared with the blade length, the previous results and formulas will be corrected in the following approximate manner.

On the basis of the formulas for the quasi-bending vibrations of a blade inherently rigid but elastically mounted (Section IV, 2,a), the frequencies of a blade mounted on a hub of length ϵ are increased as compared with the frequencies of a blade rotating about its inner edge in the ratio

$$\sqrt{\frac{\lambda_o^2 + \omega^2\left(1 + \epsilon\,\frac{S}{\theta}\right)}{\lambda_o^2 + \omega^2}} \qquad (32)$$

(See equations 13 and 14.) If it is now assumed that the

corresponding frequencies of a blade, which is elastic throughout its whole length, increase in the same ratio, then only the augmentation of the centrifugal forces due to the outward displacement of the blade is taken into consideration. No account is taken of the change in the vibration-elastic line due to the same circumstance. Though the forms of vibration previously employed as the basis of the calculations were the best of the available forms, this is no longer the case under the changed conditions. According to the Rayleigh principle, the frequencies calculated in the previously described manner for the hub effect can again be only too large.

By increasing the bending frequencies represented by equation (31) in ratio 32, we finally obtain for the bending frequencies of a propeller with respect to the effect of the hub:

$$\frac{\lambda}{\lambda_0} = \left[1 + \frac{7\left(\frac{\omega}{\lambda_0}\right)^2}{6 + 7\left(\frac{\omega}{\lambda_0}\right)^2}\right] \sqrt{\frac{1 + \left(1 + 2\,\frac{\epsilon}{l}\right)\left(\frac{\omega}{\lambda_0}\right)^2}{1 + \left(\frac{\omega}{\lambda_0}\right)^2}} \qquad (33)$$

Thereby the value $S/\theta = 2\,l$ is already introduced into equation (33) for all propellers linearly tapered in their cross section. (Cf. equation 13a.)

As soon as the frequency λ_0 of the nonrevolving blade is determined in any way, equation (33) generally yields the bending frequencies in terms of the revolution speed for any propeller with any ratio of hub radius to blade length. The formula is simple and very accurate, as we shall see. Equation (33) is plotted in Figure 9 for different values of the ratio of the hub radius to the blade length $0 \leq \epsilon/l \leq 0.5$. For ordinary propellers $\epsilon/l = 0.1$ to 0.2. There is a noticeable increase in the bending frequencies due to the consideration of the hub.

With equation (33) the mathematical determination of the bending frequencies of revolving propeller blades may be regarded as settled, as soon as the magnitude of the deviations from the true frequencies can be satisfactorily determined by accurate estimation of the errors. At present we know only that our formulas yield too great frequencies. Hence there still remains what is perhaps the most important part of the investigation, namely, the de-

termination of a limit which must not be exceeded by the true frequencies.

f) A universally valid lower limit for the frequencies of an elastic system.- For verifying the calculation made on the basis of the Rayleigh principle, the following simple theorem was first developed.

We will consider a system capable of vibrating, which is subjected to several directional forces acting either simultaneously or separately. The potential energy of the system then equals the sum of the potential energies of the n individual forces. If the potential energies freed from their constant coefficients are designated by U_i ($i = 1, 2, \ldots n$) and the kinetic energy of the system by $T = \lambda^2 T'$, the energy equation then reads

$$\lambda^2 T'(y) = a_1 U_1(y) + a_2 U_2(y) + \ldots + a_n U_n(y),$$

T' and U_i being functions of the vibration form y. From the energy equation follows the frequency

$$\lambda^2 = c_1 X_1(y) + c_2 X_2(y) + \ldots + c_n X_n(y) \qquad (34)$$

in which U_i/T' is shortened to X_i. If y is at first considered as unknown, it can be determined for the case of the fundamental vibration according to Rayleigh's theory by the requirement that the frequency must become a minimum. The true frequency is therefore represented by the variation formula

$$\lambda^2 = \left[c_1 X_1(y) + c_2 X_2(y) + \ldots + c_n X_n(y)\right] = \min \qquad (35)$$

Corresponding to the assumption that all the forces affect the considered system independently of one another, all the X_i in equation (34) can sometimes be made equal to zero. When only one of the n forces is regarded as active, this yields

$$\lambda_1^2 = c_1 X_{1\,\min}, \quad \lambda_2^2 = c_2 X_{2\,\min}; \quad \ldots, \quad \lambda_n^2 = c_n X_{n\,\min},$$

for the lowest frequencies of the system. If these values are introduced into equation (35), it becomes

$$\lambda^2 = \left[\frac{X_1}{X_{1\,\min}} \lambda_1^2 + \frac{X_2}{X_{2\,\min}} \lambda_2^2 + \ldots + \frac{X_n}{X_{n\,\min}} \lambda_n^2\right] = \min \qquad (36)$$

