TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 647

SPATIAL BUCKLING OF VARIOUS TYPES OF AIRPLANE STRUT SYSTEMS

By Alfred Teichmann

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Notation

$e_{i,h}$, $e_{i,i}$ } "unit rotation" of end h or i, respectively, of a strut "i" pin-jointed in h and i, due to a single bending moment M = 1 (in the sense of this moment), applied at the same end h or i, respectively.

(One support of member "i" is assumedly displaceable in the direction of the axis of the member. The first index at e denotes the member, the second the end, where e occurs. In reports Nos. 183 and 224 of the D.V.L. (Deutsche Versuchsanstalt für Luftfahrt) no differentiation was made between $e_{i,h}$ and $e_{i,i}$ because there the arguments pertained to members with evenly distributed stiffness.)

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*"Das räumliche Knicken einiger Stabverbindungen des Flugzeugbaus." Zeitschrift für Flugtechnik und Motorluftschiffahrt, Sept. 14, 1931, pp. 525-526.

In extension of report No. 183 of the D.V.L.:

A. Teichmann, "Effects of the End Fixation of Airplane Struts." T.M. No. 582, N.A.C.A., 1930. Zeitschrift für Flugtechnik und Motorluftschiffahrt, May 28, 1930, pp. 249-254; and D.V.L. Yearbook 1930, pp. 221-226, which also contains a list of references.

A. Teichmann, "The Spatial Buckling of Various Types of Airplane Bracing Systems." Report No. 224, D.V.L. Yearbook 1931, pp. 230-232.

$\hat{e}_{i,h}$, $\hat{e}_{i,i}$ } "unit rotation" of end h or i, respectively, of member "i" due to a single bending moment M = 1 (in the sense of this moment) at the opposite end i or h, respectively. According to Maxwell,

$$\hat{e}_{i,h} = \hat{e}_{i,i} = \hat{e}_i.$$

$d_i = e_{i,i}\, e_{i,h} - \hat{e}_i^2.$

$\bar{e}_{i,h}$, $\bar{e}_{i,i}$ } "unit rotation" of free end h or i, respectively, of a beam "i" restrained in i or h, respectively, in consequence of a twisting moment T = 1 (in the sense of this moment) applied at the free end. It is

$$\bar{e}_{i,h} = \bar{e}_{i,i} = \bar{e}_i.$$

$x_i' = \dfrac{x_i - x_h}{l_i}$, $y_i' = \dfrac{y_i - y_h}{l_i}$, $z_i' = \dfrac{z_i - z_h}{l_i}$: unit-vector-component of member "i." (The unit vector points from the lower numbered end h toward the higher end i.)

x_i'', y_i'', z_i'', x_i''', y_i''', z_i''' } unit vector components of the principal axes of inertia of the cross section of member "i." (The sense of rotation of these unit vectors to be such as to form a clockwise system with the unit vector x_i', y_i', z_i'.)

For the most frequent case of a member with evenly distributed bending stiffness EJ and of length l, stressed by a centrally applied force S, we have, with $\alpha = l\sqrt{S/EJ}$:

$$e_{i,h} = e_{i,i} = \frac{l}{EJ}\,\frac{1}{\alpha^2}\left(1 - \frac{\alpha}{\tan\alpha}\right);$$

$$\hat{e}_i = \frac{l}{EJ} \frac{1}{\alpha^2} \left(1 - \frac{\alpha}{\sin \alpha}\right)^*$$

and in the limiting case $S = 0$:

$$e_{i,h} = e_{i,i} = \frac{l}{EJ} \frac{1}{3}; \quad \hat{e}_i = \frac{l}{EJ}\left(-\frac{1}{6}\right).$$

(For tension loads we may figure with e and $\hat{e}$ for $S = 0$. D.V.L. report No. 183.)

If $G J_d$ is the evenly distributed torsional stiffness, then

$$\bar{e}_i = \frac{l}{G J_d}$$

<u>Buckling equations</u>.- In the following we give the conditions under which buckling occurs for a number of strut types consisting of members connected by joints which are bending and torsion resistant. The buckling stage is so characterized that, in the absence of transverse loading of the individual members, rotations at the strut end can occur; this possibility exists only when the denominator determinant of that linear symmetrical system of equation which with existing transverse loading would supply the rotations of the strut ends, disappears:

$$\begin{vmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} \end{vmatrix} = 0 \quad \text{(buckling condition)}$$

$(\delta_{ik} = \delta_{ki})$

The δ terms to be inserted when disregarding the strut length changes, are given in the table of the formulas.

