Second International Workshop on Squeezed States and Uncertainty Relations
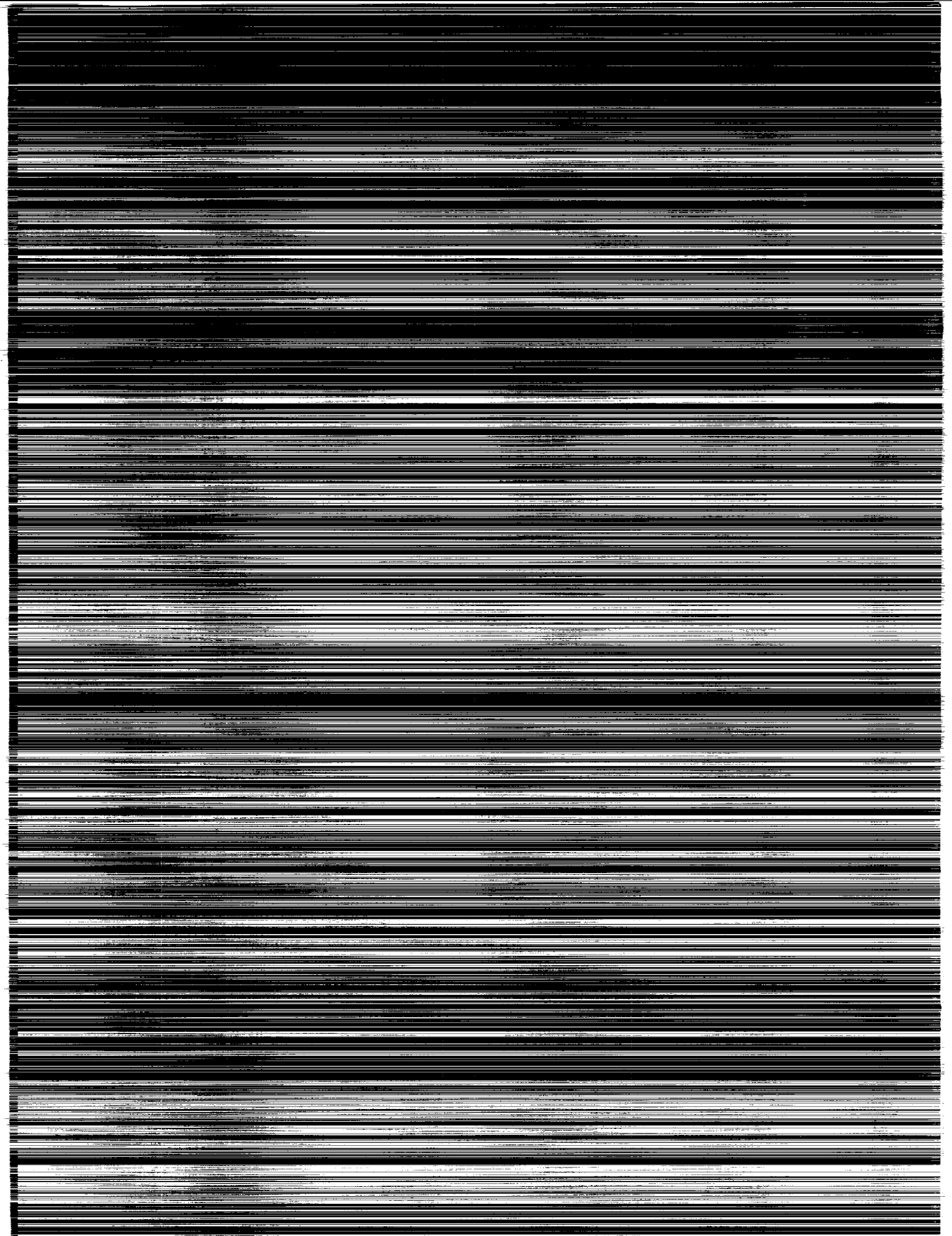
Second International Workshop on Squeezed States and Uncertainty Relations

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PREFACE

The Second International Workshop on Squeezed States and Uncertainty Relations was held in Moscow on May 25 – 29, 1992. This Workshop was organized jointly by the P. N. Lebedev Physics Institute of the Academy of Sciences of the Russian Republic and the University of Maryland at College Park. The Workshop was supported in part by the Committee of Science and Technology of the Russian Republic, the Lebedev Physics Institute, and the University of Maryland. This program was one of the scientific conferences held in accordance with the cooperative agreement between the Lebedev Physics Institute and the University of Maryland. The first meeting of this workshop in this series took place at the College Park Campus of the University of Maryland in 1991.

The purpose of this Workshop was to study possible applications of squeezed states of light. Specifically, the workshop was concerned with the following questions.

(1) What physics can we do with squeezed states?

(2) Are there squeezed states in other branches of physics?

(3) What are possible forms for the uncertainty relations?

The Workshop brought together many active researchers in squeezed states of light and those who may find the concept of squeezed states useful in their research, particularly in understanding the uncertainty relations. There were many participants from the European countries including of course Russia. There were also many from the United States.

The third meeting in this series will be held at the University of Maryland Baltimore County. The principal organizers are R. H. Rubin and Y. H. Shih. We expect that the international character of this Workshop series will be preserved and strengthened.

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INTRODUCTION

Squeezed states of light were predicted in the 1970's. They were observed in laboratories during the period 1985–1988. Indeed, the production of squeezed light is one of the landmarks in the development of laser technologies. Efforts are being made to find new experiments using this new laser technology. This new development encourages and enables us to study the forms of uncertainty relations.

The physical basis of squeezed states of light is the uncertainty relation in Fock space whose basic operation consists of creation and annihilation of photons. The uncertainty relation in this case is that of the second quantization. One of the fundamental questions in physics has been and still is how this uncertainty in second quantization is the same as or different from the position and momentum uncertainty with which we are so familiar.

Let us consider a two–dimensional space with two orthogonal axes. The word "squeeze" means that one of the coordinate variables is contracted while the other is expanded in such a way that their product remains unchanged. For Heisenberg's uncertainty relation, we can increase the uncertainty in position variable while decreasing that in the momentum variable while keeping the value of uncertainty constant. Indeed, the squeeze transformation has been one of the most important transformations in many branches of physics, including special relativity, harmonic oscillators with time-dependent frequency, canonical transformations in classical mechanics, and Bogoliubov transformations in condensed matter physics, thermofield dynamics, and symplectic transformations in mathematical physics.

Thus, the word "squeezed state" can have two different meanings. In a narrow sense, the word is applicable only to two–photon coherent states in quantum optics. There are many who say that the potential for industrial applications of the squeezed states of light is enormous. There are also many who say that the squeezed state in optics was only a fad and is no longer an interesting subject. However, we should not make a hasty judgment on this new word, because the squeezed state can have its second meaning.

The word squeeze can also have a broader implication. It does not have to be limited to quantum optics. The point is that there are many squeeze transformations in different branches of physics. Indeed, there were .and there are many who have been studying these transformations without using the word squeeze. The squeezed state of light has made a very important contribution to the physics world by giving us the word "squeeze" as one of the fundamental transformationss in physics. This word may therefore lead to an entirely new organization of physicists who are working in many different areas of physics including quantum optics.

The Workshop was attended by many researchers in the squeezed states of light as well as those who worked on related fields even before the squeezed state of light became one of the important subjects in physics. This volume contains four chapters. The first chapter contains the papers on the latest development in quantum optics. The second chapter consists of articles dealing with the forms of uncertainty relations. The articles in Chapter 3 are on theoretical developments based on the concept of squeeze transformations. It is important to note that the time-dependent problem in quantum mechanics is intimately connected to the concept of squeezed states in optics. The papers in Chapter 4 contains the papers dealing with time–dependent problems in quantum mechanics and quantum optics.

I. QUANTUM OPTICS

A PHASEONIUM MAGNETOMETER: A NEW OPTICAL MAGNETOMETER BASED ON INDEX ENHANCED MEDIA

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Abstract

An optical magnetometer based on quantum coherence and interference effects in atoms is proposed whose sensitivity is potentially superior to the present state-of-the-art devices. Optimum operation conditions are derived and a comparision to standard optical pumping magnetometers is made.

1 Optical Pumping Magnetometer

The detection of magnetic fields via optical pumping techniques was first discovered by Franken and Colegrove in helium [1]. An atomic system with three lower magnetic sublevels say, $m_J = +1, 0, -1$ and one upper level, is driven by resonant unpolarized light. A magnetic field, which for simplicity we take to be parallel to the propagation direction, splits the energies by an amount $\hbar a B$, where $a \approx 10^7$ s^{-1}/Gauss and B is the magnetic field strength.

Due to optical pumping, the population of the $m_J = \pm 1$ states is driven into the $m_J = 0$ level and the pump light will be transmitted through the otherwise absorbing gas.

Now, if there is a RF signal applied to the gas which is resonant to the sublevel transition, the atoms will be driven back to the $m_J = \pm 1$ states and the gas will again absorb the optical radiation. Thus by monitoring the transmitted pumping light while varying the RF frequency one has a sensitive measure of the spacing of the magnetic sublevels. That is, the pumping light will be "shut off" when

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$$\omega_{RF} = aB \tag{1}$$

This is summarized in Fig.1.
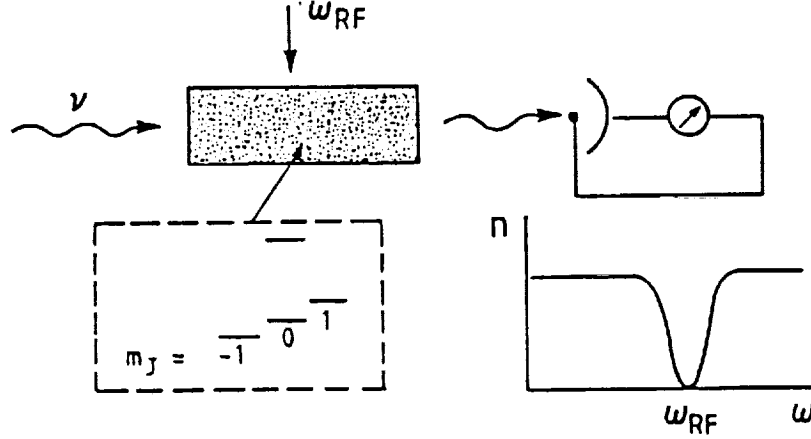
FIG.1. Optical Pumping Magnetometer Concept

The ultimate precision to which we can measure this frequency and the strength of the magnetic field is determined by intensity fluctuations in the transmitted light beam, i.e. fluctuations in the number m of observed photoelectrons. To obtain the resonance frequency, one determines the position of the half maxima. The intensity fluctuations at this point lead to an error

$$\Delta\omega_{error} = \left|\frac{\partial\omega}{\partial m}\right| \Delta m, \tag{2}$$

where $\partial m/\partial\omega$ is the slope of the transmission curve at the half maximum.

Assuming shot noise in the number of observed photoelectrons, i.e. $\Delta m = \sqrt{m}$, and 100% detection efficiency, so that $m = P_{in}t_m/\hbar\nu$, we obtain under optimum conditions for the frequency error

$$\Delta\omega_{error} = \gamma_{mag}\sqrt{\frac{\hbar\nu}{P_{in}t_m}}. \tag{3}$$

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Here P_{in} is the optical input power, ν the frequency of the pump field, and t_m is the measurement time. γ_{mag} is the width of the transmission line, which in the absence of power broadening is the transverse decay rate γ_c of the RF transition. Equating the signal frequency (1) to the error (3) we arrive at the minimum detectable change in the magnetic field for the optically pumped magnetometer

$$B_{min} = \frac{\gamma_{mag}}{a} \sqrt{\frac{\hbar\nu}{P_{in}t_m}}.$$ (4)

Increasing the power of the pump radiation obviously increases the sensitivity. However, as P_{in} grows the transmission line will get power broadened and γ_{mag} will eventually increase. In order to optimize the parameters, we calculated the width of the transmission line by solving the density matrix equations within a second order perturbation approach in the RF field. We thereby consider the level configuration shown in Fig.2.
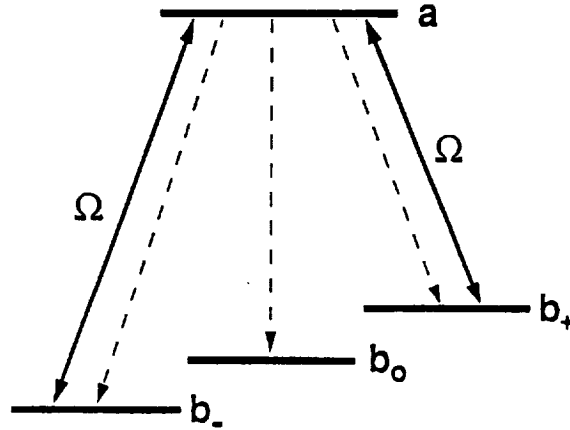
FIG.2. 4-level scheme for the optical pumping magnetometer. Since the magnetic field is parallel to the propagation axis, the unpolarized pump field drives the $m_J = \pm 1$ lower levels to the $m_J = 0$ upper state.

In the interaction picture we have the equations of motion for the populations

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$$\dot{\rho}_{b_+b_+} = \gamma_+\rho_{aa} + i\left(\Omega^*\rho_{ab_+} - c.c.\right) - i\left(\Omega^*_{RF}\rho_{b_+b_0} - c.c.\right), \tag{5a}$$

$$\dot{\rho}_{b_0b_0} = \gamma_0\rho_{aa} + i\left(\Omega^*_{RF}\rho_{b_+b_0} - c.c\right) - i\left(\Omega^*_{RF}\rho_{b_0b_-} - c.c.\right), \tag{5b}$$

$$\dot{\rho}_{b_-b_-} = \gamma_-\rho_{aa} + i\left(\Omega^*\rho_{ab_-} - c.c\right) + i\left(\Omega^*_{RF}\rho_{b_0b_-} - c.c\right), \tag{5c}$$

for the RF polarizations

$$\dot{\rho}_{b_+b_0} = -(i\Delta + \gamma_c)\rho_{b_+b_0} - i\Omega_{RF}\left(\rho_{b_+b_+} - \rho_{b_0b_0}\right) + i\Omega^*\rho_{ab_0}, \tag{6a}$$

$$\dot{\rho}_{b_0b_-} = -(i\Delta + \gamma_c)\rho_{b_0b_-} - i\Omega_{RF}\left(\rho_{b_0b_0} - \rho_{b_-b_-}\right) - i\Omega\rho^*_{ab_0}, \tag{6b}$$

and for the optical polarization

$$\dot{\rho}_{ab_0} = -\frac{\Gamma}{2}\rho_{ab_0} - i\Omega^*_{RF}\rho_{ab_-} - i\Omega_{RF}\rho_{ab_+} + i\Omega\rho_{b_+b_0} + i\Omega\rho_{b_-b_0}, \tag{7}$$

Here $\gamma_+, \gamma_-, \gamma_0$ are the longitudinal decay rates of the optical transitions, $\Gamma = \gamma_+ + \gamma_- + \gamma_0$, Ω_{RF} and Ω are the Rabi-frequencies of the RF and optical field, and Δ is the detuning of the RF-frequency from the magnetic transition frequency. In the absence of the RF-field all population is optically pumped into level b_0. Hence, in zeroth order the only non-vanishing matrix element is $\rho^{(0)}_{b_0b_0} = 1$, and the medium is totally transparent with respect to the optical field. In first order of the RF-coupling, low-frequency coherences build up. Solving Eqs. (6) and (7) we find

$$\rho^{(1)}_{b_+b_0} = -\rho^{(1)}_{b_0b_-} = \frac{\Omega_{RF}(\Delta + i\gamma_c)}{\Delta^2 + \gamma_c^2 + \dfrac{4|\Omega|^2\gamma_c}{\Gamma}}, \tag{8}$$

where we have assumed $\Omega^*_{RF} = \Omega_{RF}$. In second order of the RF-field, population in the $b_\pm$ ground levels is created and the optical field will be absorbed. Noting that $\rho^{(2)}_{aa} = 0$, we find from Eqs. (5a) and (5c) the imaginary part of the $a - b_\pm$ susceptibilities, which determine the absorption of the pump field radiation

$$\chi'' = \frac{\wp^2 N}{\hbar\epsilon_0}\frac{\Omega^2_{RF}}{\Delta^2 + \gamma_c^2 + \dfrac{4|\Omega|^2\gamma_c}{\Gamma}}\left(\gamma_c + \frac{2|\Omega|^2}{\Gamma}\right). \tag{9}$$

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As can be seen from this equation and Fig. 3, an increasing Rabi-frequency Ω leads to a power broadened transmission line with width

$$\gamma_{mag} = \gamma_c \left(1 + \frac{4|\Omega|^2}{\gamma \, \Gamma} \right)^{1/\Gamma} \tag{10}$$
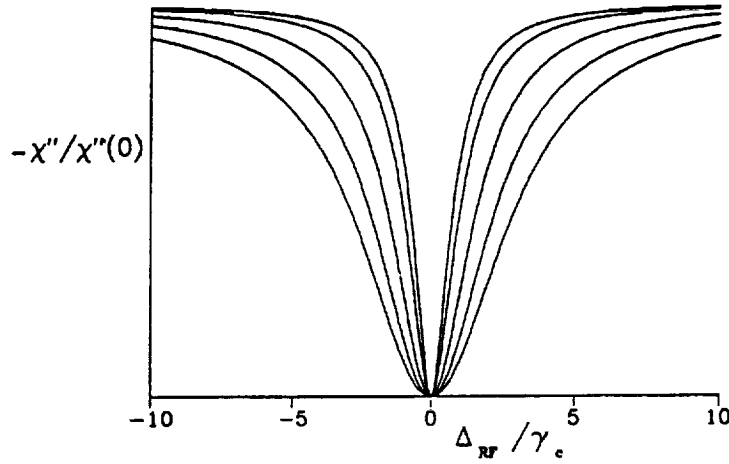
FIG.3. Normalized imaginary part of optical susceptibility as function of RF detuning Δ. The Rabi-frequency of the pump field is (from top to bottom) 0.1 , 0.5, 1, 1.5, 2 $\times$ $\gamma_c \Gamma / 4$.

For a sufficiently small input power, such that $\gamma_{mag} \approx \gamma_c$, the minimum detectable magnetic field, Eq. (4), decreases with increasing input power P_{in}. However, above a certain value P_{in}^c, corresponding to the critical value of the optical Rabi-frequency

$$\Omega^c = \sqrt{\frac{\gamma_c \Gamma}{4}}. \tag{11}$$

B_{min} attains a constant value

$$B_{min} \rightarrow \frac{1}{a}\sqrt{\frac{\gamma_c}{t_m}}\sqrt{\frac{3\lambda^2}{2\pi A}}, \tag{12}$$

where A is the pump laser cross section. For a measurement time of 1 s, $\lambda = 500$ nm, $\gamma_c = 10^3$ s^{-1}, and $A = 1$ cm^2, the rhs of Eq. (12) is of order 10^{-10} Gauss. The highest sensitivity obtained experimentally so far with an optical pumping magnetometer is of the order of 10^{-9} Gauss [2].

2 Interferometric Measurements of Magnetic Level Shifts

An alternative way of determining magnetic level shifts is to detect the change of the index of refraction near an atomic resonance.

Let us consider a simple two-level atomic absorber. If we ignore the absorption for the moment, the dispersion of such a medium near resonance is given by

$$n \approx 1 + \frac{\chi'}{2} \approx 1 + \lambda^3 N \frac{\Delta}{\gamma}, \tag{13}$$

where λ is the wavelength of the atomic transition, N the number density of atoms, $\Delta = \omega_{ab} - \nu$ is the detuning between the atomic transition frequency ω_{ab} and the probe field frequency ν. An applied magnetic field which shifts the atomic transition frequency will thus lead to a change of the index of refraction

$$\Delta n \approx \lambda^3 N \frac{a \cdot B}{\gamma}. \tag{14}$$

A probe beam transmitted through a sample of these atoms over a distance L will hence acquire a phase shift due to the magnetic field

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta n L \approx \frac{2\pi}{\lambda}\lambda^3 N L \frac{a \cdot B}{\gamma}. \tag{15}$$

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Detecting this phase shift by interferometric means, for instance in a Mach-Zehnder interferometer, thus gives a sensitive measure for the magnetic level shift. The phase measurement error is found from $\Delta\phi_{error}\Delta m \approx 1$. Assuming again shot noise, i.e. $\Delta m = \sqrt{m}$ and equating the signal and error expressions, yields the minimum detectable magnetic field

$$B_{min} = 2\pi\frac{\gamma}{a}\frac{1}{\lambda^2 LN}\sqrt{\frac{\hbar\nu}{P_{in}t_m}}. \tag{16}$$

Naturally, however, such a gaseous medium will not be useful because of the large absorption as indicated in Fig.4.
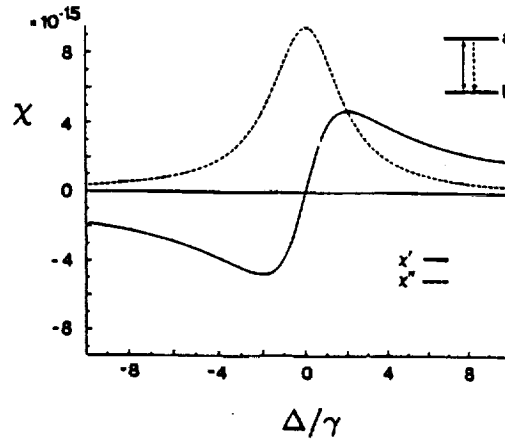
FIG.4. Real (χ') and imaginary part (χ'') of the susceptibility of a two level atom, determining the index of refraction and the absorption.

This is the point where the idea of quantum interference in atomic systems comes in. If the upper level a of an optical transition is driven by a strong driving field to an auxiliary level c, the absorption from the ground state b is essentially cancelled [3], while the index of refraction displays a large dispersion, due to quantum interference of different absorption pathways.

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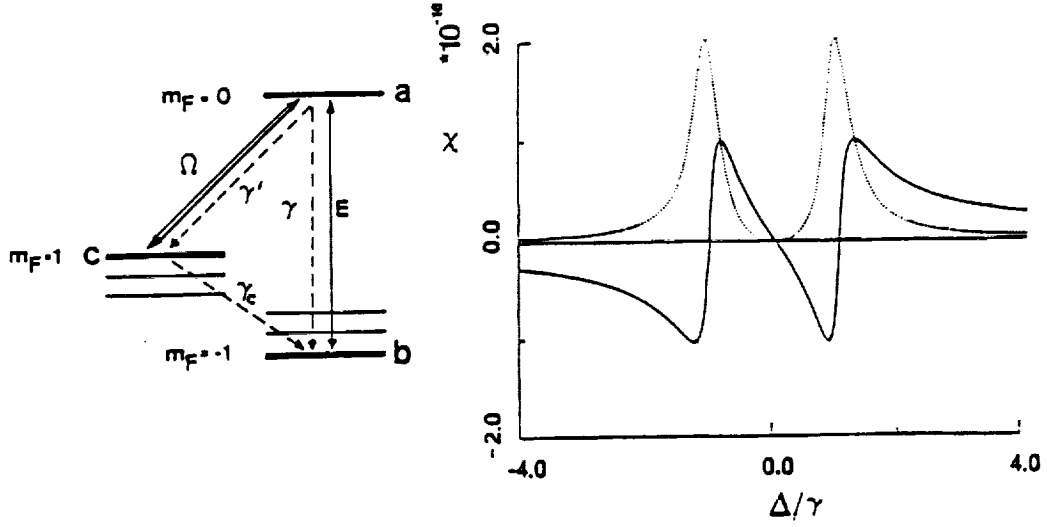
FIG.5. Left: Λ configuration in which strong driving field with Rabi-frequency Ω' on a-c transition generates transparency on the a-b transition. Right: corresponding susceptibility spectrum.

3 Optical Magnetometer Based on Electromagnetically Induced Transparency

Near a resonance of the coherent medium we have a large dispersion of the index of refraction. A probe field propagating a distance L through the phaseonium medium will acquire a phase shift

$$\Delta\phi_{sig} = -\frac{3}{4\pi}\lambda^2 N L \frac{\gamma}{|\Omega'|^2} aB \tag{17}$$

due to the magnetic field. The induced transparency is not perfect due to collisional dephasing of the c-b polarization (γ_c) and the amplitude of the transmitted field will be reduced by a factor κ

$$\kappa = \exp\left\{-\frac{3}{8\pi}\lambda^2 L N \frac{\gamma\gamma_c}{|\Omega'|^2}\right\}. \tag{18}$$

κ is, however, close to unity for sufficiently strong driving fields.

Putting a phaseonium gas cell in one arm of a Mach-Zehnder interferometer as per Fig.6, the signal phase shift (17) can be measured by a balanced detection of the intensities at the two outputs. As shown in Ref.[4], the operation of such a phaseonium magnetometer is again shot noise limited. Equating the signal and noise expressions one finds for the minimum detectable magnetic field in a phaseonium magnetometer

$$B_{min} = \frac{1}{a} \frac{4\pi}{3} \frac{1}{\lambda^2 LN} \frac{|\Omega'|^2}{\gamma} \left[\frac{1 + \kappa^2}{2\kappa^2} \right]^{1/2} \left[\frac{\hbar\nu}{P_{in} t_m} \right]^{1/2} \tag{19}$$
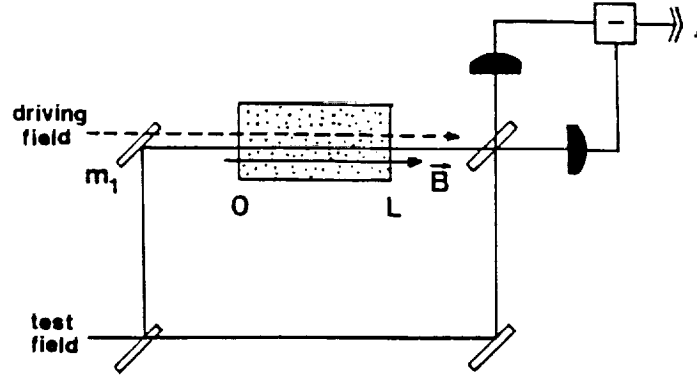
FIG.6. Mach-Zehnder interferometer

Increasing the number density N or the interaction length L enhances the signal phase shift. On the other hand the transmittivity κ decreases. An optimum value is found when

$$\frac{4\pi}{3} \frac{1}{\lambda^2 LN} \frac{|\Omega'|^2}{\gamma} \approx \gamma_c. \tag{20}$$

This gives for the minimum detectable field under optimum parameter conditions

$$B_{min} = \frac{\gamma_c}{a} \left[\frac{\hbar\nu}{P_{in} t_m} \right]^{1/2} \tag{21}$$

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which is identical to the expression found for the standard optical magnetometer for the case of small input power. However, if in the optical pumping magnetometer the input power exceeds a critical value determined by the critical Rabi-frequency (10), the sensitivity remains constant, whereas in the case of the phaseonium magnetometer much higher sensitivities are possible as can be seen in Fig. 7. Here the Rabi-frequency of the probe field, Ω. is limited only by the condition of linearity

$$\Omega \leq \gamma. \tag{22}$$
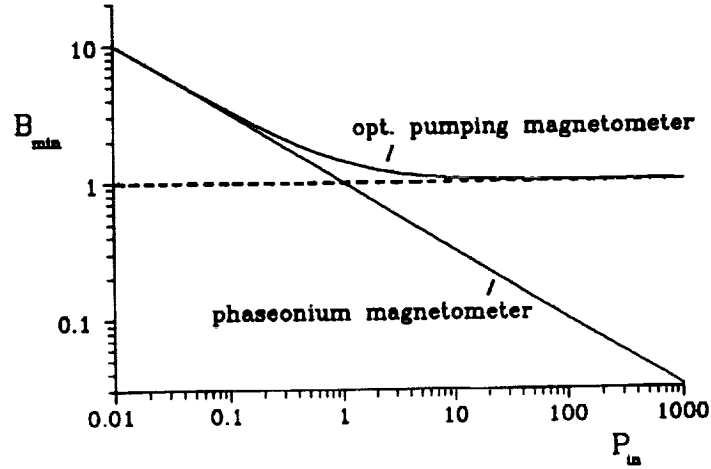
FIG.7. Minimum detectable magnetic field for the optical pumping and phaseonium magnetometers as functions of the input intensity in units of P_{in}^c

To see the potentially enhanced sensitivity, let us consider a special numerical example. Reasonable values are: $\gamma = 10^7$ s^{-1}, $\gamma_c = 10^3$ s^{-1}, $|\Omega'| = \gamma$, $\lambda = 500$ nm, $L = 10$ cm, $t_m = 1$ s, $P_{in} = 1$ mW, $a = 10^7$ s^{-1}/gauss, $N = 2 \times 10^{12}$ cm^{-3} (10^{-4} torr at room temperature). This gives a minimum detectable magnetic field strength of

$$B_{min} \rightarrow 10^{-12} \text{ Gauss}$$

which is smaller by one or two orders of magnitude than that of existing magnetometers. Thus, the phaseonium magnetometer potentially leads to much higher sensitivities than existing state-of-the-art devices.

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4 Acknowledgments

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SMOOTHLY DEFORMED LIGHT

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Abstract

A single mode cavity is deformed smoothly to change its electromagnetic eigenfrequency. The system is modelled as a simple harmonic oscillator with varying period. The Wigner function of the problem is obtained exactly starting with a squeezed initial state. The result is evaluated for a linear change of the cavity length. The approach to the adiabatic limit is investigated. The maximum squeezing is found to occur for smooth change lasting only a fraction of the oscillational period. However, only a factor of two improvement over the adiabatic result proves to be possible. The sudden limit cannot be investigated meaningfully within the model.

1 Introduction

If the length of an electromagnetic cavity is changed, there are two meanings to the concept of adiabaticity. Firstly, the movement may be so slow that the cavity eigenfrequency varies only little during one oscillational period; this is the adiabatic limit proper. However, the process of establishing the correct oscillational frequency requires that the radiation has time for many round trips in the cavity. The cavity deformation may enter another regime, the eigenfrequency does not change appreciably over a few round trips, but it may change significantly over a single oscillational period. In this limit, we still expect the cavity mode to be described by a simple harmonic oscillator, but its frequency changes smoothly with time. If the movement is rapid compared with the cavity round trip time, the complete Maxwell equations need to be used in the calculation. Solving an eigenvalue problem with a moving boundary is a tricky problem; I do not want to discuss this situation here

The theory of a harmonic oscillator with variable frequency is a paradigmatic problem in physics. Classically it appears as a case of parametric driving, and quantum mechanically it is connected to the history of adiabatic invariants. A classical discussion is found in van Kampen [1] and of the many quantum treatments I wish to mention only Dykhne[2], Popov and Perelomov [3] and Man'ko and his collaborators [4]-[5]. Because the Heisenberg equations of motion agree with the classical ones, the quantum solution can be reduced completely to solving the classical problem; this was recently shown in an elegant way by Lo [6]. The same conclusion was formulated for the Wigner function by the present author [7] albeit in a different physical context. Squeezing introduced by time evolution has been discussed for other physical situations in Refs. [8]-[10].

2 The general problem and its solution

In a cavity of length L we assume the Hamiltonian for one radiation mode to be of the form

$$H = \frac{1}{2}\left(p^2 + \Omega^2(t)q^2\right) \tag{1}$$

where the time dependent frequency is given by

$$\Omega^2(t) = \Omega_0^2 f(t) \; ; \; \Omega_0 = \frac{c\pi}{L_0} \; . \tag{2}$$

L_0 is the initial length of the cavity. If we introduce the scaled variables

$$\tau = \Omega_0 t \; ; \; \pi = \frac{p}{\Omega_0} \; , \tag{3}$$

we find the Heisenberg equations of motion using the canonical commutation relations between p and q

$$\dot{q} = \pi \; ; \; \dot{\pi} = -f(\tau)q \; , \tag{4}$$

where the dot denotes derivation with respect to τ. Integrating these equations gives the solution for the Heisenberg variables as has been discussed in the literature.

In the Schrödinger picture we obtain the equation of motion for the Wigner function in the form

$$\frac{\partial W}{\partial \tau} + \pi \frac{\partial W}{\partial q} - f(t)q\frac{\partial W}{\partial \pi} = 0 \; . \tag{5}$$

Its characteristics are the very Eqs. (4), but now they are classical relations between c-numbers. In order to solve (5) we proceed as in Ref. [7] and define the fundamental system of solutions w_1 and w_2 such that

$$w_1(0) = \dot{w}_2(0) = 1$$
$$\dot{w}_1(0) = w_2(0) = 0 \; . \tag{6}$$

Their Wronskian is a constant of the motion equalling unity. We assume the mode in the cavity to initially be in the squeezed state having the Wigner function

$$W_0(q_0, \pi_0) = C \exp\left[-\frac{(q_0 - \bar{q})^2}{s^2 b^2} - \frac{s^2}{b^2}(\pi_0 - \bar{\pi})^2\right] \; . \tag{7}$$

Expressing the general solution of (4) in terms of the solutions (6)

$$q = q_0 \, w_1(\tau) + \pi_0 w_2(\tau)$$
$$\pi = \dot{q} = q_0\dot{w}_1(\tau) + \pi_0\dot{w}_2(\tau) \tag{8}$$

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and inserting q_0 and π_0 from (8) into Eq. (7) we obtain the required solution of (5)

$$W(q,\pi,\tau) \;=\; C \exp\left[-\frac{(\dot{w}_2 q - w_2 \pi - \bar{q})^2}{s^2 b^2}\right]$$

$$\times \;\; \exp\left[-\frac{s^2}{b^2}(w_1 \pi - \dot{w}_1 q - \bar{\pi})^2\right] \;.$$

(9)

Calculating the marginal distribution for the variable q we obtain

$$W_\pi(q,\tau) \;=\; \int d\pi W(q,\pi,\tau)$$
$$=\; C \exp\left[-\frac{(q-q(\tau))^2}{b^2 \sigma^2(\tau)}\right] \;.$$

(10)

The Wigner function thus progresses along the classical trajectory according to

$$q(\tau) = w_1(\tau)\bar{q} + w_2(\tau)\bar{\pi}$$

(11)

and its spreading is given by

$$\sigma^2(\tau) = w_1^2(\tau)s^2 + \frac{w_2^2(\tau)}{s^2} \;.$$

(12)

At the initial time the squeezing is given by s^2, but at the final time, after the change of the cavity length, the result is determined by the values of w_1 and w_2 at the end of the interaction. It is generally agreed, that in the adiabatic limit proper, the change of the squeezing must be small, see e.g. Graham [11]. In the next Section we will investigate a simple model, where we can see how the situation is changed if the motion is smooth, but not necessarily adiabatic with respect to the oscillational frequency.

3 Linear change of cavity length

We now assume that the length of the cavity is changed linearly, viz

$$L(t) = L_0 + \lambda t = L_0 + \lambda\tau/\Omega_0 \;.$$

(13)

The characteristic time scale of the cavity change is given by

$$|t_0| = \frac{L_0}{|\lambda|} = \frac{\Omega_0}{|\dot{\Omega}(t)|}$$

(14)

which goes to infinity for properly adiabatic motion. Negative λ means that the cavity is made to contract.

With these definitions the function $f(t)$ becomes

$$f(t) = \frac{L_0^2}{(L_0 + \lambda t)^2} = \frac{1}{(1 + (t/t_0))^2} \;.$$

(15)

Relations (4) give the equation

$$\ddot{q} + f(\tau)q = 0 \ , \tag{16}$$

which has to be solved with the initial conditions (6). For the given function (15), this becomes a Fuchsian problem with two singularities and the solution can be obtained in a straightforward way.

We introduce the variables

$$A = \sqrt{\Omega_0^2 L_0^2/\lambda^2 - \frac{1}{4}}$$

$$T = \frac{t}{t_0} = \frac{\tau}{\Omega_0 t_0} = \frac{\lambda t}{L_0} = \frac{L(t) - L_0}{L_0} \ . \tag{17}$$

With these definitions the fundamental solutions (6) are given by the expressions

$$w_1(\tau) = \sqrt{1 + T} \left\{ \cos\left[A \log(1 + T)\right] - \frac{1}{2A} \sin\left[A \log(1 + T)\right] \right\}$$

$$w_2(\tau) = \frac{\Omega_0 t_0}{A} \sqrt{1 + T} \sin\left[A \log(1 + T)\right] \ . \tag{18}$$

Regarding T as a function of τ, we can easily see that these functions constitute a solution to the problem. Exciting the cavity state by a classical source, we will find it in a coherent state with $s = 1$ in (7). The width as a function of time becomes

$$\sigma^2(t) = w_1^2(\tau) + w_2^2(\tau) \ . \tag{19}$$

Before we proceed to consider the consequences of the exact expression (18) for the width (19), we look at the adiabatic limit proper, i.e. $\lambda \Rightarrow 0$. Then we find

$$A \log(1 + T) \Rightarrow \frac{\Omega_0 L_0}{\lambda} \frac{\lambda t}{L_0} = \Omega_0 t$$

$$\frac{\Omega_0 t_0}{A} \Rightarrow 1 \ . \tag{20}$$

With these results, the equations (18) go over into

$$w_1(\tau) = \sqrt{1 + T} \cos \Omega_0 t$$

$$w_2(\tau) = \sqrt{1 + T} \sin \Omega_0 t \ . \tag{21}$$

Remembering that Eq. (17) implies

$$\sqrt{1 + T} = \sqrt{\frac{L(t)}{L_0}} = \sqrt{\Omega_0/\Omega(t)} \tag{22}$$

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we find that the results (21) follow from a simple application of the WKB-method to the equation (16). Inserting these results into the width (19) we find

$$\sigma^2(t) = \frac{\Omega_0}{\Omega(t)} \ . \tag{23}$$

As we cannot hope to change the oscillational frequency by a large fraction, we reach the conclusion that no large amount of squeezing can be achieved in the adiabatic regime proper. This agrees with conclusions arrived at in earlier treatments, in particular the adiabatic invariance of $\Omega\sigma^2$ has been found, see e.g. Ref. [11].

Another peculiarity of the result (23) is that no trace of the oscillational behaviour survives. If the parameter A is not too large, the situation changes. Because of the second term in w_1 of Eq. (18), oscillations appear in the width. To see how much squeezing they can achieve, we write the solutions (18) in the form

$$w_1 = \sqrt{1+T}\left[\cos\varphi - \frac{1}{2A}\sin\varphi\right]$$

$$w_2 = \frac{\Omega_0 t_0}{A}\sqrt{1+T}\sin\varphi \ . \tag{24}$$

Here φ is the argument of the trigonometric functions in Eqs. (18). The width (19) then becomes

$$\sigma^2(t) = \frac{\Omega_0}{\Omega(t)}\left[1 - \frac{1}{2A}\sin 2\varphi + \frac{1}{2A^2}\sin^2\varphi\right] \ . \tag{25}$$

For $A \Rightarrow \infty$ this reproduces (23). The expression has a minimum for each fixed value of the parameter A, but for large A, this approaches the adiabatic limiting value (23). For example $A = 1$ gives the minimum value 0.69 for the expression in square brackets in (25). This occurs at the time when $\varphi = 0.55$.

The best possible values for the squeezing are obtained with a very small A, in which case the minimum occurs for early times, $\varphi \approx 0$. The expression (25) can then be written

$$\sigma^2(t) = \frac{\Omega_0}{2\Omega(t)}\left[1 + \left(\frac{\varphi}{A} - 1\right)^2\right] \geq \frac{\Omega_0}{2\Omega(t)} \ . \tag{26}$$

which is not a large improvement over (23). The minimum also occurs for a small parameter A, in which case we rapidly approach the breakdown of the validity of the theory. For very small A, the expression (17) gives

$$\Omega_0 t_0 \simeq \frac{1}{2} \ , \tag{27}$$

which is not in the adiabatic regime proper. The minimum then occurs at times when

$$\varphi \approx \Omega_0 t \approx A < \Omega_0 t_0 \approx \frac{1}{2} \ . \tag{28}$$

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Thus we have to change the cavity eigenfrequency in a time less than the oscillation period. This cannot obviously be achieved by mechanical means, and even using some electronic switching to change the effective path length through the cavity, we can attempted this only in the microwave region. However, as the advantage of the method is expected to be small, there seems to be little motivation to solve the technical problems involved.

4 Discussion

We have solved the problem of the deformation of an intracavity field during a smooth change of the cavity eigenfrequency. Even if we are allowed to depart from the strict adiabatic limit, the expected squeezing remains modest. The calculation cannot be taken to the sudden limit, because then the simple harmonic oscillator description is no longer valid. The complete Maxwell equations must be treated in that case. In this aspect our problem differs from the corresponding Schrödinger equation [12]-[14] where both the sudden and the adiabatic limit can be handled in the same way.

5 Acknowledgements

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Nonclassical states of the second optical harmonic in the presence of self-action

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Abstract

The quantum theory of coherent radiation frequency doubling in crystals with quadratic and cubic optical nonlinearities is developed. The possibility to produce the quadrature – squeezed state of the second harmonic (SH) field is shown, the nonclassical SH states arising due to self-action effect.

1. Introduction

The quantum theory of the second and higher harmonic generation has been developed in a number of works (see, for instance, Refs. 1-3) in which the possibility of obtaining the squeezed states of electromagnetic field and photon antibunching has been analyzed. It has been established that the frequency doubling is accompanied with the generation of the squeezed states at the fundamental frequency whereas the second harmonic (SH) field turns out to be in the coherent state. At the same time the frequency doubling of the squeezed light, as it was shown in Ref. 1, causes a decrease in squeezing. From the practical point of view, the methods based on the quadratic and cubic medium nonlinearities with respect to the electric field are of considerable interest. It is known[4] that in the centrosymmetric nonlinear medium, i.e. the Kerr medium, the quadrature – squeezed field can be produced due to the self-action effect. In the media mentioned above the four frequency wave processes always occur in the presence of self-action. In the media with the induced quadratic optical susceptibility the three frequency wave interactions can also occur under the evident influence of self-action.

In the present paper the quantum theory of the SH generation (SHG) in the presence of self-action is developed. In the framework of the classical approach the problem under consideration has been solved in Refs. 5 and 6. The basic equations of the process which are of interest to us are presented in Sec. 2. In the Sec. 3 SHG is analyzed for the case of low efficiency of the fundamental radiation conversion into the SH; however, we do not take into account here the SH influence on the effective refractive index of the medium. The possibility of the SH quadrature – squeezed state generation is shown in Sec. 4.

2. Basic equations

Interaction of the fundamental wave of frequency ω and second harmonic wave of frequency 2ω in an optical medium with nonlinear susceptibilities of the second $\chi^{(2)}$ and third $\chi^{(3)}$ orders is described by the Hamiltonian:

$$\hat{H} = \hbar\omega\hat{a}\hat{a}^+ + 2\hbar\omega\hat{b}\hat{b}^+ + \hat{H}_i \quad ,$$

$$\hat{H}_i = \hbar\beta(\hat{b}^+\hat{a}^2 + \hat{b}\hat{a}^{+2}) + \hbar\gamma(\hat{a}^{+2}\hat{a}^2 + \hat{b}^{+2}\hat{b}^2 + 2\hat{a}^+\hat{a}\hat{b}^+\hat{b}) \quad ,$$

$$(1)$$

where $\hat{a}^+(\hat{a})$ and $\hat{b}^+(\hat{b})$ are photon creation (annihilation) operators of the fundamental wave and SH which obey the commutation relations:

$$[\hat{a},\hat{a}^+] = 1 \ , \quad [\hat{b},\hat{b}^+] = 1, \qquad (2a,b)$$

the nonlinear parameter β is proportional to $\chi^{(2)}$, and parameter γ to $\chi^{(3)}$. The operator evolution is given by the Heisenberg equations:

$$\frac{\partial\hat{a}}{\partial z} = -i2\beta\hat{a}^+\hat{b} - i\gamma\hat{a}^+\hat{a}^2 - i2\gamma\hat{b}^+\hat{b}\hat{a} \ , \qquad (3a)$$

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$$\frac{\partial \hat{b}}{\partial z} = -i\beta\hat{a}^2 - i2\gamma\hat{a}^+\hat{a}\hat{b} - i\gamma\hat{b}^+\hat{b}^2 \qquad (3b)$$

where z is the length of the medium in the direction of wave propagation.