The minimum of a sum of functions cannot be smaller, however, than the sum of the minima of the individual summands. Since it is obvious that the minima of the summands in equation (36) have a total value of 1, we have the relation

$$\lambda^2 \geqq \lambda_1^2 + \lambda_2^2 + \ldots\ldots + \lambda_n^2.$$

The equality sign is valid only when all the summands in equation (36) assume their minima for the same value of the variable y. This happens only when the vibration form of the forces acting on the system is not affected, i.e., in the case of a vibrating system composed of rigid masses and springs. If this unimportant case is excluded, we have, for all actually elastic systems, the inequality

$$\lambda^2 > \sum_{i=1}^{n} \lambda_i^2 \qquad (37)$$

or, expressed in words, if several independent forces act simultaneously on any elastic system, the square of the frequency of the fundamental vibration is always greater than the sum of the lowest frequencies which the system would have, if only one of the forces were acting on the system.*

g) A lower limit for the bending frequencies of revolving propeller blades. Analysis of the results.- If the inequality (37) is applied to the special case of the propeller vibrations, it then reads:

$$\lambda^2 > \lambda_o^2 + \lambda_\omega^2$$

where λ_o again denotes the frequency of the propeller blade with disregard of the centrifugal force, i.e., the frequency of the blade at rest, and λ_ω the frequency

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*Inequality (37), in the special form for the present problem of propeller vibrations, was originally derived from equation (26) or (27) (see also Z. f. techn. Phys. 1929, pp. 361-369), to which formula (37) is closely related. Subsequently it was discovered that the general applicability of the lower limit (37) had been previously demonstrated by Southwell (Lamb and Southwell, Proc. Roy. Soc., Vol. 99, London, 1921). On account of the utility of the apparently little-known theory of Southwell, even in other cases, it is here given in general form, whereby, in the formal demonstration, we have again combined it with our equations (26) and (27).

with disregard of the elastic forces of the blade. In the latter case the propeller blade behaves like a perfectly flexible heavy rope. Equations (15) and (16) (Section IV, 2,a) are then valid, so that we obtain the following lower limits, which can be neither exceeded nor gone below by the bending frequencies:

For $\epsilon = 0$,

$$\lambda^2 > \lambda_0^2 + \omega^2 \tag{38}$$

For $\epsilon = 0$,

$$\lambda^2 > \lambda_0^2 + \omega^2 \left(1 + \epsilon \frac{S}{\theta}\right) \tag{39}$$

with $S/\theta = 2/l$ for propellers of conventional form with linear cross-sectional taper.

The lower limit (38) is shown in Figure 8, where it is the common lower limit for all the curves in the figure. For proving the relations with regard for the propeller effect (equation 33), both the cases $\epsilon/l = 0.1$ and $\epsilon l = 0.5$ are again represented in Figure 10 with the corresponding lower limits (39).

The greatest possible errors can now be read from Figures 8 and 10. For both large and small centrifugal forces, the percentile error tends toward zero. In the intermediate region it cannot be greater than 5 per cent according to Figure 8. In the case $\epsilon/l \neq 0$, the maximum error (as was to be expected) is about 2 per cent greater. As a matter of fact, it is considerably smaller, for the frequencies computed on the basis of the bending lines (18) by Rayleigh's method closely approximate the actual, while the lower limit only receives a more formal importance. For rough calculations, on the contrary, the simple equations (38) and (39) can be used with good results for the lower limits.

In this connection, it is of special importance that the lower limits (38) and (39) are entirely independent of any preliminary assumptions regarding the shape of the propeller blade. Inequality (37) is valid for any elastic system on the basis of its derivation. Likewise, the special lower limits (38) and (39) are valid for every revolving elastic system, which behaves like a revolving rope, when the centrifugal forces are regarded as acting

alone. The lower limits (38) and (39) can therefore be used just as well for estimating the frequencies of bent and warped propeller blades, whereby only the correct value (as determined experimentally) is to be used for λ_0. On this basis and the fact that formula (33), due to neglect of the torsion, contains an exaggeration of the bending frequency, no refinement of the theory by taking account of the exact form of a propeller is necessary from the practical standpoint.*

It is possible to go a step further. By the chosen marginal conditions (Section IV, 2,c), we assumed that the propeller blade was firmly secured at its root. This condition is not always perfectly fulfilled. If it is not fulfilled ($\delta y/\delta \xi \neq 0$ for $\xi = 0$), the frequencies of the propeller blade diminish, due to the removal of a compulsory condition. They cannot go lower, however, than indicated by the lower limits (38) and (39), which manifestly retain their validity. Hence, even in the case of the propeller blade which is not rigidly mounted, formula (33) can be successfully used, again with the assumption that the correct experimentally determined stand frequency is used for λ_0.

According to the results of the estimation of the maximum errors possible in equations (32) and (33) (as also on the basis of the comparisons made for the calculated frequencies of the nonrevolving propeller), it may be assumed that the practically requisite accuracy is satisfactorily obtained by the calculation.

Note.- In the initially cited report in "Z. f. techn. Phys. 1929," reference was made to various older formulas for the bending frequencies of revolving blades. Here it will only be mentioned that there is also given in Hütte Vol. I, ed. 25, page 407, a formula for "blades on the edge of a revolving disk," which, expressed in the here-used symbols, reads

$$\lambda^2 = \lambda_0^2 + \omega^2 \left(0.75 + 1.5 \frac{\epsilon}{l}\right).$$

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*The lower limits (38) and (39) are likewise valid (e.g.) for propellers of the Haw type, which consist of tensile rods having a metal envelope of the shape of a propeller blade.

Comparison with equation (39), where the corresponding value $S/\theta = 1.5/l$ (constant cross section) is to be inserted shows, however, that Hütte's formula yields values below the possible lower limit. For small ϵ/l ratios, λ may even be smaller than ω, according to this formula, so that, in the given case, it is possible to calculate from it resonances with the revolution speed which certainly cannot exist.