*For evaluation of these expressions, see tables in: Zimmerman, "Buckling Strength of Struts," (Ernst & Sohn, 1925); Müller-Breslau, "Graphic Statics of Building Constructions," Vol. II, 2 (Kröner, 1925); Bleich, "Theory and Calculation of Iron Bridges," (Springer, 1924).

Two-Dimensional Struts (figs 1-6)

For these strut types it is stipulated that the principal axes of inertia are perpendicular or parallel, respectively, to the plane XY, or else that circular sections or other isotropic sections are available. We differentiate between

α) buckling within the XY plane,

β) bulging out of the XY plane.

The values e, ê, and EJ, respectively, refer in case α) to bending about the cross section axes perpendicular to XY, and in case β) to bending around the cross section axes in the XY plane.

1. The strut-type connected by joints free of bending and twisting, to the spatially defined points i = 1, 2, 3 ....

a) Three or more struts with a common, but otherwise unsupported joint (for instance, K and X bracing systems, Figures 1 and 2). In cursory estimations of case β), the effect of struts "3" and "4" may be disregarded, as a rule, for K struts.

b) N strut (fig. 3).

c) V strut (fig. 4). This strut buckles out of its plane as soon as the force in a member "i" attains $S_i = \frac{E\ J_i\ \pi^2}{l_i^2}$.

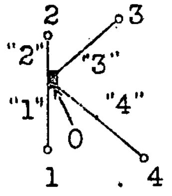

Fig.1 K strut

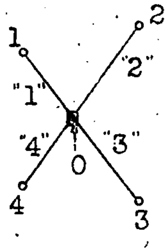

Fig.2 X strut

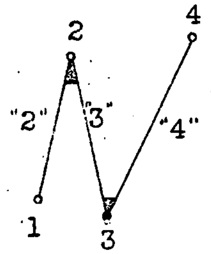

Fig.3 N strut

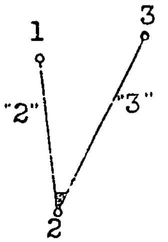

Fig.4 V strut

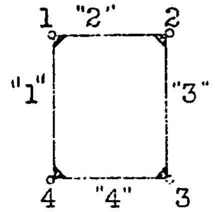

Fig.5 □ strut

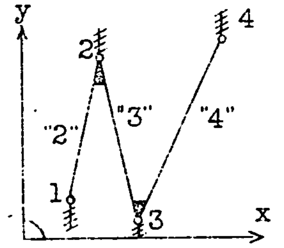

Fig.6 N strut with incomplete joints.

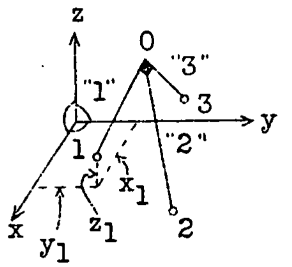

Fig.7 Three space strut system.

d) Rectangular frame system. (Fig. 5.) The δ terms here given for buckling case α) have the advantage over those given in D.V.L. report No. 183, in so far as they still hold when $S = \frac{E\,J\,\pi^2}{l^2}$ in one or more struts.

For buckling case β) we find two determinants (cases β_1 and β_2). Buckling occurs when one of these becomes zero.

2. Strut type connected by bending-free Cardanic joints to spatially defined points $i = 1, 2, 3$ ..; the Cardanic joints are designed to take twisting moments whose vector falls into direction y. The treatment is restricted to strut types conforming to 1 b and c (fig. 6); the conditions for buckling case α) are the same as under 1.

Three-Dimensional Strut System (fig. 7)

An arbitrary number of struts not lying in one plane is connected in one joint 0, stiff in bending and torsion; the other strut ends are connected by joints, free of bending and torsion, to spatially defined points $i =$ 1, 2, 3 .... $e_{i,o}''$ and $e_{i,o}'''$, respectively, refer to bending about the principal axis of inertia with the unit vector x_i'', y_i'', z_i'' and x_i''', y_i''', z_i''', respectively.

For struts whose moment of inertia is the same for all centroidal axes of its cross section, as, for instance, in round tubes, we must substitute:

$$1 - \frac{x_i'^2}{e_{i,o}} \quad \text{for} \quad \frac{x_i''^2}{e_{i,o}''} + \frac{x_i'''^2}{e_{i,o}'''}$$

$$- \frac{x_i'\,y_i'}{e_{i,o}} \quad \text{for} \quad \frac{x_i''\,y_i''}{e_{i,o}''} + \frac{x_i'''\,y_i'''}{e_{i,o}'''}$$