Let us discuss the terms on the right – hand side of Eqs. (3a) and (3b). The first terms are associated with the process of degenerated three-frequency interaction (the first term in Eq.(3b) describes the SH generation (SHG), whereas in Eq.(3a) the first term takes account of parametric interaction). The second terms in Eqs.(3a) and (3b) deal with the self-action and cross-action of the radiation of frequency ω. Finally the third terms in Eqs.(3a) and (3b) take into account the cross-action and self-action at frequency 2ω.

Assuming that at the input of the nonlinear medium the fundamental wave and second harmonic are in the coherent and vacuum states respectively, we have

$$\hat{a}(z=0) = \hat{a}_o , \qquad \hat{a}_o |\alpha\rangle = \alpha_o |\alpha\rangle ;$$

$$(4)$$

$$\hat{b}(z=0) = \hat{b}_o , \qquad \hat{b}_o |0\rangle = 0 .$$

3. SH generation in the fixed photon number approximation

The analysis given below implies the low efficiency of the conversion of the fundamental radiation into the SH. Therefore, we can neglect the last terms in Eqs. (3a) and (3b). We thus take into account the refractive index variation due to the cubic nonlinearity caused only by the intensive fundamental wave. The SHG process is analyzed in the fixed photon number approximation. Using this approximation one neglects the photon number variation of the fundamental wave, i.e. we suppose that the operator of the photon number $\hat{n}(z) = \hat{a}^+(z)\hat{a}(z)$ remains unchanged during the process of the nonlinear interaction $(\hat{n}(z) = \hat{n}_o = \hat{a}_o^+\hat{a}_o)$. It should be noted that this approximation is

in fact the quantum analog of the fixed intensity approximation (see Ref. 7).

Let us introduce the new operators $\hat{c}(z)$ and $\hat{f}(z)$ for the fundamental radiation and SH respectively:

$$\hat{c}(z) = e^{i\gamma z \hat{n}_o} \hat{a}(z) \ , \qquad \hat{f}(z) = e^{2i\gamma z \hat{n}_o} \hat{b}(z. \qquad (5a,b)$$

These operators also obey the commutation relations similar to Eq.(2) and the initial conditions similar to Eq.(4). The evolution of the new operators is given by the equations:

$$\frac{d\hat{c}(z)}{dz} = -i2\beta \ e^{i\gamma z} \ \hat{c}^+(z)\hat{f}(z) \ , \qquad (6a)$$

$$\frac{d\hat{f}(z)}{dz} = -i\beta \ e^{-i\gamma z} \ \hat{c}^2(z) \ . \qquad (6b)$$

By differentiating Eq.(6b) and using Eq.(6a), we obtain the equation for the SH operator $\hat{f}(z)$

$$\frac{d^2}{dz^2} \hat{f}(z) + 4\beta^2 (\hat{n}_o + \frac{1}{2}) \ \hat{f}(z) = 0 \qquad (7)$$

with the initial conditions

$$\hat{f}(z=0) = \hat{b}_o \ , \qquad \frac{d\hat{f}}{dz}\bigg|_{z=0} = -i\beta \hat{a}_o^2. \qquad (8)$$

Below we make use of the operator $\hat{f}(z)$ expanded into the Taylor series to within χ^2 ($\chi=\beta z$):

$$f(z) = \hat{f}_o + \frac{d\hat{f}}{dz}\bigg|_{z=0} z + \frac{1}{2} \frac{d^2\hat{f}}{dz^2}\bigg|_{z=0} z^2 + \ldots \qquad (9)$$

Returning to the primary operators of the fundamental wave and SH ($\hat{a}$ and $\hat{b}$ respectively) we obtain the expression:

$$\hat{b}(z) = e^{-2i\gamma z \hat{n}_o} \{\hat{b}_o - i\beta z \hat{a}_o^2 - \frac{1}{2} \beta\gamma z^2 \hat{a}_o^2 - 2(\beta z)^2 (\hat{n}_o + \frac{1}{2})\hat{b}_o\}. \quad (10)$$

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The evolution of the SH field operator depends on the value of the nonlinear parameters βz and γz.

By averaging over initial states of the fields, we obtain the mean value for the operator $\hat{b}$ (10) :

$$\langle \hat{b}(z) \rangle = -(i\beta z + \frac{1}{2}\beta\gamma z^2) |\alpha_o|^2 \, e^{2i(\phi_o - \Psi)} \qquad (11)$$

Here $\Psi = \gamma z |\alpha_o|^2$ is the nonlinear phase addition arising due to the self-action and $\phi_o = \arg |\alpha_o|$ is the fundamental wave phase. In the framework of the considered approximation the operators $\hat{b}^+$ and $\hat{b}$ satisfy the commutation relation (2b).

Let us turn to the analysis of the SH field photon statistics. Calculations of the Fano factor $F = \sigma_N^2 / \langle \hat{N} \rangle$ (where $\hat{N} = \hat{b}^+(z)\hat{b}(z)$) result in the following expression:

$$F(z) = 1 + (\beta z)^2 (4|\alpha_o|^2 + 2) \, . \qquad (12)$$

Thus, as one can see from Eq.(12), the photon statistics of the SH field becomes super-Poissonian.

4. Quadrature components of the SH field

In this section we dwell upon the fluctuations of the SH quadrature components described by the operators:

$$\hat{X}(z) = \frac{1}{2} \{\hat{b}(z) + \hat{b}^+(z)\}, \qquad \hat{Y}(z) = \frac{1}{2i} \{\hat{b}(z) - \hat{b}^+(z)\}. \qquad (13)$$

The quadrature components (13) are registered by the balanced homodyne detection (see Figure). The SH field being under investigation is mixed with the coherent one at the same frequency, generated in the absence of the self-action and cross-action effects. The mixed radiation of the both reference coherent wave and that of the analyzed SH is fed to the balanced detector input. Thus, we have the possibility to record one of the SH quadrature components for the field under consideration.
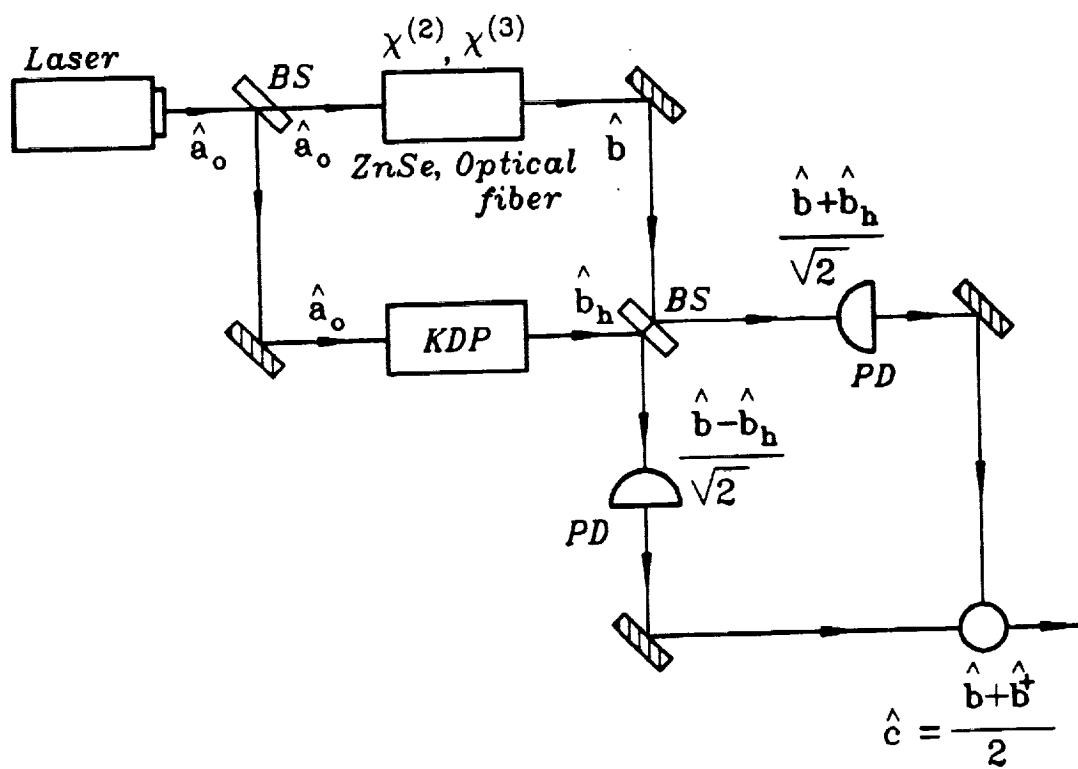
Figure: Layout of measurement of SH quadrature components.

According to Eqs.(10) and (13) the mean values of the quadrature components (13) are equal to

$$\langle \hat{X}(z) \rangle = \beta z |\alpha_o|^2 \sin[2(\phi_o - \Psi)] - \frac{1}{2} \beta z \Psi \cos[2(\phi_o - \Psi)],$$

$$(14)$$

$$\langle \hat{Y}(z) \rangle = -\beta z |\alpha_o|^2 \cos[2(\phi_o - \Psi)] - \frac{1}{2} \beta z \Psi \sin[2(\phi_o - \Psi)]$$

and determined by the value of the nonlinear parameter βz and nonlinear phase addition Ψ.

Calculations of the variances of the SH quadrature component yeild in the expressions:

$$\sigma^2_{x,y} = \frac{1}{4} + \frac{|\alpha_o|^2}{4} \{ (\frac{1}{2}(\beta\gamma)^2 z^4 - 2(\beta z)^2)(\cos[4(\phi_o - \Psi) - 2\gamma z] - \cos[4(\phi_o - \Psi)])$$

$$(15)$$

$$\mp 2\beta^2 \gamma z^3 (\sin[4(\phi_o - \Psi) - 2\gamma z] - \sin[4(\phi_o - \Psi)]) \},$$

where the upper sign is for the $\hat{X}$ quadrature and the lower is for $\hat{Y}$. Let us transform Eq.(15) by retaining only the terms of order $(\beta z)^2$ and smaller. As a result we have to within $(\beta z)^2$

$$\sigma^2_x = \frac{1}{4} - K^2 \Psi \sin[4(\phi_o - \Psi)] + 2(\beta z \Psi)^2 \cos[4(\phi_o - \Psi)],$$

$$(16)$$

$$\sigma^2_y = \frac{1}{4} + K^2 \Psi \sin[4(\phi_o - \Psi)] - 2(\beta z \Psi)^2 \cos[4(\phi_o - \Psi)],$$

where the coefficient $K^2 = (\beta |\alpha_o| z)^2$ characterizes the efficiency of the SH conversion. It follows from Eq.(16) that the variances are the oscillatory functions of the parameter Ψ due to the self-action. The oscillation amplitude depends on the SHG efficiency and the value of the phase Ψ. It is evident that the variations of the variances have the opposite tendency. The analysis of Eq.(16) is more clear provided the initial radiation phase ϕ_o is optimized:

$$\phi_o = \Psi + \frac{1}{4} \arg(-\frac{1}{2\gamma z}) \cong \Psi. \qquad (17)$$

The extremal values of the variances Eq.(16) are equal to:

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$$\sigma^2_{x,y} = \frac{1}{4} \pm K^2 \Psi \ . \tag{18}$$

One can see from Eq.(18) that it is possible to obtain the quadrature – squeezed states of SH field. In this case the predominant role is played by the self-action effect. In the absence of the self-action ($\Psi=0$) the SH field is in the coherent state ($\sigma^2_x = \sigma^2_y = 1/4$). It is obvious from Eq.(18) that the degree of squeezing can be arbitrary high and is determined by the efficiency of the SH conversion K^2 and phase Ψ .

It follows from the calculation of the uncertainty relation for the SH quadrature components that we have the ideal quadrature squeezing to within $(\beta z)^2$.

6. Conclusions

It follows from the analysis given above that SH quadrature – squeezed states are produced by frequency doubling in the presence of the self-action phenomenon which plays a predominant role. The degree of squeezing is determined by both the SHG efficiency and nonlinear phase induced by self-action. The nonlinear medium where the considered process is likely to occur can be realized in noncentrosymmetric nonlinear crystal (for example ZnSe) or centrosymmetric medium in a static electric field. It seems to be promising to use optical fibers, in which the SHG efficiency can reach 1–5% (Ref.7).

As it was mentioned above the possibility to produce the squeezed states of the fundamental radiation isusually studied in SHG process occurring in the absence of the self-action. Outside the framework of the fixed photon number approximation we also considered the fundamental field statistics, taking into account the self-action effect. We have found that the degree of the fundamental radiation squeezing depends on the SHG efficiency and nonlinear phase as in the case of the second harmonic field.

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QUANTUM NOISE AND SQUEEZING IN OPTICAL PARAMETRIC OSCILLATOR WITH ARBIRTRARY OUTPUT COUPLING

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Abstract

The redistribution of intrinsic quantum noise in the quadratures of the field generated in a sub-threshold degenerate optical parametric oscillator exhibits interesting dependences on the individual output mirror transmittances, when they are included exactly. We present here a physical picture of this problem, based on mirror boundary conditions, which is valid for arbitrary transmittances and so applies uniformly to all values of the cavity Q factor representing in the opposite extremes perfect oscillator and amplifier configurations. Beginning with a classical second-harmonic pump, we shall generalize our analysis to apply to finite amplitude and phase fluctuations of the pump.

1 Introduction

A degenerate optical parametric oscillator (DOPO) has long been considered a nearly ideal squeezing device when operated just below threshold. The quantum fluctuations of the generated sub-harmonic field are rather immune to spontaneous emission since the two-photon transition governing the parametric down-conversion process sees no resonant intermediate levels.

Nearly all prior work dealing with this problem [1,2,3] has been limited to the situation in which the DOPO cavity is nearly perfect. In a general approach [4,5] developed recently by the author and Abbott, which is based on the exact treatment of mirror boundary conditions, it has become possible to discuss cavity problems in quantum optics for the entire range of cavity transmissions possible. In the present DOPO context, this approach thus permits the extreme limits of a single-pass amplifier (cavity transmission →100%) and of a nearly perfect DOPO cavity (cavity transmission →0%), and all intermediate-Q oscillator configurations to be treated on the same footing. By employing this viewpoint (which may be viewed as a generalization of Collett and Gardiner's approach [2]), we also develop a physically insightful picture of the general squeezing

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[1]Permanent address

problem, one which emphasizes the correlations of the input, output, and intracavity fields that govern the relationship of intracavity and output field fluctuations. Any reference to modes is altogether avoided here.

After treating the DOPO problem with a perfectly monochromatic pump, we shall model realistic experiments in which the pump field has finite amplitude and phase fluctuations. Although any amplitude noise of the pump has a relatively minor impact on the squeezing of the subharmonic signal field, pump phase diffusion even when it is tracked can cause a severe degradation of that squeezing. More detailed discussions of this problem will appear elsewhere [6].

2 Mathematical Formalism

A description of the problem at hand that covers the whole gamut of cavity transmission factors is necessarily multimode in character. We avoid all reference to cavity modes by writing the fully quantized signal field inside the cavity in terms of its rightward (positive$-z$) and leftward (negative$-z$) propagating parts. For the positive-frequency part, this decomposition is written in the Heisenberg picture (HP) as

$$E^{(+)}(z,t) = (e_+(z,t)\,e^{ik_0 z} + e_-(z,t)\,e^{-ik_0 z})\,e^{-i\Omega_0 t}, \tag{1}$$

in which the operators $e_\pm(z,t)$ have expectation values that are assumed slowly varying in space and time on the scale of the central wavelength $2\pi/k_0$ and period $2\pi/\Omega_0$.

The parametric interaction of $E^{(+)}(z,t)$ with an intense quasimonochromatic is described via the interaction Hamiltonian (also written in HP) in a cavity of length ℓ filled with the parametric medium:

$$H'_{DOPO} = \frac{3A}{4}\,\chi^{(2)}\,e^*_{pump}\int_0^\ell \left[e_+^2(z,t) + e_-^2(z,t)\right]dz + \text{Hermitian Conjugate} \tag{2}$$

The complex pump amplitude e_{pump} is at most slowly varying in time. The constants A and $\chi^{(2)}$ are the cross-sectional area of the cavity and nonlinear susceptibility, respectively. The notation used is the same as in Ref. [7]. We may write the equations of propagation for $e_\pm(z,t)$ in the slowly-varying envelope approximation as

$$\left(\frac{\partial}{\partial z} \pm \frac{1}{c}\frac{\partial}{\partial t}\right)e_\pm(z,t) = \pm\frac{ik}{2\epsilon}p_\pm^{NL}(z,t), \tag{3}$$

in which the nonlinear polarization waves $p_\pm^{NL}(z,t)$ driving the parametric interaction are given by a functional differentiation of the quadratic interaction Hamiltonian (2):

$$\begin{aligned} p_\pm^{NL}(z,t) &= -\tfrac{1}{A}(\delta/\delta e_\pm^\dagger(z,t))\,H'_{DOPO} \\ &= -\tfrac{3}{4}\chi^{(2)*}e_{pump}\,e_\pm^\dagger(z,t). \end{aligned} \tag{4}$$

Thus, on combining (3) and (4), we have the following generalization of the single-mode equations describing the parametric amplification process:

$$\left(\frac{\partial}{\partial z} \pm \frac{1}{c}\frac{\partial}{\partial t}\right)e_\pm(z,t) = \pm q\,e_\pm^\dagger(z,t), \tag{5}$$

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CAVITY WITH
NONLINEAR MEDIUM

MIRROR
z=0

MIRROR
z=1

Fig. 1. The DOPO Cavity with End Mirrors at $z = 0$ and $z = \ell$.

To complete the formalism, we supplement Eq. (5) with boundary connections of the *intra-cavity* $e_{\pm}(z,t)$ fields with the *input* vacuum fields. These connections are

$$
\begin{aligned}
e_+(0,t) &= -\tilde{r}e_-(0,t) + \tilde{t}e_+^{vac}(0,t); \\
e_-(\ell,t) &= -\tilde{r}'e_+(\ell,t) + \tilde{t}'e_-^{vac}(\ell,t),
\end{aligned}
\tag{6}
$$

in which $e_{\pm}^{vac}$ are the two traveling pieces of the vacuum field entering the cavity through its mirrors at $z = 0$ and $z = \ell$ with inside-to-outside reflection and transmission coefficients $(-\tilde{r}, \tilde{t})$ and $(-\tilde{r}', \tilde{t}')$ respectively (see Fig. 1).

3 The Parametric Amplifier Problem

Without the cavity mirrors, the oscillator reduces to the amplifier configuration in which the two traveling parts e_+ and e_- are not coupled to each other. We may therefore concentrate on only one of them, say the e_+−field.

Furthermore, for simplicity, we shall assume in this section that the pump has no amplitude and phase randomness, so that it is strictly monochromatic. For this case, one may assume without any loss of generality that q is real and positive, for any constant nonzero phase ϕ_q of q may be scaled out by redefining $e_+(z,t)$ to carry a constant phase factor $\exp(i\phi_q/2)$:

$$
e_+(z,t) \rightarrow e_+(z,t)\,e^{i\phi_q/2},
\tag{7}
$$

without altering the physics.

By adding to Eq. (5) and by subtracting from it its Hermitian conjugate, one obtains the

following pair of uncoupled equations for the quadratures of e_+:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) X_+(z,t) = qX_+(z,t); \quad \left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) Y_+(z,t) = -qY_+(z,t), \quad (8)$$

where $X_+(z,t) = \frac{1}{2}\left(e_+(z,t) + e_+^\dagger(z,t)\right); Y_+(z,t) = \frac{1}{2i}\left(e_+(z,t) - e_+^\dagger(z,t)\right)$ are the in-phase and $\pi/2$ out-of-phase quadratures. The solution of Eqs. (8) is straightforward in terms of the retarded time variable, $\tau = t - z/c$:

$$X_+(z,t) = X_+(0, t - z/c)e^{qz}; \quad Y_+(z,t) = Y_+(0, t - z/c)e^{-qz}; \quad (9)$$

which represents a phase-sensitive amplification process characteristic of the parametric interaction. These solutions are entirely equivalent to the following time-evolution equations

$$X_+(z,t) = X_+(z - ct, 0)e^{qct}; \quad Y_+(z,t) = Y_+(z - ct, 0)e^{-qct}. \quad (10)$$

The linear relationships of Eqs. (9) or (10) indicate that both the expectation value and fluctuations about it of the X_+-quadrature (Y_+-quadrature) of the signal field amplify (attenuate) by the same factor. This statement, valid both classically and quantum-mechanically, clearly implies that any noise initially present in the signal is stretched along the X-quadrature and shrinks along the Y-quadrature, as shown in Fig. 2. It is in this way that quadrature squeezing comes about in a parametric amplifier.

4   The Parametric Oscillator Problem

Our treatment of the parametric *oscillator* builds upon the simple *amplifier* analysis presented in Sec. 3 by limiting z to lie between 0 and ℓ and adding mirrors at $z = 0$ and at $z = \ell$, which serve to connect e_+ and e_- and the input vacuum fields via (6). As in Sec. 3, we restrict our analysis here to a perfectly monochromatic pump wave for which Eqs. (9) describe the interaction of the e_+ wave with the medium. Similar relations may be written down for the quadratures of the e_- -field (integrated backwards from $z = \ell$):

$$\begin{array}{ll} X_-(z,t) = & X_-(\ell, t - (\ell - z)/c)\, e^{q(\ell - z)}; \\ Y_-(z,t) = & Y_-(\ell, t - (\ell - z)/c)\, e^{-q(\ell - z)} \end{array} \quad (11)$$

Since we are ultimately interested in calculating the quadrature squeezing of the intracavity field $e_\pm(z,t)$, we concentrate here onwards on the quantum fluctuations alone of the various quadratures. We first consider what the implications of the boundary connection relations (6) are for the fluctuations. Since $(\bar{r}, \bar{t})$ and $(\bar{r}', \bar{t}')$ are all real, these relations are formally the same as those obeyed by any particular quadrature of $e_\pm$ and $e_\pm^{vac}$ fields, including their $X-$ and $Y-$ quadratures separately. Furthermore, the two fields (or their quadratures) on the right-hand side (RHS) of each equation in (6) are uncorrelated at any t. To see this, we note, for example, that the $e_+^{vac}(0,t)$ field entering the $z = 0$ mirror contributes to the $e_-^{vac}(\ell, t')$ field only after a time $t' - t = 2\ell/c$ during which the former field makes a full round trip through the cavity. Thus, $e_-(0,t)$ is correlated with $e_+^{vac}(0, t - 2\ell/c)$ which is not correlated with $e_+^{vac}(0,t)$, since the vacuum field fluctuations are essentially δ-correlated in time. In view of this lack of correlation, we may
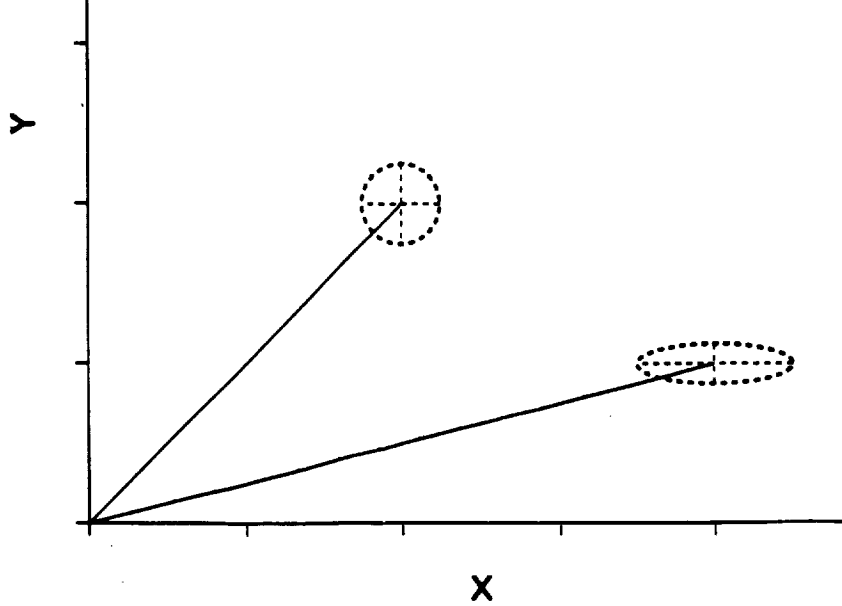
Fig. 2. The Parametric Amplification Process. The X—quadrature is amplified by a given factor (taken to be 2 here) while the Y—quadrature is attenuated by the same factor.

write for the quantum-mechanical variance of, say, the Y—quadrature of fields at the mirrors in terms of the power reflection and transmission coefficients (R, T) and (R', T') (with $R = \bar{r}^2$, etc.)

$$
\begin{aligned}
\langle \Delta Y_+(0, t)^2 \rangle &= R \langle \Delta Y_-(0, t)^2 \rangle + T \left\langle \Delta Y_+^{vac}(0, t)^2 \right\rangle ; \\
\langle \Delta Y_-(\ell, t)^2 \rangle &= R' \langle \Delta Y_+(\ell, t)^2 \rangle + T' \left\langle \Delta Y_-^{vac}(\ell, t)^2 \right\rangle ,
\end{aligned}
\tag{12}
$$

while setting $z = \ell$ in Eqs. (9) and $z = 0$ in Eqs. (11) yields for the propagation of variances through the medium

$$
\left\langle \Delta Y_+(\ell, t)^2 \right\rangle = \left\langle \Delta Y_+(0, t - \ell/c)^2 \right\rangle e^{-2q\ell}; \quad \left\langle \Delta Y_-(0, t)^2 \right\rangle = \left\langle \Delta Y_-(\ell, t - \ell/c)^2 \right\rangle e^{-2q\ell}.
\tag{13}
$$

With the aid of Eqs. (12) and (13), we may express the retarded propagation of the Y_+—variance at $z = 0$ in one complete round trip as

$$
\begin{aligned}
\langle \Delta Y_+(0, t)^2 \rangle &= R \langle \Delta Y_-(\ell, t - \ell/c)^2 \rangle \, e^{-2q\ell} + T \left\langle \Delta Y_+^{vac}(0, t)^2 \right\rangle \\
&= e^{-2q\ell} R \left[R' \langle \Delta Y_+(\ell, t - \ell/c)^2 \rangle + T' \left\langle \Delta Y_-^{vac}(\ell, t - \ell/c)^2 \right\rangle \right] \\
&\quad + T \left\langle \Delta Y_+^{vac}(0, t)^2 \right\rangle \\
&= RR' e^{-4q\ell} \langle \Delta Y_+(0, t - 2\ell/c)^2 \rangle + T \left\langle \Delta Y_+^{vac}(0, t)^2 \right\rangle \\
&\quad + RT' e^{-2q\ell} \left\langle \Delta Y_-^{vac}(\ell, t - \ell/c)^2 \right\rangle .
\end{aligned}
\tag{14}
$$

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PROPAGATE BACKWARDS

MIXING OF INPUT AND CAVITY
FIELDS AT $z = 0$ MIRROR

MIXING OF INPUT AND CAVITY
FIELDS AT $z = l$ MIRROR

PROPAGATE FORWARDS
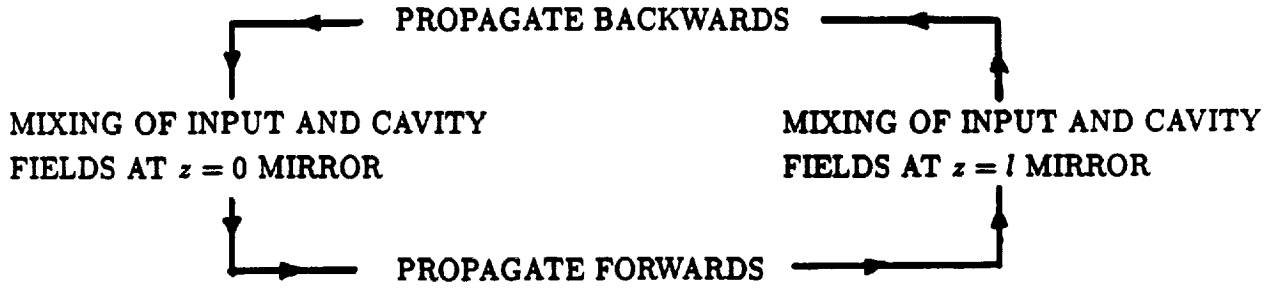
Fig. 3. Round Trip Evolution of Fields and Their Variances.

The foregoing sequence of mathematical steps in arriving at the round trip propagation of variances is shown diagrammatically in Fig. 3 to bring out the underlying physical picture.

In steady state, the quantum statistical properties of the field do not change from one round trip to the next. In this long-time limit, suppressing the time entry of each variance in Eq. (14), we get

$$\left\langle \Delta Y_+(0)^2 \right\rangle = \frac{\left[T \left\langle \Delta Y_+^{vac}(0)^2 \right\rangle + RT'e^{-2q\ell} \left\langle \Delta Y_-^{vac}(\ell)^2 \right\rangle \right]}{(1 - RR'e^{-4q\ell})}, \tag{15}$$

a result that is uniformly valid for all values of (R, T) and (R', T') pairs (with the obvious energy-conservation constraints, $R + T = R' + T' = 1$). It is also worth noting that in the derivation of (15), the only property of the input fields used was their white-noise (δ-correlated) character. Thus, (15) applies not just to vacuum-field inputs, but to arbitrary white-noise input fields.

In the good-cavity limit, $R, R' \simeq 1, q\ell \simeq 0$, we recover the result of Collett and Gardiner generalized to allow for arbitrary white-noise input fields at the two mirrors:

$$\left\langle \Delta Y_+(0)^2 \right\rangle \simeq \frac{T \left\langle \Delta Y_+^{vac}(0)^2 \right\rangle + T' \left\langle \Delta Y_-^{vac}(\ell)^2 \right\rangle}{(T + T') + 4q\ell}. \tag{16}$$

For vacuum-field inputs as explicitly indicated in Eq. (15), since the two input fields are statistically identical (except for their direction of propagation), we may write more simply

$$\left\langle \Delta Y_+(0)^2 \right\rangle = \frac{\left(T + RT'e^{-2q\ell} \right)}{(1 - RR'e^{-4q\ell})} N_{vac}, \tag{17}$$

where

$$N_{vac} \equiv \left\langle \Delta Y_+^{vac}(0)^2 \right\rangle = \left\langle \Delta Y_-^{vac}(0)^2 \right\rangle.$$

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Note that the calculation of the variance $\langle \Delta X_+(0)^2 \rangle$ of the X-quadrature of the intracavity is entirely analogous and is given by Eq. (17) provided q is replaced by $-q$ everywhere.

The degree of quadrature squeezing is the ratio $\langle \Delta Y_+(0)^2 \rangle /N_{vac}$ which is generally the factor by which two input fields with the same quadrature variance, but not necessarily vacuum fields, get squeezed on entering the cavity. Detailed discussions of this quantity in both textual and graphical forms have been presented elsewhere, where its generalization to include arbitrary relative phase between the two traveling components of the monochromatic pump has also been derived [5,6].

Having discussed the intracavity field, we now present the noise characteristics of the output field. Like the former field, the latter field is strongly correlated with the input fields as well. However, unlike the former, the output field quadratures can be easily subjected to a spectral analysis by choosing a sufficiently narrowband local oscillator field and integrating long enough in a balanced homodyne setup as was done in the original experiments [8]. We shall see that it is in this spectral sense that the output field exhibits a very high degree of squeezing.

The boundary connection of the output is similar to Eqs. (6). For example, the leftward-traveling output field at the $z = 0$ mirror is a linear superposition of the transmitted part of $e_-(0,t)$ and the reflected part of $e_+^{vac}(0,t)$. So any quadrature of the output field, say its Y_--quadrature, obeys the boundary connection formula

$$Y_{out}(0,t) = \tilde{t} Y_-(0,t) + \tilde{r} Y_+^{vac}(0,t). \tag{18}$$

However, unlike the intracavity field, we must know the full time dependence of $Y_{out}(0,t)$, not just of its variance, before it can be spectrally analyzed. Equivalently, as (18) shows, we must know how $Y_-(0,t)$ evolves in time. But, that is easy to write down over a complete round trip since we know via Eqs. (9) and (11) how the intracavity field $e_\pm$ interacts with the active medium in a single pass through it, while Eqs. (6) tell us how the input fields $e_\pm^{vac}$ leak into the cavity at the $z = 0$ and $z = \ell$ mirrors. The round trip evolution of $Y_-(0,t)$ turns out to be

$$\begin{aligned}
Y_-(0,t) = \tilde{r}\tilde{r}e^{-2q\ell}Y_-(0,t-2\ell/c) - & \ \tilde{t}\tilde{r}'e^{-2q\ell}Y_+^{vac}(0,t-2\ell/c) \\
+ & \ \tilde{t}'e^{-q\ell}Y_-^{vac}(\ell,t-\ell/c),
\end{aligned} \tag{19}$$

which could also have been written down directly based on physical arguments presented below.

If $Y_-(0,t-2\ell/c)$ is the Y-quadrature of the cavity field just before it is incident on the $z = 0$ mirror from the right then after that mirror reflection a fraction if it is reflected while a fraction $\tilde{t}$ of the input field $Y_+^{vac}(0,t-2\ell/c)$ is transmitted. The two waves propagate rightward through the medium with their Y-quadratures attenuated by factor $e^{-q\ell}$. They are then reflected at the mirror at $z = \ell$ by factor $-\tilde{r}'$ while a fraction $\tilde{t}'$ of the second input $Y^{vac}(0,t-\ell/c)$ is added to the circulating wave. The net field then propagates a distance ℓ leftward through the active medium, with its Y-quadrature attenuated further by $e^{-q\ell}$ as a result, to become the net field, given by the left-hand side of Eq. (19), a time ℓ/c later.

A Fourier analysis of Eq. (19) is straightforward. We shall focus only on the central (zero-detuning) frequency component since it has the largest noise reduction. Denoting the Fourier transform of a function $f(t)$ by $\tilde{f}(\delta\omega)$, we see that for $\delta\omega = 0$, Eq. (19) yields

$$\begin{aligned}
\tilde{Y}_-(0,0)\left[1 - \tilde{r}\tilde{r}'e^{-2q\ell}\right] &= -\tilde{t}\tilde{r}'e^{-2q\ell}\tilde{Y}_+^{vac}(0,0) \\
&+ \tilde{t}'e^{-q\ell}\tilde{Y}_-^{vac}(\ell,0),
\end{aligned}$$

while Eq. (18) yields

$$\tilde{Y}_{out}(0,0) = \tilde{t}\,\tilde{Y}_-(0,0) + \tilde{r}\tilde{Y}_+^{vac}(0,0).$$

By eliminating $\tilde{Y}_-(0,0)$ between these two relations and using the energy-conservation relation $\tilde{r}^2 + \tilde{t}^2 = 1$, one may easily show that

$$\tilde{Y}_{out}(0,0) = \frac{\left(\tilde{r} - \tilde{r}'e^{-2q\ell}\right)\tilde{Y}_+^{vac}(0,0) + \tilde{t}\tilde{t}'e^{-q\ell}\tilde{Y}_-^{vac}(\ell,0)}{\left(1 - \tilde{r}\tilde{r}'e^{-2q\ell}\right)}, \tag{20}$$

whose variance is related to the spectral variance of (uncorrelated) input-field quadratures. If we assume that the input fields have the same spectral quadrature variance at a given frequency, such as is surely true for vacuum-field inputs then the spectral squeezing of the output field at zero detuning is by the factor

$$S_{out}^{(Y)}(0) \equiv \left\langle \Delta\tilde{Y}_{out}(0,0)^2 \right\rangle / \left\langle \Delta\tilde{Y}_+^{vac}(0,0)^2 \right\rangle = \frac{\left(\tilde{r} - \tilde{r}'e^{-2q\ell}\right)^2 + \tilde{t}^2\tilde{t}'^2 e^{-2q\ell}}{\left(1 - \tilde{r}\tilde{r}'e^{-2q\ell}\right)^2}. \tag{21}$$

Just as for the cavity field, the ratio $\left\langle \Delta\tilde{X}_{out}(0,0)^2 \right\rangle / \left\langle \Delta\tilde{X}_+^{vac}(0,0)^2 \right\rangle$ for the X-quadrature is given by replacing q by $-q$ everywhere in relation (21).

It is worth noting that just below threshold $\tilde{r}\tilde{r}'e^{2q\ell} \to 1$, the X-quadrature of the output field at the $z = 0$ mirror has infinite variance in its central frequency component, while the corresponding Y-quadrature spectral component has a finite variance that depends on how large the transmission T' of the other mirror is. In particular, for $T' = 0$ regardless of the value of R (or of T), the output Y-quadrature has zero spectral variance at the line center. This is a very surprising result, implying as it does that even in a very low -Q but single-ended cavity the output field is perfectly squeezed in the spectral sense, if the parametric gain is high enough to drive the oscillator to its oscillation threshold. A more complete discussion of the output field, including the bandwidth of the squeezing spectrum, may be found in Ref. [6].

5 Squeezing in the Presence of Pump Noise

In a real experiment, pump noise is inevitable. Typically, the pump field has both amplitude and phase noise that can be described well in classical terms alone. For example, the pump amplitude may have a small fluctuating piece, described in Eq. (5) via a time dependent q,

$$q(t) = q_0 + \delta q(t), \tag{22}$$

in which $\delta q(t)$ is an Ornstein-Uhlenbeck Gaussian process with zero mean and an exponentially decaying two-time correlation

$$\langle \delta q(t)\,\delta q(t') \rangle = \alpha_0 \Gamma e^{-\Gamma|t-t'|}. \tag{23}$$

The pump phase noise, on the other hand, is ultimately limited by phase diffusion which is well described by a classical Wiener-Levy Gaussian random process with zero mean value for the time derivative of the diffusing phase, $\delta\psi(t)$, and its two-time correlations:

$$\langle \delta \dot{\psi} \rangle = 0; \quad \langle \delta \dot{\psi}(t)\, \delta \dot{\psi}(t') \rangle = 2D\delta(t - t'). \tag{24}$$

The constants 2Γ and $2D$ are the amplitude and phase-noise contributions to the total pump linewidth.

Since detailed discussions of this problem have been presented elsewhere [6], we shall restrict our derivations here to its relatively simple but physically revealing aspects. To begin with, we shall take the white-noise limit, $\Gamma \to \infty$, for the amplitude noise. In more precise terms, this is the limit in which $\Gamma \ell / c \gg 1$.

Since q in Eqs. (5) and (8) is time dependent, the exponentials in Eqs. (9) and (11) have integrals in their exponents. For example, in Eq. (9b) one must replace

$$e^{-qz} \to e^{-q_0 z - \int_0^z \delta q(t - z'/c)\, dz'}$$

for a given statistical realization of δq. This means that the Y-quadrature variance is down by the factor

$$e^{-2q_0 \ell} \left\langle e^{-2 \int_0^\ell \delta q(t - z/c)\, dz} \right\rangle = e^{-2q\ell + 4\alpha_0 \ell c}$$

in every single pass either leftward or rightward between the two mirrors. We used the familiar result that for a Gaussian random variable x,

$$\langle e^x \rangle = e^{\langle x \rangle} e^{\frac{1}{2}\langle \Delta x^2 \rangle} \tag{25}$$

and the fact that when $\Gamma \ell / c \gg 1$,

$$\langle \delta q(t)\, \delta q(t') \rangle \simeq 2\alpha_0 \delta(t - t'), \tag{26}$$

to obtain the preceding factor.

A recognition of the extra factor $e^{4\alpha_0 \ell c}$ by which the Y-quadrature variance is altered when the pump amplitude has a fluctuating piece immediately tells us that Eqs. (15) and (17) must also be altered accordingly. Thus, for example, Eq. (17) takes the form

$$\left\langle \Delta Y_+(0)^2 \right\rangle = \frac{\left(T + RT' e^{-2q_0 \ell + 4\alpha_0 \ell c}\right)}{\left(1 - RR' e^{-4q_0 \ell + 4\alpha_0 \ell c}\right)} N_{vac}.$$

Since $\alpha_0 > 0$, the net effect of the δ-correlated pump amplitude fluctuations is to merely reduce the parametric attenuation of Y-quadrature fluctuations thereby leading to a smaller intracavity squeezing.

Although we have not discussed the opposite, static pump amplitude noise limit, $\Gamma \ell / c \ll 1$, it can be seen by physical arguments that for a given amplitude noise $\langle \delta q^2 \rangle^{\frac{1}{2}}$, the static case compromises intracavity squeezing more dramatically than the white-noise case, for it is roughly the zero-frequency Fourier component of the pump noise spectrum that controls the steady state characteristics of the signal field. As the noise bandwidth Γ increases, a fixed amount of amplitude noise is partitioned into more and more Fourier components, so that the zero-frequency component (like any other) goes down.

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We turn now to the computation of spectral squeezing of the output field in the presence of a δ-correlated pump amplitude noise. This task is quite involved when compared with the derivation of the preceding intracavity variance formula. One must begin with the fluctuating analog of (19) which may be shown to be

$$Y_-(0,t) = \tilde{r}\tilde{r}'e^{-\eta(t)-\eta(t-\ell/c)}Y_-(0,t-2\ell/c) - \begin{array}{l} \tilde{t}\tilde{r}'e^{-\eta(t)-\eta(t-\ell/c)}Y_+^{vac}(0,t-2\ell/c) \\ + \tilde{t}'e^{-\eta(t)}Y_-^{vac}(\ell,t-\ell/c), \end{array} \tag{27}$$

in which

$$\eta(t) \equiv \int_0^\ell [q_0 + \delta q(t-z/c)]\,dz. \tag{28}$$

A direct Fourier transform of Eq. (27) is not possible. We must compute first the two-time correlation functions $\langle Y_-(0,t)Y_-(0,t')\rangle$, $\langle Y_-(0,t)Y_+^{vac}(0,t')\rangle$, and $\langle Y_+^{vac}(0,t)Y_+^{vac}(0,t')\rangle$ that enter the output autocorrelation function $\langle Y_{out}(0,t)Y_{out}(0,t')\rangle$ via Eq. (18). A Fourier transform of the output correlation then furnishes the spectral variance. To compute the former two correlation functions, we solve Eq. (27) for $Y_-(0,t)$ iteratively in terms of $Y_\pm^{vac}$ at successively earlier times, one differing from the next by the roundtrip time $2\ell/c$:

$$Y_-(0,t) = \begin{array}{l} -\tilde{t}\tilde{r}' \sum_{n=0}^\infty (\tilde{r}\tilde{r}')^n e^{-\eta_{2n+2}(t)}Y_+^{vac}(0,t-2\ell(n+1)/c) \\ + \tilde{t}' \sum_{n=0}^\infty (\tilde{r}\tilde{r}')^n e^{-\eta_{2n+1}(t)}Y_-^{vac}(\ell,t-\ell(2n+1)/c) \end{array} \tag{29}$$

in which

$$\eta_p(t) \equiv \int_0^{p\ell} [q_0 + \delta q(t-z/c)]\,dz. \tag{30}$$

We may use the identity (25) and the white-noise approximation (26) to obtain the useful formula

$$\begin{array}{ll} \langle e^{-\eta_p(t)}e^{-\eta_{p'}(t)}\rangle &= e^{-(p+p')q_0\ell}e^{\frac{1}{2}\left[\langle\delta\eta_p^2\rangle+\langle\delta\eta_{p'}^2\rangle+2\langle\delta\eta_p\delta\eta_{p'}\rangle\right]} \\ &= e^{-(p+p')q_0\ell}e^{\alpha_0 c\ell(p+p'+2p^<)}, \end{array} \tag{31}$$

in which $p^<$ is the smaller of (p,p').