V. PRACTICAL RESULTS

1. General Conclusions

The whole of Section IV may be summarized, from the standpoint of its practical application by equation (33) for the calculation of the bending frequencies of revolving propeller blades, in case of need with the addition of equation (25) and Figure 7 for the frequency of the propeller at rest.

It may be concluded from equation (33) and likewise from Figure 9, that the bending frequencies are always greater than the revolution speeds. Any decided resonance between the natural frequencies of the blade and the disturbances, which follow with the frequency of the simple revolution speed, is therefore excluded.

The exact resonance point is not the only thing of importance, however. It is possible to calculate with complete damping of the impulses, only when the frequency of the impulses is considerably greater than the natural frequency. The less favorable case (impulse frequency smaller than the natural frequency) occurs in the bending vibrations of propellers. Hence the chief thing to know is how closely the natural frequencies can approach the revolution speeds for possible dimensions of the propeller blades.

As an illustrative example, the bending frequencies are plotted against the revolution speed in Figure 11 for a thin metal blade on the basis of Figure 7 and equation (33), and indeed in the customary units

$$\nu = \frac{\lambda}{2\pi} 60 \text{ min}^{-1} \quad \text{and} \quad n = \frac{\omega}{2\pi} 60 \text{ min}^{-1}.$$

The dimensions and other blade constants are given in the

following table. The curves I to IV therefore correspond to four propeller blades of equal length and root cross section, but of different tapers. Curve I represents the blade of uniform cross section, while curves II to IV represent blades of rectilinear cross section, but diminishing in inertia moment, sometimes rectilinearly, quadratically and cubically.

TABLE: DIMENSIONS OF METAL BLADES

Curve	E (kg cm^{-2})	ρ_m (kg cm^{-4} s^2)	l (cm)	J_o (cm^4)	F_o (cm^2)	κ	ϑ
I						0	0
II	7.8×10^5	3×10^{-6}	150	13	40	1	1
III						1	2
IV						1	3

In order to determine the danger limits for a given propeller blade, one must first know the vibration strength of the blade, expressed by the greatest permissible amplitude, and, secondly, the magnitude of the disturbing force, both of which can be determined.* It is then possible to conclude, on the basis of the known resonance curves, that, for a certain ratio of the disturbance frequency to the natural frequency, the amplitudes exceed the value indicated by the vibration strength. If it is assumed that this limit would be reached for $n/\nu = 0.8$ then, in Figure 11, the intersection points of the line $\nu = 1.25\ n$ with the different frequency curves indicate the revolution speeds at which the given propeller blade begins to produce dangerous vibrations. The prismatic blade would already be endangered at $n \approx 700$ r.p.m., but the tapered blades only above 1500 r.p.m.

*The disturbance experienced by a propeller near a wing (variation in the air velocity and in the angle of attack of the propeller blade) can be approximately calculated on the basis of simple formulas of the wing theory. For the velocity field behind a wing we already have data obtained for another purpose on an airplane in flight. See, for example, H. Schrenk, "Ueber Profilwiderstandmessung im Fluge nach dem Impulsverfahren." Luftfahrtforschung, May 18, 1928, pp. 1-32. For translation, see T.M. Nos. 557 and 558: "Measurement of Profile Drag on an Airplane in Flight by the Momentum Method," Parts I and II, N.A.C.A., 1930.

In Figure 11, moreover, the intersection points of curves I to IV with the lines $\nu = 2n$ should be especially noted. It is obvious that all the blades may fall into pronounced resonance vibrations, even at relatively low revolution speeds, as soon as there is a possibility of disturbances with a frequency of twice the revolution speed. This case is of special practical importance, as is obvious from the general representation of the disturbance due to any kind of disturbance field.

$$\sum_{i=1}^{\infty} \{A_i \sin i \omega t + B_i \cos i \omega t\} \qquad (40)$$

Resonance then always occurs, when

$$\lambda = i \omega \qquad (i = 1.2 \ldots, \infty) \qquad (41)$$

i.e., when the natural frequency of the propeller equals an integral multiple of the angular velocity or of the revolution speed. The cases $\lambda = i\omega$ for $i \geqq 3$ for all practical propellers (fig. 11, and especially curve I, of fig. 13) lie at revolution speeds so far below the normal that they can, for this reason, represent no danger. It follows that the amplitudes A_i and B_i of the disturbance are functions of the flow velocity toward the propeller, which diminish approximately as the square of the diminishing velocity. Hence, for this reason also, the resonance cases at low revolution speeds ($\lambda = i \omega$ for large i) are of no importance. Moreover, since the case $\lambda = \omega$ is excluded (as we have seen), there remains only the pronounced resonance case $\lambda = 2\omega$, which can seriously endanger the propeller. This is the case which occurs at the maximum revolution speed and consequently exhibits the maximum vibration amplitudes, and which, at the same time, most closely approaches the cruising speed of normal propellers.

Figure 9 may be taken as the second general example. This indicates numerically the importance of a firm hub, not only for increasing the stand frequency, but also for increasing the frequencies of the revolving propeller. The well-known Reed propeller is an unfavorable example in this respect.

For illustration, see also Figure 12. If one takes blade I and inserts it unchanged in a hub (case IV), the stand frequency and the flight frequency are both increased. If care is taken to restore the stand frequency

to that of the original blade, perhaps by making it thinner (case III), the frequencies of the revolving blade III still exceed the corresponding values of the original blade I. These cases can be supplemented by a few practical examples.