Table of Formulas

Expressions δ for Various Groups of Bars

	I, 1a		I, 1b	
	α	β	α	β
δ_{11}	$i\Sigma \frac{1}{e_{i,o}}$	$i\Sigma \frac{x_i'^2}{e_{i,o}}$	$\frac{1}{e_{2.2}} + \frac{e_{3.3}}{d_3}$	$\frac{x_2'^2}{e_{2.2}} + \frac{x_3'^2 e_{3.3}}{d_3} + \frac{y_3'^2}{\bar{\bar{e}}_3}$
δ_{22}	–	$\Sigma \frac{y_i'^2}{e_{i,o}}$	$\frac{1}{e_{4.3}} + \frac{e_{3.2}}{d_3}$	$\frac{y_2'^2}{e_{2.2}} + \frac{y_3'^2 e_{3.3}}{d_3} + \frac{x_3'^2}{\bar{\bar{e}}_3}$
δ_{33}	–	$\Sigma \frac{1}{l_i^2}\left(\frac{1}{e_{i,o}} - l_i s_i\right)$	–	$\frac{x_4'^2}{e_{4.3}} + \frac{x_3'^2 e_{3.2}}{d_3} + \frac{y_3'^2}{\bar{\bar{e}}_3}$
δ_{44}	–	–	–	$\frac{y_4'^2}{e_{4.3}} + \frac{y_3'^2 e_{3.2}}{d_3} + \frac{x_3'^2}{\bar{\bar{e}}_3}$
$\delta_{12}=\delta_{21}$	–	$\Sigma \frac{x_i'y_i'}{e_{i,o}}$	$\frac{\hat{e}_3}{d_3}$	$\frac{x_2'y_2'}{e_{2.2}} + x_3'y_3'\left(\frac{e_{3.3}}{d_3} - \frac{1}{\bar{\bar{e}}_3}\right)$
$\delta_{13}=\delta_{31}$	–	$\Sigma \frac{1}{l_i} \frac{x_i'}{e_{i,o}}$	–	$\frac{x_3'^2 \hat{e}_3}{d_3} + \frac{y_3'^2}{\bar{\bar{e}}_3}$
$\delta_{14}=\delta_{41}$	–	–	–	$x_3'y_3'\left(\frac{\hat{e}_3}{d_3} - \frac{1}{\bar{\bar{e}}_3}\right)$
$\delta_{23}=\delta_{32}$	–	$\Sigma \frac{1}{l_i} \frac{y_i'}{e_{i,o}}$	–	$x_3'y_3'\left(\frac{\hat{e}_3}{d_3} - \frac{1}{\bar{\bar{e}}_3}\right)$
$\delta_{24}=\delta_{42}$	–	–	–	$\frac{y_3'^2 \hat{e}_3}{d_3} + \frac{x_3'^2}{\bar{\bar{e}}_3}$
$\delta_{34}=\delta_{43}$	–	–	–	$\frac{x_4'y_4'}{e_{4.3}} + x_3'y_3'\left(\frac{e_{3.2}}{d_3} - \frac{1}{\bar{\bar{e}}_3}\right)$

Table of Formulas (contd.)
Expressions δ for Various Groups of Bars

	I, 1c		I, 1d		
	α	β	α	β_1	β_2
δ_{11}	$\frac{1}{e_{2.2}}+\frac{1}{e_{3.2}}$	$\frac{x_2'^2}{e_{2.2}}+\frac{x_3'^2}{e_{3.2}}$	$\frac{e_{1.1}}{d_1}+\frac{e_{4.3}}{d_4}$	$\frac{e_{1.1}}{d_1}+\frac{1}{\bar{e}_4}$	$\frac{e_{2.2}}{d_3}+\frac{1}{\bar{e}_1}$
δ_{22}	—	$\frac{y_2'^2}{e_{2.2}}+\frac{y_3'^2}{e_{3.2}}$	$\frac{e_{2.2}}{d_2}+\frac{e_{1.4}}{d_1}$	$\frac{e_{1.4}}{d_1}+\frac{1}{\bar{e}_2}$	$\frac{e_{2.1}}{d_2}+\frac{1}{\bar{e}_3}$
δ_{33}	—	—	$\frac{e_{3.3}}{d_3}+\frac{e_{2.1}}{d_2}$	$\frac{e_{3.3}}{d_3}+\frac{1}{\bar{e}_2}$	$\frac{e_{4.4}}{d_4}+\frac{1}{\bar{e}_3}$
δ_{44}	—	—	$\frac{e_{4.4}}{d_4}+\frac{e_{3.2}}{d_3}$	$\frac{e_{3.2}}{d_3}+\frac{1}{\bar{e}_4}$	$\frac{e_{4.3}}{d_4}+\frac{1}{\bar{e}_1}$
$\delta_{12}=\delta_{21}$	—	$\frac{x_2'y_2'}{e_{2.2}}+\frac{x_3'y_3'}{e_{3.2}}$	$\frac{\hat{e}_1}{d_1}$	$\frac{\hat{e}_1}{d_1}$	$\frac{\hat{e}_2}{d_2}$
$\delta_{13}=\delta_{31}$	—	—	—	—	—
$\delta_{14}=\delta_{41}$	—	—	$\frac{\hat{e}_4}{d_4}$	$\frac{1}{\bar{e}_4}$	$\frac{1}{\bar{e}_1}$
$\delta_{23}=\delta_{32}$	—	—	$\frac{\hat{e}_2}{d_2}$	$\frac{1}{\bar{e}_3}$	$\frac{1}{\bar{e}_3}$
$\delta_{24}=\delta_{42}$	—	—	—	—	—
$\delta_{34}=\delta_{43}$	—	—	$\frac{\hat{e}_3}{d_3}$	$\frac{\hat{e}_3}{d_3}$	$\frac{\hat{e}_4}{d_4}$