When combined with the δ-correlated nature of the vacuum fields, relation (29) enables one to secure the needed correlations from which the following output quadrature autocorrelation function is obtained [6]:

$$\begin{array}{ll} \langle Y_{out}(0,t) \quad Y_{out}(0,t')\rangle &= \frac{C}{\tilde{r}^2}\Big[\tilde{t}^2\left(\tilde{t}^2 + \tilde{r}^2\tilde{t}'^2 e^{-2(q_0-2\alpha_0 c)\ell}\right) \\ &+ \sum_{n=0}^\infty \sum_{n'=0}^\infty (\tilde{r}\tilde{r}')^{n+n'}\,\delta(t-t'-2(n-n')\ell/c)\,e^{-2(n+n')q_0\ell}e^{2\alpha_0 c\ell(n^>+3n^<)} \\ &- \tilde{t}^2 \sum_{n=0}^\infty (\tilde{r}\tilde{r}')^n\left[\delta(t-t'-2n\ell/c) + \delta(t-t'+2n\ell/c)\right]e^{-2nq_0\ell+2\alpha_0 nc\ell} + \delta(t-t')\Big] \end{array} \tag{32}$$

in which

$$\langle Y_+^{vac}(0,t)Y_+^{vac}(0,t')\rangle = \langle Y_-^{vac}(\ell,t)Y_-^{vac}(\ell,t')\rangle \equiv C\delta(t-t').$$

As before, we are only interested here in the central frequency component of the quadrature spectrum. This is obtained from Eq. (32) by integrating it over $(t - t')$ in the range $(-\infty, \infty)$, which is a trivial task due to the presence of a δ-function in every term. The resulting infinite sums are related to the geometric series and can be carried out in closed form. The net result of these straightforward steps is the following noise reduction factor at line center:

$$S_{out}^{(Y)}(0) = \frac{1}{\bar{r}^2} \left[1 + \frac{\bar{t}^2 \left(\bar{t}^2 + \bar{r}^2 \bar{t}'^2 e^{-2(q_0 - 2\alpha_0 c)\ell} \right) \left(1 + \bar{r}\bar{r}' e^{-2(q_0 - \alpha_0 c)\ell} \right)}{\left(1 - \bar{r}^2 \bar{r}'^2 e^{-4(q_0 - 2\alpha_0 c)\ell} \right) \left(1 - \bar{r}\bar{r}' e^{-2(q_0 - \alpha_0 c)\ell} \right)} - \frac{2\bar{t}^2}{1 - \bar{r}\bar{r}' e^{-2(q_0 - \alpha_0 c)\ell}} \right]. \quad (33)$$

When the pump fluctuations are absent ($\alpha_0 = 0$), this expression naturally reduces to result (21). In general, however, a graphical presentation of (33) is imperative for physical insights. This is done in Fig. 4 for a symmetric cavity ($R = R'$). It is no surprise that as the pump amplitude fluctuations increase in strength, the amount of squeezing reduces for any fixed value of R (i.e. along a vertical line on the figure). For a fixed fluctuation strength, on the other hand, the higher its value the slower the squeezing increases, with increasing R, to its maximum value at oscillation threshold.

A reduction of the amplitude-noise bandwidth, so that $\Gamma \ell / c$ is no longer large compared to 1, leads to reduced output squeezing for the same reasons as for the intracavity field. It is worth noting that amplitude noise, being essentially multiplicative in nature (see Eq. (5)), is less important than pump phase noise which unavoidably couples the squeezed quadrature to the highly fluctuating quadrature, thereby seriously undermining squeezing.

6 Pump Phase Fluctuations

Even the quietest pump, such as one generated by a highly stable laser, has intrinsic random phase diffusion arising from the purely quantum mechanical process of spontaneous emission. This means that squeezing in the sub-harmonic signal field when measured relative to a fixed (or independently fluctuating) phase will show a time-dependent behavior as both the squeezed and unsqueezed orthogonal quadratures with phases slaved to the pump mix. However, if both the local oscillator (LO) and pump are derived from the same laser, then the reference LO phase and the phase of the ideally squeezed quadrature track each other. In spite of this phase tracking, there is a residual effect on squeezing, due to the time dependence of the pump phase diffusion [9], which we consider here.

In the presence of a finite $\delta\psi(t)$, as described by a Wiener-Levy Gaussian random process with moments (24), Eq. (5) has q replaced by $qe^{i\delta\psi(t)}$, and the signal quadratures $X_\pm(z, t)$ and $Y_\pm(z, t)$ are defined relative to the phase $\delta\psi(t)/2$:

$$\begin{aligned} X_\pm(z, t) &= \tfrac{1}{2} \left[e_+(z, t)e^{-i\delta\psi(t)/2} + e_-(z, t)e^{i\delta\psi(t)/2} \right]; \\ Y_\pm(z, t) &= \tfrac{1}{2i} \left[e_+(z, t)e^{-i\delta\psi(t)/2} - e_-(z, t)e^{i\delta\psi(t)/2} \right]. \end{aligned} \quad (34)$$

These quadratures evolve according to the matrix equation

$$\left(\frac{\partial}{\partial z} \pm \frac{1}{c} \frac{\partial}{\partial t} \right) V_\pm(z, t) = \left[q\sigma_3 \pm \frac{i}{2c} \delta\dot{\psi}(t)\sigma_2 \right] V_\pm(z, t), \quad (35)$$

45
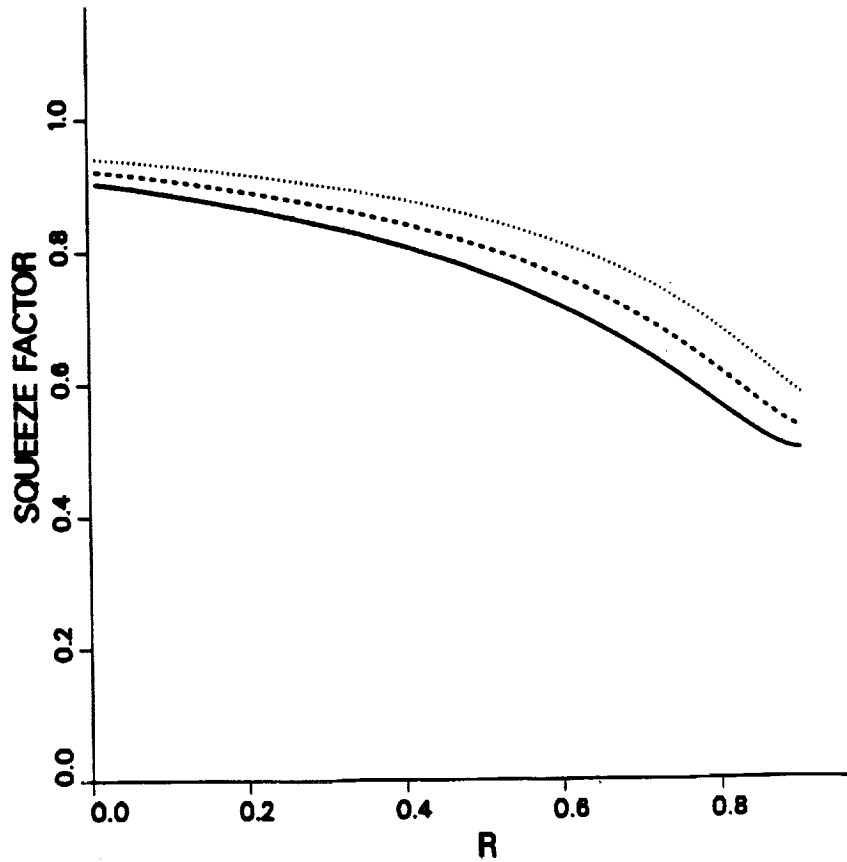
Fig. 4. Squeezing of the Central Frequency Component of the Output Field Quadrature in a Symmetric Cavity. The full, dashed, and dotted curves represent values of the fluctuation parameter $\alpha_0 c\ell$ equal to 0, 0.005, and 0.01, respectively, while the roundtrip gain coefficient $q_0\ell$ is 0.05 in each case.

in which the column vector $V_{\pm}(z,t)$ is $(X_{\pm}(z,t), Y_{\pm}(z,t))^T$ and the σ's are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{36}$$

Although Eq. (35) is a first order equation, it is a matrix equation with the coefficient matrix on the RHS at any time not committing with itself at another time. This renders the solution a formal one in terms of time-ordered or path-ordered exponentials. The path-ordering (or time-ordering) has however the advantage that successive path-ordered (or time ordered) exponentials from one roundtrip to the next may be easily multiplied. One first combines the solution of Eq. (35) with the boundary connections (9) to determine the single roundtrip evolution of $V_+(0,t)$ to obtain a matrix analog of Eg. (19). Iterative processing of such equation leads to a formal solution that can, via the simplicity of writing products of time (or path) ordered exponentials with contiguous limits as single time (path) ordered exponentials over the entire time (or path) interval, be expressed in the form

$$V_+(0,t) = \sum_{n=0}^{\infty} (\tilde{r}\tilde{r}')^n \, C(0, 2\ell, n; t) e^{2q\ell n\sigma_3} W^{vac} \left(t - 2n\ell/c \right). \tag{37}$$

In Eq. (37), W^{vac} is a column vector related to the quadratures of the two known input fields and $C(0, 2\ell n; t)$, a path-ordered matrix exponential involving an integral over $\delta\dot\psi(t)$, represents the residual effect of pump phase diffusion over signal noise.

In Ref. [6], solution (37) serves as the starting point for computing the various variances and correlations needed for determining the steady-state intracavity quadrature variances and output-field quadrature noise spectrum. Eq. (37) is sufficiently complex that a statistical averaging over the phase noise $\delta\dot\psi$, in spite of its Gaussian and δ-correlated nature, cannot be exactly performed in the involved integrals. One must settle for a series expansion of intracavity and output squeezing in powers of the phase diffusion constant D, which has been determined to $O(D^2)$ [6]. We refer the interested reader to that reference for more details. It suffices here to state that pump phase diffusion seems to be most important near threshold where the fluctuations in the X−quadrature of the cavity field have a highly slowed relaxation rate.

7 Conclusions

We have presented here an analysis of squeezing in a degenerate parametric oscillator that lends itself to an easy physical interpretation for the most part. For completeness, we have also summarized the impact of pump amplitude and phase noises of sorts encountered in a real experiment on the observed degrees of cavity and output squeezing. An exact analysis for the case of a finite pump-phase diffusion noise $\delta\dot\psi(t)$ is beset by the difficulties of computing the statistical averages of path-ordered integrals involving $\delta\dot\psi(t)$.

8 Acknowledgments

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Disturbance, the Uncertainty Principle and Quantum Optics

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Abstract

It is shown how a disturbance-type uncertainty principle can be derived from an uncertainty principle for joint measurements. To achieve this, we first clarify the meaning of "inaccuracy" and "disturbance" in quantum mechanical measurements. The case of photon number and phase is treated as an example, and applied to a quantum non-demolition measurement using the optical Kerr effect.

1 Introduction

One of the most appealing aspects of quantum optics is that within its domain of application experiments can be realized that used to be confined to the domain of *Gedanken* experiments. The proposed [1] quantum non-demolition (QND) schemes for photon number measurement are such fundamental measurements. In fig. 1 we have sketched a simple setup [1, 2]. A signal beam S is mixed with a probe beam P in a non-linear Kerr medium. The refraction index of this medium is intensity dependent. Accordingly, the probe's phase will depend on the number of photons in the signal beam. By coupling the outgoing probe beam with a reference beam, the probe phase can be detected and thus the signal photon number can be deduced. However, this is not the only consequence of the interaction between signal and probe beams. Also the S-phase will be influenced.
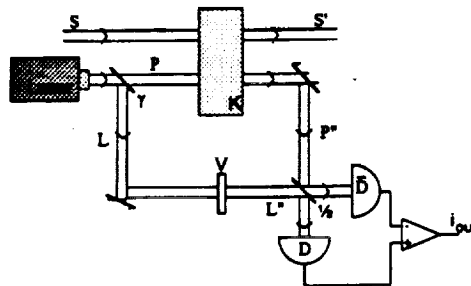
FIG. 1 Basic QND scheme, using the Kerr effect.

The experiment can be seen as an analog of Heisenberg's γ microscope experiment [3]. Ther‹ a particle's position is measured in a non-destructive way. In the γ-microscope, it is argued momentum is disturbed by an amount Dp as a result of measuring position, satisfying ($\hbar = 1$)

$$\delta q Dp \gtrsim 1, \tag{1}$$

δq representing the microscope's resolution, i.e., its inaccuracy in determining position. Anal‐ ogously, in the Kerr device, where photon number N is measured, the probe effect on signa‐ phase can be expected to take the form of a disturbance, in size reciprocally related to the $\hat{N}$‐ measurement inaccuracy. In order to avoid certain ambiguities (cf. [4]) we shall give a formal definition of this disturbance notion. In particular we show how relations like (1) can be derived in a precise way from uncertainty relations for the inaccuracy, achievable in joint measurements of incompatible observables. Such relations have become available relatively recently [1, 6, 7].

2 Inaccuracy

We represent measurements by *positive operator-valued measures* (POVM's) [5, 8], a notion gener‐ alizing von Neumann's projection-valued measures (PVM's). For a discrete set of outcomes K, a POVM $\mathcal{M} = \{\hat{M}_k, k \in K\}$ generates the probability of outcome k by $Tr\hat{\rho}\hat{M}_k$, when the object is in state $\hat{\rho}$. Hence $\mathcal{M}$ must satisfy $\sum_{k \in K} \hat{M}_k = \hat{1}, \hat{M}_k \geq \hat{0}$. A second POVM, $\mathcal{O} = \{\hat{O}_l\}$, is then said to represent a *non-ideal* measurement [7] of $\mathcal{M}$ if there is a stochastic matrix λ_{lk} ($\sum_l \lambda_{lk} = 1; \lambda_{lk} \geq 0$) such that

$$\hat{O}_l = \sum_k \lambda_{lk} \hat{M}_k. \tag{2}$$

We use the shorthand $\mathcal{M} \to \mathcal{O}$ for this relation. The $\mathcal{O}$-distribution is a smeared version of the $\mathcal{M}$-distribution. Finally we need to characterize the amount of inaccuracy by a real number. Clearly, if $\lambda_{lk} = \delta_{lk}$, the Kronecker-delta, $\mathcal{O}$ is equal to $\mathcal{M}$: then a measurement of $\mathcal{O}$ is a perfect measurement of $\mathcal{M}$. Thus we need to quantify how much λ_{lk} deviates from δ_{lk}. Consider again the QND scheme of fig.1. Given that the incoming probe beam is described by a coherent state $|\beta >$ and the signal beam by $\hat{\rho}$, it can be shown that the outcome probabilities $P(q)$ of the outgoing probe phase measurement are given by [2, 9]

$$\begin{aligned} P(q) &= \sum_{n_S} P(q|n_S) < n_S|\hat{\rho}|n_S >, \\ P(q|n_S) &= \tfrac{1}{\sqrt{2\pi}\sigma}\exp\left[-\tfrac{1}{2}\left(\tfrac{q-\mu}{\sigma}\right)^2\right], \sigma = |\beta|, \ \mu = |\beta|^2 \sin\left(\chi_\tau n_S\right), \end{aligned} \tag{3}$$

where we have taken the initial beam splitter's transmittivity $\gamma = \tfrac{1}{2}$. The constant χ_τ depends on the non-linearity coefficient of the medium. Defining the POVM $\mathcal{O}$ by the requirement $P(q) = Tr\hat{\rho}\hat{O}(q)$, (2) is satisfied if $\{\hat{M}_k\}$ represents the photon number observable. The measurement inaccuracy is characterized by the width of $P(q|n_S)$, and can be interpreted as being due to excess noise inherent in the measurement. For low photon numbers the response is approximately linear: $\mu \approx |\beta|^2 \chi_\tau n_S$. Hence this measurement can be characterized by parameters quantifying *noise* (σ) and *gain* ($\partial\mu/\partial n_S$). A suitable inaccuracy measure is the ratio of these two:

$$\delta_{N_S} = \frac{\sigma}{\partial\mu/\partial n_S} = \frac{1}{|\beta|\chi_\tau}. \tag{4}$$

In deriving (3) we ignored self-phase-modulation (SPM) [1]. It can be shown that, beyond a certain probe photon number, SPM has a strongly adverse effect on the measurement quality [2] Refining the setup, however, can compensate for SPM to a large extent [1, 9].

3 Joint Measurements and the Uncertainty Principle

For finite-dimensional Hilbert spaces a general proof has been given that joint non-ideal measurements of incompatible observables are possible, but that their quality is limited by an uncertainty relation [7]. In the present paper we will focus on the phase-number observable pair. In this infinite dimensional case no completely general result is known, but the special results obtained are nevertheless quite convincing. Consider the (*non*-Hermitean) phase observable,

$$\hat{e^{i\phi}} = \sum_n |n><n+1|,$$ (5)

given by Lévy-Leblond, $|n>$ denoting the number states[10]. Not only is it incompatible with $\hat{N}$, but the pair forms à perfect analog of the position-momentum pair, cf.

$$e^{\hat{im\phi}}e^{ia\hat{N}} = e^{ia\hat{N}}e^{\hat{im\phi}}e^{ima}, m \in Z, \alpha \in R.$$ (6)

Next consider a second (ancillary) system, being in state $\hat{\rho}'$, and having similar observables $\hat{\phi}'$ and $\hat{N}'$ defined on its Hilbert space $\mathcal{H}'$. Then the composite observables

$$e^{\hat{i\phi_t}} := e^{\hat{i\phi}}e^{-\hat{i\phi}'}, \quad \hat{N}_t := \hat{N} + \hat{N}'$$ (7)

are compatible, as evaluation of their Weyl commutation relation, using (6), shows. Hence, $\hat{N}_t$ and $\hat{\phi}_t$ can be measured jointly. Then the POVM $\{\hat{M}(\phi, n) = Tr_{\mathcal{H}'}(\hat{\rho}'|\phi, n><\phi, n|)\}$, $|\phi, n>$ being the common eigen-states of $\hat{\phi}_t$ and $\hat{N}_t$, describes a joint non-ideal measurement of $\hat{\phi}$ and $\hat{N}$. Indeed, for the relation between the probability distributions of $\hat{N}_t, \hat{N}$, and $\hat{N}'$ we find

$$P_{\hat{N}_t}(n_t) = \sum_{n=0}^{n_t} P_{\hat{N}'}(n_t - n)P_{\hat{N}}(n), \quad P_{\hat{N}'}(n') = <n'|\hat{\rho}'|n'>.$$ (8)

Comparing this with (2) we see that the $\hat{N}_t$ measurement is a non-ideal measurement of $\hat{N}$, i.e., $\hat{N} \rightarrow \hat{N}_t$, the stochastic matrix λ_{lk} being given by $P_{\hat{N}'}(n_t - n)$. Therefore the inaccuracy of the non-ideal $\hat{N}$-measurement is determined by the spread in the number n' present in the state $\hat{\rho}'$ of the ancillary system. Similarly the $\hat{\phi}_t$-measurement can be seen to be a non-ideal measurement of $\hat{\phi} : \hat{\phi} \rightarrow \hat{\phi}_t$, the inaccuracy being determined in an analogous way by the phase spread of the ancillary system. As a measure δ_ϕ of the inaccuracy of the $\hat{\phi}$-measurement we may take [8] $\delta_\phi^2 = -1 + |<e^{\hat{i\phi}'}>|^{-2}$. In this way we have a formal scheme of generating joint non-ideal measurements of incompatible observables. Indeed for position-momentum this scheme has long been known (e.g. [11]). From an uncertainty relation derived for observables $\hat{N}'$ and $\hat{\phi}'$ in state $\hat{\rho}'$ [8, 10], the following inequality now straightforwardly follows for the inaccuracies of the jointly performed $\hat{N}$-and $\hat{\phi}$-measurements:

$$\delta_N \delta_\phi \geq \frac{1}{2}.$$ (9)

This relation is of the same kind as (1). These were termed inaccuracy relations in [7]. In this special case this relation is a consequence of the restrictions in preparing the ancillary object state.

4 From Inaccuracy to Disturbance

Neither the γ-microscope nor the QND measurement referred to in sect. 1 are joint measurements: in the first momentum is not actually measured; neither is phase in the second. Yet, with however good a measuring instrument we try to measure the signal's initial phase, we can never quite remove the inaccuracy from this measurement. It appears that there is a limiting inaccuracy present already in the outgoing signal beam in the form of a phase *disturbance* caused by the presence of a measurement arrangement for measuring photon number. In order to be able to obtain a quantitative expression for this phase disturbance we first consider the general description of measurements once again. In sect. 2 we saw that the outcome probabilities of measurement results (i.e., the determinative aspect of measurement) in general are described by POVM's . Now we also need to take into account the object state after the measurement i.e., the preparative aspect of the measurement. In the von Neumann framework a *measurement transformation of the first kind* leaves the object in an eigenstate of the measured observable. In realistic cases this should be generalized to *operation valued measures* (OVM's) [5]. If a measurement yields outcome k, the output state will be $\hat{\mu}_k(\hat{\rho})$, given that the object started out in state $\hat{\rho}$. The probability of k is then given by $Tr[\hat{\mu}_k(\hat{\rho})]$. Accordingly, the mapping $\hat{\rho} \rightarrow \hat{\mu}_k[\hat{\rho}]$ should satisfy

$$\sum_{k \in K} Tr[\hat{\mu}_k(\hat{\rho})] = Tr[\hat{\rho}], \ \hat{\rho} > \hat{0} \rightarrow \hat{\mu}_k(\hat{\rho}) \geq 0. \tag{10}$$

The POVM $\mathcal{M} = \{\hat{M}_k\}$ corresponding to the OVM $\{\hat{\mu}_k\}$ is therefore given by

$$\forall_{\hat{\rho}} Tr[\hat{\mu}_k(\hat{\rho})] = Tr[\hat{\rho}\hat{M}_k] \Leftrightarrow \hat{M}_k = \hat{\mu}_k^\dagger[\hat{1}]. \tag{11}$$

For every OVM there is only one POVM, whereas many measurement transformations may realize a given POVM. Now consider the outgoing object. Suppose we measure some POVM $\mathcal{O} = \{\hat{O}_l\}$ on it. Then the probabilities are given by

$$P_O(l) = Tr[\hat{\mu}_K(\hat{\rho})\hat{O}_l] = Tr[\hat{\rho}\hat{\mu}_K^\dagger(\hat{O}_l)], \ \hat{\mu}_K = \sum_{k \in K} \hat{\mu}_k. \tag{12}$$

Hence a measurement of $\mathcal{O}$ in the final state can be seen as a measurement of $\bar{\mathcal{O}} = \{\hat{\bar{O}}_l\} = \{\hat{\mu}_K^\dagger(\hat{O}_l)\}$ in the initial state. Moreover, every repetition of the experiment yields values for *both* l and k. Therefore we have a joint measurement, characterized by the bivariate POVM $\{\hat{\mu}_k^\dagger(\hat{O}_l)\}$, of which $\bar{\mathcal{O}}$ is one marginal and $\mathcal{M}$ is the other one. Summarizing, we see that consecutive measurements of $\mathcal{M}$ and $\mathcal{O}$ may be seen as joint measurements of $\mathcal{M}$ and $\bar{\mathcal{O}}$.

Let us apply this to the QND scheme. Suppose we want to look at the outgoing signal beam S' in order to find out the *initial* signal phase. Then we must not measure the phase of the outgoing state $\hat{\rho}_{S'}$, i.e., *not* $\hat{\phi}_{S'} \rightarrow \mathcal{O}$ in $\hat{\rho}_{S'}$, but we must have $\hat{\phi}_S \rightarrow \bar{\mathcal{O}}$ in $\hat{\rho}_S$. We should build the $\mathcal{O}$-device such that $\bar{\mathcal{O}}$ is related to $\hat{\phi}_S$ by (2), rather than that $\mathcal{O}$ itself is thus connected to $\hat{\phi}_{S'}$. In this way possible distortions in the medium are compensated for. Since SPM has the effect that $\hat{\phi}_S$ and $\hat{\phi}_{S'}$ are incompatible [2], this difference is not quite trivial here.

If $\hat{\phi}_S \rightarrow \bar{\mathcal{O}}$, however, we have a joint measurement of $\hat{N}_S$ and $\hat{\phi}_S$. The former, the QND POVM, measures $\hat{N}_S$, the latter we must choose so as to measure $\hat{\phi}_S$. Accordingly, (9) is applicable. The phase inaccuracy thus achievable is limited by $(2\delta_{N_S})^{-1}$.

Note that we have made no assumption about the nature of POVM $\mathcal{O}$ whatsoever. The above reasoning holds quite generally. Define therefore

$$\epsilon_{\phi_S} := \inf_{\mathcal{O}}(\delta_{\phi_S}),$$
(13)

where the infimum is taken in the set of all POVM's $\tilde{\mathcal{O}}$ satisfying $\hat{\phi}_S \to \tilde{\mathcal{O}}$. Assuming $\hat{\phi}_S$ to be optimal, for all such POVM's $\tilde{\mathcal{O}}$ the bound (9) must hold, so that [12]

$$\epsilon_{\phi_S}\delta_{N_S} \geq \frac{1}{2}.$$
(14)

The quantity (13) does not depend on $\tilde{\mathcal{O}}$ (which is a variable in a set of POVM's), but on the meter's transformation $\hat{\mu}_K$, which is implicitly contained in the condition $\hat{\phi}_S \to \tilde{\mathcal{O}}$. Thus ϵ_{ϕ_S} is a property of the $\hat{N}_S$-meter, known once the OVM $\{\hat{\mu}_k\}$ has been calculated from the device's blueprint. ϵ_{ϕ_S} characterizes how much initial phase information can be retrieved from the outgoing signal. In that sense the term *disturbance* is apt [12]. If all phase information is lost (e.g., if $\{\hat{\mu}_k\}$ is a measurement of the first kind), the disturbance ϵ_{ϕ_S} is maximal. If, on the other hand, the meter measures nothing (e.g., if $\hat{\mu}_K(\hat{\rho}) = \hat{\rho}$ for all $\hat{\rho}$), there is no disturbance at all, and $\epsilon_{\phi_S} = 0$.

5   Phase Disturbance in the QND-Scheme

Finally, we study the phase disturbance in the Kerr-setup of fig.1. Define generalized phase states

$$|\phi; \nu > := \sum_n (2\pi)^{-1/2} e^{\left(i\phi n + \frac{1}{2}i\nu n(n+1)\right)}|n > .$$
(15)

For $\nu = 0$ these reduce to the eigenstates of (5). Then it can be shown that we need to measure the POVM $\{|\phi; -\frac{1}{2}\chi_\tau > < \phi; -\frac{1}{2}\chi_\tau|\}$ on S' in order to get information on $\hat{\phi}_S$. In fact, [2, 13]

$$\phi_K(|\phi; -\tfrac{1}{2}\chi_\tau >_{S'} < \phi; -\tfrac{1}{2}\chi_\tau|) = \int_{-\pi}^{\pi} \mu(\phi - \phi')|\phi' >_S < \phi'|d\phi',$$
$$\mu(\phi) \approx \frac{1}{2\pi}\Theta_3[\tfrac{1}{2}\phi; e^{-\frac{1}{2}|\beta|^2\chi_\tau^2}],$$
(16)

the latter approximation being valid for low photon numbers. Here Θ_3 denotes the third of Jacobi's Θ-functions. The smearing function μ is plotted in fig.2. Note that the convolution form of (16)
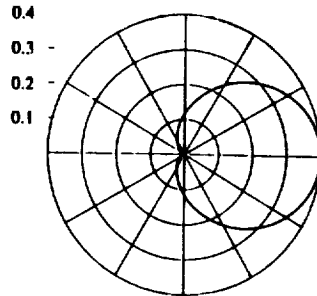
FIG. 2 Polar plot of the phase smearing function $\mu(\phi)$ (linear regime, $|\beta|^2\chi_\tau^2 = 8$).

is in agreement with (2). We have discarded an uninteresting phase bias term in (16).

Calculating δ_{ϕ_S} from (16), we get $\delta_{\phi_S}^2 \simeq -1 + \exp(\frac{1}{4}|\beta|^2\chi_7^2)$, implying (cf. (4)):

$$\log\left(1 + \delta_{\phi_S}^2\right)\delta_{N_S}^2 \simeq \frac{1}{4}. \tag{17}$$

This is only slightly worse than the bound set by the uncertainty principle (14), indicating that the measurement procedure described by (16) is optimal in the sense that $\delta_{\phi_S} \simeq \epsilon_{\phi_S}$.

As said before, the disturbance concept evades distortions in the medium, and therefore phase disturbance is unaffected by SPM, contrary to photon number inaccuracy (but see [9]).

6   Acknowledgments

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ONE DIMENSIONAL REPRESENTATIONS IN QUANTUM OPTICS

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Abstract

The possibility to represent the quantum states of a harmonic oscillator not on the whole α-plane but on its one dimensional manifolds is considered. It is shown that a simple Gaussian distribution along a straight line describes a quadrature squeezed state while a similar Gaussian distribution along a circle leads to the amplitude squeezed state. The connection between the one dimensional representations and the usual Glauber representation is discussed.

1 Introduction

There are several widely used representations to describe a state of a quantum oscillator in the Hilbert space. The most natural one is the expansion of the state into the number state

$$| c >= \sum_{n=0}^{\infty} c_n | n > . \tag{1}$$

Another well known possibility is the coherent state representation [1,2]

$$| f >= \frac{1}{\pi} \int f(\alpha^*) \exp\left(- | \alpha |^2 /2\right) | \alpha > d^2\alpha, \quad d^2\alpha = d(Re\alpha)d(Im\alpha), \tag{2}$$

$f(\alpha^*)$ being an analytical function of α^*. Here the state is represented by a superposition of nonorthogonal coherent states all over the complex α-plane.

As already Glauber pointed out, there is an infinite number of ways of expanding any state in terms of coherent states due to the overcompleteness of the latter states

$$| f >= \frac{1}{\pi} \int G(\alpha,\alpha^*) | \alpha > d^2\alpha, \tag{3}$$

here the expansion function $G(\alpha,\alpha^*)$ may be a rather general function of α and α^*. Being confined to some given class of functions the uncertainty in finding $G(\alpha,\alpha^*)$ can be reduced. In

this paper we shall deal with such representations that correspond to kern functions $G(\alpha, \alpha^*)$ leading to integration over a one dimensional manifold of the α-plane in Eq. (3). The possibility to represent any state on a subspace of the complex plane comes from Cahill's theorem on overcompleteness [3,4]. We shall show that such nonclassical states as the quadrature and amplitude squeezed states can be represented very naturally by superposition of coherent states along a straight line or along a circle in the α-plane correspondingly.

2 Representation along a straight line

The most simple states emerging from superposition of coherent states are the even $| x, + >$ and the odd $| x, - >$ states

$$| x, + >= c_+ (| x > + | -x >), \tag{4}$$

$$| x, - >= c_- (| x > - | -x >), \tag{5}$$

where $| x >$ is a usual coherent state with real eigenvalue of the annihilation operator $a | x >= x | x >$. It is remarkable that the even state $| x, + >$ being a superposition of two classical states is squeezed [5]

$$(\Delta a_2)^2 = \frac{1}{4} - x^2 / [1 + \exp(2x^2)]. \tag{6}$$

where a_1 and a_2 are the Hermitian quadratures of the annihilation operator.

The squeezing can be further enhanced by adding the vacuum state to $| x, + >$

$$| x, p >= c_p (| x > + p | 0 > + | -x >). \tag{7}$$

This way one can achieve a variance $(\Delta a_2)^2 = 0.0651$ instead of 0.111 for $| x, + >$ or 0.25 for the vacuum state. Superposing more and more even states to it one can get even more squeezing

$$| f >= \int_{-\infty}^{\infty} f(x) | x > dx. \tag{8}$$

In fact for any positive even function $f(x)$, but for the $f(x) = \delta(x)$ describing the vacuum, the state defined by Eq. (8) is squeezed. A most important particular case is the Gaussian superposition function [5,6]

$$f(x) = c \, \exp(-x^2/\gamma^2), \quad c = \sqrt{\sqrt{(1 + \gamma^2)}/\gamma^2 \pi}, \tag{9}$$

describing the usual squeezed vacuum state with uncertainties of the quadratures

$$(\Delta a_1)^2 = (1 + \gamma^2)/4, \quad (\Delta a_2)^2 = 1/4(1 + \gamma^2). \tag{10}$$

Similar distributions can be constructed not only along the real axis but along any straight line. For example the squeezed coherent state with coherent signal α and squeezing parameter $\varsigma = r\exp(i\theta)$ can be written in the form [7,8]

$$| \alpha,\varsigma >= \int_{-\infty}^{\infty} f(x,\alpha,\varsigma) \mid \alpha + \exp(i\theta/2)x > dx, \tag{11}$$

$$f(x,\alpha,\varsigma) = c \exp(-x^2/\gamma^2 - i\delta x), \quad \delta = Im[\alpha^* \exp(i\theta/2)], \quad \gamma = \sqrt{e^{2r}-1}. \tag{12}$$

As the Gaussian superposition of coherent states of Eq. (8,9) was a useful generalisation of the even states of Eqs. (4,7) analogously one can build an odd state $\mid \gamma, 1 >$ resembling Eq. (5)

$$| \gamma,1 >= \int_{-\infty}^{\infty} G(x,\gamma,1) \mid x > dx, \tag{13}$$

$$G(x,\gamma,1) = c_1 x \exp(-x^2/\gamma^2), \quad c_1 = \frac{2}{\gamma^3}\sqrt{\sqrt{(1+\gamma^2)^3}/\pi}. \tag{14}$$

The mean photon number and the uncertainty of the quadrature a_2 in this state are

$$< \gamma,1 \mid a^\dagger a \mid \gamma,1 >= 1 + \frac{3\gamma^4}{4(1+\gamma^2)}, \tag{15}$$

and

$$(\Delta a_2)^2 = \frac{3}{4(1+\gamma^2)}. \tag{16}$$

We can see that the state $\mid \gamma, 1 >$ coincides with the one photon state $\mid 1 >$ in the limit $\gamma = 0$ and with increasing γ at $< \gamma,1 \mid a^\dagger a \mid \gamma,1 >= 2$, $\gamma = \sqrt{2}$ it becomes squeezed.

Similarly one can define states $\mid \gamma, n >$ with x^n instead of x in their weight function $G(x,\gamma,n)$. Superpositions of such states leading to Hermite polynomial weight functions are rather remarkable

$$| h_n >= \int_{-\infty}^{\infty} h_n(x) \mid x > dx, \tag{17}$$

$$h_n(x) = \sqrt{\sqrt{3}/(2^n n!)} H_n(\frac{\sqrt{3}x}{2}) \exp(-x^2/2). \tag{18}$$

The states $\mid h_n >$ are orthonormalized $(< h_n \mid h_m >= \delta_{nm})$, satisfy the relation

$$a \mid h_n >= \sqrt{\frac{n+1}{3}} \mid h_n + 1 > + \sqrt{\frac{4n}{3}} \mid h_n - 1 >, \tag{19}$$

and correspondingly

$$< h_n \mid a \mid h_n >= 0, \quad < h_n \mid a^2 \mid h_n >= \frac{2}{3}(2n+1), \quad < h_n \mid a^\dagger a \mid h_n >= \frac{5n+1}{3}. \qquad (20)$$

The projection operator constructed from the Hermite states

$$P_h \equiv \sum_{n=0}^{\infty} \mid h_n >< h_n \mid, \qquad (21)$$

is a unity operator both for the coherent and photon number states

$$< x \mid P_h \mid y >=< x \mid y >= \exp[-(x-y)^2/2], \qquad (22)$$

$$< n \mid P_h \mid m >= \delta_{nm}, \qquad (23)$$

which shows that any state can be represented by them. For example one can expand a $\mid f >= \int_{-\infty}^{\infty} f(x) \mid x > dx$ state into the $\mid h_n >$ states

$$\mid f >= \sum_{n=0}^{\infty} f_n \mid h_n >, \qquad (24)$$

where

$$f_n = \int \int_{-\infty}^{\infty} f(x) h_n(y) \exp[-(x-y)^2/2] dx dy. \qquad (25)$$

3 Representation along a circle

Let us now consider a state emerging from superposition of coherent states with the same amplitude $\mid \alpha \mid = R$ i. e. we choose only those coherent states which lie on the same circle in the α-plane [5].

$$\mid F,R >= \frac{\exp(R^2/2)}{2\pi} \oint F(\phi) \mid R\exp(i\phi) > d\phi. \qquad (26)$$

If the radius of the circle is chosen big enough so that Eq. (2) can be replaced by

$$\mid f >= \frac{1}{\pi} \int_{\mid \alpha \mid < R} f(\alpha^*) \exp(- \mid \alpha \mid^2 /2) \mid \alpha > d^2\alpha, \qquad (27)$$

then we can find connections between the distribution function $F(\phi)$ and Glauber's weight function $f(\alpha^*)$

$$F(\phi) = \frac{R\exp(i\phi)}{\pi} \int_{\mid \alpha \mid < R} f(\alpha^*) \frac{\exp(- \mid \alpha \mid^2)}{z - \alpha} d^2\alpha, \quad z = R\exp(i\phi) \qquad (28)$$

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and

$$f(\alpha^*) = \oint F(\phi)exp(\alpha^* z)d\phi. \tag{29}$$

We note that if one knows the time behaviour of the annihilation operator $a(t)$ the analytic expansion function $f(\alpha^*,t)$ can be found from the expression [9]

$$f(\alpha^*,t) = \int d^2\eta \chi(\eta,t)\exp(-\mid\eta\mid^2 -\eta\alpha^*)/\sqrt{\pi\int d^2\eta \chi(\eta,t)\exp(-\mid\eta\mid^2)}, \tag{30}$$

where $\chi(\eta,t)$ is the normally ordered characteristic function (ρ being the density operator)

$$\chi(\eta,t) = Tr[\rho\exp(\eta a^\dagger(t))\exp(-\eta^* a(t))]. \tag{31}$$

Using Eq. (28) we find for the n-photon state and the coherent state correspondingly

$$F(\phi,n) = \sqrt{n!}R^{-n}\exp(-in\phi), \tag{32}$$

$$F(\phi,\alpha) = \frac{z\exp(-\mid\alpha\mid^2/2)}{z-\alpha}, \quad \mid\alpha\mid < R. \tag{33}$$

According to Eq. (32) we can obtain the coefficients of the n-photon representation c_n of Eq. (1) if we know the distribution function $F(\phi)$

$$c_n = R^n F_n/\sqrt{n!}, \tag{34}$$

where F_n are Fourier coefficients of $F(\phi)$

$$F(\phi) = \sum_{n=0}^{\infty} \exp(-in\phi)F_n. \tag{35}$$

An interesting state is the state with Gaussian distribution function $\mid u >$

$$F(\phi,u) = c_u\exp(-i\delta\phi - \frac{u^2}{2}\phi^2). \tag{36}$$

In case of extremely large u it describes the usual coherent state while in the opposite limit it is the n-photon state ($n = \delta$). Between these states it will be an amplitude squeezed banana state. Graphically it can be understood if one visualizes how with decreasing u the muffin-like coherent state going through a squeezed crescent-like state deforms along the circle into the donut-like number state.

It is also worth mentioning that the Gaussian superposition of coherent states along an arc are not only describe amplitude squeezing [10,11] but they are also approximate number-phase intelligent states [10] associated with the Pegg-Barnett phase operator [12].

Remarkable feature of this state is the complete analogy with the usual quadrature squeezed state discussed in the previous Section, as the Gaussian arc distribution is amplitude squeezed while the Gaussian straight line distribution is quadrature squeezed. Moreover, as the even

superposition of two coherent states from Eq. (4) can be derived by truncation from the straight line Gaussian state of Eqs. (5,9) so Schleich's superposition state [13]

$$| \psi >= c_{\alpha\phi}(| \alpha e^{i\phi/2} > + | \alpha e^{-i\phi/2} >), \qquad (37)$$

similarly can be considered as a truncated arc Gaussian state of Eqs. (26,36).

A physical example, the so called phonon squeezing [14], where an arc distributed state occurs is the Franck-Condon transition induced by short coherent light pulse in a molecule [5,14,15]. It is worth mentioning that using Eq. (28) one can to some extent purposefully shape the molecular vibrational state by special choice of the characteristics of the exiting light pulse. For example we showed that by appropriate linear chirp the vibrational state can be turned in the α -plane while using nonlinear chirp the amplitude squeezed vibrational state can be deformed into a typical quadrature squeezed form.

4 Acknowledgments

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HARMONIC OSCILLATOR INTERACTION
WITH SQUEEZED RADIATION

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Although the problem of the electromagnetic radiation by a quantum harmonic oscillator is considered in textbooks on quantum mechanics (see, e.g., [1]) some its aspects seem to be not clarified until now. By this we mean that usually the initial quantum states of both the oscillator and the field are assumed to be characterized by a definite energy level of the oscillator and definite occupation numbers of the field modes. In connection with growing interest in squeezed states it would be interesting to analize the general case when, the initial states of both subsystems are arbitrary superpositions of energy eigenstates. This problem was considered partly in Refs. 2-4, where the power of the spontaneous emission was calculated in the case of an arbitrary oscillator's initial state (but the field was supposed to be initially in a vacuum state). In the present article we calculate the rate of the oscillator average energy and squeezing and correlation parameter change under the influence of an arbitrary external radiation field. Some other problems relating to the interaction between quantum particles (atoms) or oscillators with the electromagnetic radiation being in arbitrary (in particular, squeezed) state were investigated, e.g., in Refs 5-7.

Let us describe a charged harmonic oscillator by a Hamiltonian

$$H_o = \hbar\omega \, a^+a \qquad (1)$$

and the field by a hamiltonian

$$H_R = \hbar \sum_j \omega_j b_j^+ b_j. \tag{2}$$

here ω is the frequency of the oscillator, ω_j — ones of field modes, a, b — corresponding destruction operators.

In a rather general case interaction can be described in a form

$$H_1 = \hbar \sum_j \left[\mu_j a^+ b_j^+ + \lambda_j a^+ b_j + H.c. \right] \tag{3}$$

(H.c. means hermitian conjugated part, μ_j and λ_j are constants).

In Schrodinger picture an arbitraty initial state vector $|\psi(0)\rangle$ evolves into a state vector $|\psi_S(t)\rangle$ as predicted by Schrodinger equation with Hamiltonian $H = H_O + H_R + H_1$.

In interaction picture any Schrodinger operator Q changes according to evolution operator U_O corresponding to $H = H_O + H_R$

$$Q(t) = U_O^+(t) Q U_O(t). \tag{4}$$

For example

$$a(t) = a \exp(-i\omega t), \qquad b_j(t) = b_j \exp(-i\omega_j t). \tag{5}$$

The interaction Hamiltonian in this picture

$$H_I = \hbar \sum_j \left[\mu_j a^+ b_j^+ \exp(i\omega_j t + i\omega t) + \lambda_j a^+ b_j \exp(-i\omega_j t + i\omega t) + H.c. \right] \tag{6}$$

generates evolution operator U(t) so that a state vector in this picture defined as

$$|\psi(t)\rangle = U_O^+(t) |\psi_S(t)\rangle \tag{7}$$

will variate according to

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle. \tag{8}$$

Expectation value in this picture

$$\langle Q \rangle_I = \langle \psi(t) | Q | \psi(t) \rangle \tag{9}$$

variates slowly, only due to interaction. On the other hand, it is related to the conventional expectation value as follows

$$\langle Q \rangle_I = \langle \psi_S | U_O Q U_O^+ | \psi_S \rangle. \tag{10}$$

After introducing designations we can pose several questions to answer:

1. Can absorption and emission be distinguished in a general case ?
2. Then how to calculate the rates of these processes ?