2. Practical Examples

Some time ago the Reed propellers were chiefly used on three-engine German commercial airplanes. Propellers of this type have, in numerous instances, suffered rupture while running.

For such a Reed propeller of 3 m (9.84 ft.) diameter, the cross section and inertia moment of which at every point of the blade was known, the stand frequency according to Figure 7 and the bending frequencies according to equation (33) were calculated in terms of the revolution speed and plotted in Figure 13.

From the course of the frequency curve it is obvious that, in the region of normal revolution speeds, the natural frequencies of the propeller do not fall in the vicinity of the revolution speed. They do, however, assume values equal to twice the revolution speed, this decided resonance occurring just in the region of the cruising revolution speeds of 1000 to 1200 r.p.m.

If we consider the field of flow of a three-engine airplane (fig. 14), it is obvious that the propellers must pass through two disturbance regions during each revolution. For the side propellers, both disturbances occur in passing the wing at a distance of about 1/3 of the wing chord. On the inside they simultaneously cut the slipstream of the middle propeller, which is spread rearward by the fuselage. The middle propeller revolves at about twice the distance of the side propellers from the wing. It is only slightly affected by the wing and by the side propellers back of it. As a matter of fact, centrally located propellers are less disturbed, though not altogether free from disturbance.

On the whole, there is found in this example a confirmation of the here-developed theory of the resonance vibrations of propellers. Moreover, Figure 13 shows that, in a case like the one mentioned, it would be better, under some conditions, to use a more flexible propeller having such natural frequencies that, under the ordinary op-

erating conditions, the vibrations exceed the natural frequencies. The vibrations will then be well damped.

In addition to the above-mentioned explanatory test for the occurrence of resonance vibrations in the practical case described, there is also the following circumstance. On various available propellers it was observed that, when one blade was set in vibration in any way, the other undisturbed blade also began to vibrate. Apparently there is no hub rigid enough to prevent the transmission of vibrations from one blade to the other. This fact, which also explains the occurrence of resonance vibrations when the natural frequency of the propeller blades is twice the revolution speed, was apparently operative in the above practical example.

Likewise, in the well-known case of the Koolhoven airplane, on which the propeller during every revolution had to pass once through a cutaway in the upper half of the fuselage, the alternate disturbance of the separate blades produced a resonance case with a frequency of twice the revolution speed. It cannot be assumed, however, that the different propellers used on this airplane were so flexible that their natural frequencies were in the vicinity of the revolution speed. There is no other explanation, however, for the propeller injuries on this airplane.

As the last example, we will consider the case of a propeller model consisting of two very thin flat boards (l = 91 cm, F = constant = 10 × 1 cm², ϵ/l = 0.235). Its previously calculated bending frequencies are likewise plotted as curve II in Figure 13. In the stand test, noticeable bending vibrations were produced at about 420 r.p.m., which can also be explained by the doubling of the vibrations per revolution through transmission from one blade to the other, if the velocity field in front of the test stand is not already so constituted that the corresponding disturbance function in formula (40) also contains the second harmonic term (i = 2).

In any case the observed vibration phenomena in all the examples considered are traceable to a bending-resonance case $\lambda = 2\omega$, in which the bending frequency of the propeller blade equals twice the revolution speed.

Note.- Attention is here called to the following point, which might lead to errors. If the investigation

of a propeller break resulting from fatigue does not indicate simple bending stress, it cannot be concluded from this fact alone that there could have been no "bending resonance" in the above-mentioned sense, but that, on the contrary, an essentially different kind of vibrations, namely "torsional vibrations" must have been produced by some unknown cause.

As already explained (Section II, 2,a), an aircraft propeller, owing to its bent and warped shape, can develop neither simple bending vibrations nor simple torsional vibrations (bending and torsion not being the normal coordinates of an aircraft-propeller blade). What are here called, according to custom, "bending" and "torsional" frequencies, might be more correctly designated respectively as the smaller and the greater of the two natural frequencies of a propeller blade. The justification for the division into the two kinds, bending and torsional, was derived from the fundamental assumption that, under ideal conditions, the calculable simple torsional frequencies differ but little from the greater, and the simple bending frequencies differ but little from the smaller of the natural fundamental frequencies of the propeller. In actual natural vibrations the propeller blade sometimes develops bending and torsional vibrations with the same small frequencies and at other times with the same large frequencies. In the former case the bending amplitudes are large, while in the latter case the torsional amplitudes are large.

In any case, noticeable torsional stresses can occur at the usual breaking point of a propeller blade near the hub, even when the outer portion of the blade is subjected chiefly to bending stresses. If the more probable cause, according to the here-developed calculation, namely, the bending resonance, is eliminated, then the fatigue stresses in the form of torsion also disappear.