Table of Formulas (contd.)
Expressions δ for Various Groups of Bars

	I, 2b	I, 2c	II
	β	β	
δ_{11}	$\frac{y_2'^2 e_{2\cdot 2}}{d_2} + \frac{x_2'^2}{\bar{\bar{e}}_2}$	$\frac{y_2'^2 e_{2\cdot 2}}{d_2} + \frac{x_2'^2}{\bar{\bar{e}}_2}$	$i\,\Sigma\left(\frac{x_i''^2}{e''_{i,o}} + \frac{x_i'''^2}{e'''_{i,o}}\right)$
δ_{22}	$\frac{y_2'^2 e_{2\cdot 1}}{d_2} + \frac{y_3'^2 e_{3\cdot 3}}{d_3} + \frac{x_2'^2}{\bar{\bar{e}}_2} + \frac{x_3'^2}{\bar{\bar{e}}_3}$	$\frac{y_2'^2 e_{2\cdot 1}}{d_2} + \frac{y_3'^2 e_{3\cdot 3}}{d_3} + \frac{x_2'^2}{\bar{\bar{e}}_2} + \frac{x_3'^2}{\bar{\bar{e}}_3}$	$\Sigma\left(\frac{y_i''^2}{e''_{i,o}} + \frac{y_i'''^2}{e'''_{i,o}}\right)$
δ_{33}	$\frac{y_3'^2 e_{3\cdot 2}}{d_3} + \frac{y_4'^2 e_{4\cdot 4}}{d_4} + \frac{x_3'^2}{\bar{\bar{e}}_3} + \frac{x_4'^2}{\bar{\bar{e}}_4}$	$\frac{y_3'^2 e_{3\cdot 2}}{d_3} + \frac{x_3'^2}{\bar{\bar{e}}_3}$	$\Sigma\left(\frac{z_i''^2}{e''_{i,o}} + \frac{z_i'''^2}{e'''_{i,o}}\right)$
δ_{44}	$\frac{y_4'^2 e_{4\cdot 3}}{d_4} + \frac{x_4'^2}{\bar{\bar{e}}_4}$	—	—
$\delta_{12}=\delta_{21}$	$\frac{y_2'^2 \hat{e}_2}{d_2} + \frac{x_2'^2}{\bar{\bar{e}}_2}$	$\frac{y_2'^2 \hat{e}_2}{d_2} + \frac{x_2'^2}{\bar{\bar{e}}_2}$	$\Sigma\left(\frac{x_i'' y_i''}{e''_{i,o}} + \frac{x_i''' y_i'''}{e'''_{i,o}}\right)$
$\delta_{13}=\delta_{31}$	—	—	$\Sigma\left(\frac{x_i'' z_i''}{e''_{i,o}} + \frac{x_i''' z_i'''}{e'''_{i,o}}\right)$
$\delta_{14}=\delta_{41}$	—	—	—
$\delta_{23}=\delta_{32}$	$\frac{y_3'^2 \hat{e}_3}{d_3} + \frac{x_3'^2}{\bar{\bar{e}}_3}$	$\frac{y_3'^2 \hat{e}_3}{d_3} + \frac{x_3'^2}{\bar{\bar{e}}_3}$	$\Sigma\left(\frac{y_i'' z_i''}{e''_{i,o}} + \frac{y_i''' z_i'''}{e'''_{i,o}}\right)$
$\delta_{24}=\delta_{42}$	—	—	—
$\delta_{34}=\delta_{43}$	$\frac{y_4'^2 \hat{e}_4}{d_4} + \frac{x_4'^2}{\bar{\bar{e}}_4}$	—	—

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.