3. Is time ordering important in perturbation calculation for this case ?

4. Does stimulated emission manifest itself ?

5. How does squeezing parameters of the oscillator and the field vary ?

To calculate the rates of the processes we need to consider infinitely long time intervals $\tau \to \infty$ in comparison with oscillation period. But they must be much shorter than damping time. Then the evolution operator has meaning of scattering matrix S transforming initial state $|\psi(0)> \equiv |i>$ to resulting one $|r>$. From Heisenberg equation one gets

$$S = \exp_T(-iT), \tag{11}$$

where all products are believed time-ordered (designated with subscript T), and T - matrix is given by

$$T = \int_{-\infty}^{\infty} H_I(t)/\hbar \; dt. \tag{12}$$

For our particular case

$$T = 2\pi\hbar \sum_j \left[\lambda_j a^+ b_j \delta(\omega_j - \omega) + H.c. \right], \tag{13}$$

here the terms with μ vanish because of a factor $\delta(\omega_j + \omega)$. Further $\delta_j^- \equiv \delta(\omega_j - \omega)$. Delta function originates as a limit of an integral

$$Int = \int_{-\tau/2}^{\tau/2} \exp(i\Omega t) \; dt \tag{14}$$

(here the initial instant in time re-designated as $-\tau/2$). Limits of this integral are

$$Int \to \tau, \qquad \text{if } \Omega \to 0, \tag{14'}$$

$$Int \to 2\pi \; \delta(\Omega), \; \text{if } \tau \to \infty. \tag{14''}$$

Conventional techique in quantum electrodynamics is as follows [8]. T - matrix is splitted into two parts - absorption part

$$T^- = 2\pi \sum_j \lambda_j a^+ b_j \delta(\omega_j - \omega) \tag{15}$$

and hermitian conjugated emission part T^+. Then probability for

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time τ of absorption (and similarly emission) is declared as

$$P = \langle i | T^{+}T^{-} | i \rangle \equiv \sum_{f} | \langle f | T^{-} | i \rangle |^{2}, \tag{16}$$

where summation is performed over a complete set of possible final states. If rewritten in a form

$$T^{-} = 2\pi M^{-} \delta(E_{f} - E_{i}), \tag{17}$$

where E_{f} and E_{i} are energies of final and initial states, it shows employing (14) that (16) expresses the well-known Fermi's rule

$$P = \frac{2\pi \tau}{\hbar} \sum_{f} | \langle f | M^{-} | i \rangle |^{2} \delta(E_{f} - E_{i}). \tag{18}$$

But is it always valid and why probability is defined in this manner ?

The expansion of S - matrix (11) is as follows

$$S = 1 - i(T^{+} + T^{-}) - (T^{2})_{T}/2 + \cdots \tag{19}$$

The identity of normalization must be valid in all orders of perturbation, i.e. for all powers of T as it is proportional to the first power of coupling constant :

$$1 = \langle r | r \rangle = \langle i | i \rangle + \langle i | T^{+}T^{+} | i \rangle + \langle i | T^{-}T^{+} | i \rangle$$
$$+ \langle i | T^{+}T^{-} | i \rangle + \langle i | T^{-}T^{-} | i \rangle - \langle i | (T^{2})_{T} | i \rangle + \cdots \tag{20}$$

Then terms from second to fifth can be interpreted as a probability of transitions in the second order, since the first and the sixth will be probability to stay in the initial state. So conventional procedure ignores the second and the fifth terms. It is possible only if $T^{-}|i\rangle$ is orthogonal to $T^{+}|i\rangle$. It can happen when either field or the oscillator is in energy eigenstate. Then actually only two levels are involved in any sort of transitions. In this case emission and absorption can be distinguished. That is on obtaining after measurement one of $|f\rangle$ states we can tell a result of absorption from a result of emission.

For arbitrary initial state they cannot be distinguished experimentally. But the total probability of emission and absorption together in (20) does not have physical meaning. Therefore we have to revise our approach. More well-grounded

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procedure is to calculate not probabilities but observable variations :

$$\Delta \langle Q \rangle_I = \langle i | S^\dagger Q S | i \rangle - \langle i | Q | i \rangle. \tag{21}$$

Besides, we do not need to introduce the Fock basis $|f\rangle$, but deal only with the initial state.

Since the observable variation is expected to grow with time, to calculate the rates of the processes we need to consider only terms proportional to long time τ. We will see later that expressions like (21) contain terms with factor $\delta(\omega_j - \omega)$ and terms with $\delta^2(\omega_j - \omega)$ under a sign of summation. One power of delta function disappear because of summation over the continuum of modes. The rest one power will transform to factor τ. So terms with delta function of infinitely little difference to the first and zeroth powers will give non-growing with time observable variation. Consequently, these terms represent dressing bare states by virtual quanta. Terms with the second powers of delta function will give time-proportional variations of observables. Just these terms correspond to transitions with creation of real quanta.

For our case we need S - matrix up to the second order of perturbation. In this order a time-ordered product

$$\langle T^2 \rangle_T = \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \left[H_I(t_1) H_I(t_2) \right]_T / \hbar^2, \tag{22}$$

where

$$\left[H_I(t_1) H_I(t_2) \right]_T = \begin{cases} H_I(t_2) H_I(t_1), & \text{if } t_2 > t_1, \\ H_I(t_1) H_I(t_2), & \text{if } t_1 > t_2, \end{cases} \tag{23}$$

is different from non-ordered product

$$\langle T^2 \rangle_T = T \times T + T^2_{dif} \tag{24}$$

by a term

$$T^2_{dif} = \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 \left[H_I(t_2) H_I(t_1) \right] / \hbar^2, \tag{25}$$

The latter expression depends on time like

$$\int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 \exp(i \Lambda t_2 + i \Omega t_1) = 4\pi \, \delta(\Lambda + \Omega) \lim_{z \to \infty} \frac{\exp[i(\Lambda - \Omega)Z] - 1}{i(\Lambda - \Omega)}. \tag{26}$$

Such terms do not vanish only if $\Lambda = -\Omega$. If $\Lambda = \omega + \omega_j$ then the last factor in (26) is not singular. Terms with $\Lambda = \omega_j - \omega$ or opposite give a contribution to T^2_{dif}

$$\sum |\lambda_j|^2 [ab_j^+, a^+b_j] \frac{\exp(2i\Lambda Z)-1+\exp(-2i\Lambda Z)-1}{2i\Lambda} \, , \qquad (27)$$

which is not singular either. So T^2_{dif} contains first powers of delta functions and can be neglected compared to T (the former is coupling constant λ_j times less).

We arrive to an assumption

$$S = 1 - iT - T \times T/2, \qquad (28)$$

that leads to ,

$$\Delta\langle Q \rangle_I = \langle i | \, i[T,Q] - \frac{1}{2}[T,[T,Q]] \, | i \rangle \qquad (29)$$

the first term being virtual and the second — real. Straightforward calculation using (29) gives for example

$$\Delta\langle a \rangle_I = - \sum_j i\lambda_j 2\pi\delta_j \langle b_j \rangle - \frac{1}{2} \sum_j |\lambda_j|^2 (2\pi\delta_j)^2 \langle a \rangle, \qquad (30a)$$

$$\Delta\langle b_k \rangle_I = -i\lambda_k^* 2\pi\delta_k \langle a \rangle - \frac{1}{2} \lambda_k^* 2\pi\delta_k \sum_j \lambda_j 2\pi\delta_j \langle b_j \rangle, \qquad (30b)$$

$$\Delta\langle a^+a \rangle_I = i \sum_j (\lambda_j^* \langle b_j^+a \rangle - \lambda_j \langle a^+b_j \rangle) 2\pi\delta_j - \sum_k |\lambda_k|^2 (2\pi\delta_k)^2 \langle a^+a \rangle$$

$$+ \frac{1}{2} \sum_{k,j} (\lambda_j \lambda_k^* \langle b_k^+ b_j \rangle + \lambda_j^* \lambda_k \langle b_j^+ b_k \rangle) \, (2\pi)^2 \delta_j \delta_k \, . \qquad (30c)$$

$$\Delta\langle b_k^+b_k \rangle_I = i(\lambda_k \langle a^+b_k \rangle - \lambda_k^* \langle b_k^+a \rangle) 2\pi\delta_k + |\lambda_k|^2 \langle a^+a \rangle (2\pi\delta_k)^2$$

$$- \pi\delta_k \left[\lambda_k \sum_j \lambda_j^* 2\pi\delta_j \langle b_j^+b_k \rangle + \lambda_k^* \sum_j \lambda_j 2\pi\delta_j \langle b_k^+b_j \rangle \right]. \qquad (30d)$$

These variations are expressed in terms of expectation values in the initial state (designated with triangle brackets). One can define quadrature component variances by

$$D(P,Q) = \frac{1}{2} \left[\langle PQ \rangle + \langle QP \rangle \right] - \langle P \rangle \langle Q \rangle. \qquad (31)$$

Their variations can be expressed similar to

$$\Delta D(a,a)_I = \Delta\langle aa \rangle_I - 2\langle a \rangle \Delta\langle a \rangle_I - (\Delta\langle a \rangle_I)^2. \qquad (32)$$

This kind of variance is important because in canonical

coordinate-momentum space $ImD(a,a)$ corresponds to correlation and $ReD(a,a)$ - to squeezing. In Schrodinger picture they rapidly transfer from each to other.

Retaining in (30) and (32) only terms proportional to τ and dividing by τ we obtain time derivative equations. From them we clearly see that radiation damping

$$\gamma = \sum_j |\lambda_j|^2 2\pi\delta_j \qquad (33)$$

determines variation of amplitudes

$$\frac{d}{dt} \langle a \rangle_I = -\frac{\gamma}{2} \langle a \rangle, \qquad (34a)$$

$$\frac{d}{dt} \langle b_k \rangle_I = -\frac{1}{2} |\lambda_k|^2 2\pi\delta_k \langle b_k \rangle. \qquad (34b)$$

These equations coincide with those obtained usually in the frame of Wigner - Weisskopf approximation. Field modes and the oscillator exchange their energies. As a result there is no effect of stimulated emission but only two independent fluxes of energy:

$$\frac{d}{dt} \langle a^+ a \rangle_I = -\gamma \langle a^+ a \rangle + \sum_k |\lambda_k|^2 2\pi\delta_k \langle b_k^+ b_k \rangle, \qquad (34c)$$

$$\frac{d}{dt} \langle b_k^+ b_k \rangle_I = |\lambda_k|^2 2\pi\delta_k \langle a^+ a \rangle + |\lambda_k|^2 2\pi\delta_k \langle b_k^+ b_k \rangle. \qquad (34d)$$

Squeezing-correlation parameter behaves in a similar way :

$$\frac{d}{dt} D(a,a)_I = -\gamma D(a,a) - \sum_k \lambda_k^2 2\pi\delta_k D(b_k,b_k), \qquad (34e)$$

$$\frac{d}{dt} D(b_k,b_k)_I = -\lambda_k^{*2} 2\pi\delta_k D(a,a) - |\lambda_k|^2 2\pi\delta_k D(b_k,b_k). \qquad (34f)$$

Further development can be made for the specific expressions of coefficients in Hamiltonian (3). For the continuum of modes summation is substituted by integration over phase space and summation over polarization indexes r

$$\sum_j \rightarrow \sum_r \int V\rho \, d\omega \, d\Omega \qquad (35)$$

with volume V, solid angle element $d\Omega$, mode frequency density

$$\rho = \frac{\omega^2}{8\pi^3 c^3} . \qquad (36)$$

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Decomposition of vector potential $A(r,t)$ over mode variables is

$$A(r,t) = \sum_j \sqrt{\frac{\hbar}{2\omega_j \varepsilon_0 V}} \; e_j \; (b_j(t)\exp(ik_j r) + H.c.).$$ (37)

where e_j is a polarization vector and k_j - a wave vector of j-th mode.

Gauge invariance substitution of oscillator momentum $p \to p - eA$ leads to the interaction Hamiltonian (3)

$$H_i = \frac{-epA}{m} + \frac{e^2 A^2}{2m}.$$ (38)

Here e,m are the charge and the mass of the oscillator. The second term in this case proves to be a unity operator in state-space of the oscillator. Hence it results in an infinitely little renormalization of field energy because of a factor $1/V$ (for infinitely large volume V). The coupling constant will be

$$\lambda_j = -i \frac{e}{2} \sqrt{\frac{\omega}{\omega_j m \varepsilon_0 V}} \cos\theta_j,$$ (39)

where θ_j is the angle between a polarization vector and the oscillation direction.

On the other hand, from the Hamiltonian in another gauge form

$$H_i = - eqE,$$ (40)

where q is a coordinate of the oscillator and E is the electric field vector, it follows that the coupling constant

$$\lambda'_j = \lambda_j \frac{\omega_j}{\omega}.$$ (41)

But as all expressions contain delta functions $\delta(\omega_j - \omega)$, constants (39) and (41) coincide. We see that it is one of the cases when gauge transform, performed over state vectors in the absence of vector potential and corresponding to a change from gauge form (40) to (38), does not make any difference. These transforms were considered in detail in Ref. 10.

Einstein's stimulated coefficient can be also introduced. However it is different from a common one - it depends on the angle and expresses radiation power instead of probability :

$$B = 2\pi V |\lambda|^2 = \frac{\pi e^2 \cos^2 \theta}{2m\varepsilon_0} \ . \tag{42}$$

The spontaneous emission coefficient is obtained from (33). Integration should be performed over solid angles of polarisation vectors (they are also isotropically distributed), not wave vectors of modes :

$$\gamma = \frac{\omega^2}{4\pi^3 c^3} \int B \ d\Omega = \frac{e^2 \omega^2}{6\pi m \varepsilon_0 c^3} \ . \tag{43}$$

A light beam containing several close modes has an energy density

$$W_\omega = \sum_r \rho \langle b^+ b \rangle \hbar \omega \tag{44}$$

$$\text{or} \qquad W = \int W_\omega \ d\Omega. \tag{45}$$

It will allow us to express eqs.(34) through physically meaningfull values.

$$\frac{d}{dt} \left[\hbar\omega \langle a^+ a \rangle \right] = - \gamma \hbar\omega \langle a^+ a \rangle + \int B W_\omega d\Omega, \tag{46a}$$

$$\frac{d}{dt} \left[WV \right] = B \left[\rho \hbar\omega \langle a^+ a \rangle - W_\omega \right], \tag{46b}$$

$$\frac{d}{dt} D(a,a) = - \gamma D(a,a) + \int B W_\omega \frac{D(b,b)}{\hbar\omega \langle b^+ b \rangle} \ d\Omega, \tag{46c}$$

$$\frac{d}{dt} \left[WVD(b,b) \right] = B \left[\rho \hbar\omega \langle b^+ b \rangle D(a,a) - W_\omega D(b,b) \right]. \tag{46d}$$

All above discussed enables us to answer posed questions :

1. In general absorption and emission can not be distinguished.

2. So not Fermi's rule but expectation values should be used to calculate the rates of these processes.

3. Time ordering in this case is not important up to the second order of perturbation.

4. Stimulated emission does not manifest itself in the final result.

5. Energy and squeezing-correlation parameters behave in a similar way : there are independent interchange fluxes of them proportional to their current values.

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PHOTON NUMBER AMPLIFICATION/DUPLICATION THROUGH PARAMETRIC CONVERSION

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ABSTRACT: The performance of parametric conversion in achieving number amplification and duplication is analyzed. It is shown that the effective maximum gains G_- remain well below their integer ideal values, even for large signals. Correspondingly one has output Fano factors F_- which are increasing functions of the input photon number. In the inverse (deamplifier/recombiner) operating mode, on the contrary, quasi ideal gains G_- and small factors $F_- \simeq 10\%$ are obtained. Output noise and nonideal gains are ascribed to spontaneous parametric emission.

1. INTRODUCTION

The ultimate transparency of optical networks is essentially quantum-limited and any improvement beyond the standard performance depends on availability of nonstandard high quality quantum amplifiers. The *photon number amplifier* (PNA) and the *photon number duplicator* (PND) are the quantum devices which are needed in direct detection.[1] The PNA ideally should affect the state transformation

$$|n\rangle \longrightarrow |Gn\rangle \qquad (1)$$

for integer gains G and input eigenstates $|n\rangle$ of the number. Similarly, the PND, instead of amplifying the photon number, produces two copies of the same input state for eigenstates of the number, namely

$$|n\rangle \longrightarrow |n, n\rangle . \qquad (2)$$

Both devices are particularly suited to local area network environments, where the minimum loss for user-derivation is 3dB (in average), and transparency rapidly degrades with the increasing number of users. In such situation the PNA represents the ideal preamplifier to be inserted before each derivation, whereas the PND—which ideally realizes the quantum nondemolition measurement of the number—could itself be used as an ideal lossless optical tap.

The PNA and PND could also be profitably used in the inverse operating mode, namely the PNA as a *number deamplifier* and the PND as a *number recombiner*. The number deamplifier could be used as a *number squeezer*, allowing production of subpoissonian states from coherent light; the

number recombiner, on the other hand, could produce novel nonclassical radiation from input twin-beams [an example of such application in production of phase-coherent states[2] is proposed in Ref. [3]].

The concrete realization of high quality PNA and PND for practical applications is an arduous task. As explained in Ref. [4], number conversion, in a way similar to the customary conversion, requires a medium with a $\chi^{(2)}$ or $\chi^{(3)}$ susceptibility, but here with a phase-dependent polarizability. More precisely, almost ideal number conversion can be achieved upon modulating the nonlinear susceptibility at a $(G-1)$-submultiple of the wavelenght of the amplified mode, G being the integer gain [feasibility studies of number conversion using multiple quantum wells heterostructures are currently in progress[6]]. The required phase-dependent polarizability in a $\chi^{(2)}$ or $\chi^{(3)}$ medium may also be regarded as an intensity-dependent coupling for a $\chi^{(G)}$ or $\chi^{(G+1)}$ medium (simply from polar decomposition of the boson field operators). This suggests that a gain-two PNA should be simpler to realize than a generic $G > 2$ amplifier. However, as also explained in this paper, the intensity dependent coupling should follow the power low $(a^{\dagger}a + 1)^{-1/2}$, $a^{\dagger}a$ being the number operator of the amplified mode. Such a decreasing factor is essentially the $(1 + I)^{-1/2}$ saturating behaviour of a two level system effective susceptibility in the inhomogeneous-broadening limit,[5] but it is not obvious that this power low—which is obtained in a semiclassical context—could survive in the quantization procedure.

The previous observations quite naturally lead to ask if the conventional conversion could somehow simulate the number conversion, and what would be the range of physical parameters where ideal behaviour is better approximated: this is the subject of the present paper. Quite unexpectedly (see for example Ref. [1]) we find that ideal behaviour is never approached, even in the limit of large input signals. The most striking result is that conversion is never complete and, therefore, the effective maximum gains G_{-} remain well below their integer ideal values, even for large input photon numbers: quantum mechanics thus reveals its subtle nature even for large quantum numbers, here in form of noise in amplifiers [for a discussion on applicability of the correspondence principle in a different context, see Ref. [7]].

The inverse devices—namely the number deamplifier and the number recombiner—are better approximated by parametric conversion than the direct ones. We will show that ideal gains are achieved in the large-n limit, whereas Fano factors F_{-} remain nonvanishing but small ($F_{-} \simeq 10\%$). Therefore, it seems that at present the devices which are simplest to realize concretely should be the number deamplifier and the number recombiner (even though probably the limited output noise of the deamplifier could not be satisfactory for applications as number squeezer).

After presenting the theory of the ideal devices in Sect.2, the connections between the conventional and the number conversions are explained in Sect.3, where a simple mean field approach for analytical evaluation of the conversion time is also given. In Sect.4 the announced numerical results on conversion times, effective gains and Fano factors are presented. In Sect.5 we conclude with some remarks on the physical interpretation of the nonideal behaviour in terms of spontaneous parametric emission.

2. THE IDEAL NUMBER AMPLIFIER/DUPLICATOR

In the Heisenberg picture the ideal PNA corresponds to multiplication of the number operator by the integer gain G

$$a^{\dagger}a \rightarrow Ga^{\dagger}a , \qquad (3)$$

a being the annihilator of the amplified mode of the field. Because of the integer nature of $a^{\dagger}a$,

the deamplifier does not trivially correspond to replace G into G^{-1} in Eq. (3). Actually the ideal deamplification is the following

$$a^\dagger a \rightarrow [G^{-1}a^\dagger a] \, , \tag{4}$$

where $[x]$ denotes the integer part of x. As a consequence, even in the ideal case, the deamplification has an input-dependent effective gain G_*.

$$G_* = \frac{[G^{-1}n]}{n} \leq G^{-1} \tag{5}$$

for n input photons, and $G_* \simeq G^{-1}$ for large n. [As an example, the case $G = 2$ is depicted in Fig. 4.] In terms of the shift operator $\hat{e}_+ : \hat{e}_+|n\rangle = |n+1\rangle$, the transformation (4) is obtained as follows

$$\hat{e}_+ \rightarrow (\hat{e}_+)^G \, , \tag{6}$$

where now $(\hat{e}_+)^G|n\rangle = |n+G\rangle$. In fact, the map (6) corresponds to the following[3]

$$a^\dagger \rightarrow a^\dagger_{(G)} \, , \tag{7}$$

where $a^\dagger_{(G)}$ is a boson operator creating G photons at a time,[8] namely

$$a^\dagger_{(G)}|n\rangle = \sqrt{[G^{-1}n]+1}\,|n+G\rangle \, , \tag{8}$$

$$\left[a_{(G)}, a^\dagger_{(G)}\right] = 1 \, , \quad \left[a_{(G)}, a^\dagger a\right] = Ga_{(G)} \, . \tag{9}$$

The explicit form of $a^\dagger_{(G)}$ is

$$a^\dagger_{(G)} = \left\{ \frac{[G^{-1}\hat{n}](\hat{n}-G)!}{\hat{n}!} \right\}^{1/2} (a^\dagger)^G \tag{10}$$

and from Eq. (10) it follows that

$$a^\dagger_{(G)}a_{(G)} = [G^{-1}a^\dagger a] \, , \tag{11}$$

which is the deamplification (4). The direct amplification (3) corresponds to the inverse transformation

$$a^\dagger_{(G)} \rightarrow a^\dagger \tag{12}$$

[see Ref. [3] for more details about these maps]. The transformations (7) and (12) are essentially permutations of two different types of boson. For commuting modes $[a, c] = [a, c^\dagger] = 0$ the permuting map $a \leftrightarrow c$ is realized by the Heisenberg evolution

$$PaP = c, \qquad PcP = a \, , \tag{13}$$

where

$$P = P^\dagger = \exp\left(i\frac{\pi}{2}c^\dagger c\right) \exp\left[-i\frac{\pi}{2}\left(a^\dagger c + c^\dagger a\right)\right] \exp\left(i\frac{\pi}{2}c^\dagger c\right) \, . \tag{14}$$

However, as a and $a_{(G)}$ do not commute, it is convenient to consider a simultaneous change of the field mode (namely the amplifier also converts the frequency or changes the field polarization). In this case the amplifying operator is

$$P_{(G)} = \exp\left(i\frac{\pi}{2}c^\dagger c\right) \exp\left[-i\frac{\pi}{2}\left(a^\dagger_{(G)}c + c^\dagger a_{(G)}\right)\right] \exp\left(i\frac{\pi}{2}c^\dagger c\right) . \tag{15}$$

The operator (15) now attains the transformations

$$P_{(G)}(\hat{1} \otimes \hat{e}_+)P_{(G)} = (\hat{e}_+)^G \otimes \hat{1} , \qquad P_{(G)}\left[(\hat{e}_+)^G \otimes \hat{1}\right]P_{(G)} = \hat{1} \otimes \hat{e}_+ , \tag{16}$$

where in the tensor notation $\hat{O}_1 \otimes \hat{O}_2$ the first entry is for the a mode and the second entry for the c mode. The Schrödinger evolutions of the number eigenstates corresponding to the amplifying and deamplifying operating modes are

$$P_{(G)}|0,n\rangle = |Gn,0\rangle , \tag{17}$$

$$P_{(G)}|n,0\rangle = |G\langle G^{-1}n\rangle , [G^{-1}n]\rangle , \tag{18}$$

where $\langle x \rangle = x - [x]$ denotes the fractional part of x and $|n,m\rangle = |n\rangle_a \otimes |m\rangle_c$. If one would consider only one mode in the above transformations—say a—a frequency conversion $P_{(1)}$ is needed. In this case the evolutions (17-18) rewrite

$$P_{(G)}P_{(1)}|n,0\rangle = |Gn,0\rangle \tag{19}$$

$$P_{(1)}P_{(G)}|0,n\rangle = |G\langle G^{-1}n\rangle , [G^{-1}n]\rangle , \tag{20}$$

whereas totally ignoring the mode c corresponds to trace the transformations (19-20) over this mode, adopting a density matrix representation for states. In this way nonunitary transformations for the reduced density matrix of the signal mode a are obtained, which do not preserve the Newmann-Shannon entropy: these are the 'photon fractioning' and 'multiphoton' transformations of Refs.[3,9]. The mode c is responsible of the added noise which is present even in the ideal case (see Eq. (5)) and corresponds to the 'idler mode' of the customary linear amplification.[3]

Apart from the $\frac{\pi}{2}$ phase shift—which can be obtained by changing the optical path of the b mode and which, however, for an input number eigenstate corresponds to an irrelevant overall phase factor—the evolution operator (15) comes from the interaction Hamiltonian in the Dirac picture

$$\hat{H}_I = a^\dagger_{(G)}c + \text{h.c.} \tag{21}$$

for a dimensionless evolution time

$$\tau_* = \frac{\pi}{2} . \tag{22}$$

The Hamiltonian (21) has the following constants of motion

$$\hat{s}_A = a^\dagger a + Gc^\dagger c , \tag{23}$$

$$\hat{d}_A = G\langle G^{-1}a^\dagger a\rangle = G\langle G^{-1}\hat{s}_A\rangle , \tag{24}$$

and, because of identity (24), only $\hat{s}_A$ must be specified. In the following evaluations we use the basis of the Hilbert subspace corresponding to fixed $\hat{s}_A$ eigenvalues

$$|n\rangle_{s_A} = |s_A - Gn, n\rangle , \qquad n = 0, 1, \ldots [G^{-1}s_A] \tag{25}$$

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In this basis Eqs. (17-18) rewrite

$$P_{(G)}|n\rangle_{Gn} = |0\rangle_{Gn} ,$$ (26)

$$P_{(G)}|0\rangle_n = |[G^{-1}n]\rangle_n ,$$ (27)

whereas the Hamiltonian (21) takes the tridiagonal form

$$\hat{H}_I|n\rangle_s = \alpha_n^{(s)}|n-1\rangle_s + \alpha_{n+1}^{(s)}|n+1\rangle_s ,$$ (28)

$$\alpha_n^{(s)} = \sqrt{n([G^{-1}s] - n + 1)} .$$ (29)

Conservation of the interaction Hamiltonian (21) itself corresponds to the resonance condition $G\omega_a = \omega_c$. In the nonresonating case a third pump mode d is needed with $\omega_d = G\omega_a - \omega_c$: Eq. (21) is obtained from the interaction Hamiltonian in the Schrödinger picture

$$\hat{H}' = a_{(G)}^\dagger cd + \text{h.c.}$$ (30)

in the parametric approximation of classical undepleted pump, namely with d in a highly excited coherent state.

The photon number duplicator in some respect is similar to the gain-2 photon number amplifier. Instead of amplifying the number of photons, it produces two copies of the same input state for eigenstates of the number operator. If the input copies are carried by the modes a and b whereas the output by c, the duplication map reads

$$|0, 0, n\rangle \rightarrow |n, n, 0\rangle$$ (31)

and is trivially inverted for $n_a = n_b$ [for the general case see Ref. [3]]. The state transformation (31) corresponds to the Heisenberg evolution

$$\hat{e}_+ \otimes \hat{e}_+ \otimes \hat{1} \rightarrow \hat{1} \otimes \hat{1} \otimes \hat{e}_+ ,$$ (32)

which is obtained as permutation of the boson operators $a_{(1,1)}$ and c, where $a_{(1,1)}^\dagger$ now denotes the two-mode creator

$$a_{(1,1)}^\dagger|n_a, n_b\rangle = \sqrt{(\min\{n_a, n_b\}) + 1}|n_a + 1, n_b + 1\rangle ,$$ (33)

$$[a_{(1,1)}, a_{(1,1)}^\dagger] = 1 , \qquad [a_{(1,1)}, a^\dagger a + b^\dagger b] = 2a_{(1,1)} .$$ (34)

The following realization of $a_{(1,1)}^\dagger$ is obtained in Ref. [3]

$$a_{(1,1)}^\dagger = a^\dagger b^\dagger \frac{1}{\sqrt{\max\{a^\dagger a, b^\dagger b\} + 1}} .$$ (35)

In a way analogous to the PNA, the Dirac picture interaction Hamiltonian of the PND is

$$\hat{H}_I = a_{(1,1)}^\dagger c + \text{h.c.} ,$$ (36)

with constants of motion

$$\hat{s}_D = \frac{1}{2}\left(a^\dagger a + b^\dagger b + 2c^\dagger c\right) ,$$ (37)

$$\hat{d}_D = a^\dagger a - b^\dagger b .$$ (38)

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The Hilbert subspace of interest for duplication corresponds to $d_D = 0$; the subspaces for fixed eigenvalues s_D are spanned by the eigenvectors

$$|n\rangle_{s_D} = |s_D - n, s_D - n, n\rangle . \tag{39}$$

For fixed s_D the Hamiltonian (36) has the tridiagonal form

$$\hat{H}_I|n_s\rangle = \beta_n^{(s)}|n - 1\rangle_s + \beta_{n+1}^{(s)}|n + 1\rangle_s , \tag{40}$$

$$\beta_n^{(s)} = \sqrt{(s - n + 1)n} . \tag{41}$$

Frequency conversion and simultaneous duplication require a classical undepleted pump mode d at frequency $\omega_d = \omega_a + \omega_b - \omega_c$, with interaction Hamiltonian

$$\hat{H}' = a_{(1,1)}^\dagger cd + \text{h.c.} \tag{42}$$

3. NUMBER-OPTIMIZED DOWNCONVERSION

The Hamiltonians (30) and (42) are complicated by the occurrence of the multiboson operators $a_{(G)}^\dagger$ and $a_{(1,1)}^\dagger$. An outlook at Eqs. (10) and (33) reveals that the G-photon amplification corresponds to a $\chi^{(G+1)}$ susceptibility and the duplication to a $\chi^{(3)}$. In the followings the $G = 2$ case—the simplest to attain in practice—will be considered only. For $\langle a^\dagger a\rangle \gg 2$ the two photon operator $a_{(2)}^\dagger$ can be approximated as follows

$$a_{(2)}^\dagger \simeq a^{\dagger 2}\left[2(a^\dagger a + 1)\right]^{-\frac{1}{2}} , \qquad (\langle a^\dagger a\rangle \gg 2) . \tag{43}$$

On the other hand, for $d_D = 0$ the two-mode operator $a_{(1,1)}^\dagger$ is simply

$$a_{(1,1)}^\dagger = a^\dagger(a^\dagger a + 1)^{-\frac{1}{2}}b^\dagger , \qquad (a^\dagger a = b^\dagger b) . \tag{44}$$

Hence the Hamiltonians (30) and (42) become

$$\hat{H}' \simeq a^{\dagger 2}\left[2(a^\dagger a + 1)\right]^{-\frac{1}{2}} cd + \text{h.c.} \qquad (\text{PNA}) , \tag{45}$$

$$\hat{H}' = a^\dagger(a^\dagger a + 1)^{-\frac{1}{2}}b^\dagger cd + \text{h.c.} \qquad (\text{PND}) . \tag{46}$$

As a crude approximation we substitute the intensity-dependent factors in Eqs. (45-46) with their constant average values and use the customary four wave mixing Hamiltonians

$$\hat{H}_{FWM} = a^{\dagger 2}cd + \text{h.c.} \qquad (\text{PNA}) , \tag{47}$$

$$\hat{H}_{FWM} = a^\dagger b^\dagger cd + \text{h.c.} \qquad (\text{PND}) . \tag{48}$$

In the parametric approximation of undepleted classical pump d, Eqs. (47-48) correspond to the interaction Hamiltonians

$$\hat{H}_I = a^{\dagger 2}c + \text{h.c.} \qquad (\text{PNA}) , \tag{49}$$

$$\hat{H}_I = a^\dagger b^\dagger c + \text{h.c.} \qquad (\text{PND}) . \tag{50}$$

The interaction time is rescaled by $\sqrt{I_d}$, I_d being the intensity of the pump d: the relation between the dimensionless time τ and the real time t (namely the length of the nonlinear medium) now reads

$$\tau = \chi^{(3)}\sqrt{I_d}\, t \,. \tag{51}$$

The Hamiltonians (47) and (48) were already suggested by Yuen[1], who inferred the amplifying performance from the conservation laws (23-24) and (37-38), with the assumption of complete conversion of the input signal. However, we will show that complete conversion is never achieved, apart from the case of one input photon. As an example in Fig. 1 the average output photon number is plotted for Hamiltonian (48) versus the interaction time τ, for both cases: number duplicator ($\langle n_a \rangle_0 = \langle n_b \rangle_0 = 0, \langle n_c \rangle_0 = n_i$) and number recombiner ($\langle n_a \rangle_0 = \langle n_b \rangle_0 = n_i, \langle n_c \rangle = 0$). An oscillatory quasiperiodic (or long-time periodic) behavior is evident, conversion never being complete at any time: the ideal gain is not reached, and the unconverted photons contribute to the output noise. Therefore, the saturating factors in Eqs. (45-46) are crucial to get complete conversion. Semiclassically a similar saturating behavior $\propto (1 + I)^{-\frac{1}{2}}$ is obtained for interaction of radiation with a two level system in the inhomogeneously-broadening limit or in the adiabatic-following regime:[5] however, a full quantum treatment is still lacking and would require a wideband analysis. Here we only consider the performance of parametric Hamiltonians (49-50) in achieving approximate PNA and PND. In this case the interaction time $\tau = \tau_*$ for conversion depends on the input photon number n_i

$$n_i = \begin{cases} \langle n_c \rangle_0 \,, & (\langle n_a \rangle_0 = 0 \,:\, \text{direct operating mode}) \,, \\ \langle n_a \rangle_0 \,, & (\langle n_c \rangle_0 = 0 \,:\, \text{inverse mode}) \,, \end{cases} \tag{52}$$

which, in order to simulate the intensity-saturating low in Eqs. (45-46), should behave as follows

$$\tau_* \sim n_i^{-\frac{1}{2}} \,. \tag{53}$$

The conversion time (53) could be obtained tuning the pump intensity on the input photon number n_i: for n_i varying in a wide range, this would require a suited feedback mechanism based on a quantum nondemolition measurement of n_i. In the following we give more accurate evaluations of τ_*, using either analytical methods (a mean field approximation) and numerical calculations. The results obtained in the two ways will be compared and discussed in the end.

3.1 *A mean field approximation*

In Ref. [10] a linearization procedure for parametric conversion has been proposed, where Hamiltonians (49-50) are approximated in a selfconsistent way by the ideal ones (21) and (36). As we will see in the followings, this approach is correct only in the limit of large input photons numbers in the amplified/duplicated channels (i.e. a and b modes), namely it is suited to describe the inverse operating mode only. The method allows evaluation of the conversion time τ_*: its major limitation is that it leads to exact conversion and, therefore, there is no systematic way to estimate quantum fluctuations and nonideal gains. As a consequence, the direct operating mode cannot be described in terms of the time-reversed transformation of the inverse mode, because in this case knowledge of the output noise become essential. Therefore, in this section we analyze only the deamplifier/recombiner case.

The starting point of the method is to rewrite Hamiltonians (49-50) in a form similar to the ideal ones (21) and (36), namely

$$\hat{H}_I = f(a^\dagger a)Ac + A^\dagger f(a^\dagger a)c \,, \tag{54}$$

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where

$$A = \begin{cases} a_{(2)} & \text{(PNA)}, \\ a_{(1,1)} & \text{(PND)} \end{cases} \tag{55}$$

and

$$f(x) = \begin{cases} (2x + 3 - (-1)^x)^{\frac{1}{2}} & \text{(PNA)}, \\ (x + 1)^{\frac{1}{2}} & \text{(PND)}. \end{cases} \tag{56}$$

The operator function $f(a^{\dagger}a)$ will be treated as a c-number time-dependent effective coupling, to be determined selfconsistently *a posteriori*. The Hamiltonian (54) is rewritten as

$$\hat{H}_I = f(a^{\dagger}a)Ac^{\dagger} + f(a^{\dagger}a - \nu)A^{\dagger}c , \tag{57}$$

where, in order to have a unified description of the two devices, the integer number ν is used

$$\nu = \begin{cases} 1 & \text{(PNA)}, \\ 2 & \text{(PND)}. \end{cases} \tag{58}$$

We write a mean field Hamiltonian taking the intermediate value $\bar{f}(a^{\dagger}a) = f(a^{\dagger}a - \frac{\nu}{2})$ between the two forms in Eq. (57) and averaging on the input state. One obtains

$$\hat{H}_{MF} = \bar{f}(n_a(\tau)) \left[Ac^{\dagger} + \text{h.c.} \right] , \tag{59}$$

where

$$n_a(\tau) = \langle a^{\dagger}(\tau)a(\tau) \rangle_0 , \tag{60}$$

and

$$\bar{f}(x) = \sqrt{\nu \left(x + \frac{1}{2} \right)} \tag{61}$$

(the oscillating $(-1)^{n_a(\tau)}$ term in Eq. (56) is neglected). In the Dirac picture the time evolution of an operator $\hat{O}$ is written as follows

$$\hat{O}(\tau) \simeq \exp\left(i\hat{H}_{av}\tau \right) \hat{O} \exp\left(-i\hat{H}_{av}\tau \right) , \tag{62}$$

using the time-averaged Hamiltonian $\hat{H}_{av}$

$$\hat{H}_{av} = \frac{1}{\tau} \int_0^{\tau} \hat{H}_{MF}(\tau)d\tau = \frac{\theta(\tau)}{\tau} \left(Ac^{\dagger} + A^{\dagger}c \right) , \tag{63}$$

$$\theta(\tau) = \int_0^{\tau} \bar{f}(n_a(\tau'))d\tau' . \tag{64}$$

The evolution of the operators A and c takes the simple form

$$A(\tau) = A\cos\theta(\tau) + ic\sin\theta(\tau) \tag{65}$$

$$c(\tau) = c\cos\theta(\tau) + iA\sin\theta(\tau) . \tag{66}$$

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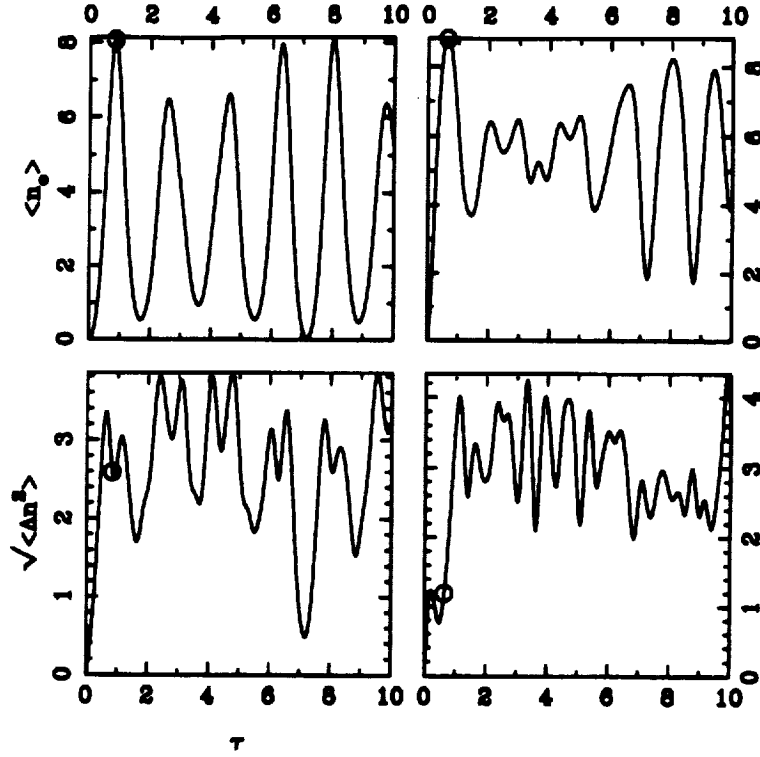
Figure 1: Time evolution of the output signal $\langle \hat{n}_o \rangle$ (figures on the top) and of the r.m.s. output noise $\sqrt{\langle \Delta \hat{n}^2 \rangle}$ (figures on the bottom) for parametric conversion (Hamiltonian (50)) of input number states with $n_i = 10$. The two figures on the left refer to the number duplicator ($\langle n_a \rangle_0 = \langle n_b \rangle_0 = 0, \langle n_c \rangle_0 = n_i$); those on the right to the number recombiner ($\langle n_a \rangle_0 = \langle n_b \rangle_0 = n_i, \langle n_c \rangle = 0$). The small circles enclose the conversion point corresponding to $\tau = \tau_*$.

We are now in a position to evaluate $n_a(\tau)$ selfconsistently. From Eq. (65) one has

$$n_A(\tau) = \langle A^\dagger(\tau) A(\tau) \rangle_0 = \langle A^\dagger A \rangle_0 \cos^2 \theta(\tau) . \tag{67}$$

For large input photons n_i and $\nu = 1$ the expectation n_A can be approximated as

$$n_A = [n_a/2] \simeq n_a/2 . \tag{68}$$

From Eqs. (64) and (67) we obtain the following integral equation for $n_a(\tau)$

$$n_a(\tau) = n_i \cos^2 \theta(\tau) = n_i \cos^2 \int_0^\tau \bar{f}(n_a(\tau')) d\tau' . \tag{69}$$

Differentiation of Eq. (69) leads to

$$\sqrt{\nu} d\tau = d\theta \left(n_i \cos^2 \theta + \frac{1}{2} \right)^{-\frac{1}{2}} \tag{70}$$

From Eqs. (65) and (69) one can see that complete conversion occurs at $\tau = \tau_*$ such that

$$\theta(\tau_*) = \frac{\pi}{2} . \tag{71}$$

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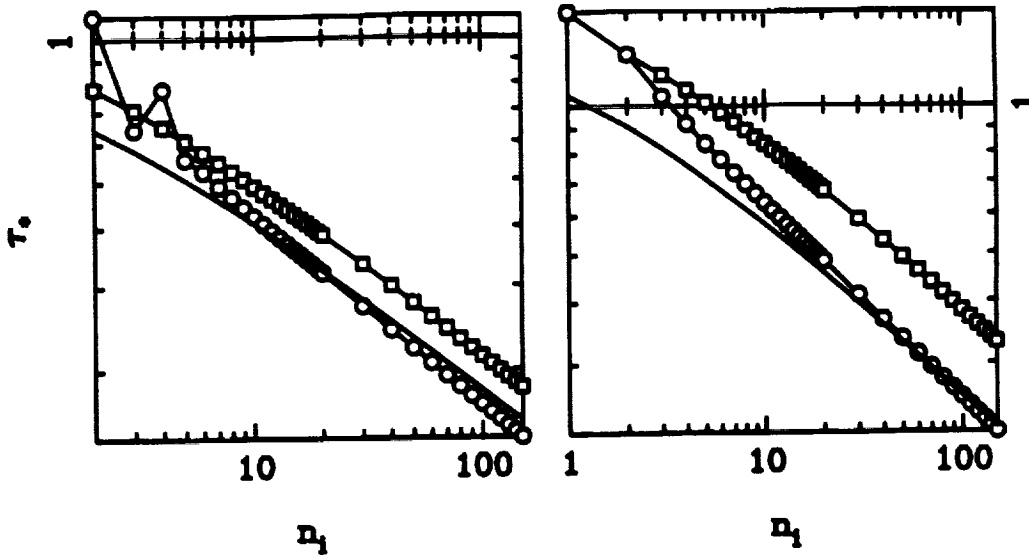
Figure 2: The best conversion time τ_* for parametric Hamiltonians (49-50) (PNA on the left and PND on the right). The squares are for the amplifier/duplicator, the circles for the deamplifier/recombiner. The lines without dots represent the mean-field approximation.