VI. SUMMARY

On the basis of the consideration of various possible kinds of propeller vibrations, the resonance vibrations caused by unequal impacts of the propeller blades appear to be the most important. Their theoretical investigation is made by separate analysis of torsional and bending vibrations. This method is justified by the very great dif-

ference in the two natural frequencies of aircraft propeller blades. A mathematical estimation of the torsional frequencies shows:

1. That they are not noticeably affected by the influence of the aerodynamic forces acting on the propeller;

2. That their order of magnitude is so far above the revolution speed of the propeller, that any danger from resonance in the form of torsional vibrations seems to be excluded.*

For the bending vibrations of a propeller blade, it appears, on the contrary, that the frequencies, under the action of the centrifugal force, are of the order of magnitude of the revolution speed. The bending frequencies are calculated in terms of the revolution speed of the propeller, on the basis of the Rayleigh principle, from the minimum natural frequency of an elastic system. According to Rayleigh's theory, the values thus obtained represent an upper limit for the bending frequencies. On the basis of a generally valid theorem (Section IV, 2,f) concerning the frequencies of an elastic system simultaneously exposed to several forces (in the present case to elastic and centrifugal forces), a lower limit for the bending frequencies can also be established. The upper and lower limits approach each other so closely in the calculation, that the calculated bending frequencies adequately fulfill the accuracy requirements. The calculation yields the following practical results:

1. The aerodynamic forces are of much less importance for the bending vibrations of aircraft propellers than for the torsional vibrations.

2. Two aircraft propellers, which may differ in all their dimensions, material constants and (within wide limits) also in their taper, which, however, must have the same ratio of the hub radius to the blade length ϵ/l and the same natural stand frequency λ_0, have practically

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*No decision can be made on the basis of this analysis, regarding torsional vibrations due to other outside disturbances than those connected with an irregular field of flow.

the same bending frequencies, even when revolving.

3. Hence it is possible, for all propellers of whatever dimensions, with any ratio of hub radius to blade length, to adopt a single formula for calculating the bending frequencies in terms of the revolution speed ω. (See also figs. 8 and 9.)

$$\frac{\lambda}{\lambda_0} = \left[1 + \frac{7\left(\frac{\omega}{\lambda_0}\right)^2}{6 + 7\left(\frac{\omega}{\lambda_0}\right)}\right] \sqrt{\frac{1 + \left(1 + 2\,\frac{\epsilon}{l}\right)\left(\frac{\omega}{\lambda_0}\right)^2}{1 + \left(\frac{\omega}{\lambda_0}\right)^2}}$$

The stand frequency λ_0 is best obtained experimentally. However, λ_0 can also be read from Figure 7 for the practically important cases of linear cross-sectional tapering of the propeller blade with various reductions in the cross-sectional inertia moment.

4. The lower limit of the bending frequencies is represented by the formula

$$\lambda^2 > \lambda_0^2 + \omega^2\left(1 + \epsilon\,\frac{S}{\theta}\right)$$

where S is the statical moment and θ the inertia moment of the propeller blade with reference to the axis of fixation. For the case of linear cross-sectional taper, as approximated by nearly all aircraft propellers, the limit of the bending frequencies may be more simply expressed by the formula

$$\lambda^2 > \lambda_0^2 + \omega^2\left(1 + 2\,\frac{\epsilon}{l}\right).$$

For $\epsilon = 0$ (vanishingly small hub) the lower limit becomes simply

$$\lambda^2 > \lambda_0^2 + \omega^2.$$

5. The indicated lower limits hold good for every propeller blade of any shape. Hence they are also valid for the bending frequencies of bent and warped propeller blades. Moreover, they are independent of the manner of mounting on the hub. Hence they are also valid for blades whose mounting cannot be termed perfectly rigid ($\delta y/\delta \xi \neq 0$ for $\xi = 0$).

6. The presence of a large hub in the ratio to the blade length favorably affects the increase in the bending frequencies of the revolving blade.

7. The bending frequencies are always higher than the revolution speed of the propeller. Pronounced resonance vibrations due to external disturbances of the frequency of the simple revolution speed are therefore excluded.

8. On the other hand, pronounced resonance cases are possible when the bending frequency of a propeller is a low multiple of the revolution speed. This case is of practical importance, since many propellers are so mounted with reference to other airplane parts (e.g., the wings), that more disturbances than one (generally two) are produced during each revolution. Resonances of twice the revolution speed can also be produced by transmission of the vibrations from one blade to the other.

The calculated data are illustrated by practical examples. Thereby the observed vibration phenomenon in the given examples is explained by a bending resonance, for which the bending frequency of the propeller is equal to twice the revolution speed.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

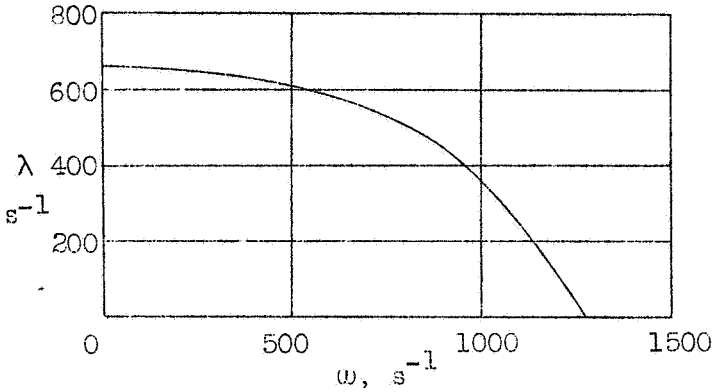

Fig. 1 Example of torsional frequencies plotted against the angular velocity.

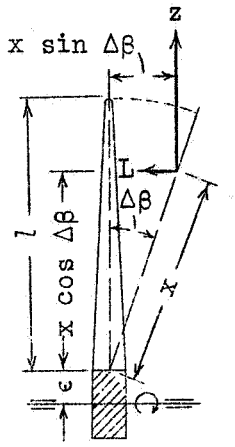

Fig. 2 Diagram for illustrating equations 7 to 10.