After integrating Eq. (70) from $\theta = 0$ to $\theta = \frac{\pi}{2}$ we find the conversion time as a function of the input photon number n_i

$$\tau_* = \nu^{-\frac{1}{2}} \left(n_i + \frac{1}{2} \right)^{-\frac{1}{2}} K \left(\frac{n_i}{n_i + \frac{1}{2}} \right) , \tag{72}$$

where $K(k)$ denotes the complete Jacobian elliptic integral

$$K(k) = \int_0^{\frac{\pi}{2}} (1 - k \sin^2 x)^{-\frac{1}{2}} dx . \tag{73}$$

For large numbers n_i, using the asymptotic behavior $K(k) \sim -\log \sqrt{1-k}$ for $k \to 1$, one obtains

$$\tau_* \sim \frac{1}{2} \nu^{-\frac{1}{2}} n^{-\frac{1}{2}} \log n , \tag{74}$$

which, a part from a logarithmic correction, has the same form of the preliminary result (53).

5. NUMERICAL RESULTS

The quantum evolution of input number eigenstates for the Hamiltonians (49-50) is evaluated numerically, taking advantage of the tridiagonal forms (28) and (40), which now read

$$\alpha_n^{(s)} = \sqrt{n(s - 2n + 1)(s - 2n + 2)} , \quad (s \equiv s_a) , \tag{75}$$

$$\beta_n^{(s)} = \sqrt{n}(s - n + 1) , \quad (s \equiv s_D) . \tag{76}$$

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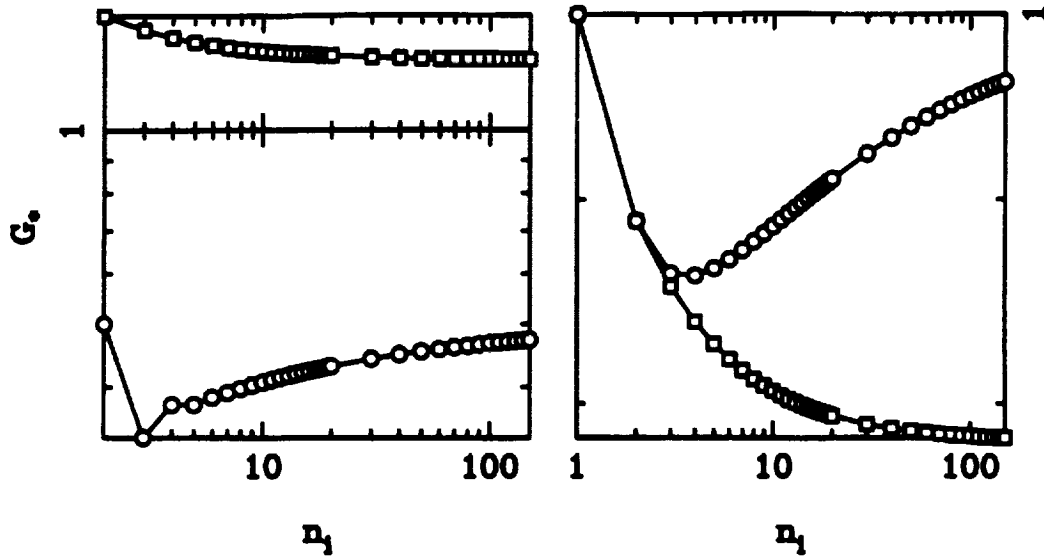
Figure 3: The maximum effective gain G_- (corresponding to the conversion time τ_- of Fig. 2) for parametric Hamiltonians (49-50) (PNA on the left, PND on the right). The squares are for amplification/duplication, the circles for the inverse operating mode.

The evolution of the output signal has been checked using the numerical results in Ref. [11]. In Fig. 1 we report a sample of the evolution for the PND. The time-dependence is periodic or nearly periodic for very low input photon numbers n_i, whereas it becomes more and more irregular (essentially irreversible) for increasing n_i. Qualitative differences between the direct and the inverse operating cases are evident. In the direct case the output signal exhibits maxima corresponding to high noise level, whereas low noise occurs only for depleted signal. In the inverse case, on the contrary, the first occurrence of a local maximum for the signal coincides with the absolute maximum, whereas the relative noise is always well below the subsequent values (this gap being an increasing function of n_i). The conversion is never complete in both cases, however, it is more efficient in the inverse operating mode, due to the low noise at the output. The conversion time τ_- has been identified as the time corresponding to the first local maximum of the signal (in the direct operating mode this could be slightly lower than the absolute maximum). The same features in the time evolution can be found for the PNA approximated by the conversion Hamiltonian (49), with analogous differences between the direct and inverse operating modes.

In Fig. 2 the conversion time τ_- is plotted against the input number n_i, for both Hamiltonians (49) and (50). The direct and inverse operating modes lead to two different curves, the former corresponding to longer conversion times τ_- (a part from some features which are peculiar of the deamplifier for low inputs n_i, and are reminiscent of the fractional behaviour (68)). The mean field approximation, which is pertinent only to the inverse operating mode, is reported for comparison. A good agreement is found for large n_i, better for the PND than for the PNA. For large n_i numerical best fits give power-low behaviours of the form $\tau_- \sim n^{-\alpha}$, with $\alpha \sim .4$ or smaller.

In Fig. 3 the maximum effective gain G_- (corresponding to the conversion time τ_- in Fig. 2) is reported. One can see that parametric conversion when used as a gain-two number amplifier

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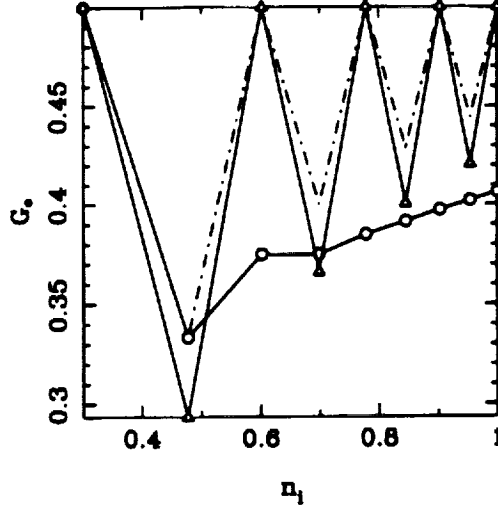
Figure 4: Effective gain G_* for the deamplifier with $G = 2$: circles and full line describe the parametric conversion (Hamiltonian (49)); triangles and full line describe the intensity-saturating Hamiltonian (45); dot-dashed line corresponds to the ideal deamplifier (5).

leads to an effective gain G_* which is a decreasing function of the input signal n_i, approaching the value $G_* \simeq 1.28$ for large n_i, well below the ideal gain. In a similar fashion the effective gain of the duplicator $G_* = \langle n_a(\tau_*) \rangle / \langle n_c(0) \rangle$ tends asymptotically to $G_* \simeq .78$. The inverse operating mode, on the contrary, behaves quite well, the deamplifier achieving the ideal $G_* = 1/2$ gain and the recombiner $G_* = 1$ in the large n_i limit. The deamplifier gain is compared with the ideal one (5) in Fig. 4, where also the intensity-saturating case (45) is reported [notice that in the direct operating mode the intensity-saturating Hamiltonians (45-46) lead to ideal behaviour].

Finally, in Fig. 5 the output Fano factors F_* at the conversion time τ_* are plotted. It is evident that parametric conversion lead to noisy PNA and PND, with $F_* \sim n^{-\beta}$ and exponent β slightly lower then 1: this corresponds to an output signal-to-noise ratio which is slowly (logarithmically) vanishing. The number deamplifier and recombiner are better approximated, with $F_* \simeq .13$ for large n_i: on the other hand, the intensity-saturating Hamiltonian (45) leads to vanishing F_* for large n_i (F_* is exactly zero for even n_i.)

6. CONCLUSIONS

We end with some remarks on physical interpretation of numerical results. We have seen that parametric Hamiltonians (45-46) are not good candidates for number amplification/duplication devices, whereas they could be profitably used to achieve approximate number deamplification/recombination. Here we emphasize that the source of noise in the simulated number devices is the socalled *spontaneous parametric emission*.[12] As a matter of fact, as explained in Ref. [11], the Hamiltonians (45-46) are formally similar to the Hamiltonian of a laser amplifier: in particular, Eq. (46) can be put in correspondence with the Hamiltonian describing a cluster of N two-level atoms
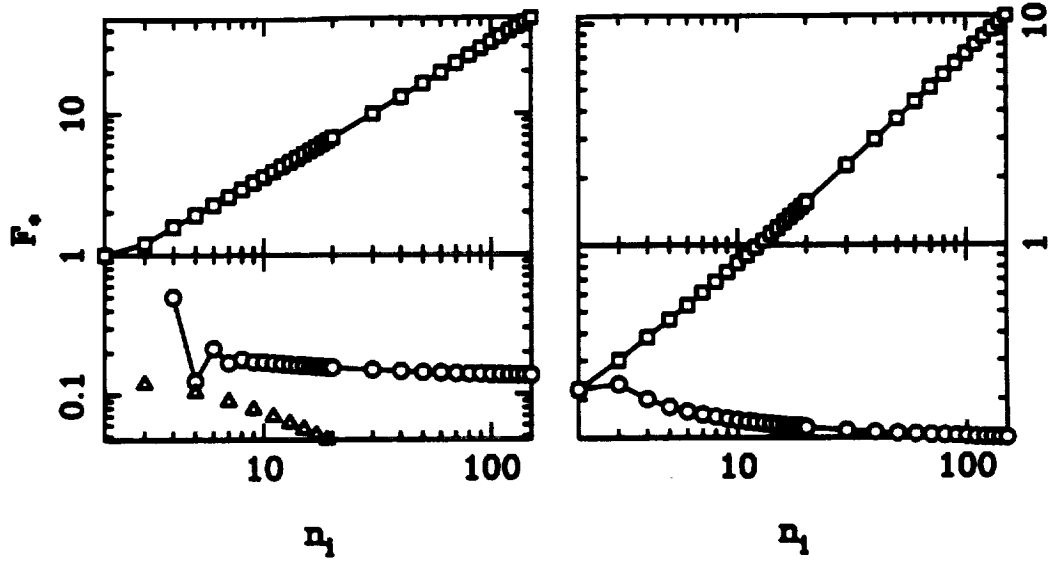
Figure 5: The Fano factor F_a at the conversion time τ_a of Fig. 2 for parametric Hamiltonians (49-50) (PNA on the left and PND on the right). The squares are for amplification/duplication, the circles for the inverse operating case. The triangles correspond to the intensity-saturating Hamiltonian (45).

interacting with one (resonant) mode of radiation

$$\hat{H}_I \propto a^\dagger \hat{J}_- + a \hat{J}_+ , \tag{77}$$

where $\hat{J}_\alpha = \sum_{i=1}^N \sigma_i^\alpha$ are the collective spin-flip operators for atoms. In fact, the angular momentum operators can be represented in terms of the two mode-operators b and c as follows

$$\hat{J}_+ = bc^\dagger , \quad \hat{J}_- = b^\dagger c , \quad \hat{J}_z = \frac{1}{2}\left(c^\dagger c - b^\dagger b\right) ,$$

$$\hat{J} = \frac{1}{2}\left(c^\dagger c + b^\dagger b\right) , \quad J = \frac{N}{2} . \tag{78}$$

When operating as a PND the Hamiltonian (46) acts on input states with $n_a = n_b$: in the direct operating mode one has $\langle n_a \rangle_0 = 0$ and $\langle n_c \rangle_0 = n_i$, whereas in the inverse $\langle n_a \rangle_0 = n_i$ and $\langle n_c \rangle_0 = 0$, namely $|M| = J$ in both case: this is exactly the spontaneous emission limit for the parametric converter (as opposed to the noiseless coherent superradiant limit corresponding to $M = 0$). Thus, in conclusion, both the output noises and the nonideal effective gains are signs of the spontaneous parametric emission in the converter.

ACKNOWLEDGMENTS

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AN EXACTLY SOLVABLE MODEL OF AN OSCILLATOR
WITH NONLINEAR COUPLING AND ZEROS OF BESSEL FUNCTIONS

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Abstract

We consider a model of oscillator with nonpolynomial interaction admitting exact solutions both for energy eigenvalues in terms of zeros of Bessel functions considered as functions of the continuous index, and for the corresponding eigenstates in terms of Lommel polynomials

Let us consider the following Hamiltonian,

$$H = \omega a^{+} a + \lambda \left\{ a^{+} \left[a^{+} a + 1 \right]^{-1/2} + \left[a^{+} a + 1 \right]^{-1/2} a \right\} . \qquad (1)$$

Here a and a^{+} are usual boson annihilation and creation operators, ω and λ are positive real parameters (the generalization to complex coupling constant λ does not lead to any new result, since the phase of λ is trivially eliminated by the canonical transformation $a \to a e^{i\varphi}$ preserving the energy spectrum). If the mean number of quanta is close to zero, then (1) turns into the Hamiltonian of usual forced oscillator. In the opposite quasiclassical regime of large mean number of excitations $N = \langle a^{+} a \rangle \gg 1$ the substitution $a \approx N^{1/2} e^{i\varphi}$ leads to the energy-independent interaction Hamiltonian

$$H_{int} = \lambda \cos\varphi, \qquad (2)$$

which is in fact exact, since the expression inside the figure brackets is nothing but the Susskind-Glogower cosine phase operator [1] which properties were discussed in detail in the known review by Carruthers and Nieto [2].

Expanding the energy eigenstate $|E\rangle$ over the Fock states

$$|E\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \qquad (3)$$

and taking into account the known matrix elements of operators a and a^+ one can easily reduce the stationary Schroedinger equation to the following set of coupled linear algebraic equations,

$$Ec_0 = \lambda c_1 \, ,$$
$$Ec_n = \omega n + \lambda \left(c_{n-1} + c_{n+1} \right), \qquad n \geq 1 \qquad (4)$$

It is convenient to introduce dimensionless variables

$$z = \lambda / \omega \, , \qquad \mu = E / \omega \, . \qquad (5)$$

Then normalized energy μ is determined from the equation $\Phi(z,\mu)=0$, where function Φ is the characteristic determinant of system (4):

$$\Phi(z,\mu) = \begin{vmatrix} -\mu & z & 0 & 0 & 0 & \cdots \\ z & 1-\mu & z & 0 & 0 & \cdots \\ 0 & z & 2-\mu & z & 0 & \cdots \\ 0 & 0 & z & 3-\mu & z & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (6)$$

Expanding this determinant over the elements of the first row one can easily obtain the following recurrence relation,

$$\Phi(z, \mu) = -\mu \Phi(z, \mu-1) - z^2 \Phi(z, \mu-2) . \qquad (7)$$

Introducing new function

$$F(z,\mu) = z^{-\mu} \Phi(z,\mu) \qquad (8)$$

one can rewrite (7) as follows,

$$F(z, \mu) + F(z, \mu-2) = -\frac{2\mu}{2z} \cdot F(z, \mu-1) . \qquad (9)$$

But this is the well known relation for Bessel functions [3,4]. Consequently, the energy levels are determined by zeros of Bessel functions in accordance with the equation

$$z^{\mu} J_{-1-\mu}(2z) = 0. \qquad (10)$$

For small values of parameter z the well-known power series expansion of the Bessel function leads to the equation

$$z^{-1} \sum_{m=0}^{\infty} \frac{(-z^2)^m}{m! \, \Gamma(m-\mu)} = 0. \qquad (11)$$

For $z \to 0$ the solutions of this equation with respect to μ are determined by the poles of gamma-function. Evidently, they reproduce

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approximately the equidistant harmonic oscillator spectrum: $\mu_n \approx n$; $n = 0,1,2,\ldots$ Since all poles of gamma-function are simple, with the residues $(-1)^n/n!$, the correction to the n-th energy level has the order of $z^{2(n+1)}$:

$$\mu_0 \approx -z^2, \qquad \mu_n = n - z^{2(n+1)}/n + \ldots, \quad n \geq 1. \qquad (12)$$

Note that all corrections are negative.

For large values of the coupling constant we can use the known asymptotic formula

$$J_{-1-\mu}(2z) \approx (\pi z)^{-1/2}\cos(2z + \pi\mu/2 + \pi/4). \qquad (13)$$

Then for $|\mu| \ll |z|$ the spectrum is equidistant **again**, but with the twice distance between the neighbouring energy levels:

$$\mu_n \approx 1/2 + 2n - 4z/\pi + O(z^{-1}). \qquad (14)$$

Here n is an arbitrary integer having the same order of magnitude as the large parameter z. Note that energy values depend on the coupling constant in a specific almost periodic manner:

$$\mu_n(z) \approx \mu_{n+1}(z + \pi/2). \qquad (15)$$

Now let us look again at eq. (4). Comparing it with different recurrence relations for special functions given in [3,4], one can recognize that it is nothing but the equation for Lommel's polynomials (which are in fact polynomials with respect to $1/z$)

$$R_{n+1,\nu}(z) + R_{n-1,\nu}(z) = \frac{2(\nu + n)}{z} R_{n,\nu}(z) \qquad (16)$$

Consequently,

$$c_n^{(\mu)} = N(\mu,z) R_{n,-\mu}(-2z) = N(\mu,z)\sum_{l=0}^{n/2} \frac{(-1)^{n-l}(n-l)!\Gamma(n-l-\mu)}{l!(n-2l)!\Gamma(l-\mu)} z^{2l-n}, \qquad (17)$$

where $N(\mu,z)$ is the normalizing factor. For example, the first three coefficients $\tilde{c}_n^{(\mu)} = c_n^{(\mu)}/N(\mu,z)$ are as follows,

$$\tilde{c}_0^{(\mu)} = 1, \qquad \tilde{c}_1^{(\mu)} = \mu/z, \qquad \tilde{c}_2^{(\mu)} = \mu(\mu-1)/z^2 - 1 \qquad (18)$$

Taking into account (12) we have, e.g., for the ground state

$$\tilde{c}_0^{(0)} = 1, \qquad \tilde{c}_1^{(0)} = -z, \qquad \tilde{c}_2^{(0)} = z^2, \qquad (19)$$

In conclusion let us discuss the correspondence between the quantum problem under study and its classical counterpart described

in the energy – phase canonical variables

$$E = \tfrac{1}{2}(p^2 + q^2), \qquad \varphi = \arccos\left[q/(2E)^{1/2}\right] \tag{20}$$

with Hamiltonian

$$H = E + \lambda\cos\varphi. \tag{21}$$

Since this Hamiltonian depends *linearly* on the energy variable E, the canonical equations of motion

$$\partial E/\partial t = \partial H/\partial\varphi, \qquad \partial\varphi/\partial t = -\partial H/\partial E \tag{22}$$

can be found without difficulty for an arbitrary "potential" $f(\varphi)$:

$$\varphi(t) = -t, \qquad E(t) = E_0 + f(0) - f(\varphi). \tag{23}$$

However, in the quantum case just the "potential" $\cos\varphi$ seems distinguished. For example, if one takes instead of (2) the interaction Hamiltonian

$$\hat{H}_{int} = \lambda\left[\cos\hat{\varphi}\right]^2, \tag{24}$$

then instead of (6) and (7) one gets $(\tilde{\mu} = \mu - 2z)$

$$\Phi(z,\mu) = \begin{vmatrix} -\tilde{\mu} & 0 & z & 0 & 0 & \cdots \\ 0 & 1-\tilde{\mu} & 0 & z & 0 & \cdots \\ z & 0 & 2-\tilde{\mu} & 0 & z & \cdots \\ 0 & z & 0 & 3-\tilde{\mu} & 0 & z\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{vmatrix}, \tag{25}$$

$$\Phi(z,\mu) = -\tilde{\mu}\Phi(z,\mu-1) + z^2(\tilde{\mu}-1)\Phi(z,\mu-3) + z^4\Phi(z,\mu-4) \tag{26}$$

with unknown solution.

Although the physical meaning of the quantum model with Hamiltonian (1) is not clear at the moment (its "nearest neighbour" $H = E^2/2 + \lambda\cos\varphi$ describes the Josephson junction), we hope that due to its beauty it will find applications in future.

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NON CLASSICAL EFFECTS IN PLANAR WAVEGUIDES

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Abstract

The quantum description of ligth propagation inside a planar waveguide is given , looking in particular at the behaviour of the field inside a directional coupler . Nonclassical effects are presented and discussed.

Introduction

Electromagnetic fields in optical guided wave systems are usually described simply by using classical Maxwell's equations, but there are cases in which a quantum treatment is necessary.Three purely quantum phenomena are known having no classical analogous; namely photon antibunching, sub-poissonian photon statistics, and squeezing of optical fields. If problems

connected with these phenomena or the evolution of photon statistics are to be dealt with,a quantum mechanical treatment must be used.

Of course a linear system is not able to produce or change these properties but a nonlinear one is. For this reason,in the following the Hamiltonian for a nonlinear optical waveguide will be derived,and its application to some propagation problems will be considered.Although the Hamiltonian is quite general,emphasis is given to planar structures only and a more suitable approach to describe propagation fenomena is discussed.

One of the results of having the propagation problem treated in quantum mechanical form is to allow for the possibility of studying how purely quantum effects propagate in linear systems.We will show ,for example,that a quantum effect as squeezing is affected by the operation of switching in a linear structure because of the phase changes involved in the operation.

2. Quantization of the radiation field

The recent experiments on nonclassical states of light have called for a full quantum analysis of the electromagnetic field [1] especially in the cases of propagation of the fields inside dipersive media.

We remember that the standard quantization method consists of writing the Hamiltonian in a given volume V , demanding periodicity in space. For propagating fields, the space evolution is then replaced by a time evolution, by linking the space and time variables by the equation $z = ct$. The length of the nonlinear medium is then replaced by an effective interaction time. Of course this method has two main limitations. The firt one is that , by identifying the space evolution with time evolution we lose one variable and this formalism can describe only c.w. operation : the second problem is that this procedure cannot

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be applied rigorously to a dispersive medium, where each frequency propagates with different velocity.

However we remember that by using the Hamiltonian formalism and working in the Heisenberg picture, the time evolution of $\hat{E}(z,t)$ operator is given by

$$i\hbar \frac{\partial \hat{E}(z,t)}{\partial t} = [\hat{E}(z,t), \hat{H}] \quad ,$$

(1)

so that the generator for time evolution is the Hamiltonian $\hat{H}$, while the generator for space propagation is the momentum operator $\hat{G}$:

$$-i\hbar \frac{\partial \hat{E}(z,t)}{\partial z} = [\hat{E}(z,t), \hat{G}]$$

(2)

The $\hat{G}$ operator is related to the wave flux of the Poynting vector [2].

Therefore a suitable way to quantize the radiation field to describe the propagation fenomena is the one starting from the flux of the Poynting vector. This leads us to the realization that the important quantity is the flux and not , as usually is assumed with the Hamiltonian formalism , the energy density.

In this way instead of quantizing the field in a large volume and demanding for spatial periodicity , it is necessary to assume a time periodicity T of the field , with the requirement that T must be large with respect to any relevant time . Then instead of writing the field in term of spatial modes (thus performing a Fourier analysis of the space variable z into the wave vector K_m) it is possible to write it in term of temporal modes (thus performing a Fourier analysis of the time variable t into discrete frequencies ω_m) and space dependent operators. The advantage is that the temporal modes remain the same inside and outside the dielectric medium [3] and the space evolution of the mode operator can now be obtained by means of the momentum operator; moreover dispersion of the material can be included.

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By the help of this formulation the expression of the electric field is (inside a dielectric)

$$\hat{E}(z,t) = \hat{E}^+(z,t) + \hat{E}(z,t),$$

where the cross stands for c.c. and

$$\hat{E}^+(z,t) = \sum_m [\frac{\hbar \omega_m}{2\varepsilon_0 c T n(\omega)}]^{1/2} [\hat{a}(z,\omega_m)\exp(-i\omega_m t)] \tag{3}$$

being $\hat{a}(z,\omega_m)$ and their conjugates form a set of localized creation and the annihilation operators , ω_m the field frequency , $n(\omega)$ the refractive index at the ω frequency , ε_0 the dielectric constant and c the ligth velocity.
The number operator for the field becomes

$$\hat{N}(z_0,\omega_m) = \hat{a}^+(z_0,\omega_m)\,\hat{a}(z_0,\omega_m) \tag{4}$$

which represents the number operator of the photons of frequency ω_m passing through the plane $z = z_0$ during a period T , and the commutation rules now become commutation at "equal space":

$$[\hat{a}(z,\omega_i),\hat{a}^+(z',\omega_j)] = \delta_{i,j}e^{jk_i(z-z')} \tag{5}$$

and the $\hat{G}$ operator is defined as

$$\hat{G}(z) = \sum_m (\hbar k_m)\,\hat{a}^+(z,\omega_m)\hat{a}(z,\omega_m) \tag{6}$$

where $k_m = n(\omega_m)\frac{\omega_m}{c}$ is the wavevector of the field.

3. Quantum mechanical description of propagation in a planar waveguide

A planar dielectric waveguide is a medium whose dielectric permittivity depends on one direction,parallel to which we shall assume the x-axis (s.Fig.1).

If this medium does not contain absorbing centers,if there is no amplification of radiation,and if the permittivity is weakly dependent on the field frequency ω_l ,the electromagnetic field inside the guide is expressed in terms of normal modes in the following form [4,5]

$$A (r) = \sum_j A_0 f (x) \exp(i \beta_j .r),\tag{6}$$

where β_j is the wave-vector with components y and z $(\beta_j.r= k_y\ y + k_z\ z)$ of the j-th mode propagating inside the waveguide and $f(x)$ is a function dependent only on x, defined over all space, and determined by the waveguide structure. Therefore each guided mode is defined by a β_j vector at each ω_j frequency.

From the quantum thery point of view (as pointed out in the previous paragraph ,if the operator describing the field mode in a free space is given by (note the operator is the one that obeys at the equal space commutation rules)

$$\hat{a}(z,\omega_l) = \hat{a}(z)e^{-i\omega_l t} = \hat{a}(0)\ e^{-ik_z z}e^{-i\omega_l t}\tag{7}$$

in a guided strucure it can be described as

$$\hat{a}(z,\omega_l) = \hat{a}(z)e^{-i\omega_l t} = \hat{a}(0)\ e^{-\beta_j z}e^{-i\omega_l t}\tag{8}$$

and eq. (6) written in operatorial form becomes

$$\hat{A}(r,t) = \sum_j A_0 [\hat{a}(z,\omega_j) \exp(-i\omega_j t) + c.c.]$$ (9)

where A_0 is a constant .
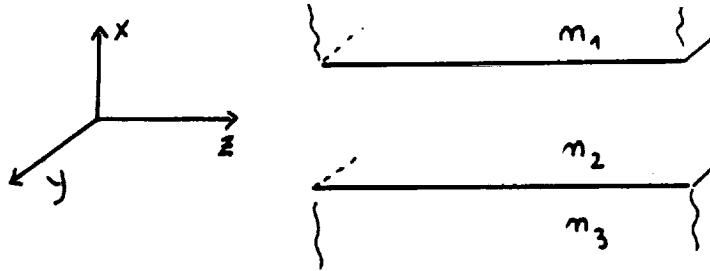
Fig.1 A planar waveguide

We would like to remark that in the case of dispersionless material the time evolution of the field operators(Heisenberg equation) is the same as the one in space , i.e. the Hamiltonian operator and the momentum operator approach provide the same results. This remark is particularly important when we consider the quantum treatmen of a guided mode inside a guiding structure, due to the fact that in general we propagate different spatial modes of equal frequency and we are not obliged to take into account dispersion of the medium if we assume a c.w. propagation.

The same kind of consideration is still valid if we study the propagation of single or more modes inside a nonlinear planar waveguide with third order nonlinearity. These cases have been extensively studied in the paper [5].

In the following we analyze the case of propagation in a directional coupler , which is one of the most interesting guiding devices , very important from the point of view of its switch properties.

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4. Behaviour of a linear directional coupler when nonclassical states are involved (squeezing)

The directional linear coupler consists in two adjacent and parallel waveguides [5,6](channels).When radiation goes through the structure,exchange of power between the channels is possible because of the evanescent field which is present in the region between them.

In the frame of classical theory the coupler is studied by using the coupled mode theory [5,6],in which a perturbation polarization responsible for the coupling contains the refractive index of the guides. Complete power transfer occurs in a distance L = (π /2)K , where K is the coupling constant determined by the refractive indices of the structure; if the detuning parameter δ is zero,that is in the case of complete phase matching [6] ,being

$$\delta = \frac{1}{2}(\beta_a - \beta_b) \tag{10}$$

where β_a and β_b are the wavevectors of two modes of equal frequency propagating in channel a and b respectively .If δ is not zero the maximum fraction of power that can be trasferred is proportional to $\frac{K^2}{K^2 + \delta^2}$.

From the classical equations for the complex amplitude for the directional coupler,in the frame of the coupled-mode theory,with obvious generalization we have the following Heisenberg equations for the operators,

$$\begin{cases} d\hat{a}/dz = -iK\hat{b}\exp(i2\,\delta\,z), \\ \\ d\hat{b}/dz = -iK\hat{a}\exp(-2i\,\delta\,z), \end{cases} \tag{11}$$

where $\hat{a}$ and $\hat{b}$ are the field annihilation operators in channel a and b respectively .For the sake of simplicity we neglect damping terms because we are interested in the coupling effect only;this is a good approximation in the region of low temperature and optical frequencies. Without taking into account dispersion the set of eqs. (11) is the same that we can write startimg from the momentum operator with the substitutions t -> z/c and $\hat{H}$ -> c $\hat{G}$.

In this way we get the following solutions of eqs.(11)

$$\hat{a} = C_a \hat{a}_0 + G_a \hat{b}_0 \qquad\qquad \hat{b} = C_b \hat{b}_0 + G_b \hat{a}_0 \qquad (12)$$

where $\hat{a}_0$ and $\hat{b}_0$ are the input annihilation operators and

$$C_a = e^{i\delta z}[cos(\gamma z) - i\delta/\gamma \, sin(\gamma z)]$$

$$C_b = e^{-i\delta z}[cos(\gamma z) + i\delta/\gamma \, sin(\gamma z)]$$

$$G_a = -i K/\gamma \, sin(\gamma z) \, e^{i\delta z}$$

$$G_b = -i K/\gamma \, sin(\gamma z) \, e^{-i\delta z}$$

where $\gamma^2 = K^2 + \delta^2$.

To study the propagation of nonclassical field through the structure we use the following characteristic function

$$C_N(\beta) = Tr(\rho \, exp[\beta \hat{a}^+] \, exp[-\beta^* \hat{a}] = exp[-M|\beta|^2 + \frac{1}{2}(S^*\beta^2 + S\beta^{*2}) + \beta W^* - \beta^* W]$$

$$(13)$$

which is able to describe a field which is not a pure coherent or squeezed state,but has simultaneously squeezed ,coherent and chaotic features [7].In eq.(13) W = W exp(i ϕ) is the coherent signal,and M and S are related to the

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noncoherent part of the field.So for the vacuum state we have M= S= W= 0 ;in the pure coherent state M=S=0, and for the chaotic field W=S=0.

The state is a pure squeezed state if

$$M = Q = 0.5((4 S + 1) - 1)$$

$$S = \exp(i \Phi) \cosh(r) \sinh(r) \tag{14}$$

r being the squeezing parameter [13].

A mixed state is given from a superposition of a pure squeezed state with coherent signal W and a cahotic field described by the normally ordered characteristic function given by eq.(13) if

$$M = Q + N, \tag{15}$$

where N is the noise photon number.

We shall suppose that the input statistics of light in both modes can be described by the normally ordered characteristic function (13).Putting solutions (12) into eq.(13) we can see that the truncated normally ordered output characteri- stic functions will have the same functional form as the input ones with new terms

$$M_a = M_a^{(0)} |C_a|^2 + M_b^{(0)} |G_a|^2$$

$$S_a = S_a^{(0)} C_a^2 + S_b^{(0)} G_a^2$$

$$W_a = W_a^{(0)} C_a + W_b^{(0)} G_a \tag{16}$$

where the superscript (o) labels the input quantities,and similar expressions can be found for the b-mode,by interchanging the subscripts a and b.

We are interested in finding expressions for the variances $< (\Delta Q)^2 >$ and $<(\Delta P)^2 >$,(where $\hat{Q} = \hat{a} + \hat{a}^+$ and $\hat{P} = -i(\hat{a} - \hat{a}^+))$ .In ref.(14) it is shown that

$$< (\Delta Q)^2 > = 1 + 2M + S + S^*$$
$$< (\Delta P)^2 > = 1 + 2M - S - S^* \qquad (17)$$

Several interesting cases can be considered which depend on the way the coupler is feeded.

Let us suppose first that a pure squeezed state enters channel b and a coherent state (or vacuum) channel a. It can be shown in this case that for $L = \pi /2K$ and $\delta = 0$ we have

$$< (\Delta \hat{Q})^2 >_a = < (\Delta \hat{P})^2 >_{b0} , < (\Delta \hat{P})^2 >_a = < (\Delta \hat{Q})^2 >_{b0} ,$$
$$< (\Delta \hat{Q})^2 >_a = < (\Delta \hat{P})^2 >_b = 1.$$

$$(18)$$

This means that at the output of channel a we have an opposite squeezing than at the input of channel b,while the output in channel b shows no squeezing.

A related situation is obtained when two opposite squeezed fields enter the two channels in the same conditions as in the previous case.In this case squeezing is preserved in both channels because the field entering channel a comes out of channel b with opposite squeezing and the same happens with field entering channel b.At intermediate lengths of the coupler the squeezing is not completely preserved.

A very interesting role is played by the detuning parameter . In general [14] if $\delta \neq 0$ some noise is added to both channels and squeezing is reduced, and

for some special values of δ, noise is absent. Let us consider for example the case in which $\delta = \sqrt{3}$ K. . In this case $\gamma = 2k$ and for the same coupler length $L = \pi /2k$ if a squeezed field entered channel b and a coherent or vacuum field channel a then we have at the output

$$< (\Delta \hat{Q})^2 >_a \ = \ < (\Delta \hat{P})^2 >_a = \ 1.$$
$$< (\Delta \hat{Q})^2 >_b \ = \ < (\Delta \hat{Q})^2 >_{b0}.$$
$$< (\Delta \hat{P})^2 >_b \ = \ < (\Delta \hat{P})^2 >_{b0}. \qquad . \tag{19}$$

We can see that changing the detuning parameter from zero to $\sqrt{3}K$ switches from one channel to another.This result is rather interesting for the purpose of measurement.The squeezed state is detected by interfering it with a coherent reference light and looking at fluctuations.The switching behaviour just described allow to preserve both the squeezed state and its reference beam.

5 Directional and contradirectional coupler with modes with small different frequency propagating inside

We have studied the problem of propagation of radiation in a coupler assuming two different frequencies inside the channels , with the hypothesis that each channel can support one only guided mode : this is possible if the two frequencies are quite similar. In general for a coupler the more realistic description of the field propagating inside all the structure is the one which takes into account the superposition of the single modes propagating in each channel (so called supermodes); in the case of different frequencies this approach is particularly convenient and it is the one that we have adopted but in its quantum analogous, i.e. introducing this concept in the statistical dependence of the modes supported by the structure.

The motion equation of the operator describing the propagation inside the structure is given by (Heisenberg form)

$$\frac{\partial \hat{a}_j(z)}{\partial z} = \frac{i}{h} [\hat{a}_j(z), \hat{G}(z)] , \qquad (20)$$

where j is the index corresponding to the mode (j= 1,2) and this equation is related to the momentum operator $\hat{G}$, that for this case is given by

$$\hat{G}(z) = h \sum_{j=1}^{2} k_j \hat{a}_j^\dagger (z) \hat{a}_j(z) + h (\chi^* \hat{a}_1^+ (z) \hat{a}_2(z) + h.c.), \qquad (21)$$

where $k_j = \beta_j = \frac{n_{eff}}{\lambda_j}$, is the mode wave vector , χ is the coupling constant , which depends on the refractive index $n(\omega_j)$ distribution inside the coupler . It is very interesting to observe that the $\hat{G}$ operator looks like the one of a second order nonlinearity for a bulk material.

Using the approach of the supermodes we can describe the two fields of different frequency supported by the strucure as

$$| 1 \rangle = 2^{-1/2} (| 1_{in} \rangle + u_{11} | 1_{out} \rangle + u_{21} | 2_{out} \rangle)$$

$$| 2 \rangle = 2^{-1/2} (| 2_{in} \rangle + u_{12} | 1_{out} \rangle + u_{22} | 2_{out} \rangle) \qquad (22)$$

being $u_{j,k}$ a function related to the transformation law of the coupler , containing all the informations about the structure , such as the coupling constant , the detuning parameter , etc. (see functions C $_{a,b}$ and G $_{a,b}$ of the provious paragraph).

To follow the statistics of the field we start from the characteristic function (the antinormal one) from which it is possible to derive all the factorial moments and the photon counting distribution. As in the previous paragraph we suppose the input state is a superposition of a coherent state and noise , including squeezing .

Due to to the hypothesis of the supermode we can write the charateristic function for all the fields β_1 and β_2 , $C_A(\beta_1,\beta_2)$, but we can follow also the behaviour of each separate mode $C_A(\beta_j)$ [8] :

$$C_{A_{in}}(\beta_1,\beta_2) = \exp\left\{\sum_{j=1}^{2}\left[- B_{j_{in}} | \beta_j |^2 + \frac{1}{2}(C^*_{j_{in}} \beta_j^2 + c.c.) + (\xi^*_{j_{in}} \beta_j - c.c.) + \right.\right.$$

$$\text{(noise)} \qquad \text{(squeezing)} \qquad \text{(coherent)}$$

$$\left.\left.+ (- B_{12_{in}}\beta_1^* \beta_2 + C^*_{12_{in}} \beta_1\beta_2 + c.c.)\right]\right\}.$$

$$\text{(interference of noise)} \qquad\qquad\qquad (23)$$

The output characteristic function is of the same form as the input, where all the features of the coupler are inside the B and C coefficients of the eq.(23).

Several cases of inputs states have been studied, such as coherent , two-photon coherent , two-mode squeezed states and all factorial moments have been calculated [8] finding as the detuning parametr plays a very important role on the evolution of the fileds : it adds addidional noise if it is non zero [8].

It is interesting also to follow the photon counting distribution which put into evidence the switch properties of the structure always starting from the hypotesis of supermode supported by the coupler . An example is shown in Fig.2 , where the detuning parameter δ is zero, the input state in the first channel is a two photon coherent state in the first mode , with a " small amount of squeezing" ,and a coherent state in the other mode . The picture shows the marginal photon number distribution in the channel 2 ; at a suitably distances the sub-Poissonian behaviour turns super-Poissonian, which characterizes the field in the squeezed vacuum state: this confirms the switching of ligth of certain photon statistics from one mode to the other one.
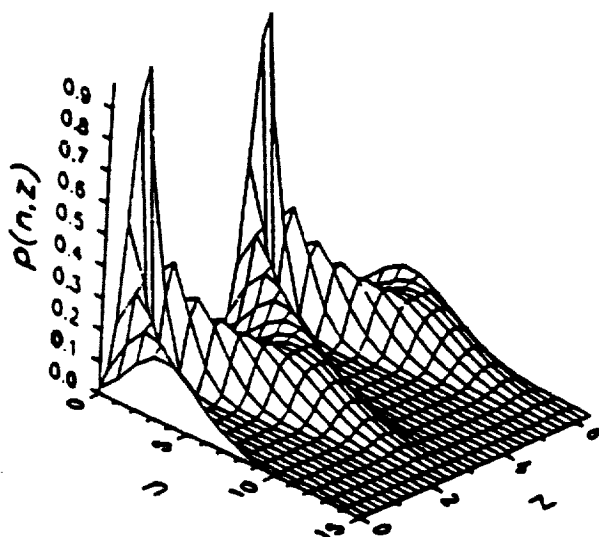
Fig.2 - The marginal photon number distribution for δ =0 and K =i for the
channel 2

Conclusions

The propagation of ligth in a linear directional coupler can be studied without
taking into account the dispersion of the dielectric constant until c.w. field
propagation is considered; of course dispersion must be taken into account in
non stationary cases and when the structure of the propagating device supports
different frequencies.

The ability of the coupler to switch from one channel to the other by introducing a
phase lag allows to change the squeezing directions, until the δ parameter
is of suitably values; in general a detuning different from zero reduces the switch
properties of the coupler and adds additional noise to the propagating fields. This
effect is in turn evident also on the photon counting distribution.

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NONPERTURBATIVE METHODS IN THE PROBLEM OF MULTIPHOTON EXCITATION OF ATOM BY SQUEEZED LIGHT

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Abstract

The multiphoton detectors for the strong squeezed light vacuum are considered. Te result is compared with the perturbation theory. It is shown that as the degree of squeezing is increased the statistical factor decreases.

Multiphoton transitions in atoms due to squeezed light were analyzed for the first time by Yansky and Yushin [1] by using perturbation theory. On the other hand, at present parametric generators of squeezed light are discussed [2]. They allow us to obtain high density of photons $N \sim 10^{20} - 10^{21}$ in resonator with volume $V \sim 1 cm^3$ for stored energy density $\geq 1J$. Although experimentally such photon densities are not reached, it is of interest to describe physical processes in atoms interesting with intensive squeezed light. For the squeezed vacuum $|0>_s$, as is known, $N =_s< 0|a^+a|0>_s = |\nu|^2(a^+(a)-$ are operators of appearing and disappearing of quantum of electromagnetic field), $\nu = |\nu|e^{i\phi}$ is squeezing parameter of Stoler unitary transformation [3, 4] of operators $a^+(a)$ to the new variables of squeezed field $b^+(b)$:

$$
\begin{aligned}
b &= \mu a + \nu a^+ \\
b^+ &= \mu^* a^+ + \nu^* a; |\mu|^2 - |\nu|^2 = 1
\end{aligned}
\tag{1}
$$

For the squeezing degree $\nu \sim 10^{10} - 10^{11}$ the criteria for application of perturbation theory methods are not satisfied. In fact, let us coincides two level system with nonzero average dipole moment d in the excited state (2) (neglect for simplicity the dipole moment in the ground state (1)). The characteristic theory parameter ρ appearing due to multiphoton transition on the degenerate level (2) has the form [5]

$$
\rho = Fd/\hbar\omega
\tag{2}
$$

where F is the amplitude of the intensity of electromagnetic field with frequency ω. Parameter $\rho \geq q_0$ (q_0 is the number of photons participating in the transition) is reached for $N \sim 10^{20} - 10^{21}(q_0 \sim 3 - 5, d \sim 10D)$.

In the paper [6] the statistical factor $\chi_{(G\delta)} = W^{(G)}/W^{(\delta)}$ was calculated for the multiphoton transition on the degenerated level of hydrogen atom for the source of gauss electromagnetic field (G) and pure coherent source (δ). It was shown that with the increase of radiation intensity the difference in statistical properties of multiphoton excitation of atom disappear. The expression was

received for the probability of coherent multiphoton transition in the presence of probe radiation with intensity $\mathcal{F}$ and frequency $\Omega \gg \omega$:

$$W^{(\delta)}(\Omega) = \frac{d_{12}^2 \mathcal{F}^2}{\hbar^2 \gamma} J_{q_0}^2(\rho) \tag{3}$$

where $q_0 = (\Delta - \hbar\Omega)/\hbar\omega$, Δ is electron excitation energy, γ is damping constant of excited electron state, $J_n(x)$ is the Bessel function of real argument. Using the formula (3) gives us methodical advantage because it permits to realize the rearrangement of multiphoton process with the frequency of probe radiation. Let us consider the statistical factor $\chi_{(S\delta)} = W^{(S)}/W^{(\delta)}$, where W^S is the transition probability under the action of squeezed light. S - matrix formalism is used for calculating W^S. Confining to the second order of perturbation theory on the probe radiation. We have:

$$W^S(\Omega) = \frac{d_{12}^2 \mathcal{F}^2}{\hbar^2} \int_{-\infty}^{+\infty} dt \exp[iq_0\omega t - \gamma t] I^*(t) \tag{4}$$

where $I^*(t)$ is generating function of transition probability:

$$I^*(t) = <G(t)>,$$

The evolution operator $G(t)$ satisfied the motion equation:

$$i\hbar \dot{G}(t) = [g(t)a + g^*(t)a^+]G(t); G(0) = 1$$

$$\tag{5}$$

$$g(t) = ive^{-i\omega t}; v = d_{22}(2\pi\hbar\omega/V)^{1/2}$$

The brackets $< \cdots >$ in (5) denote the averaging over squeezed state, d_{22} in (5) is dipole moment in electronic state (2), $d_{22} \approx 10e_0a_0$ for the level with the main quantum number $n = 3$(a_0 is the Bohr radius, e_0 is the electron charge). The solution to (5) may be presented in the following normally ordered form [7]:

$$G(t) = e^{A(t)}e^{-B^*(t)a^+}e^{B(t)a}$$

$$B(t) = -\frac{i}{\hbar}\int_0^t d\tau g(\tau) \tag{6}$$

$$A(t) = -\frac{1}{\hbar^2}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 g(\tau_1)g^*(\tau_2)$$

Let us use back transition to (1):

$$a = \mu^*b - \nu b^+$$
$$a^+ = \mu b^+ - \nu^*b$$

With the Backer - Hausdorff transformation

$$e^A \cdot e^B = e^{A+B} \cdot e^{\frac{1}{2}[A,B]}$$
$$[[A, B], A] = [[A, B], B] = 0$$

it is easy to recieve the following expression for the generating function $\Gamma^{(s)}$ of quantum transition under the action of squeezed light

$$\Gamma^{(s)}(t) = \Gamma_0^{(s)}(t), \, <\beta|e^{-(B\nu+B^*\mu)b^+} \cdot e^{(B^*\nu^*+B\mu^*)b}|\beta>,$$

$$\Gamma_0^{(s)}(t) = \exp\left\{-|\nu|^2|B|^2 - \frac{1}{2}(B^2\mu\nu^* + c.c.)\right\} \tag{7}$$

The value β characterizes the initial coherent state $|\beta>$. In the case of squeezed vacuum we have $\Gamma^{(s)}(t) = \Gamma_0^{(s)}(t)$. The received exact expression for the generating function $\Gamma^{(s)}(t)$ does not permit to make analytical calculation of the transition probability and creates certain difficulties for numerical calculations. This expression differs from the known formulas in [7,8] obtained in perturbation theory in two positions. Firstly, in (7) the reemitting of photons is taken into account, secondly, anomaly correlation functions with nonequal number of operators a and a are not discarded. The first condition for the strong field is strictly necessary. The second condition may be used for both weak and strong fields, as will be shown below. Taking into consideration the remarks let us simplify the common expression for the transition probability. Present formula (6) in antinormal form and rewrite $\Gamma_0^{(s)}(t)$:

$$\Gamma_0^{(s)}(t) = e^{-|B|^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}|B|^{2m} , \, <0|a^m a^{+m}|0>, +$$

$$+ \, e^{-|B|^2} \sum_{m\neq n}^{\infty} \frac{(-1)^m}{m!n!} B^{*m} B^n , \, <0|a^n a^{+m}|0>, \tag{8}$$

The presentation of the evolution operator $G(t)$ in antinormal form is caused by simplicity of calculations, for example:

$$, <0|aa^+|0>,= 1 + |\nu|^2 = |\mu|^2 = N + 1$$

The last term in (8) is the contribution of anomaly correlation functions and do not gives the contribution in multi-photon processes. Thus, we leave the first member in (8). We find:

$$\Gamma_0^{(s)}(t) = e^{-|\nu|^2|B|^2} \cdot I_0(|B|^2\mu|\nu|) \tag{9}$$

where $I_0(x)$ is modified Bessel function. Let us consider the photon density $|\nu| \gg 1$ corresponding to perturbation theory. In this case (see Appendix) it may be shown that statistical factor $(\chi_{(s\delta)})$

$$\chi_{(s\delta)} = \frac{W^{(s)}}{W^{(\delta)}} = (2q_0 - 1)!! \tag{10}$$

This result coincides with the known conclusion in [1]. In Fig.1 the calculation of statistical factor $\chi_{(s\delta)}$ in nonperturbative approach is given. Dashed line corresponds to perturbation theory. For comparison the same Fig.1 gives the statistical factor $\chi_{(G\delta)}$. Naturally, near field intensity which corresponds to the suppression coherent multiphoton excitation effect [10], the statistical factor increases drastically, which creates additional possibilities for experiment.
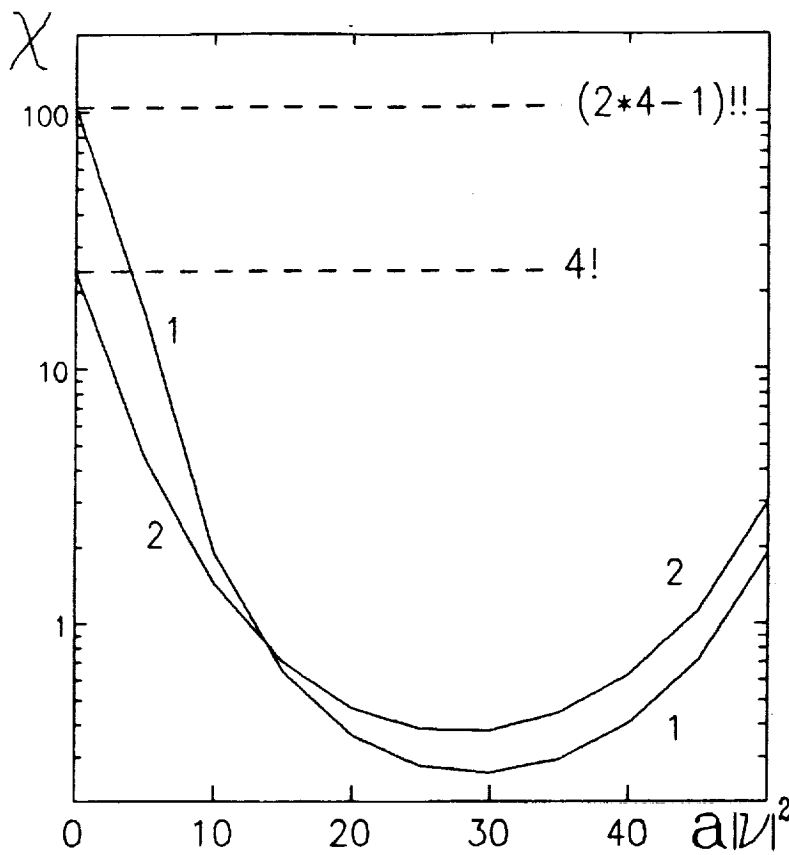
Fig.1. Dependence of χ on $a|\nu|^2$. Curve 1 corresponds $\chi_{(s\delta)}$; 2 -$\chi_{(G\delta)}$

Appendix

Let us use the expression for the multiplication of coherent state $|\alpha>$ and squeezed state $|\beta>$,

$$< \alpha|\beta >_s = \frac{1}{\sqrt{\mu}} \exp\left\{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 + \frac{\nu}{\mu}\alpha^{*2} + \frac{\nu^*}{\mu}\beta^2) + \frac{1}{\mu}\alpha^*\beta\right\}$$

Rewrite A_{mm} as:

$$
\begin{aligned}
A_{mm} &= \frac{1}{\pi}\int d^2\alpha |\alpha|^{2m}|_s <0|\alpha>|^2 = \\
&= \frac{1}{\mu}\int_0^\infty dx e^{-s} x^m I_0(x\frac{|\nu|}{\mu})
\end{aligned}
$$

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$(d^2\alpha = d(Re\alpha)d(Im\alpha)$ - is the measure of integration in complex plane α. So, the generating function may have the form:

$$\Gamma^{(s)}(t) = \frac{1}{\mu}\int_0^\infty dx e^{-x} I_0(x\frac{|\nu|}{\mu})J_0(2|B|\sqrt{x})$$

After calculating this integral we obtain the generating function (9). Let us use the summation formula for the Bessel function [11]:

$$J_0(2a\sin x/2) = \sum_{k=-\infty}^{+\infty} J_k^2(a)e^{ikx}$$

Let write the expression for the multiphoton transition probability $W^{(s)}$ under the action of the squeezed light:

$$W^{(s)} \sim \frac{1}{\mu}\sum_m F_m\delta(q_0 - m) \simeq \frac{1}{\mu}F_{q_0},$$

here we denote:

$$F_{q_0} = \int_0^\infty dx e^{-x} I_0(x\frac{|\nu|}{\mu})J_{q_0}^2(2\sqrt{ax})$$

$$a = v^2/(\hbar\omega)^2$$

The last integral is known [11]. We receive:

$$W^{(s)} \sim \frac{1}{\mu}\frac{a^{q_0}}{(q_0!)^2}\sum_{k=0}^\infty \frac{1}{2^{2k}(k!)^2}(\frac{|\nu|}{\mu})^{2k}(2k+q_0)! \times$$
$$\times \ _2F_2(q_0 + 1/2, q_0 + 2k + 1; q_0 + 1, 2q_0 + 1; -4a),$$

where $_2F_2$ is the common hipergeometrical series. At $a \ll 1$, $_2F_2 \sim 1$. We use the integral representation for the factorial. It is possible to sum up the series:

$$W^{(s)} \sim \frac{(a\mu)^{q_0}}{q_0!}\cdot P_{q_0}(\mu)$$

$P_{q_0}(\mu)$ is the Legandre polynomial. In the approximation $|\nu| \gg 1$ ($\mu \gg 1$). Let us use the asymptotic expression [11]:

$$P_{q_0}(\mu) \approx \frac{(2q_0 - 1)!!}{q_0!}\mu^{q_0}$$

We receive:

$$W^{(s)} \sim \frac{(a\mu^2)^{q_0}}{(q_0!)^2}(2q_0 - 1)!! = W^{(\delta)}(2q_0 - 1)!!$$

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SQUEEZING IN PHASE-CONJUGATED RESONANCE FLUORESCENCE

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Emission of resonance fluorescence by an atom near the surface of a four-wave mixing phase conjugator is considered. The dipole radiation field, regarded as a Heisenberg-operator field, is decomposed into plane waves with the aid of Weyl's representation of the Green's function for the wave equation. Each plane-wave component which is incident on the surface of the nonlinear medium, is reflected as its phase-conjugate image. Summation of all reflected plane waves then yields the phase-conjugate replica of the incident dipole radiation. This field adds to the radiation which is emitted by the atom into the direction away from the medium. The condition under which squeezing occurs in the emitted resonance fluorescence is investigated.

I. INTRODUCTION

Squeezing in resonance fluorescence from a two-state atom was first considered by Walls and Zoller.[1] They derived conditions on the optical parameters for which the emitted radiation would exhibit squeezing, and it appeared that only for a very limited range of the parameters squeezing could occur. On the other hand, squeezed states of the free electromagnetic field can be generated through four-wave mixing as two-photon coherent states.[2] In this paper we consider a combination of these two processes: a two-state atom with transition frequency ω_0 is close to the surface of a four-wave mixer in the phase conjugation setup. The nonlinear transparent crystal is pumped by two counterpropagating laser beams with frequency ω_p, as shown in Fig. 1. Then, an incident plane wave with frequency ω is reflected as a wave with frequency $2\omega_p - \omega$, and this wave counterpropagates the incident wave. This device will be referred to as a phase conjugator (PC). When an atom in the neighborhood of this PC emits fluorescence, then part of this radiation will be incident on the PC, and will be reflected as its phase-conjugate replica. The total radiation field then is the sum of regular fluorescence, which is emitted directly into the direction of the detector, and the phase-conjugate image of the incident field. In addition, we shall assume that the atom is

driven by a laser with frequency ω_L, and this field propagates parallel to the surface of the crystal.
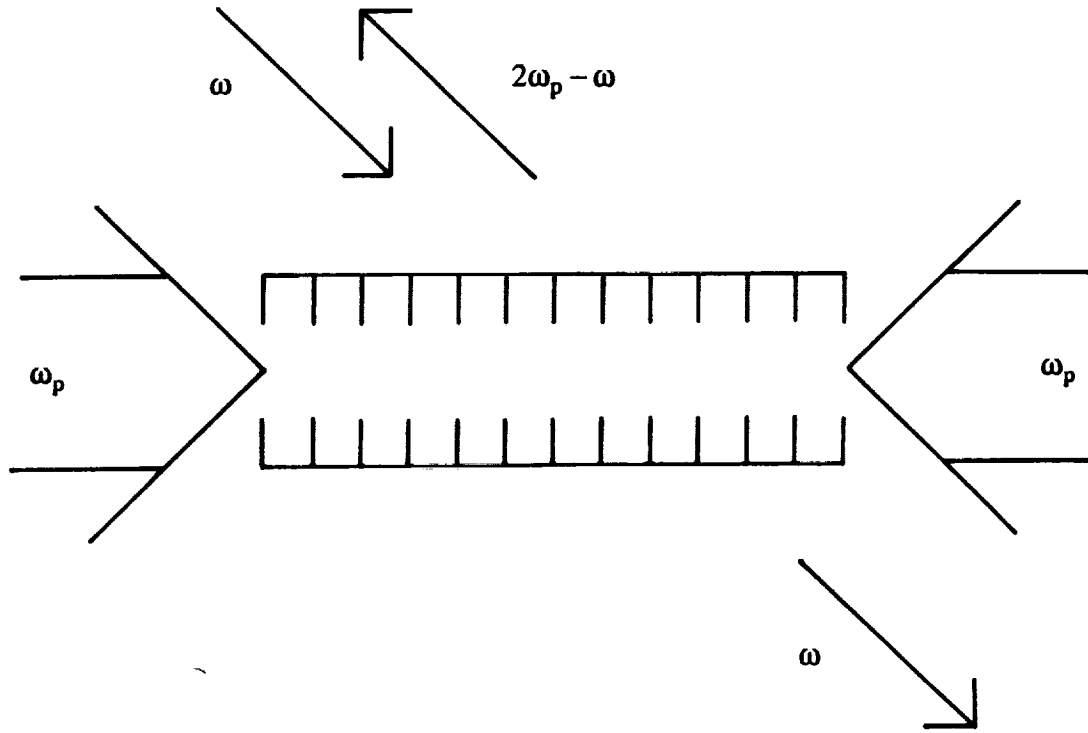
Fig. 1. Geometry of a four-wave mixing phase conjugator.

II. DIPOLE RADIATION

An electric field $\vec{E}(\vec{r},t)$ has a Fourier transform, defined as

$$\hat{\vec{E}}(\vec{r},\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, \vec{E}(\vec{r},t) \tag{1}$$

In terms of this transform, the positive-frequency part of $\vec{E}(\vec{r},t)$ is defined as

$$\vec{E}(\vec{r},t)^{(+)} = \frac{1}{2\pi} \int_{0}^{\infty} d\omega\, e^{-i\omega t}\, \hat{\vec{E}}(\vec{r},\omega) \tag{2}$$

and the total field can then be written as

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r},t)^{(+)} + \text{H.c.} \tag{3}$$

Here, the field is a quantum operator field, and the t-dependence signifies the Heisenberg picture.

For a (quantum) dipole $\vec{\hat{\mu}}(t)$, with Fourier transform $\vec{\hat{\mu}}(\omega)$, which is located at position $\vec{h}$, the Fourier transform of its electric field is given by

$$\vec{\hat{E}}_p(\vec{r},\omega) = \frac{1}{4\pi\varepsilon_o}\{k^2\vec{\hat{\mu}}(\omega) + [\vec{\hat{\mu}}(\omega)\bullet\nabla]\nabla\}\frac{e^{ik|\vec{r}-\vec{h}|}}{|\vec{r}-\vec{h}|} \quad , \tag{4}$$

with $k = \omega/c > 0$. The subscript p indicates that this field is the particular solution for a dipole in empty space. We shall assume that the plane $z = 0$ is the surface of the medium, and that the atomic dipole position vector is given by $\vec{h} = h\vec{e}_z, h > 0$. In order to obtain the field reflected by the PC, we expand the dipole field into plane waves. Then for each wave its phase-conjugate image is a counterpropagating wave, multiplied by the appropriate Fresnel coefficient, and shifted in frequency according to the rule of Fig. 1. The decomposition of the field $\vec{\hat{E}}_p(\vec{r},\omega)$ is accomplished by using Weyl's representation of the Green's function for the scalar wave equation:

$$\frac{e^{ik|\vec{r}-\vec{h}|}}{|\vec{r}-\vec{h}|} = \frac{i}{2\pi}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta\frac{1}{\gamma}e^{i\alpha x+i\beta y+i\gamma|z-h|} \quad , \tag{5}$$

where γ is given by

$$\gamma = \begin{cases} \sqrt{k^2-\alpha^2-\beta^2} \\ i\sqrt{\alpha^2+\beta^2-k^2} \end{cases} \tag{6}$$

It is understood that we take the form for which the argument of the square root is positive. When we substitute (5) into (4) and carry out the ∇ operations, then the result is the desired expansion into plane waves. The polarization of the waves is determined by the dipole operator, and this has to be decomposed into surface- and plane polarization components. The details of this lengthy calculation can be found in Refs. 3 and 4. Furthermore, we have to make an asymptotic expansion in order to find the field in the radiation zone. This was done with the method of stationary phase.[5] Subsequently, the inverse Fourier transform has to be calculated, to obtain the positive-frequency part of the field. The final result for the radiation field, evaluated at the position of a detector, located under an angle θ with the normal to the surface, is

$$\vec{E}(\vec{r},t)^{(+)} = \frac{\omega_o^2 e^{-i\omega_o\tau}}{4\pi\varepsilon_o rc^2}\{\vec{M}-\hat{r}(\hat{r}\bullet\vec{M})\} \tag{7}$$

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Here, $\tau = (h/c)\cos\theta$, ω_o is the atomic transition frequency, and the Heisenberg operator $\vec{M}(t)$ is given by

$$\vec{M}(t) = \vec{\mu}(t)^{(+)} - P^* e^{-2i\omega_p t} \vec{\mu}(t)^{(-)} \qquad (8)$$

The positive-frequency part of the dipole operator is proportional to the atomic lowering operator, and the negative-frequency part is proportional to the raising operator.

III. DRIVEN ATOM

Now assume that the atom is irradiated by a nearly-monochromatic laser beam, with an electric field of the form

$$\vec{E}_L(t) = E_o \, \mathrm{Re}\, \vec{\epsilon}_L \, e^{-i(\omega_L t + \phi(t))} \qquad (9)$$

The phase $\phi(t)$ is a random process, which accounts for the laser linewidth. We take the phase to be the independent-increment process, leading to a Lorentzian laser lineshape with a width equal to λ. This field couples to the atomic dipole as $-\vec{\mu} \bullet \vec{E}_L$ in the Hamiltonian, giving rise to stimulated transitions between the two levels. The equation of motion for the atomic density operator σ in the rotating frame, and averaged over the stochastic laser phase, can readily be solved. For the matrix elements we obtain:

$$<e|\sigma|e> = \frac{1}{2} \frac{\Omega_o^2 \eta + A P_o^2 (\Delta^2 + \eta^2)}{\Omega_o^2 \eta + A(1 + P_o^2)(\Delta^2 + \eta^2)} \qquad , \qquad (10)$$

$$<e|\sigma|g> = -\frac{1}{2}\Omega \frac{A(\Delta - i\eta)}{\Omega_o^2 \eta + A(1 + P_o^2)(\Delta^2 + \eta^2)} \qquad (11)$$

Here we introduced the notations: $\Delta = \omega_L - \omega_o$, $\eta = \lambda + A(1 + P_o^2)/2$, and $\Omega_o = |\Omega|$, with Ω the (complex) Rabi frequency of the transition, A the Einstein coefficient for spontaneous decay, and P_o the absolute value of the Fresnel reflection coefficient.

IV. DEFINITION OF SQUEEZING

The electric field of the emitted radiation is given by Eq. (7). The slowly-varying amplitude of the resonance fluorescence, with respect to the incident field, is given by[6]

$$E_\alpha(t) = E(t)^{(+)} e^{i(\omega_L t + \phi(t) - \alpha)} + \mathrm{H.c.} \qquad (12)$$

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with $E(t)^{(+)}$ the projection of the field from Eq. (7) onto a fixed polarization direction. Angle α can be varied in an experiment. For $\alpha = 0$ or $\alpha = \pi/2$ this corresponds to the in-phase and out-of-phase quadrature component of the field, respectively. The Heisenberg uncertainty relation for quadrature fields with different values of α is

$$\Delta E_\alpha(t) \Delta E_{\alpha'}(t) \geq \frac{1}{2} |<[E_\alpha(t), E_{\alpha'}(t)]>| \quad , \tag{13}$$

and with Eq. (12) this becomes

$$(\Delta E_\alpha(t))^2 (\Delta E_{\alpha'}(t))^2 \geq <[E(t)^{(+)}, E(t)^{(-)}]>^2 \sin^2(\alpha - \alpha') \quad . \tag{14}$$

Then we define the field $E_\alpha(t)$ squeezed, if

$$(\Delta E_\alpha(t))^2 < |<[E(t)^{(+)}, E(t)^{(-)}]>| \quad , \tag{15}$$

holds. From Eq. (14) it follows that when $E_\alpha(t)$ is squeezed for a certain value of α, then the quadrature component of the field which is 90° out of phase with this $E_\alpha(t)$ must have enhanced fluctuations.

As a measure for the amount of squeezing we introduce the normalized quantity

$$s = \frac{(\Delta E_\alpha)^2 - |<[E(t)^{(+)}, E(t)^{(-)}]>|}{<E_\alpha^2>} \quad , \tag{16}$$

so that squeezing occurs under condition

$$s < 0 \tag{17}$$

V. CONDITION FOR SQUEEZING

The squeezing parameter s can readily be evaluated, given the solution for the atomic density operator σ. It appears that parameter α can be chosen, such that it minimizes s, but this choice depends in a complicated way on the phase of the atomic transition dipole moment, the phase of the Rabi frequency, and the normal distance between the atom and the surface of the medium.[7] For this optimum value of α, parameter s is found to be

$$s = 1 - \frac{A(\Delta^2 + \eta^2)}{(1 + P_o^2)[\Omega_o^2 \eta + A(1 + P_o^2)(\Delta^2 + \eta^2)]^2}$$

$$\times \left[\Omega_0^2 (A + |1 - P_0^2| \eta) + A |1 - P_0^4| (\Delta^2 + \eta^2) \right] \qquad (18)$$

Therefore, squeezing occurs when the following condition on the optical parameters holds:

$$(1 + P_0^2)[\Omega_0^2 \eta + A(1 + P_0^2)(\Delta^2 + \eta^2)]^2 <$$
$$< A(\Delta^2 + \eta^2)\left[\Omega_0^2 (A + |1 - P_0^2| \eta) + A |1 - P_0^4| (\Delta^2 + \eta^2) \right] \qquad (19)$$

If we set $P_0^2 = 0$ in Eq. (19), then we recover the result for a free atom.[8] When we set $\Omega_0^2 = 0$, which corresponds to the case without the driving laser, then it is easy to verify that in this situation squeezing never occurs. Figure 2 shows the region were squeezing occurs, as a function of the laser power and the phase-conjugate reflectivity, and for zero detuning Δ and laser linewidth λ.
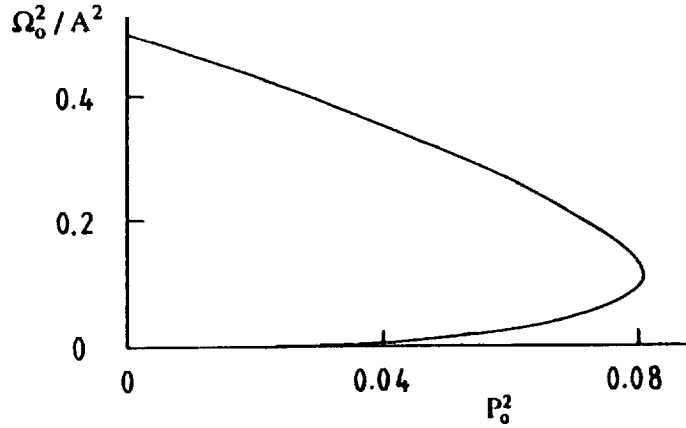
Fig. 2. Squeezing occurs when the reflectivity and the laser power are such that the corresponding point in this plane is within the loop.

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II. UNCERTAINTY RELATIONS

THE LEGACY OF UNCERTAINTY

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Abstract

An historical account is given of the circumstances whereby the uncertainty relations were introduced into physics by Heisenberg. The criticisms of QED on measurement-theoretical grounds by Landau and Peierls are then discussed, as well as the response to them by Bohr and Rosenfeld. Finally, some examples are given of how the new freedom to advance radical proposals, in part the result of the revolution brought about by "uncertainty," was implemented in dealing with the new phenomena encountered in elementary particle physics in the 1930s.

1 Introduction

I must thank the organizers of this conference on Squeezed States and Uncertainty Relations for the kind invitation to speak here. For some years I have studied and written on the history of modern physics, and so I assumed that I was to speak on some topic in that field. Let me say why a talk on the history of physics may be relevant, and why I have chosen the title as I have. According to a Greek historian of the period just before the Christian era, Dionysius of Halicarnassus, "History is philosophy from examples." But why should physicists care anything about philosophy, by examples or otherwise? Because physics was and is natural philosophy, and never more so than when we deal with uncertainty relations.

I will begin by discussing the general significance of the Heisenberg uncertainty relations, how they entered physics, and what interpretational (i.e., philosophical) problems they were intended to solve. I will then mention the criticisms that Lev Landau and Rudolph Peierls addressed to the measurement problem in QED, criticisms which led Niels Bohr and Lon Rosenfeld to attempt to justify the real existence of quantized electromagnetic fields. But I will not be so foolhardy as to review this subject in technical detail, when I am in the presence of so many experts on quantum optics. Instead, I shall ask how the establishment of a quantum mechanics that accepts the impossibility of exactly describing an atomic system in classical terms, influenced the thinking of physicists as they tried to understand the phenomena of subatomic, i.e., nuclear and subnuclear, systems. For, after the introduction of "uncertainty," physicists felt permitted to advance hypotheses that would have been unthinkable before the quantum mechanical revolution of 1925-26.

In particular, I shall discuss some bold developments during the 1930s in quantum field theory and in nuclear and cosmic ray physics, three subjects whose confluence gave rise to the new field that is now called elementary particle physics. [1]

2 The Origin of the Uncertainty Relations and of Complementarity

In a recent biography of Heisenberg by David Cassidy, entitled Uncertainty, the author begins a chapter, which is called "Certain of Uncertainty," as follows:

> On March 22, 1927 Werner Heisenberg submitted a paper to the Zeitschrift fuer Physik entitled "On the perceptual [anschaulich] content of quantum theoretical kinematics and mechanics." The 27-page paper, forwarded from Copenhagen, contained Heisenberg's most famous and far-ranging achievement in physics–his formulation of the uncertainty, or indeterminacy, principle in quantum mechanics. Together with Bohr's complementarity principle, enunciated later that year, and Born's statistical interpretation of Schrdinger's wave function, Heisenberg's uncertainty principle formed a fundamental component of the so-called Copenhagen interpretation of quantum mechanics–an explication of the uses and limitations of the mathematical apparatus of quantum mechanics that fundamentally altered our understanding of nature and our relation to it ... [This] marked the end of a profound transformation in physics that has not been equalled since. [2]

The development of quantum mechanics by Heisenberg, Born, Jordan, Bohr, Schroedinger, Dirac, and others in 1925-26 marked the end of a period, beginning with Planck's introduction of the quantum of action in 1900, that was characterized by efforts, sometimes described as "desperate," to apply the well-established Newtonian particle and Maxwellian wave concepts, even if modified by Einstein's relativity and restricted by the quantum rules of Bohr-Sommerfeld. But quantum mechanics entailed a whole new epistemology. Common-sense notions of causality, separability, locality, visualizabilty, and measurability demanded, at the least, reinterpretation, and perhaps utter abandonment at the quantum level. Heisenberg's uncertainty principle lay at the very heart of all this consternation and excitement. How did it first appear?

After Born, Jordan, and Heisenberg set out the principles and methods of matrix mechanics, Schroedinger introduced wave mechanics, and soon proved that the two very different approaches would always lead to the same predictions. (The equivalence of wave and quantum mechanics was independently shown by Wolfgang Pauli.) This immediately raised the old spectre of the wave-particle paradox in a new context, as did the experiments of 1927 on electron diffraction. (However, histories of quantum mechanics emphasize the theory, and they do not seem to take much notice of the latter.) After 1927 it became necessary to take seriously the matter waves of Louis de Broglie, and to explain how the de Broglie-Schrdinger wavelike electron could be the same object that leaves a well-defined track in a Wilson cloud chamber. Bohr and Heisenberg, then Bohr's assistant in Copenhagen, had been very concerned about this paradox the previous year, and to help clear up the matter, Bohr invited Schroedinger to visit them. Accordingly, Schroedinger took the train to Copenhagen from his post in Zurich, in October 1926. The Austrian physicist still adhered to a "realist" view of electron waves, and rejected any notion of "quantum-jumping," that is, the transfer of energy in discrete amounts, rather than continuously.

By all accounts [4], poor Schroedinger was attacked so vigorously by the usually congenial Bohr that he became ill and took to bed. Bohr, however, pursued him even into the sickroom,

and would not allow him to rest. Nevertheless, Schroedinger left Copenhagen without giving up the reality of his waves and still refused to concede the existence of quantum jumps.[4] According to Heisenberg, the result of the visit was a continued preoccupation by Bohr and himself with the problem of interpreting the quantum theory. As Heisenberg described it:

> For all that, we in Copenhagen felt convinced toward the end of Schroedinger's visit that we were on the right track, though we fully realized how difficult it would be to convince even leading physicists that they must abandon all attempts to construct perceptual models of atomic processes. During the next few months the physical interpretation of quantum mechanics was the central theme of all conversations between Bohr and myself ... Since our talks often continued till long after midnight ... both of us became utterly exhausted and rather tense. Hence Bohr decided in February 1927 to go skiing in Norway, and I was quite glad to be left behind in Copenhagen, where I could think about these hopelessly complicated problems by myself. [5]

Recalling a conversation with Einstein, who had maintained that it was only the theory which decides what we can observe, Heisenberg began to question what we really see when we examine an electron track in a cloud chamber:

> In fact, all we do see in the cloud chamber are individual water droplets which must certainly be much larger than the electron. The right question should therefore be: Can quantum mechanics represent the fact that an electron finds itself approximately in a given place and that it moves approximately with a given velocity, and can we make these approximations so close that they do not cause experimental difficulties? A brief calculation ... showed that one could indeed represent such situations mathematically, and that the approximations are governed by what would later be called the uncertainty principle of quantum mechanics.[6]

Upon Bohr's return to Copenhagen, there was "a fresh round of difficult discussions," in which Bohr insisted that the correct solution was to be given by the principle of complementarity. "But he soon realized," said Heisenberg, " ... that there was no serious difference," and that the main problem remaining was how to convince other physicists of the new way of looking at the world. That would not be easy. Comparing the scientist's voyage of discovery with that of Columbus, Heisenberg said:

> In science, too, it is impossible to open up new territory unless one is prepared to leave the safe anchorage of established doctrine and run the risk of a hazardous leap forward ... When it comes to entering new territory, the very structure of scientific thought may have to be changed, and that is far more than most men are prepared to do. [7]

For a brief period, Bohr and Heisenberg had had a falling-out, since Heisenberg wished to base his uncertainty relations entirely upon the particle viewpoint of matrix mechanics, while to Bohr the indeterminacy was related to the necessity of including in the discussion the complementary wave aspect of matter and of radiation. However, the two had reconciled their views

by October 1927, when Bohr gave a major address at the Fifth Solvay Congress in Brussels, essentially repeating a speech that he had made a month earlier at Como, Italy at a conference on the centenary of Alessandro Volta. At the Solvay conference, there began the famous and long-lasting Bohr-Einstein debates on the interpretation of quantum mechanics, the forerunner of the Einstein-Podolsky-Rosen arguments and Bohr's reply. [8]

3   The Measurability of Quantum Fields

In the spring of 1929, Heisenberg gave a set of lectures on quantum theory at the University of Chicago, a major portion of the lectures being concerned with a critique of the wave and particle concepts in interpreting experiments on Wilson photographs, x-ray and electron diffraction, etc. He also analyzed the spreading of wave packets, and he obtained uncertainty relations for electromagnetic fields, e.g., those holding for the simultaneous measurement of a component of the electric and a compnent of the magnetic field, both being measured in the same volume element. His conclusion was that: "After a critique of the wave concept has been added to that of the particle concept all contradictions between the two disappear, provided only that due regard is paid to the limits of applicability of the two pictures." [9]

In his Chicago lectures, Heisenberg gave three "proofs" of the relation

$$\Delta E_x \Delta H_y \ \geq \ \frac{hc}{(\delta l)^4}, \tag{1}$$

for the fields averaged over a cubic cell of side δl. However, as shown later by Bohr and Rosenfeld, due to the presence of a δ- function involving the time difference in the commutator of two field components, the inequality (1) is ambiguous. When the averaging is more appropriately done over a space-time region, rather than space only, the right-hand side of (1) becomes zero. [10] Bohr and Rosenfeld concluded: "From this it follows that the averages of all field components over the same space-time region commute, and thus should be exactly measurable, independently of each other." [11]

The work of Bohr and Rosenfeld was in large part a response to a criticism of QED, based on measurement theory, that had been made by two very young (and rather brash) theorists, namely Lev Landau and Rudolph Peierls. In 1929 Landau was visiting physics centers in Western Europe on a grant from the Soviet Union, spending some time with each of Ehrenfest, Pauli, Heisenberg, Rutherford, Kapitza, and Born. However, for the most part he stayed in Copenhagen with Bohr, who (we know from his correspondence) was at that time concerned and, rather uncertain, about the uncertainty relations for two electromagnetic field components. [12] Visiting Zurich at the beginning of 1930, Landau began working on problems of QED with Peierls, who was then Pauli's assistant. In December of 1930, Landau again visited Zurich, and he and Peierls wrote a paper arguing that QED was essentially meaningless, because a fundamental limitation made the measurement of electromagnetic fields impossible in the context of quantum theory. Obviously, this paper was intended to (and did) generate a major controversy. [13]

According to Rosenfeld:

There was indeed reason for excitement, for the point raised by Landau and Peierls was a very fundamental one. They questioned the logical consistency of quantum electrodynamics by contending that the very concept of electromagnetic field is not susceptible, in quantum theory, to any physical determination by means of measurements. The measurement of a field component requires determinations of the momentum of a charged test-body; and the reaction from the field radiated by the test-body in the course of these operations would (except in trivial cases) lead to a limitation of the accuracy of the field measurement, entirely at variance with the premises of the theory ... On the other hand, ... the occurrence of irregular fluctuations in the value of any field component ... was known to be responsible for one of the divergent contributions to the self-energy of charged particles. Landau and Peierls, somewhat illogically, tried to bring it into relation with their alleged limitation of measurability of the field, and this only further confused an already tangled issue. [14]

As noted above, after two years of soul-searching, and by the use (in thought experiments) of classical test bodies, the consistency of QED as regards measurements was proven, for, again according to Rosenfeld [15]:

So long as we treat all sources of electromagnetic fields as classical distributions of charge and current, and only quantize the field quantities themselves, no universal scale of space-time dimensions is fixed by the formalism. It is then consistent to disregard the atomistic structure of the test-bodies and there is no restriction to the logically admissable values of the charge density. [16]

Surely this is one of the few examples of a problem of physics reduced to one of mere logic. As in much of Bohr's work on measurement theory, a great deal of effort went into assuring readers that they need not worry further about the puzzling issues that gave rise to the paper. Abraham Pais quotes approvingly a friend's remark on Bohr-Rosenfeld: "It is a very good paper that one does not have to read. You just have to know it exists." [17]

4 The Legacy of Uncertainty: The Positron and the Neutrino Conjectured

After the probability interpretation and the (quite separate) measurement problems of nonrelativistic quantum mechanics had been "solved," or at least put in abeyance for a time, most thoughtful physicists felt that the first order of business was to look at other fundamental issues of the theory, especially those related to the striking new phenomena then being revealed by experiment. At least one important era of research had been successfully concluded; Dirac in 1929 expressed it as follows:

The general theory of quantum mechanics is now almost complete, the imperfections that still remain being in connection with the fitting of the theory with relativity ideas ... The underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known. [18]

Dirac admitted that there remained great practical difficulty in actually solving the complicated equations for atomic and molecular systems, but he failed to mention that there were also "fundamental" questions remaining even in nonrelativistic quantum mechanics; for example, the treatment of collective behavior like superconductivity. As the quotation above shows, the fundamental problem that concerned Dirac at this time was the relativistic theory of the electron itself, and this also appeared to be implicated in at least three other problematic areas, namely: quantum field theory, nuclear physics, and the cosmic rays. [19]

Problems associated with the theory of the electron had been present almost since the turn of the century. The existence of a finite-sized concentration of electric charge appeared to require a new stabilizing force to prevent its explosion. As a constraint on the structure, physicists (notably, H.A. Lorentz) advanced the hypothesis that all the mass of the electron was electromagnetic in origin. In classical models, this required the (spherical) electron to have a radius

$$r_0 = a\frac{e^2}{mc^2},$$

(2)

e and m, being the mass and charge of the electron, c the velocity of light, and a a dimensionless constant of order unity, whose value depended on the assumed structure of the electron. (We shall assume in what follows that $a = 1$). Letting the radius tend to zero gave the electron an infinite self-energy, i.e., an infinite mass. There was difficulty in reconciling a finite electron with the theory of relativity, and Lorentz had suggested that within the electron radius r_0, physical laws that were different from the usual ones might apply. [20]

The problem became acute with the advent in 1925 of quantum mechanics, in which the electron was treated as a point particle. The most obvious relativistic generalization of the Schroedinger equation, the Klein-Gordon equation ("the equation of many fathers"), did not give the correct fine structure of the hydrogen spectrum, which Arnold Sommerfeld had somehow managed to obtain (without electron spin!) by using the Bohr-Sommerfeld "old" quantum theory. The problem in quantum mechanics was that the electron spin was not properly taken into account. Dirac set out to find an equation that would give the right spin and magnetic moment to the electron (he referred to these as "duplexity phenomena") by remedying the "incompleteness of the previous theories lying in their disagreement with relativity, or alternatively, with the general transformation theory of quantum mechanics." [21]

Dirac's new electron theory was spectacularly successful in treating the fine structure of hydrogen, Compton scattering, the electron's magnetic moment, and other important physics–but it also gave rise to new puzzles. The chief difficulty was the presence of negative energy states, which were meaningless in a relativistic theory, since an electron in such a state would have a negative mass. Dirac tried to prevent electron transitions to these negative energy states by declaring that they were all filled, and hence, by the exclusion principle, unavailable in practice. If occasionally "holes" did occur, they would act in every way as positive electrons.

Later, the one-electron theory of Dirac, with filled vacuum states, was supplanted by a quantum field theory, which was then combined with the quantum field theory of the electromagnetic field that Dirac (and also Pascual Jordan) had pioneered in 1927. The theory of the two fields in interaction became known as quantum electrodynamics (QED). [22] However, this completely

relativistic theory was itself plagued by serious inconsistencies, of which the most egregious were the so-called "divergences," namely, infinite predictions for the physical mass and charge of the electron. These divergences arose when virtual (i.e., energy-nonconserving intermediate) states were summed over, according to the rules of perturbation theory. The lowest approximations did give finite results, and were in surprisingly good accord with experiment. It was, therefore, assumed that the theory was correct at lower energies, but that it broke down above some critical interaction energy. In a suitably modified QED, it was argued, the small value of the expansion parameter (the dimensionless fine-structure constant, $\alpha = 1/137$) would validate the perturbation expansion.

Quantum mechanics can be expressed either in configuration space or in its complementary energy-momentum space, the two spaces being related by the Fourier transform theorem. Thus, a critical *high energy* can be related to a critical *small distance*. QED was working well at the energy scale that corresponds to the Compton wavelength, but it was assumed that it would very likely break down at the classical electron radius r_0, which is 137 times smaller than the Compton wavelength. That might account, it was thought, for the apparent contradictions to accepted laws of physics that were puzzling physicists around 1930, especially in the higher energy nuclear and cosmic ray phenomena, since $r_0 = 10^{-13} cm$ is almost identical with the known range of nuclear forces. [23] This distance was also a "natural" fundamental length at which to expect a breakdown in the classical theory, as Lorentz had, in fact, predicted at the beginning of the century. One of the principles guiding the development of quantum theory had been that classical physics is a limiting case of quantum physics (Bohr's Correspondence Principle); it was not forgotten in the 1930s.

Bohr suggested just such a breakdown of known laws in his Faraday lecture to the British Association in London in 1930, and repeated the idea at a conference in Rome in October 1931. [24] To Dirac he wrote: "I ... believe firmly that the solution to our present troubles will not be reached without a revision of our general ideas still deeper than that contemplated in the present quantum mechanics." [25] Heisenberg, who adopted the same belief as Bohr, tried to make a theory involving a minimum length, introducing a space that was a lattice-world, rather than a continuum, a concept to which he returned several times later on in his life. As the appropriate lattice spacing he proposed the distance h/2cMc, where M is the mass of the proton. Thus this distance is about 2000 times smaller than the electron's Compton wave length. He motivated his choice by the argument that distances smaller than the uncertainty inherent in a measurement with the most massive known elementary particle, the proton (i.e., the uncertainty in position determination by an ideal hypothetical proton microscope) were meaningless. This, then, was one legacy of the uncertainty relations. [26]

Let us now leave aside the problems of QED and consider the conventional picture of the structure of the nucleus around 1930. In 1930 it was believed that there were only two elementary particles, the proton and the electron (described in an Encyclopedia Britannica article by Robert Millikan as negative and positive electrons). These particles interacted according to the laws of Maxwell and of quantum mechanics to produce ordinary matter. Thus all matter, atoms *and* their nuclei were supposed to be electrical in nature. (The only additional fundamental interaction was gravity–curved space-time perhaps–although if all mass were truly electromagnetic, then perhaps gravity itself was intimately entangled with electromagnetism. (The notion of a unified field, was

considered by Einstein, Hermann Weil, Theodor Kaluza and Oskar Klein, and others.)

One of the most immediate difficulties with the electron-proton nuclear model was β-decay. Without the neutrino, not yet postulated by Pauli, any theory of β-decay inevitably violated energy and momentum conservation. These days we may find it surprising that the generation of quantum revolutionists did not insist upon the preservation of the basic conservation laws. (Indeed, Bohr rather preferred the idea that energy was not conserved in individual elementary processes, but only statistically. He argued that in that case, the first and second laws of thermodynamics would have a comparable statistical foundation. [27])

Some other difficulties of the electron-proton model were [28] :

- The symmetry character of the nuclear wave function depends upon the parity of the atomic mass number A, not Z, as the model predicted. [The number of fermions in the nucleus in the model is 2A-Z]; when A-Z is odd the spin and statistics of the nucleus were given incorrectly. For example, nitrogen (Z = 7, A = 14) was known, from the molecular band spectrum of N_2, to have spin 1 and Bose-Einstein statistics. In the e-p model, it was composed of 21 fermions–so it should have had half-integer spin and should have obeyed Fermi-Dirac statistics.]