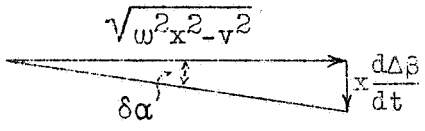

Fig. 3 Velocity components of vibrating propeller blade.

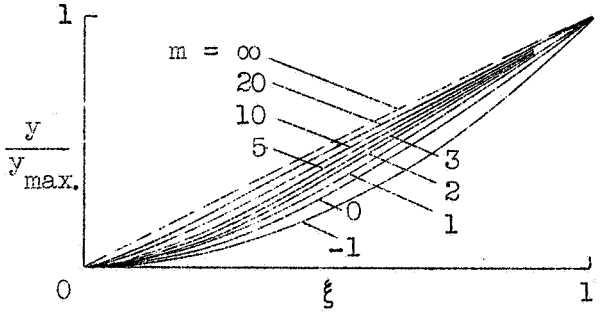

Fig. 4 Vibration-elastic lines according to equation 18.

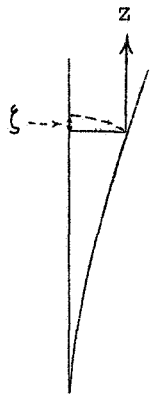

Fig. 5 Radical displacement ζ of a blade element during a vibration.

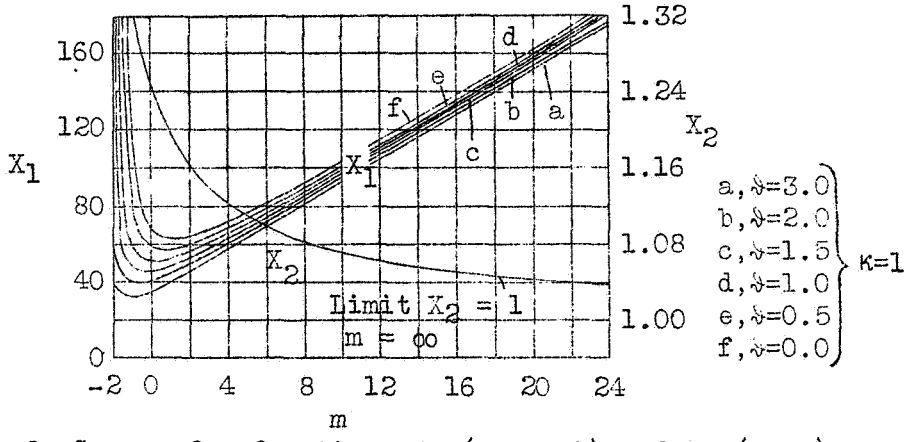

Fig. 6 Curves for functions X_1 (m, κ, ϑ) and X_2 (m, κ) in equations 23 to 26.

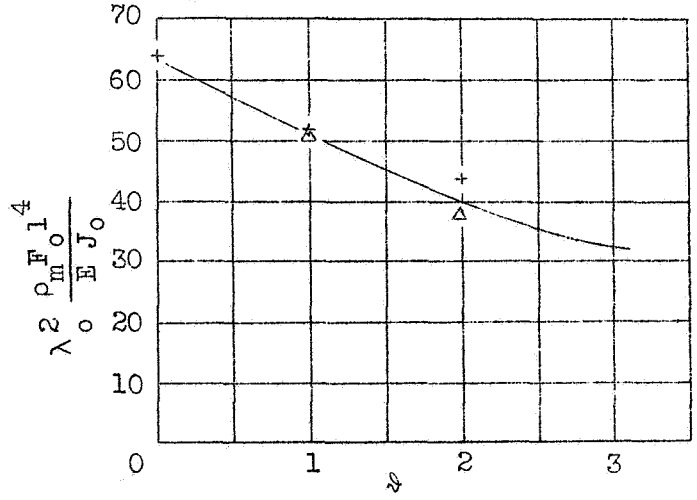

Fig. 7 Bending frequencies of nonrevolving propeller blades with linear cross-sectional taper (κ = 1) and various reductions of the inertia moment. (Variable ϑ).

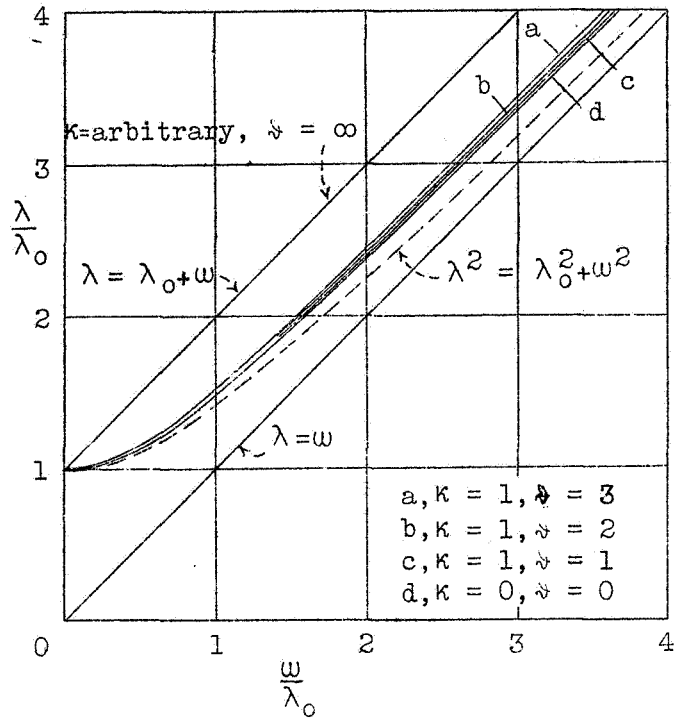

Fig. 8 Nondimensional representation of the bending frequencies of revolving propel - ler blades plotted against the angular velocity. The different κ and ν curves refer to different reductions in the cross section and inertia moment of the blade.(cf. equation 17)