- No potential well is deep enough and narrow enough to confine a particle as light as an electron to a region the size of the nucleus. [The argument for this is based on the uncertainty principle and on the relativistic electron theory.]

- It is hard to see how to "suppress" the very large (on the nuclear scale) unpaired magnetic moments of the electrons in the nucleus, which would conflict with the data on the hyperfine structure of atomic spectra.

The great *attraction* of the electron-proton model was that it was a unified theory. Indeed, no more unified theory has existed between that of Thales of Miletus (who is said to have believed that everything is made of water) and modern string theory. The only problem was that the electron-proton model could not coexist with quantum mechanics. But could it be that quantum mechanics was the correspondence limit of some more general dynamical theory that might relinquish even more of measurability than quantum mechanics did? For example, the observables in the new theory might be represented by operators that were non-associative, as well as non-commutative.

Such was the thinking as the thirties began: A new physics *was* in the offing, a new revolution in physics as one penetrated below some minimum distance. In part that thinking was correct–a new physics was in the offing. But it was not to be a physics of new laws, but one of new particles! The particles were new, but they obeyed the known laws of relativity, quantum mechanics, and quantum field theory.

The first of the new particles, the neutrino, was proposed by Pauli in a famous letter, dated 4 December 1930 and addressed to a meeting on radioactivity in Tuebingen (via Hans Geiger and Lise Meitner). The letter began: "Dear radioactive ladies and gentlemen." The new proposal had probably more a conservative than a radical character. One of the suggested neutral fermions was supposed to sit with each electron in the nucleus, thus solving the spin-statistics difficulty. In β-decay, it would accompany the emitted electron, thus permitting the conservation of energy and momentum. Pauli called the particle a "neutron," and indeed it was meant to accomplish

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a part of what was later done by the neutron *and* the neutrino together. (Of course, it still did nothing to help the "confinement" and the hyperfine structure difficulties of the electrons in the nucleus. It should also be noted that Pauli's neutron was a purely conjectural particle, designed to be almost undetectable. The actual neutrino was detected only on the 1950s.)

Pauli was very uncertain about his neutron-neutrino idea, and while he told people about it privately, he did not want the idea to be published. One of the first times it was mentioned in print was in a report given by S.A. Goudsmit at an international conference in Rome in October 1931. [29] However, at the same meeting, Bohr discussed "Atomic stability and conservation laws," saying about β-decay:

> If energy were conserved in these processes, it would imply that the individual atoms of a given radioactive product were essentially different, and it would be difficult to understand their common rate of decay. If, on the other hand, there is no energy balance, it is possible to explain the law of decay by assuming that all nuclei of the same element are essentially identical. This conclusion would also be in accord with the general evidence on the nuclear statistics of non-radioactive elements, which has revealed the essential identity of any two nuclei containing equal numbers of protons and electrons. [30]

A proposal rather close to our present idea of the neutrino was first presented by Pauli at the Seventh Solvay Conference in October 1933, a year and a half after the neutron had been discovered. A few months later, Fermi made his β-decay theory, conserving all important physical quantities and fitting the β- decay lifetimes very well. Nevertheless, in October 1934 at an international conference in London-Cambridge, the preferred theory presented was not Fermi's, but a non-conserving theory proposed by Guido Beck and Kurt Sitte and openly advocated by Bohr. [31]

5   The Legacy of Uncertainty: The Neutron and the Fermi-Field

The *annus mirabilis* of elementary particle physics was the year 1932. Here is how the discoveries went: January, deuterium (Urey et al.); February, the neutron (Chadwick); April, the first accelerator induced nuclear reactions (Cockroft and Walton); August, the positron (Anderson); September, the cyclotron (Lawrence). In the same year, 1932, Heisenberg wrote a three-part paper which introduced a neutron-proton model of the nucleus. [32]

Heisenberg's model is widely praised in nuclear physics textbooks, and some of the physicists who were active in nuclear theory during the 1930s (e.g., Bethe) have said that it allowed them to use quantum mechanics, because it effectively took electrons out of the nucleus. In Heisenberg's model, nuclei are built of protons and neutrons interacting through charge-exchange forces. In the Hamiltonian describing the nucleus, only neutron and proton space and spin coordinates appear, and the isospin operators are introduced to change the nucleon type. Thus, if one ignores the frequent mention and use of nuclear electrons in the Heisenberg paper, treating it as pure

phenomenology of nuclear systematics, it is possible to argue that Heisenberg's model makes quantum methods available to nuclear physics (although the usefulness of such a partial approach had already been demonstrated by Gamow in his α-particle model of the nucleus).

However, there are still electrons, and they play an important role, in Heisenberg's "neutron-proton model" of the nucleus. For example, the neutron is an electron-proton compound; the charge that is exchanged to provide the attractive binding force is an electron; in β-decay radioactivity, the electron is emitted without a neutrino (and it is thus an energy, momentum, and angular momentum non-conserving theory); in addition to the electrons bound in neutrons and particles, there are other "free" nuclear electrons to account for the frequent occurrence of interactions involving high energy radiation, e.g., Bremsstrahlung.

It is difficult for us to see how so radical a departure from physical norms could have been tolerated. It is, in fact, so difficult that most textbook authors are embarrassed to reveal that Heisenberg's fundamental theory violated almost all conservation laws (charge is an exception to this rule), or that half of the Heisenberg work consisted of wrestling with this devil! In the Hamiltonian, one sees that the neutron is treated as an electron-proton composite of spin 1/2, obeying Fermi statistics, while the proton is an elementary fermion. The p-p interaction is pure Coulomb; the n-n interaction is a double exchange, as in the hydrogen molecule, or more generally, as in covalent bonding; the n-p interaction is one-electron exchange, as in the ion H_2^+. It was only after the success of the Fermi β- decay theory that Heisenberg accepted the idea of the neutrino and the "elementary" neutron, and he was one of the first to do so! [33]

Fermi's theory of β-decay contributed much to the solution of the difficulties of nuclear structure theory, aside from being a good account of this special form of radioactivity. Embracing Pauli's neutrino (so christened by Fermi after Chadwick's discovery of the proton's neutral partner), the theory treated the emission of an electron-neutrino pair, coupled in a "four-vector" state, much like the emission of a photon from an excited atom. The photon was not "in the atom" to begin with, but it was created in the transition. Thus electrons and neutrinos need not be inside nuclei. Advances in radiation theory using QED also showed that the large observed radiative interactions were made by virtual electron-positron pairs in the nuclear Coulomb field– these were the "low-mass" radiating charges of the nucleus. The radiative processes consisted of, besides Bremsstrahlung, pair production and pair annihilation. [34]

The upshot was that it became *unnecessary* to postulate the existence of electrons in any nucleus, even those that β-decay. Heisenberg enthusiastically accepted the idea of the Fermi-field, not only for β-decay, but also as the nuclear analog of the electromagnetic field. Thus, much as atoms were held together by the exchange of electromagnetic quanta, the photons, nuclear forces were to be carried by the quanta of the nuclear field, i.e., electron-neutrino pairs. The small value of the Fermi coupling constant, fitted at low energies to the observed rates of β-decay, would be compensated in the case of nuclear binding, where higher virtual energies were dominant, by large matrix elements of the interaction. Indeed, these matrix elements were more than large–they were infinite! Thus, if the integrations in calculating the matrix elements were cut off at a suitably chosen high energy (again implying a characteristic length), it was possible to fit the required strength of nuclear binding forces. [35]

Unfortunately for the many physicists who had been attracted by the high degree of unification presented by the Fermi-field theory of nuclear forces, it was not possible to fit both the strength

and the *range* of nuclear forces simultaneously by the choice of cutoff. That such a procedure would fail by many orders of magnitude became clear to Heisenberg when he worked out the details; independently, this result was found and published by two Russians, Igor Tamm and (once again) Dmitri Iwanenko. [36]

Meanwhile, in far-off Japan, a young physicist of the next scientific generation, Hideki Yukawa, advanced boldly to the next step. Challenging the new orthodoxy of quantum mechanics and quantum fields, just as the previous generation had done in postulating and developing those new dynamical systems, Yukawa decided that a *new* field should have a *new* quantum, not the electron, not the electron-neutrino pair, but a quantum all of its own. He called this the "heavy quantum," or the "U-quantum," of the nuclear force field, which he called the U-field.

This scientific revolution that has been called, by Yoichiro Nambu, the paradigm of modern elementary particle theory, namely, the identification of forces and their representation by quantum fields, having their characteristic quanta, came about this way, as Yukawa relates it:

> The crucial point came to me one night in October [1934]. The nuclear force is effective at extremely small distances, on the order of 2×10^{-15} cm. That much I knew already. My new insight was the realization that this distance and the mass of the new particle that I was seeking are inversely related to each other. Why had I not noticed that before? The next morning, I tackled the problem of the mass of the new particle and found it to be about two hundred times that of the electron. It also had to have the charge of plus or minus that of the electron. Such a particle had not, of course, been found, so I asked myself, "Why not?" The answer was simple: an energy of 100 million electron volts would be needed to create such a particle, and there was no accelerator, at that time, with that much energy available. [37]

After presenting this paper at a physics meeting, and after submitting the article with his theory to a journal in November 1934, Yukawa felt that his struggle with the problem of nuclear forces had been, for the time being at least, resolved. He concluded his account of his scientific life up to that time as follows:

> I felt like a traveler who rests himself at a small tea shop at the top of a mountain slope. At that time I was not thinking about whether there were any more mountains ahead. [38]

I too feel that it is time now to rest, without proceeding further with this description of the legacy of uncertainty.

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QUANTUM NON DEMOLITION MEASUREMENT
OF CYCLOTRON EXCITATIONS
IN A PENNING TRAP

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Abstract

The quantum non-demolition measurement of the cyclotron excitations of the electron confined in a Penning trap could be obtained by measuring the resonance frequency of the axial motion, which is coupled to the cyclotron motion through the relativistic shift of the electron mass.

1 Introduction

The process of making a measurement on a quantum mechanical system introduces quantum noise to that system. A quantum non-demolition measurement (QND) scheme seeks to make a measurement of an observable by feeding all the introduced noise into a conjugate variable to that under consideration. An ideal QND observable is one which has always the same values in repeated series of measurements. It means that the total Hamiltonian of the system plus the interaction with the measurement device must commute with the observable to be measured at given times, for a stroboscopic QND, observable or at any times for a continuous QND observable [1].

Recently there has been a number of theoretical papers [2, 3, 4, 5, 6] proposing schemes for QND measurements and fewer experimental realizations mainly in the optical regime [7, 8, 9]. In this paper we present another scheme which could be easily verified because the system is well known and studied. The system is an electron confined in a Penning trap [10]. Penning traps for electrons, protons and ions have been extensively used for high precision measurements of fundamental constants and laws of Nature, like for instance the *g*-factor of the electron and the CPT invariance [11]. In this paper we will show that it could also be used to give a QND measurement of the excitation number of the cyclotron motion.

2 The Penning trap

A Penning trap consists of a combination of constant magnetic field and quadrupolar electrostatic potential in which a charged particle, for instance an electron, can be confined. It is composed by two end-cap and one ring electrodes to which a static potential V_0 is applied [11]. There is also a homogeneous magnetic field $\vec{B}_0$ along the symmetry axis of the trap assumed as the z-axis. Neglecting the contribution of the spin, which we keep locked, the Hamiltonian for the electron of charge e and rest mass m_0 in the trap is given by the following expression

$$H = \frac{1}{2m_0}\left(\vec{p} - \frac{e}{c}\vec{A}\right) + eV \tag{1}$$

with

$$\vec{A} = \left(-\frac{y}{2}B_0, \frac{x}{2}B_0, 0\right) \tag{2}$$

$$V = \frac{V_0}{2z_0^2}\left(\frac{x^2 + y^2}{2} - z^2\right) \tag{3}$$

The typical experimental values are

$$
\begin{aligned}
B_0 &\simeq 58100 \text{ G} \\
V_0 &\simeq 10 \text{ V} \\
z_0 &\simeq 3.3 \times 10^{-3} \text{ m}
\end{aligned}
$$

where z_0 specifies the dimension of the trap. It is easy to show that in terms of rising and lowering operators the Hamiltonian (1) becomes [10]

$$H = \hbar\omega_c'\left(a_c^\dagger a_c + \frac{1}{2}\right) + \hbar\omega_z\left(a_z^\dagger a_z + \frac{1}{2}\right) - \hbar\omega_m\left(a_m^\dagger a_m + \frac{1}{2}\right) \tag{4}$$

with

$$a_c = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m_0\omega_c}{2\hbar}}(x - iy) + \sqrt{\frac{2}{m_0\hbar\omega_c}}(p_y + ip_x)\right] \tag{5}$$

$$a_m = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m_0\omega_c}{2\hbar}}(x + iy) - \sqrt{\frac{2}{m_0\hbar\omega_c}}(p_y - ip_x)\right] \tag{6}$$

$$a_z = \sqrt{\frac{m_0\omega_z}{2\hbar}}z + i\sqrt{\frac{1}{2m_0\hbar\omega_z}}p_z. \tag{7}$$

The displaced cyclotron angular frequency is

$$\omega_c' \simeq \omega_c\left[1 - \frac{1}{2}\left(\frac{\omega_z}{\omega_c}\right)^2\right] \tag{8}$$

with $\omega_c = |e|B_0/m_0c$ the bare cyclotron frequency.

The axial angular frequency is given by

$$\omega_z = \sqrt{\frac{|e|V_0}{m_0 z_0^2}}$$ (9)

and the magnetron frequency by

$$\omega_m \simeq \frac{\omega_c}{2}\left(\frac{\omega_z}{\omega_c}\right)^2.$$ (10)

The ranges of frequency in the experimental situation are:

$$\omega_c'/2\pi \sim 164 \text{ GHz}$$
$$\omega_z/2\pi \sim 64 \text{ MHz}$$
$$\omega_m/2\pi \sim 11 \text{ kHz}.$$

Thus each frequency belongs to a very different band of the electromagnetic field.

3 The measurement model

The question one can rise is: how can we measure the various frequencies of oscillation? In order to make a measurement we need to couple the system to what Feynman called the "rest of Universe" [12]. It turns out that the best way of measuring the properties of the various motions of the electron is to measure the current induced by the axial motion of the electron along the z-axis [13]. Indeed, the electric charges induced by the oscillatory motion on the end-cap generate a current which can be measured.

The system plus the measurement device is represented in Fig 1. Here L is the inductance of the measurement device and R its resistance. The induced current dissipates on the resistor R which is in thermal equilibrium at temperature $T \simeq 4$ K. $u(t)$ represents a stochastic potential which gives the effect of thermal fluctuations or Johnson noise.

The axial motion plus the read-out are described by the following Hamiltonian

$$H' = \frac{p_z^2}{2m_0} + \frac{m_0\omega_z^2}{2}z^2 + \frac{1}{2C}(az + Q)^2 + \frac{\phi^2}{2L} +$$
$$+ \int_0^{+\infty} d\Omega\left[(p(\Omega) + k(\Omega)Q)^2 + \Omega^2 q^2(\Omega)\right]$$ (11)

where we have considered a thermal bath with a continuous distribution of modes linearly coupled to the electronic circuit; ϕ is the electric flux in the inductance L, Q is the electric charge on the capacitor C which is the capacity of the trap; az represents the induced charge due to the axial motion of the electron [14] with $a = \alpha e/2z_0$ where $2z_0$ is the distance between the two end-caps and α is a constant of order of unity which takes into account the curvature of the capacitor surfaces.

However, if we wish to measure the properties of the cyclotron motion we need a coupling between the axial motion and the cyclotron motion. In earlier experiments with the Penning trap [10] this coupling was introduced by adding an inhomogeneity on the magnetic field B_0 by means of a "magnetic bottle".
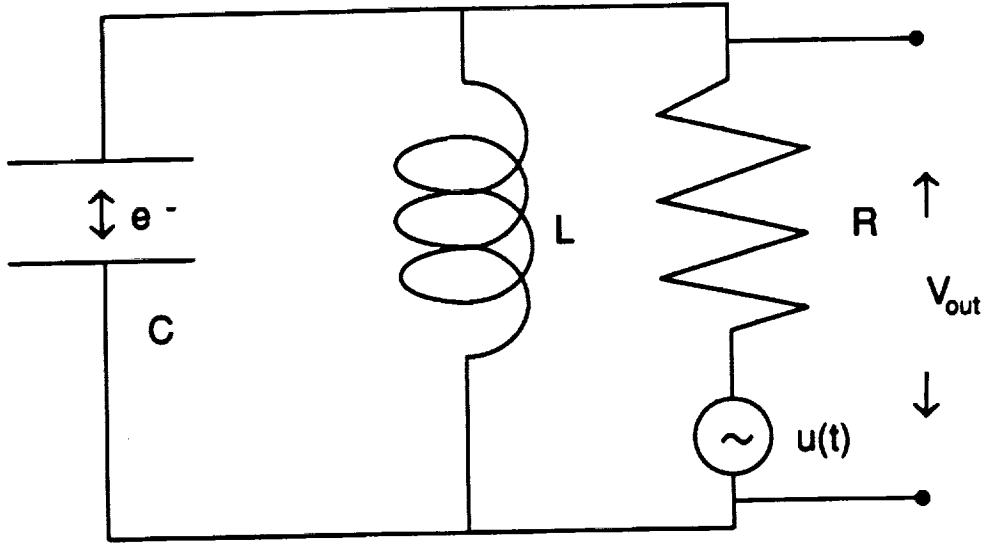
FIG. 1 The axial motion of the electron coupled to the read-out apparatus.

4 The Hamiltonian of the system

The precision of the measurements is, however, so high that we cannot get rid of the relativistic corrections; then, the coupling between the two modes is also given by the relativistic shift of the electron mass [15]. In such a case the system's Hamiltonian we have to consider is

$$H_{sys} = H_{NR} + H_{RC} \tag{12}$$

$$H_{NR} = \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + eV \tag{13}$$

$$H_{RC} = -\frac{1}{8m_0^3 c^2}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^4 \tag{14}$$

Finally we can write the following Hamiltonian of the quantum system:

$$H_{sys} = \hbar\omega_c a_c^\dagger a_c \left[1 - \frac{1}{2}\left(\frac{\omega_z}{\omega_c}\right)^2 - \frac{\hbar\omega_c}{2m_0c^2}\right] - \frac{\hbar^2\omega_c^2}{2m_0c^2}(a_c^\dagger a_c)^2 +$$
$$+ \frac{p_z^2}{2m_0}\left[1 - \frac{\hbar\omega_c}{m_0c^2}\left(a_c^\dagger a_c + \frac{1}{2}\right)\right] - \frac{p_z^4}{8m_0^3c^2} + \frac{m_0\omega_z^2}{2}z^2 \tag{15}$$

where we have completely neglected the magnetron motion which is not coupled to other motions. It is now easily seen that the coupling between the axial motion and the cyclotron motion is due to the relativistic shift of the mass.

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5 The QND observable

If we now introduce as before the coupling with the external world, the total hamiltonian becomes

$$
\begin{aligned}
H =\ & \hbar\omega_c'' a_c^\dagger a_c + \hbar\mu (a_c^\dagger a_c)^2 + \\
& + \frac{p_z^2}{2m_0}\left[1 - \frac{\hbar\omega_c}{m_0 c^2}\left(a_c^\dagger a_c + \frac{1}{2}\right)\right] - \frac{p_z^4}{8m_0^3 c^2} + \frac{m_0\omega_z^2}{2}z^2 + \\
& + \frac{(az+Q)^2}{2C} + \frac{\phi^2}{2L} + \int\limits_0^{+\infty} d\Omega\left[(p(\Omega) + k(\Omega)Q)^2 + \Omega^2 q^2(\Omega)\right]
\end{aligned}
\tag{16}
$$

with

$$
\omega_c'' = \omega_c\left[1 - \frac{1}{2}\left(\frac{\omega_z}{\omega_c}\right)^2 - \frac{\hbar\omega_c}{2m_0 c^2}\right]
\tag{17}
$$

$$
\mu = -\frac{\hbar\omega_c^2}{2m_0 c^2}.
\tag{18}
$$

It is evident that $a_c^\dagger a_c = \hat{n}_c$ is a QND observable because

$$
[\hat{n}_c, H] = 0.
\tag{19}
$$

The axial motion of the electron represents the probe that enables us to measure the properties of the cyclotron motion. Indeed, the axial frequency now depends on the cyclotron excitation quantum number $\hat{n}_c$, which is a constant of the motion, at least as long as we can neglect the spontaneous emission of the cyclotron motion. It has been measured [10] that the spontaneous emission coefficient is $\gamma_c^{-1} \simeq 1$ s thus, if the measurement is performed in a time much shorter than γ_c^{-1} we can neglect the spontaneous emission of the cyclotron motion and perform a QND measurement of the excitation number $\hat{n}_c$. It has also been shown [16] that γ_c could be reduced by the cavity effect [17]. Indeed, when the characteristic length of the cavity of the trap is shorter than half wavelength of the cyclotron motion, the cyclotron spontaneous emission should be inhibited.

One can also show that the anharmonicity of the axial motion is very small and can be neglected. It turns out that it is $(\omega_z/\omega_c)^2$ times smaller than the anharmonicity of the cyclotron motion. Thus the equations of motion now are:

$$
\left\{
\begin{aligned}
\dot{z} &= \frac{i}{\hbar}[H, z] \\
\dot{p}_z &= \frac{i}{\hbar}[H, p_z] \\
\dot{Q} &= \frac{\partial H}{\partial \phi} = \frac{\phi}{L} \\
\dot{\phi} &= -\frac{\partial H}{\partial Q} = -\frac{Q}{C} - \frac{az}{C} + \int\limits_0^{+\infty} d\Omega\left[p(\Omega) + k(\Omega)Q\right]k(\Omega).
\end{aligned}
\right.
\tag{20}
$$

By making a Markov approximation in the equation of motion for the variables of the thermal bath, we can write the following equations [18]

$$
\begin{cases}
\dot{z} & = \dfrac{p_z}{m_0}\left[1 - \dfrac{\hbar\omega_c}{m_0 c^2}\left(\hat{n}_c + \dfrac{1}{2}\right)\right] \\[2mm]
\dot{p}_z & = -m_0\omega_z^2 z - \dfrac{a}{C}Q \\[2mm]
\dot{Q} & = \dfrac{\phi}{L} \\[2mm]
\dot{\phi} & = -\dfrac{Q}{C} - \dfrac{az}{C} - \gamma\dot{Q} + \xi(t)
\end{cases}
\tag{21}
$$

where γ represents the rate at which the axial motion dissipates its energy due to the coupling with the rest of Universe represented by the read-out apparatus. Of course, in such a case one has to sustain the axial oscillation with an oscillating external potential $V(t)$ tuned at the axial frequency of the electron. In the experimental situation is always

$$
\hbar\omega_z << k_B T
$$

with k_B the Boltzman's constant. Then, it is possible to show [18] that the statistics of the noise term $\xi(t)$ is that of a white noise with expectations

$$
\begin{aligned}
\langle \xi(t) \rangle & = 0 \\
\langle \xi(t)\xi(t') \rangle & = 2\gamma k_B T \, \delta(t - t').
\end{aligned}
\tag{22}
$$

By introducing the Fourier transforms defined by

$$
f(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} d\omega \, \tilde{f}(\omega) e^{i\omega t}
\tag{23}
$$

we can write the linear system:

$$
\begin{cases}
i\omega\tilde{z}(\omega) & -\dfrac{\tilde{p}(\omega)}{m} & & & = 0 \\[2mm]
m\bar{\omega}_z^2\tilde{z}(\omega) & +i\omega\tilde{p}(\omega) & +\dfrac{a\tilde{Q}(\omega)}{C} & & = 0 \\[2mm]
& & i\omega\tilde{Q}(\omega) & -\dfrac{\tilde{\phi}(\omega)}{L} & = 0 \\[2mm]
\dfrac{a\tilde{z}(\omega)}{C} & & +\left(\dfrac{1}{C} + i\omega\gamma\right)\tilde{Q}(\omega) & +i\omega\tilde{\phi}(\omega) & = \tilde{\xi}(\omega) - \tilde{V}(\omega)
\end{cases}
\tag{24}
$$

with

$$
m = \frac{m_0}{1 - \frac{\hbar\omega_c}{m_0 c^2}\left(\hat{n}_c + \frac{1}{2}\right)}
\tag{25}
$$

$$
\bar{\omega}_z = \omega_z\sqrt{1 - \frac{\hbar\omega_c}{m_0 c^2}\left(\hat{n}_c + \frac{1}{2}\right)}.
\tag{26}
$$

The determinant of the homogeneous system is

$$\Delta = (\bar{\omega}_z^2 - \omega^2)(\omega_e^2 - \omega^2 + i\gamma_e\omega) - \frac{a^2}{mc}\omega_e^2 \qquad (27)$$

with

$$\omega_e = \sqrt{\frac{1}{LC}} \quad, \quad \gamma_e = \frac{\gamma}{L}$$

i.e., the characteristic frequency and the bandwidth of the electronic circuit respectively.

The solution of the algebraic system is easily obtained and we get:

$$\tilde{z}(\omega) = -\frac{a\omega_e^2}{m\Delta}\left(\tilde{\xi}(\omega) - \tilde{V}(\omega)\right) \qquad (28)$$

$$\tilde{p}(\omega) = -\frac{ia\omega_e^2}{\Delta}\left(\tilde{\xi}(\omega) - \tilde{V}(\omega)\right) \qquad (29)$$

$$\tilde{Q}(\omega) = \frac{\bar{\omega}_z^2 - \omega^2}{L\Delta}\left(\tilde{\xi}(\omega) - \tilde{V}(\omega)\right) \qquad (30)$$

$$\tilde{\phi}(\omega) = \frac{\bar{\omega}_z^2 - \omega^2}{\Delta}\left(\tilde{\xi}(\omega) - \tilde{V}(\omega)\right). \qquad (31)$$

We see that at $\omega = \bar{\omega}_z$ both $\tilde{Q}$ and $\tilde{\phi}$ are zero and the current which dissipates energy on the resistor is only due to the induced charge on the end-caps.

6 Output statistics

The signal to be measured by the read-out is the voltage at the extremes of the resistor R which is proportional to the induced current. The induced current is proportional to the axial velocity of the electron through

$$I(t) = a\dot{z}(t) = \frac{ap(t)}{m} \qquad (32)$$

thus the fluctuations of the measured potential are directly connected with the fluctuations of the axial momentum of the electron:

$$\tilde{V}_{out}(\omega) = \tilde{I}(\omega)R + \tilde{\xi}(\omega) \qquad (33)$$

where $\tilde{\xi}(\omega)$ takes into account the Johnson noise on the resistance R. The spectral density of the output voltage is given by:

$$\langle\tilde{V}_{out}(\omega)\tilde{V}_{out}(\omega')\rangle - \langle\tilde{V}_{out}(\omega)\rangle\langle\tilde{V}_{out}(\omega')\rangle =$$
$$\left(\frac{Ra}{m}\right)^2\langle\tilde{p}(\omega)\tilde{p}(\omega')\rangle + \frac{Ra}{m}\left(\langle\tilde{p}(\omega)\tilde{\xi}(\omega')\rangle + \langle\tilde{\xi}(\omega)\tilde{p}(\omega')\rangle\right) + \langle\tilde{\xi}(\omega)\tilde{\xi}(\omega')\rangle. \qquad (34)$$

For simplicity we take the driving potential $V(t)$ noiseless then we get:

$$\langle\tilde{p}(\omega)\tilde{p}(\omega')\rangle = -\left(\frac{a}{LC}\right)^2\frac{\omega\omega'\left[\langle\tilde{\xi}(\omega)\tilde{\xi}(\omega')\rangle + \tilde{V}(\omega)\tilde{V}(\omega')\right]}{\Delta(\omega)\Delta(\omega')} \qquad (35)$$

$$\langle\tilde{p}(\omega)\tilde{\xi}(\omega')\rangle = -\frac{a}{LC}\frac{i\omega\langle\tilde{\xi}(\omega)\tilde{\xi}(\omega')\rangle}{\Delta(\omega)} \qquad (36)$$

$$\langle\tilde{\xi}(\omega)\tilde{\xi}(\omega')\rangle = 2L\gamma_e k_B T \delta(\omega + \omega'). \qquad (37)$$
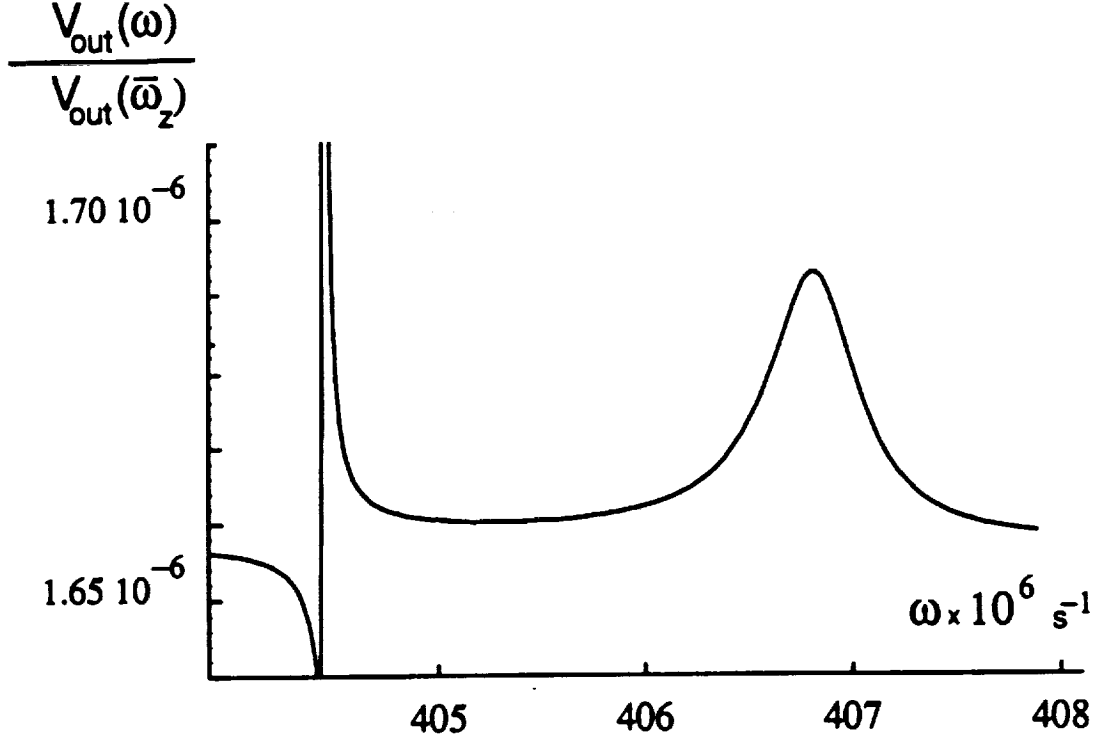
FIG. 2 The two resonances of the normalized output variances of the signal for $\omega_e \neq \bar{\omega}_z$. The value of the maximum at $\omega \simeq \bar{\omega}_z$ is not shown because it goes out of the scale. Its value is 80.

Thus eq. (34) becomes

$$\langle \tilde{V}_{out}(\omega)\tilde{V}_{out}(\omega')\rangle - \langle \tilde{V}_{out}(\omega)\rangle\langle \tilde{V}_{out}(\omega')\rangle = V_{out}(\omega)\,\delta(\omega + \omega') \tag{38}$$

with

$$V_{out}(\omega) = 2L\gamma_e k_B T \left\{ 1 + \frac{(a^2 R/m)\omega_e^2\omega^2\,[(a^2 R/m)\omega_e^2 - 2\gamma_e(\bar{\omega}_z^2 - \omega^2)]}{[(\omega_e^2 - \omega^2)(\bar{\omega}_z^2 - \omega^2) - a^2\omega_e^2/mC]^2 + [\gamma_e\omega(\bar{\omega}_z^2 - \omega^2)]^2} \right\}. \tag{39}$$

7 Conclusions

In Fig. 2 we plot $V_{out}(\omega)/V_{out}(\bar{\omega}_z)$ versus ω for a given value of $\omega_e \neq \bar{\omega}_z$. We see two maxima for $\omega > 0$; one is for $\omega = \omega_e$ and the other for $\omega \simeq \bar{\omega}_z$. As soon as we tune the electronic frequency ω_e in resonance with $\bar{\omega}_z$, we obtain only one maximum for $\omega = \bar{\omega}_z$ (Fig. 3). From eq. (26) we see that the resonance frequency depends on the quantum number $\hat{n}_c$ of the cyclotron motion. In Fig. 4 we show the top of the curves obtained with $\hat{n}_c = 0$ and $\hat{n}_c = 1$. In order to discriminate between the two maxima we need a sensitivity $\Delta\bar{\omega}_z/\omega_z \sim 7 \times 10^{-10}$ which is slightly above the experimental limit, as long as we know, which is extimated to be $\Delta\omega_z/\omega_z \sim 10^{-9}$ [10].
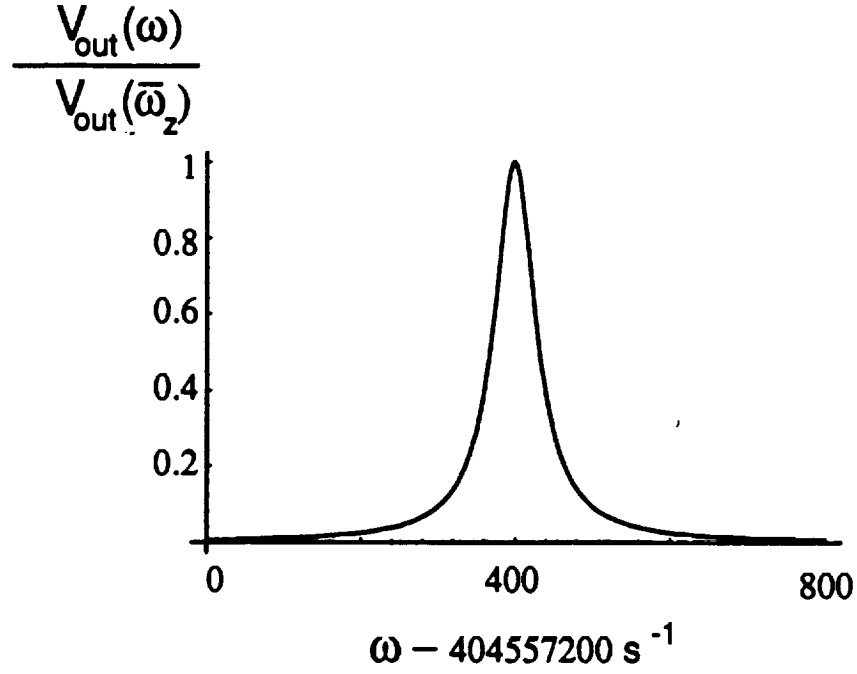
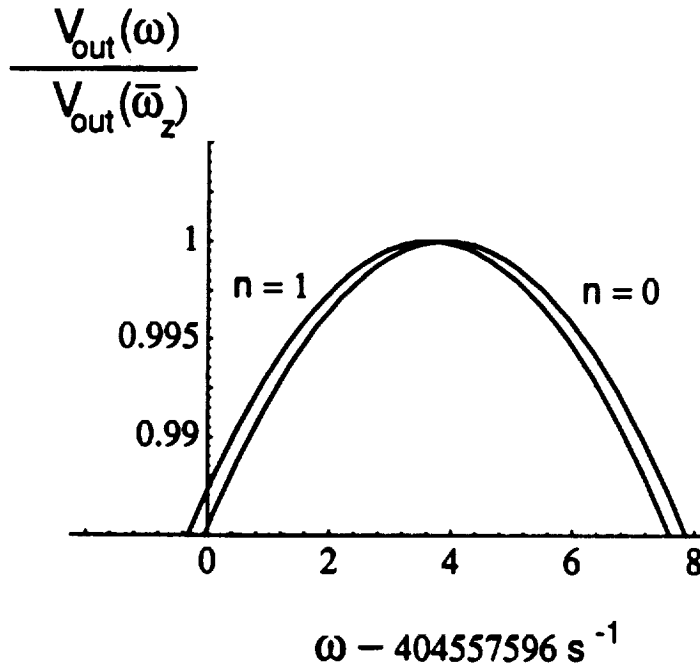FIG. 3 The resonance of the normalized output variance of the signal for $\omega_e = \bar{\omega}_z$.

FIG. 4 The amplified top of the resonance of the output variances of the signal for $\hat{n}_c = 1$ and $\hat{n}_c = 0$.

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8 Acknowledgments

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"PHASING" OPERATOR FOR TWO OSCILLATORS IN CLASSICAL FIELD

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Abstract

The origin of the Dicke cooperative states, ad hoc assumed for superradiance in the system of molecules where no mutual interactions exist but all encountering the same field of radiation, was studied by considering two harmonic oscillators driven by a common field of radiation. A phasing operator as $\Phi_N = D(\alpha)^+ P_N D(\alpha)$, where $D(\alpha)$ is the displacing operator and P_N the projection operator for constant energy N for two oscillators, was derived. The eigen states of the phasing operator Φ_N are found to show a finite correlation as for the Dicke cooperative states.

1 Introduction

The important notion put forward by Dicke [1], that molecules can not be treated as independent when the molecules are interacting each with a common field of radiation, introduced the well known phenomena of superradiance with the ad hoc assumption of cooperative states. This ad hoc assumption of the Dicke's cooperative states may be a natural consequence of the fact that a forced quantum oscillator can be described in terms of the Glauber's coherent states [2]. However, it may be more enlightening to examine in rigorous quantum mechanics how two independent molecules (harmonic oscillators) are getting correlated simply by having separate interactions with a common field of radiation.

Furthermore, it may be more interesting if we restrict the interaction between the radiation and the molecular system to be "elastic", that is , no net transfer of energy between the molecular system and the radiation field.

Naively this restriction may correspond to an elastic light scattering from the two molecule system and a possibility of phasing or correlation of the two molecules by this continuous scattering of light (more correctly a driving field of radiation in the elastic channel of interaction).

2   Two Driven Oscillators

We consider two harmonic oscillators driven by a classical field :

$$H = \hbar\omega(a_1^+ a_1 + a_2^+ a_2) - x_0(a_1^+ + a_1 + a_2^+ + a_2)E(t) \tag{1}$$

where we do not have a direct coupling between the two oscillators. This system seems to be considered as trivial because a single harmonic oscillator driven by a classical field is fully understood [3].

Since we will be extending the solutions of the single harmonic oscillator driven by a classical field to the two oscillator problem of Eq.(1), we want to recollect here some important results of a driven harmonic oscillator [3] :

$$D(\alpha)|0\rangle = \exp(\alpha a^+ - \alpha^* a)|0\rangle = |\alpha\rangle \tag{2}$$

$$D(\alpha)|n\rangle = ((a^+ - \alpha^*)^n/\sqrt{n!})|\alpha\rangle \tag{3}$$

where D(α) is the displacement operator, $|\alpha\rangle$ coherent state of Glauber,

$$\alpha = (ix_0/\hbar) \int_{-\infty}^{\infty} E(t) \exp i\omega t dt, \quad \exp(\alpha a^+ - \alpha^* a) = \exp(-|\alpha|^2/2) \exp(\alpha a^+) \exp(-\alpha^* a).$$

We address now to the two oscillator problem of Eq.(1).

Suppose the two oscillators are prepared in a state $|N\rangle$ of total energy N = n_1+n_2, then we let $|N\rangle$ be driven by a classical field D(α) but we restrict the driven system D(α)$|N\rangle$ to remain at the same energy of $|N\rangle$.

The quantum mechanical matrix element corresponding to this process may be written as

$$\langle N|D^+(\alpha)P_N D(\alpha)|N\rangle \tag{4}$$

where P_N represents the projection operator for states of total energy N :

$$P_N = \sum_{n=0}^{N} |N-n\rangle|n\rangle\langle n|\langle N-n| \tag{5}$$

Alternatively we may define a new operator :

$$\Phi_N \equiv D^+(\alpha)P_N D(\alpha) \tag{6}$$

and Eq.(4) can be written as

$$\langle N|\Phi_N|N\rangle \tag{7}$$

This implies some particular $|N\rangle$ states can become eigen states of the operator Φ_N.