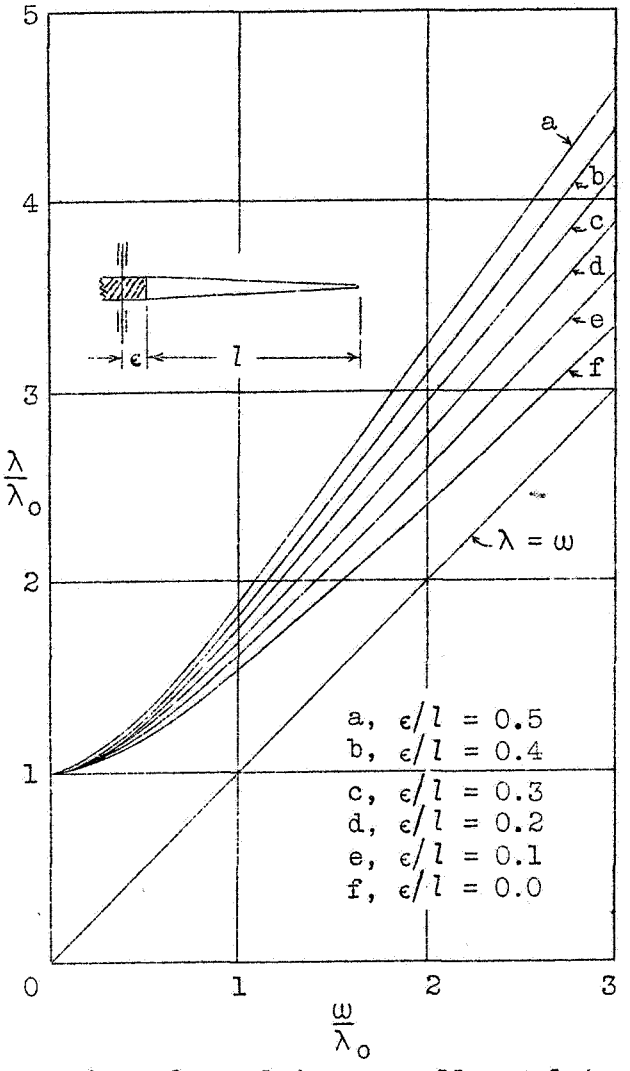

Fig. 9 Representation of equation 33. Bending frequencies of revolving propellers plotted against angular velocity for different ratios ε/l of hub radius to blade length.

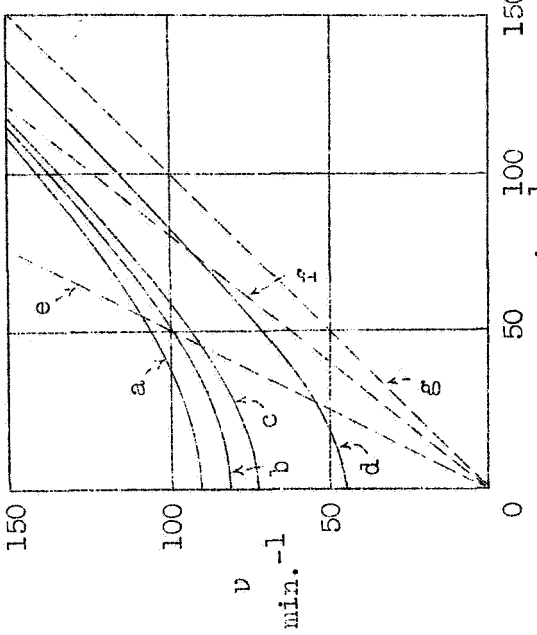

Fig. 11 Bending frequencies of four like propellers with like root cross sections, but different tapers. (See table) All the blades come into pronounced resonance with vibrations of twice the revolution speed. (See intersection points of curves with lines ν=2n)

a, κ = 1, ϰ = 1
b, κ = 1, ϰ = 2
c, κ = 1, ϰ = 3
d, κ = 0, ϰ = 0
e, ν = 2n
f, ν = 1.25n
g, ν = n

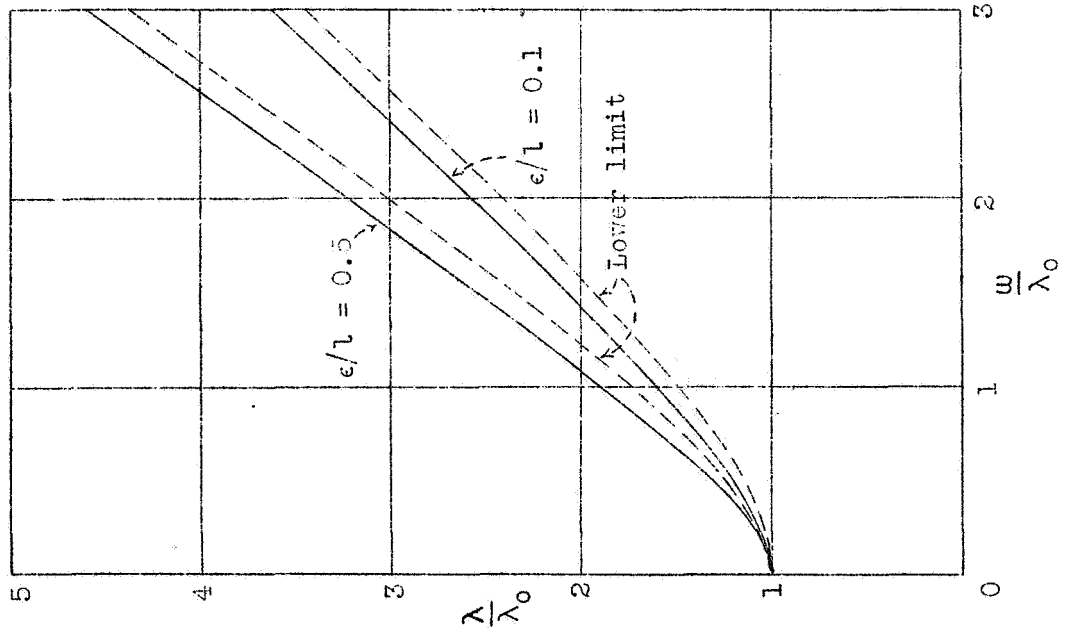

Fig. 10 Verification of equation 53 for the bending frequencies of revolving propeller blades. The errors for both the given cases (ratio of hub radius to blade length ε/l = 0.5 and ε/l = 0.1) must be smaller than the deviations between each of the plain curves and the corresponding dash curves.

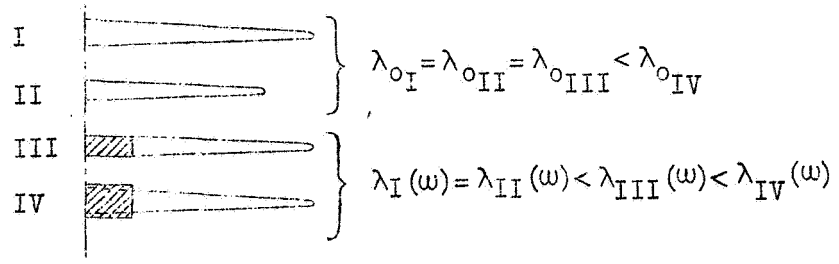

Fig. 12 Comparison of similar propellers with reference to their bending frequencies on stand and in flight.

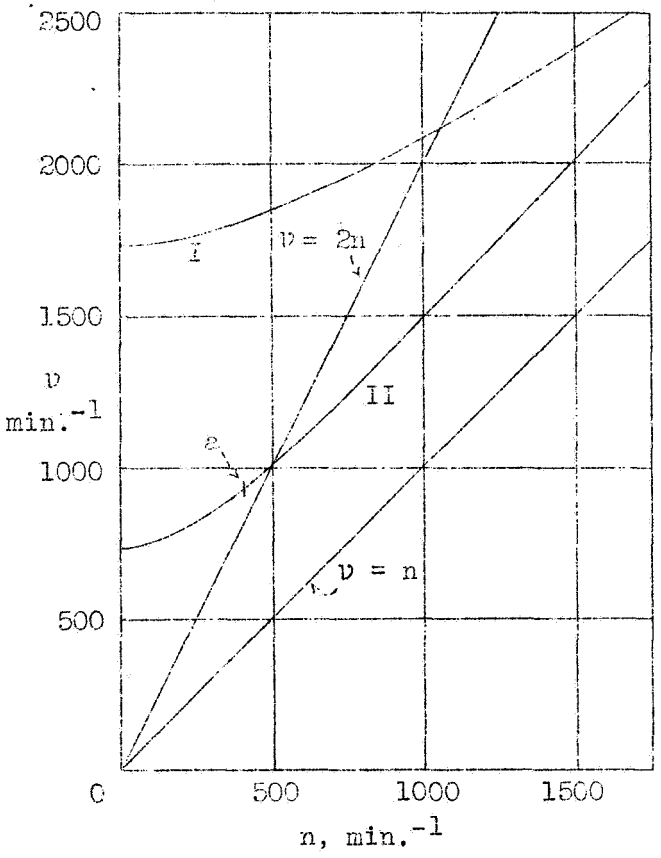

Fig. 13 Curve I. Bending frequencies of a Reed propeller. This propeller comes into resonance at the revolution speeds of cruising flight, when vibrations with a frequency of twice the revolution speed are imparted to it.

Curve II. Bending frequencies of a propeller model made from 2 flat boards. Comparison between calculation and experiment.

a, Beginning of vibrations in experiment.

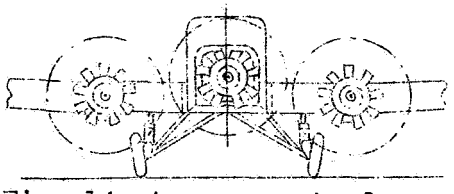

Fig. 14 Arrangement of propellers on a three-engine airplane