3   Evaluation of $\langle N|\Phi_N|N\rangle$

For two harmonic oscillator states of total energy N we can write

$$|N\rangle = \sum_{n=0}^{N} c_n |N-n\rangle |n\rangle \tag{8}$$

From eq.(3) and eq.(8) we obtain

$$D(\alpha)|N\rangle = \sum_{n=0}^{N} c_n \left((a_1^+ - \alpha_1^*)^{N-n} (a_2^+ - \alpha_2^*)^n / \sqrt{(N-n)!}\sqrt{n!} \right) |\alpha\rangle_{(1)}|\alpha\rangle_{(2)} \tag{9}$$

where the suffix (1) and (2) refer to the oscillator index. Equation (5) can be rewritten as

$$P_N = \sum_{n=0}^{N} \left((a_1^+)^{N-n}(a_2^+)^n / \sqrt{(N-n)!}\sqrt{n!} \right) |0\rangle_{(2)}|0\rangle_{(1)}\langle 0|_{(1)}\langle 0|_{(2)} \left(a_2^n a_1^{N-n} / \sqrt{n!}\sqrt{(N-n)!} \right) \tag{10}$$

From eqs.(6), (9) and (10) we obtain

$$\langle N|\Phi_N|N\rangle = \sum_{n=0}^{N} | \sum_{m=0}^{N} c_m \langle 0|_{(2)}\langle 0|_{(1)} \{ a_2^n (a_2^+ - \alpha_2^*)^m a_1^{N-n}(a_1^+ - \alpha_1^*)^{N-m}$$
$$/ \sqrt{n!}\sqrt{m!}\sqrt{(N-n)!}\sqrt{(N-m)!} \} |\alpha\rangle_{(1)}|\alpha\rangle_{(2)} |^2 \tag{11}$$

Making use of $a|\alpha\rangle = \alpha|\alpha\rangle$ and $a^n(a^+ - \alpha^*)^m = (a^+ - \alpha^*)^m a^n + m!(a^+ - \alpha^*)^{m-n}/(m-n)!$
we can obtain

$$\langle N|\Phi_N|N\rangle = \sum_{n=0}^{N} | \sum_{m=0}^{N} c_m \langle 0|_{(2)}\langle 0|_{(1)} \{ \sqrt{m!}(a_2^+ - \alpha_2^*)^{m-n}/\sqrt{n!}(m-n)!$$
$$+ (a_2^+ - \alpha_2^*)^m \alpha_2^n / \sqrt{m!}\sqrt{n!} \} \{ \sqrt{(N-m)!}\, (a_1^+ - \alpha_1^*)^{n-m}/\sqrt{(N-n)!}(n-m)!$$
$$+ (a_1^+ - \alpha_1^*)^{N-m} \alpha_1^{N-n} / \sqrt{(N-m)!}\sqrt{(N-n)!} \} |\alpha\rangle_{(1)}|\alpha\rangle_{(2)} |^2 \tag{12}$$

Since we are dealing with two identical molecules and the same common field of radiation we may set $\alpha_1 = \alpha_2 = \alpha$ and make use of the following [3],

$$a|0\rangle = 0, \quad \langle 0|\alpha\rangle = \exp(-|\alpha|^2/2) \tag{13}$$

to obtain

$$\langle N|\Phi_N|N\rangle = \sum_{n=0}^{N}\sum_{m=0}^{N}\sum_{l=0}^{N} c_m c_l^* \{ \exp(-2|\alpha|^2)/n!(N-n)! \} \{ \sqrt{m!}\sqrt{(N-m)!}\, \delta_{mn}$$
$$+ (-1)^{N-n} \sqrt{m!}\, |\alpha|^{2(N-n)}/(m-n)!\sqrt{(N-m)!}$$
$$+ (-1)^n \sqrt{(N-m)!}\, |\alpha|^{2n}/(n-m)!\sqrt{m!}$$
$$+ (-1)^N |\alpha|^{2N}/\sqrt{m!}\sqrt{(N-m)!} \} \{ \sqrt{(N-l)!}\sqrt{l!}\, \delta_{ln}$$
$$+ (-1)^{N-n}\sqrt{l!}\, |\alpha|^{2(N-n)}/(l-n)!\sqrt{(N-l)!}$$
$$+ (-1)^n \sqrt{(N-l)!}\, |\alpha|^{2n}/(n-l)!\sqrt{l!}$$
$$+ (-1)^N |\alpha|^{2N}/\sqrt{l!}\sqrt{(N-l)!} \} \tag{14}$$

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Since $|N\rangle$ of eq.(8) can be represented by a (N+1)–dimensional state vector, Φ_N can be defined as a (N+1)×(N+1) matrix operator to give

$$\langle N|\Phi_N|N\rangle = \sum_{m=0}^{N}\sum_{l=0}^{N} c_m c_l^* \Phi_N^{lm} \tag{15}$$

where we find, from eq.(14), the (l,m) matrix element of the operator Φ_N as follows:

$$
\begin{aligned}
\Phi_N^{lm} = \sum_{n=0}^{N} (\exp -2|\alpha|^2/n!(N-n)!)\{&\sqrt{m!}\sqrt{(N-m)!}\,\delta_{mn} \\
& + (-1)^{(N-n)}\sqrt{m!}\,|\alpha|^{2(N-n)}/(m-n)!\sqrt{(N-m)!} \\
& + (-1)^n\sqrt{(N-m)!}\,|\alpha|^{2n}/(n-m)!\sqrt{m!} \\
& + (-1)^N|\alpha|^{2N}/\sqrt{m!}\sqrt{(N-m)!}\}\{\sqrt{(N-l)!}\sqrt{l!}\,\delta_{ln} \\
& + (-1)^{N-n}\sqrt{l!}\,|\alpha|^{2(N-n)}/(l-n)!\sqrt{(N-l)!} \\
& + (-1)^n\sqrt{(N-l)!}\,|\alpha|^{2n}/(n-l)!\sqrt{l!} \\
& + (-1)^N|\alpha|^{2N}/\sqrt{l!}\sqrt{(N-l)!}\}
\end{aligned}
\tag{16}
$$

We can check for the correct limiting values:

$$
\begin{aligned}
\lim_{\alpha\to 0}\Phi_N^{lm}(\alpha) &= \delta_{lm} \\
\lim_{\alpha\to 0}\langle N|\Phi_N|N\rangle &= \sum_{n=0}^{N}\left|\sum_{m=0}^{N} c_m\left(\sqrt{m!}\sqrt{(N-m)!}/\sqrt{n!}\sqrt{(N-n)!}\right)\delta_{mn}\right|^2 \\
&= \sum_{n=0}^{N}|c_n|^2 = 1
\end{aligned}
$$

4 Φ_N As Phasing Operator

In order to probe into the physical meaning of the operator Φ_N we illustrate for the simplest nontrivial case of $N = n_1 + n_2 = 1$.

The Φ_N operator is then given in the form

$$\Phi_{N=1} = \begin{pmatrix} \Phi_1^{00} & \Phi_1^{01} \\ \\ \Phi_1^{10} & \Phi_1^{11} \end{pmatrix} \tag{17}$$

and the matrix elements are obtained from eq.(16) as follows:

$$
\begin{aligned}
\Phi_1^{00} &= \Phi_1^{11} = \exp(-2|\alpha|^2)(4 - 8|\alpha|^2 + 8|\alpha|^4) \\
\Phi_1^{01} &= \Phi_1^{10} = \exp(-2|\alpha|^2)(-8|\alpha|^2 + 8|\alpha|^4)
\end{aligned}
$$

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Diagonalizing the matrix of eq.(17) gives the eigen values λ_1 and λ_2 as

$$\lambda_1 = \Phi^{00} + \Phi^{01},$$
$$\lambda_2 = \Phi^{00} - \Phi^{01} \tag{18}$$

The corresponding eigen states $|\lambda_1\rangle$ and $|\lambda_2\rangle$ are obtained as follows:

$$|\lambda_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{(1)}|1\rangle_{(2)} + |1\rangle_{(1)}|0\rangle_{(2)}),$$
$$|\lambda_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{(1)}|1\rangle_{(2)} - |1\rangle_{(1)}|0\rangle_{(2)}) \tag{19}$$

Correlation or phasing of two oscillators can be measured by a value of $< x_1 \cdot x_2 >$ where x_1 and x_2 are the two harmonic oscillator coordinates of displacement [4].

We can see easily

$$\langle 0|_{(2)}\langle 1|_{(1)} x_1 \cdot x_2 |1\rangle_{(1)}|0\rangle_{(2)} = \langle 0|_{(1)}\langle 1|_{(2)} x_1 \cdot x_2 |1\rangle_{(2)}|0\rangle_{(1)} = 0$$

for the case of $\alpha = 0$, but for our eigen states of Φ_N we obtain

$$\langle \lambda_1 | x_1 \cdot x_2 | \lambda_1 \rangle = \langle \lambda_2 | x_1 \cdot x_2 | \lambda_2 \rangle$$
$$= \frac{1}{2} x_0^2 \{\langle 0|_{(2)}\langle 1|_{(1)} a_1^+ a_2 |0\rangle_{(1)}|1\rangle_{(2)} + \langle 1|_{(2)}\langle 0|_{(1)} a_1 a_2^+ |1\rangle_{(1)}|0\rangle_{(2)}\}$$
$$= x_0^2$$

We can thus see that the two noninteracting molecules in the common driving field of radiation find themselves as correlated. The correlated eigen states of the concerning Hamiltonian of eq.(1) can be found as the eigen states of the phasing operator Φ_N as introduced in eq.(6).

The existence of the correlated eigen states of the two oscillator Hamiltonian thus justifies the ad hoc assumption of the Dicke cooperative states for independent molecules all in the same common field of radiation.

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Generalised squeezing and information theory approach to quantum entanglement

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Abstract

It is shown that the usual one and two-mode squeezing are based on reducible representations of the SU(1,1) group. Generalised squeezing is introduced with the use of different SU(1,1) rotations on each irreducible sector. Two-mode squeezing entangles the modes and information theory methods are used to study this entanglement. The entanglement of three modes is also studied with the use of the strong subadditivity property of the entropy.

1. Introduction

In a recent paper [1] we have explained that two-mode squeezing is based on reducible representations of the SU(1,1). The various irreducible sectors have been identified and different SU(1,1) rotations have been performed on each of them, generalizing in this way the concept of squeezing. In this paper we extend these ideas. In section 2 we consider one mode squeezing and prove that it is also based on reducible representations of SU(1,1). The two irreducible sectors are identified and different SU(1,1) rotations are applied on each of them, generalising in this way the concept of one-mode squeezing. In section 3 the two-mode case is considered in connection with both the SU(1,1) and SU(2) groups. Some of the results presented in [1] are briefly reviewed here. Each irreducible sector of the SU(1,1) (or SU(2)) group is squeezed independently and the generalised squeezed state is characterised by an infinite number of squeezing parameters. Hamiltonians which will lead to this type of squeezing, are presented.

Two-mode squeezing entangles the two modes. Especially our generalised squeezing entangles them in a very complicated way. One approach to study this entanglement is by using information theory methods. In section 4 we use the subadditivity and strong subadditivity properties of the entropy to define quantities which express the entanglement of two and three quantum systems. Especially interesting are the results for three entangled systems, because they indicate that this case is a non-trivial generalisation of the two system entanglement. The latter case has of course been discussed since the beginning of quantum mechanics; but it is only recently that some preliminary discussion of the former case has appeared [2]. Our results based on information theory methods suggest that the three system entanglement is a very interesting problem that requires further study.

2. Generalised one-mode squeezing

We consider the harmonic oscillator Hilbert space H and express it as

$$H = H_0 + H_1 \tag{1}$$

where H_0 is the subspace spanned by the even number eigenstates and H_1 the subspace spanned by the odd number eigenstates. We also consider the corresponding projection operators to these subspaces:

$$\pi_0 = \sum_{N=0}^{\infty} |\, 2N > < 2N \,|$$

$$\pi_1 = \sum_{N=0}^{\infty} |\, 2N + 1 > < 2N + 1 \,| \tag{2}$$

$$\pi_0 + \pi_1 = 1$$

The one mode squeezing operators are defined as:

$$S(r, \theta, \lambda) = \exp\left[-\frac{1}{2} r\, e^{-i\theta} K_+ - \frac{1}{2} r\, e^{i\theta} K_-\right] \exp(i\lambda K_0)$$

$$K_0 = \frac{1}{2} a^+ a + \frac{1}{4} \quad ; \quad K_+ = \frac{1}{2} a^{+2} \quad ; \quad K_- = \frac{1}{2} a^2$$

$$[K_0, K_{\pm}] = \pm K_{\pm} \quad ; \quad [K_-, K_+] = 2 K_0 \tag{3}$$

$$K^2 = K_0^2 - \frac{1}{2}(K_+ K_- + K_- K_+) = k(k-1) = -\frac{3}{16}$$

They form a reducible representation of SU(1,1). More specifically, they form the $k = 1/4$ irreducible representation when they act on H_0 only; and the $k = 3/4$ irreducible representation when they act on H_1 only[3]. Related to this is the fact that:

$$[S(r, \theta, \lambda), \pi_0] = [S(r, \theta, \lambda), \pi_1] = 0 \tag{4}$$

The following unitary operator squeezes independently each irreducible sector:

$$U(r_0\, \theta_0\, \lambda_0 \; ; \; r_1\, \theta_1\, \lambda_1) = S(r_0, \theta_0, \lambda_0)\, \pi_0 + S(r_1, \theta_1, \lambda_1)\, \pi_1 \tag{5}$$

This is more general than the operator of equ.(3). Only in the special case

$$r_0 = r_1 \quad ; \quad \theta_0 = \theta_1 \quad ; \quad \lambda_0 = \lambda_1 \tag{6}$$

the operator (5) reduces to the operator (3). Acting with the operator (5) on a coherent state $|\, A >$, we get a generalised squeezed state:

$$|\, A \; ; \; r_0\, \theta_0\, \lambda_0 \; ; \; r_1\, \theta_1\, \lambda_1 > = U(r_0\, \theta_0\, \lambda_0 \; ; \; r_1\, \theta_1\, \lambda_1)\, |\, A >$$

$$= S(r_0, \theta_0, \lambda_0)\, \pi_0\, |\, A > + S(r_1, \theta_1, \lambda_1)\, \pi_1\, |\, A > \tag{7}$$

In the special case of equ.(6) this reduces to the usual squeezed states.

In systems described by the Hamiltonian

$$H = \omega' a^+ a + (\mu_o a^{+2} + \mu_o^* a^2) \pi_o + (\mu_1 a^{+2} + \mu_1^* a^2) \pi_1 \tag{8}$$

ordinary coherent states will evolve into the generalised squeezed states (7). In the special case $\mu_o = \mu_1$ the Hamiltonian (8) reduces to the Hamiltonian

$$H = \omega a^+ a + \mu a^2 + \mu a^{+2} \tag{9}$$

which is associated to the usual squeezed states.

3. Generalised two-mode squeezing

The appropriate group for the study of two-mode quadratic Hamiltonians is $Sp(4,R)$ [4]. In this paper we shall only consider its subgroups $SU(1,1)$ and $SU(2)$ in connection with the Hamiltonians:

$$H_1 = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \mu a_1 a_2 + \mu^* a_1^+ a_2^+ \tag{10}$$

$$H_2 = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \mu a_1 a_2^+ + \mu^* a_1^+ a_2 \tag{11}$$

correspondingly. Both of these Hamiltonians have been used extensively in quantum optics problems [5].

Starting with the $SU(1,1)$ group we express the two-mode Hilbert space as

$$H_A \times H_B = \sum_{k=-\infty}^{\infty} H_k \tag{12}$$

where H_k is the subspace spanned by the number eigenstates

$$H_k = (\mid N + k, N > \quad ; \quad N = \max (o, - k), \ldots \infty) \tag{13}$$

We also introduce the corresponding projection operators

$$\pi_k = \sum \mid N + k, N> < N + k, N \mid$$

$$\sum \pi_k = 1 \tag{14}$$

The two-mode $SU(1,1)$ squeezing operators are defined as

$$S(r, \theta, \lambda) = \exp [- \frac{1}{2} r e^{-i\theta} K_+ + \frac{1}{2} r e^{i\theta} K_-] \exp (i \lambda K_o)$$

$$K_+ = a_1^+ a_2^+ \quad ; \quad K_- = a_1 a_2, \quad K_o = \frac{1}{2} (a_1^+ a_1 + a_2^+ a_2 + 1)$$

$$K^2 = \frac{1}{4} (a_1^+ a_1 - a_2^+ a_2)^2 - \frac{1}{4} \tag{15}$$

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They form a reducible representation of SU(1,1). More specifically, when they act on the space H_k only, they form the

$$\ell = \frac{1 + |k|}{2} \tag{16}$$

irreducible representation of SU(1,1) which belongs in the discrete series. Note also that

$$[S(r, \theta, \lambda), \pi_k] = 0 \tag{17}$$

The following unitary operator squeezes independently each irreducible sector:

$$U(\{r_k, \theta_k, \lambda_k\}) = \sum S(r_k, \theta_k, \lambda_k) \pi_k \tag{18}$$

In the special case

$$\begin{aligned}
\cdots &= r_{-1} = r_0 = r_1 = \cdots \\
\cdots &= \theta_{-1} = \theta_0 = \theta_1 = \cdots \\
\cdots &= \lambda_{-1} = \lambda_0 = \lambda_1 = \cdots
\end{aligned} \tag{19}$$

the operators (18) reduce to the operators (15).

Acting with the operators (18) on two-mode coherent states we get generalised two-mode squeezed states:

$$U(\{r_k \theta_k \lambda_k\}) | A_1, A_2 > = \sum_k S(r_k, \theta_k, \lambda_k) \pi_k | A_1, A_2 > \tag{20}$$

In the special case of equ.(19) they reduce to the usual two-mode squeezed states. In systems described by the Hamiltonian

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \sum_k (\mu_k a_1 a_2 + \mu_k^* a_1^+ a_2^+) \pi_k \tag{21}$$

ordinary coherent states will evolve into the states of equ.(20). In the special case that all the μ_k are equal to each other, the Hamiltonian (21) reduces to the Hamiltonian (10).

In the case of the SU(2) group we express the two-mode Hilbert space as

$$H_A \times H_B = \sum_j H_{2j+1} \tag{22}$$

$$j = 0, \tfrac{1}{2}, 1, \ldots .$$

where H_{2j+1} is the subspace spanned by the number eigenstates

$$H_{2j+1} = \{ | N, 2j-N > ; N = 0, \ldots, (2j) \} \tag{23}$$

We also introduce the corresponding projection operators

$$\pi_{2j+1} = \sum_{N=0}^{2j} | N, 2j-N> < N, 2j-N|$$

$$\sum \pi_{2j+1} = 1 \tag{24}$$

The SU(2) squeezing operators are defined as:

$$T(r, \theta, \lambda) = \exp[-\frac{1}{2} r e^{-i\theta} J_+ + \frac{1}{2} r e^{i\theta} J_-] \exp(i\lambda J_0)$$

$$J_+ = a_1^+ a_2 \quad ; \quad J_- = a_1 a_2^+ \quad ; \quad J_0 = \frac{1}{2}(a_1^+ a_1 - a_2^+ a_2)$$

$$J^2 = [\frac{1}{2}(a_1^+ a_1 + a_2^+ a_2)][\frac{1}{2}(a_1^+ a_1 + a_2^+ a_2 + 1)] \tag{25}$$

They form a reducible representation of SU(2). When they act on the space H_{2j+1} only, they form the j irreducible representation of SU(2). Note also that:

$$[T(r, \theta, \lambda), \pi_{2j+1}] = 0 \tag{26}$$

The following unitary operator performs SU(2) rotations independently on each irreducible sector:

$$U((r_{2j+1}, \theta_{2j+1}, \lambda_{2j+1})) = \sum T(r_{2j+1}, \theta_{2j+1}, \lambda_{2j+1}) \pi_{2j+1} \tag{27}$$

In the special case:

$$\begin{array}{l} r_1 = r_2 = \ldots \\ \theta_1 = \theta_2 = \ldots \\ \lambda_1 = \lambda_2 = \ldots \end{array} \tag{28}$$

The operators (27) reduce to the operators (25). Acting with the operators (27) on two-mode coherent states we get the states:

$$U((r_{2j+1}, \theta_{2j+1}, \lambda_{2j+1})) | A_1, A_2 > =$$
$$\sum_j T(r_{2j+1}, \theta_{2j+1}, \lambda_{2j+1}) \pi_{2j+1} | A_1, A_2 > \tag{29}$$

They will be formed during the time evolution of ordinary coherent states in systems described by the Hamiltonian:

$$H = \omega_1 a_1^+ a_1 + \omega_2 a_2^+ a_2 + \sum \pi_{2j+1} (\mu_{2j+1} a_1 a_2^+ + \mu_{2j+1}^* a_1^+ a_2) \tag{30}$$

In the special case that all the μ_{2j+1} are equal to each other, the Hamiltonian (30) reduces to the Hamiltonian (11).

The uncertainty properties of the states (20), (29) have been studied in [1]. The results presented there show that both of these states exhibit squeezing.

4. Information theory approach to quantum entanglement

In this section we use quantum information theory methods for the study of two- and three-mode correlated systems. Let ρ be a two-mode density matrix and $\langle N_1 \rangle$, $\langle N_2 \rangle$ the average number of photons in the two modes. As in our previous work [6] we define the information contained in this density matrix as

$$I = S_{max} - S(\rho) = S[\rho_1^{th}(\langle N_1 \rangle) \times \rho_2^{th}(\langle N_2 \rangle)] - S(\rho)$$

$$S(\rho) = - Tr\, \rho \ln \rho$$

$$\rho_1^{th}(\langle N_1 \rangle) = \frac{\langle N_1 \rangle^{Ni}}{(1 + \langle N_1 \rangle)^{1+N_i}} \; |N_i\rangle\langle N_i| \quad ; \quad i = 1, 2 \qquad (31)$$

Following the negentropy ideas of Brillouin we subtract here the entropy of the system from the maximum entropy that the system could have had, with the average number of photons in the two modes been kept fixed. The maximum entropy is equal to the entropy of a thermal system with an average number of photons in the two modes $\langle N_1 \rangle$, $\langle N_2 \rangle$. Taking partial traces, we define:

$$\rho_1 = Tr_2\, \rho \quad ; \quad \rho_2 = Tr_1\, \rho \qquad (32)$$

and express the information (31) as [7, 8]

$$I = I_1 + I_2 + I_{12}$$

$$I_1 = S[\rho_1^{th}(\langle N_1 \rangle)] - S(\rho_1)$$

$$I_{12} = S(\rho_1) + S(\rho_2) - S(\rho) \qquad (33)$$

I_i is the information in the mode i; and I_{12} is the information in the correlation between the two modes. The subadditivity property ensures that the I_{12} is non-negative. Numerical evaluation of the quantities I_1, I_2, I_{12} for several examples has been presented in [1].

A non-trivial extension of these ideas occurs in the case of three correlated modes. The information in this case is given by

$$I = S(\rho^{th}) - S(\rho)$$

$$\rho^{th} = \rho_1^{th}(\langle N_1 \rangle) \times \rho_2^{th}(\langle N_2 \rangle) \times \rho_3^{th}(\langle N_3 \rangle) \qquad (34)$$

We define

$$\rho_{ij} = Tr_k\, \rho \quad , \quad \rho_i = Tr_{jk}\, \rho$$

$$I_i = S[\rho_i^{th}(\langle N_i \rangle)] - S(\rho_i)$$

$$I_{ij} = S(\rho_i) + S(\rho_i) - S(\rho_{ij}) \qquad (35)$$

I_i is the information in the mode i. I_{ij} is the information in the correlation between the modes (i,j). We then express the total information in the three-mode system as:

$$I = I_1 + I_2 + I_3 + I_{12} + I_{23} + I(12 ; 23) \qquad (36)$$

where

$$I(12 ; 23) = S(\rho_{12}) + S(\rho_{23}) - S(\rho) - S(\rho_2) \qquad (37)$$

The strong subadditivity property [9] ensures that the quantity $I(12 ; 23)$ is non-negative. For symmetry reasons, somebody might be tempted to split $I(12 ; 23)$ as:

$$I(12 ; 23) = I_{13} + A \qquad (38)$$

so that he can express the information I of equ.(36), as the sum of the three informations in the three modes; the three informations in the correlated pairs of modes; and the quantity A characterising the correlation between all modes. However the quantity A is not necessarily positive and its interpretation as information would be incorrect. Therefore, the information I of a three-mode system is the sum of the three informations in the three modes; the two correlation informations in two of the pairs; and the information $I(12 ; 23)$ of equ.(37) which describes new types of correlations in the three-mode systems. This result can be used as a "guide" of how to study the entanglement of three systems. It is seen that three system entanglement is a non-trivial generalisation of two system entanglement.

5.   **Discussion**

In many cases the concept of squeezing is based on reducible representations of the SU(1,1) (or SU(2)) group. In these cases different SU(1,1) (or SU(2)) rotations on each irreducible sector lead to generalised squeezing. These ideas have been applied to both one-mode and two-mode squeezing.

Two-mode squeezing correlates the two-modes and information theory methods have been used for the study of these correlations. The subadditivity and strong subadditivity properties of the entropy have been used for the study of two and three correlated systems, correspondingly. It has been shown that the entanglement of three systems is a non-trivial generalisation of the entanglement of two systems. Further work is required in this direction.

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COMPLETENESS PROPERTIES OF THE MINIMUM UNCERTAINTY STATES

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Abstract

The completeness properties of the Schrödinger minimum uncertainty states (SMUS) and of some of their subsets are considered. The invariant measures and the resolution unity measures for the set of SMUS are constructed and the representation of squeezing and correlating operator and SMUS as superpositions of Glauber coherent states on the real line is elucidated.

1 Introduction

In the present paper we consider the completeness properties of the set (and some subsets) of the states, which minimize the Schrödinger–Robertson uncertainty relation [1]

$$\sigma_q^2 \sigma_p^2 \geq \frac{1}{4}(1 + 4c^2), \tag{1}$$

where σ_q and σ_p are the dispersions of the quadrature operators Q and P ($[Q, P] = i$),

$$\sigma_q = \langle X^2 \rangle - \langle X \rangle^2, \qquad X = Q, P,$$

and c is their covariation,

$$c = (1/2)\langle QP + PQ \rangle - \langle Q \rangle \langle P \rangle.$$

We call such states Schrödinger minimum uncertainty states (SMUS). In fact they were introduced by Dodonov, Khurmyshev and Man'ko [2] and studed as correlated states (see Ref. [3] and references therein). When the covariation is zero, $c = 0$, one gets the Heizenberg minimum uncertainty states (HMUS) and when in addition to this the dispersions are equal, $\sigma_q = \sigma_p$, the corresponding MUS are the Glauber coherent states (CS) [4].

From the group–theoretical point of view SMUS are equivalent [5] to the group–related CS [6] with maximal symmetry [7], the group in this case being the semidirect product $H_w \otimes SU(1,1)$ (see also [8]) of the Heizenberg–Weyl group H_w and the quasiunitary group $SU(1,1) \sim Sp(2, R)$. Up to a phase factor they coinside [5] with the Stoler states [9], known also as squeezed states or two–photon CS [10] widely used in quantum optics (see for example the review papers [11, 12] and references therein). The stable time–evolution of SMUS, which is important for the squeezing and correlating processes, is considered in [5]. In other notations it was in fact obtained in [13].

SMUS are continuous set of states, which is clear from the definition. For such sets of states the completeness properties (in the Hilbert space $\mathcal{H}$) are very important for the applications in

mathematical and theoretical physics. In the weak sense [6] the completeness of a continuous set of states $|x\rangle$ is defined as a dense subset in $\mathcal{H}$, while in the strong sense it is defined as the (integral) resolution of the unity operator

$$1 = \int |x\rangle\langle x|\, d\mu(x), \qquad (2)$$

where $d\mu(x)$ is a positive measure in the label space $X \ni x$. Such complete set of states $|x\rangle$ is called (in general sense) CS [6]. The group–related CS are always complete in the weak sense, while the resolution of unity has to be proved in every case. A sufficient conditions is the square integrability of the corresponding representation of the group involved against the invariant measure.

In this paper we consider the resolution of unity (2) for the set of SMUS and for some of their subsets. First we construct the corresponding invariant measures and check the square integrability against them. Since the latter failed to be valid we look and find the noninvariant measures, which provide the resolution (2). We call such measures the resolution unity measures (RUM). In other notations (i.e. in no relation to SMUS) for the $H_w \otimes SU(1,1)$–CS RUM were considered in [8],

According to the definition of CS they are always over complete (at least in the weak sense [6]) in $\mathcal{H}$ family of states. Then it worth looking for a more simple subset of CS which is also complete in $\mathcal{H}$ or in some subspace (or even subset) of $\mathcal{H}$. We consider this problem in the last section. In particular we construct the squeezing and correlating operators as integral along the real line of projectors on the Glauber CS and reproduce the result of Janszky and Vinogradov [14] for the superpositions of Glauber CS along the real axis.

2   The Invariant Measures and RUM for SMUS

Up to a phase factor SMUS can be written in the form [5] of the $H_w \otimes SU(1,1)$–CS with maximal symmetry

$$\begin{aligned}
|\xi;\eta\rangle &= N(\xi;\eta)\exp\left[\frac{\xi}{2}a^{\dagger^2} + \eta a^{\dagger}\right]|0\rangle, \\
N(\xi;\eta) &= (1-|\xi|^2)^{1/4}\exp\left[-\frac{1}{2}\frac{|\eta|^2 + \mathrm{Re}(\bar{\xi}\eta^2)}{1-|\xi|^2}\right],
\end{aligned} \qquad (3)$$

where $a^{\dagger} = (1/\sqrt{2})(Q - iP)$ is the boson creation operator, $[a, a^{\dagger}] = 1$, η is arbitrary complex number and ξ belongs to the unit disk, $|\xi| \leq 1$. One also has the relation to the Stoler states $|z;\alpha\rangle$ (i.e. the squeezed states or the two–photon CS)

$$|\xi;\eta\rangle = |z;\alpha\rangle = \exp\left[\frac{1}{2}\left(za^{\dagger^2} - \bar{z}a^2\right)\right]|a\rangle.$$

where $|\alpha\rangle$ is the Glauber CS and

$$\xi = e^{i\phi}\tanh|z|, \qquad \eta = a' - \xi\bar{a}', \qquad \alpha' = \alpha\cosh|z| + \bar{\alpha}e^{i\phi}\sinh|z|.$$

The second momenta σ_q, σ_p and c are expressed in terms of ξ in [5, 15] and in terms of z in [8].

The $H_w \textcircled{S} SU(1,1)$-CS (3) are related to the representation $T(g)$, generated by the semidirect sum algebra $h_w \textcircled{D} su(1,1)$ (known also as the one mode two–photon algebra)

$$
\begin{aligned}
h_w &= \text{lin. env. } \{1, a, a^\dagger\}, \\
su(1,1) &= \text{lin. env. } \left\{ K_- = \frac{1}{2}a^2, \quad K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_0 = \frac{1}{2}(a^\dagger a + \frac{1}{2}) \right\}; \\
T(g) &= \exp(\gamma K_+ - \overline{\gamma} K_- + i\omega K_0) \exp(it + \alpha a^\dagger - \bar{\alpha} a) \\
&\equiv T(\gamma, \omega) T(t, \alpha), \qquad g \equiv g(\gamma, \omega; t, \alpha).
\end{aligned}
\tag{4}
$$

In terms of the above group parameters the invariant measure is a product of the $SU(1,1)$– and the H_w–invariant measures,

$$
d\mu(\gamma, \omega; t, \alpha) = \frac{\sinh^2 \Lambda}{\Lambda^2} d^2\gamma \, d^2\omega \, d^2\alpha \, dt, \qquad \Lambda^2 = 4|\gamma|^2 - \frac{1}{4}\omega^2.
\tag{5}
$$

But the representation (4) is not square integrable against the invariant measure (5) on the group manifold. Then we have to look for the invariant measure $d\mu(\xi; \eta)$ on the factor space $G/K \ni (\xi; \eta)$, which is a label space for the SMUS $|\xi; \eta\rangle$, Eq. (3),

$$
d\mu(\xi; \eta) = \frac{d^2\gamma \, d^2\eta}{(1 - |\xi|^2)^3}.
\tag{6}
$$

This measure is not a product of the $SU(1,1)$–invariant measure on the label space $\mathbb{D}_1 \ni \xi$ and the H_w–invariant measure $d^2\eta$ on the label space $\mathbb{C} \ni \eta$. And we still do not have the square integrability, i.e. the right hand side of the Eq. (2) with $|x\rangle = |\xi; \eta\rangle$ and $d\mu(x) = d\mu(\xi; \eta)$ goes to infinity.

Let us now look for the noninvariant resolution unity measure (RUM). The noninvariant RUM if exists is highly nonunique. It is clear from the definition of RUM as a measure providing the resolution (2), that if $d\mu(x)$ is a RUM for a group–related CS $|x\rangle$ then

$$
d\mu_g(x) = d\mu(g \cdot x),
\tag{7}
$$

where $g \cdot x$ denotes the action of the group element on $x \in X$, is a set of RUM. It is an open problem whether the noninvariant RUM exists simultaneously with the invariant one. For the Glauber CS $|\alpha\rangle$ the invariant measure $d^2\alpha$ is the only RUM. In our case of SMUS the simplest noninvariant RUM reads (in Stoler parameters)

$$
d\mu_0(z, \alpha) = \frac{1}{\pi^2} e^{-z\bar{z}} d^2z \, d^2\alpha,
\tag{8}
$$

which can be expressed in terms of ξ, η by means of the relations obtained above. The other measures $d\mu_g(\xi; \eta)$, Eq. (7), are obtained by means of the group action

$$
g \cdot (\xi; \eta) = \left(\frac{\bar{u}\xi - v}{u - \bar{v}\xi}, \frac{\eta - \bar{\alpha}\xi + \alpha}{u - \bar{v}\xi} \right),
\tag{9}
$$

where $g = g(\gamma, \omega; t, \alpha)$ and u, v are the new $SU(1,1)$ parameters,

$$
u = \cosh \Lambda - \frac{i\omega}{2\Lambda} \sinh \Lambda, \qquad v = -\frac{2\gamma}{\Lambda} \sinh \Lambda, \qquad \Lambda^2 = 4|\gamma|^2 - \frac{1}{4}\omega^2.
$$

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As we have already noted the RUM for the $H_w \otimes SU(1,1)$-CS were constructed in Ref. [8]. With the Note added in proof in [8] their measure should read (however we were not able to obtain the resolution of unity by means of this measure)

$$d\mu\,(\xi;\eta) = \frac{2^2\Gamma(2)\exp\left[\mathrm{Re}\,(\bar\xi\eta^2)/(1 - |\xi|^2)\right]}{(1 - |\xi|^2)^{3/2}}\,d^2\xi\,d^2\eta.$$

3 Completeness of Some Subsets of SMUS

The two parameters subset $|\xi_0;\eta\rangle$ of SMUS with fixed ξ_0 (i.e. with fixed second momenta of the quadrature operators) forms a strongly complete system in $\mathcal{H}$

$$\int_{\mathbb{C}} d\nu(\eta)\,|\xi_0;\eta\rangle\langle\eta;\xi_0| = 1, \qquad d\nu(\eta) = \frac{1}{\pi}\left(1 - |\xi_0|^2\right)^{-1} d^2\eta, \tag{10}$$

which in Stoler parameters is known [16] and corresponds to the generalized Glauber CS (i.e. to the H_w-CS with the squeezed and correlated vacuum as the initial vector). Such resolution of unity was used in [16] for construction of new quasi probabilities "based on squeezed state". Note that the RUM in (10) is H_w-invariant and is obtained (up to a constant factor) from the $H_w \otimes SU(1,1)$-invariant measure (6) by fixing $\xi = \xi_0$. If we fix the other complex parameter $\eta = \eta_0$ we get the subset $\{|\xi;\eta_0\rangle\}$ (this is $SU(1,1)$-CS with Glauber CS as initial vector) which however is not complete even in the weak sense in $\mathcal{H}$ since the $SU(1,1)$ representation involved here is not irreducible. If we put $\eta_0 = 0$ we obtain the complete (but only in the weak sense) set of even $SU(1,1)$-CS $|\xi;+\rangle$ [15] in the subspace $\mathcal{H}_+$ of even functions. The state $|\xi;+\rangle$ is in fact squeezed (and/or correlated) vacuum. In the subspace $\mathcal{H}_-$ of odd states we have the strongly complete system of the odd $SU(1,1)$-CS $|\xi;-\rangle$ [15],

$$|\xi;-\rangle = \left(1 - |\xi|^2\right)^{3/4}\exp\left[\xi(a^\dagger)^2/2\right]|1\rangle, \tag{11}$$

$$\int_{\mathbf{D}_1} d\nu(\xi)|\xi;-\rangle\langle-;\xi| = 1_-, \qquad d\nu(\xi) = \frac{1}{2\pi}\frac{d^2\xi}{(1 - |\xi|^2)^2}, \tag{12}$$

where $|1\rangle$ is the first exited state and $d\nu(\xi)$ is the $SU(1,1)$-invariant measure. The state $|\xi;+\rangle$ is the squeezed vacuum, and $|\xi;-\rangle$ is the squeezed one-photon state. Note that $|\xi;-\rangle$ is not SMUS. The second momenta σ_q, σ_p and c in this state obey the equality

$$\sigma_q^2\sigma_p^2 = \frac{1}{4}\left(1 + 4c^2 + 8\right), \tag{13}$$

i.e. $|\xi;-\rangle$ is another type of MUS. As in the case of squeezed ground state it is correlated when $\mathrm{Im}\,\xi \neq 0$ and $\sigma_q \to 0$ when $\xi \to 1$. In the subspaces $\mathcal{H}_\pm$ there are also strongly complete sets of even and odd CS $|\alpha\rangle_\pm$ [17], which are linear combinations of two Glauber CS $|\alpha\rangle$ and $|-\alpha\rangle$.

Let us consider the subset of SMUS $|\xi;\eta\rangle$ with fixed $\xi = \xi_0$ and $\mathrm{Im}\,\eta = \eta_{2,0}$, that is with fixed second momenta $\sigma_q = \sigma_{q,0} \equiv \sigma_0$ and $c = c_0$ and fixed first momentum $\langle P\rangle \equiv p = p_0$. This is the one parameter set of states $|q;\xi_0,p_0\rangle$, $q \sim \mathrm{Re}\,\eta \in \mathbb{R}$. It is the set of CS for the commutative subgroup generated by the unit operator and by P. It is also the subset of general Glauber CS

along the real axis, the initial vector being the squeezed and correlated vacuum, displaced by p_0. The unitary representations of the group of translations (by q along the real line) are highly reducible thereby the set $\{|q; \xi_0, p_0\rangle\}$ is not complete in $\mathcal{H}$ even in the weak sense. Let for simplicity $p_0 = 0$ and consider the operators

$$B(\xi_0) = \int_{\mathbf{R}} |q; \xi_0\rangle\langle\xi_0; q|\, dq. \tag{14}$$

$B(\xi_0)$ is an unbounded (Hermitean) operator, well defined in the Hilbert space $\mathcal{H}$ with the following property: it leaves the set of SMUS stable, that is the states $|\psi'\rangle = B(\xi_0)|\psi\rangle$ is SMUS if $|\psi\rangle$ is. Moreover if $|q\rangle$ is the Glauber CS on the real line then (one can calculate that) $B(\xi_0)|q\rangle$ is an arbitrarily squeezed and correlated state. Thus $B(\xi_0)$ is an (one dimensional) integral representation of the squeezing and correlating operator. One can also get an arbitrary SMUS by means of a fixed operator

$$B = \int_{\mathbf{R}} |q\rangle\langle q|\, dq = B(\xi_0 = 0),$$

but acting on different states $|\psi\rangle$. The obtained state $B|\psi\rangle$ is clearly a superposition of the CS $|q\rangle$ with the weights $\psi(q) = \langle q|\psi\rangle$. If (but not only if) $|\psi\rangle$ is SMUS then $B|\psi\rangle$ is also SMUS with arbitrary c and $\sigma_q > 1$. The representation of squeezed states as superpositions of Glauber CS on the real line was recently considered by Janszky and Vinogradov [14] in the form $\int_{\mathbf{R}} |q\rangle G(q)\, dq$, $G(q)$ being the Gaussian weight function.

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STUDY OF EINSTEIN-PODOLSKY-ROSEN STATE FOR SPACE-TIME VARIABLES IN A TWO PHOTON INTERFERENCE EXPERIMENT

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Abstract

A pair of correlated photons generated from parametric down conversion was sent to two independent Michelson interferometers. Second order interference were studied by means of a coincidence measurement between the outputs of two interferometers. The reported experiment and analysis studied this second order interference phenomena from the point of view of Einstein-Podolsky-Rosen paradox. The experiment was done in two steps. The first step of the experiment used 50 psec and 3 nsec coincidence time window simultaneously. The 50 psec window was able to distinguish a 1.5 cm optical path difference in the interferometers. The interference visibility was measured to be 38% and 21% for 50 psec time window and 22% and 7% for 3 nsec time window, when the optical path difference of the interferometers were 2 cm and 4 cm, respectively. By comparing the visibilities between these two windows, the experiment showed the non-classical effect which resulted from an E.P.R. state. The second step of the experiment used a 20 psec coincidence time window, which was able to distinguish a 6 mm optical path difference in the interferometers. The interference visibilities were measured to be 59% for an optical path difference of 7 mm. This is the first observation of visibility greater than 50% for a two interferometer E.P.R. experiment which demonstrates nonclassical correlation of space-time variables.

1 Introduction

Two photon interferometry has drawn a great deal of attention recently because it provides a tool to study the foundation of quantum mechanics and the fundamental properties of the electromagnetic field. A two photon interference experiment using two independent interferometers was proposed by J. D. Franson[1] which constituted a new type of E.P.R. experiment for space-time variables. Since then several experiments have reported the second order (second order in intensity, fourth order in field) interference effect.[2]-[5] These experiments have shown visibility less than 50% when the optical path difference of the interferometers are greater than the coherence length of the optical beam. The reason that the visibilities are less than 50% is due to the use of large coincidence time windows in these experiments. It has been pointed out that classical models predict a maximum of 50% visibility for these experiments.[2][3][6] Quantum theory predicts visibility greater than 50% for certain entangled states we called E.P.R. state. To make the

type of argument presented by E.P.R.[7] this state must be produced. For this experiment a short coincidence time window is needed to prepare an E.P.R. state.

Recently, a large set of measurements for a two photon interference experiment have been carried out in our laboratory. In this experiment parametric down conversion is used to produce the correlated two photons. The intensity of the down converted radiation used for the experiment is sufficiently low so that a two photon state is produced such that each beam contain at most one photon. Each photon is passed through an independent Michelson interferometer and is then detected by a coincidence counter. If the interferometers are set so that the optical path differences are longer than the coherence length of the fields, there is no first order interference (first order in intensity, second order in field). However, there is second order interference if the optical paths of the two interferometers are approximately equal. The interference arises from the frequency and wave number correlation in a given pair generated by the phase matching conditions, $\omega_1 + \omega_2 = \omega_p$ and $k_1 + k_2 = k_p$, where ω_p and k_p are the pump frequency and wave number. The second order interference is measured by studying the visibility of the interference fringes that are generated by varying the optical path difference of the interferometers. The visibility of the interference can be estimated by classical and quantum models. The classical model never predicts visibility greater than 50%. However, for idealized condition, the quantum model predicts a 100% visibility when the coincidence time window is shorter than the optical path difference. In this case, the registration time of one photon traversing the long path and the other following the short path of the interferometers is outside the coincidence window and will not be registered by the coincidence counter. As shall be explained below, the use of a short coincidence time window is equivalent to preparing a type of entangled state discussed in the original E.P.R. paper.[7]

We report in this paper an experiment which for the first time shows second order interference visibility greater than 50% for two independent interferometers. We also show in detail how the E.P.R. state is generated for the coincidence counting experiment.

2 E.P.R. Paradox and E.P.R. State

The E.P.R. paradox was based on the argument that non-commuting observables can have simultaneous reality.[7] E.P.R. first gave their criterion: if, without in any way disturbing the system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of reality corresponding to this physical quantity. The gedankan experiment discussed by Einstein, Podolsky and Rosen was modified by Bohm in 1951.[8] In Bohm's version a singlet state $| \psi \rangle$ of two spin $\frac{1}{2}$ particles is produced by some source,

$$| \psi \rangle = \frac{1}{\sqrt{2}}[| \hat{n}_1^+ \rangle \otimes | \hat{n}_2^- \rangle - | \hat{n}_1^- \rangle \otimes | \hat{n}_2^+ \rangle] \tag{1}$$

where $| \hat{n}j^\pm \rangle$ quantum mechanically describe a state in which particle j has spin "up" or "down" along the direction $\hat{n}$. For this state, if the spin of particle 1 is measured along the x-axis, particle 2 will be found to have its spin oppositely aligned along the x-axis with unit probability. Thus, the x-component of the spin of particle 2 can be measured without in any way disturbing it and so is an element of reality according to the E.P.R. criterion. It is similarly found that the other components of the spin of particle 2 can be determined as elements of physical reality and must exist without considering which component is being measured. Of course, this point of view is

different from that of quantum mechanics. Philosophical arguments aside, the predictability of the spin of particle 2 with 100% certainty after measuring the spin of particle 1 is a mathematical consequence of quantum theory applied to state of the form (1). States of the type (1) are a particular type of entangled state,[9][10] which will be called E.P.R. state. It is the E.P.R. state which leading to the nonclassical interference behavior of the two particle system. It is the E.P.R. state has no classical analog.

The existence of polarization E.P.R. states have been experimentally demonstrated.[11]-[14] The new type of E.P.R. experiment considers the measurement of position and time correlation in contrast to the historical measurement of polarization correlation. The key element is to seek an E.P.R. state for space and time variables. This is closer to the original E.P.R. gedankan experiment for the determination of position and momentum of a photon. In this case, see FIG. 1, the two-photon E.P.R. state sought is of the form,

$$\Psi_{EPR} = \Psi(L_1, L_2) + \Psi(S_1, S_2) \tag{2}$$

where the first amplitude corresponds to the photons both passing along the longer arms of the interferometers and the second amplitude corresponds to them both following the shorter arms. It is clear that this is an E.P.R. state of the type defined above, if photon 1 is determined in the long (short) arm, then, photon 2 follows the long (short) path. The photon path is then an element of physical reality according to the E.P.R. criterion. In practice state (2) is produced by parametric down conversion. If we assume perfect phase matching, then because $k_1 + k_2 = $ constant, a momentum measurement of one photon determines the momentum of the other. So the momentum of the photon is also an element of physical reality. If this state does exist, in idealized conditions, its signature is an interference visibility of 100% when the optical path difference of the two independent interferometers are equal.

However, the output of the interferometers is not state (2), but rather the state

$$\Psi = \Psi(L_1, L_2) + \Psi(S_1, S_2) + \Psi(L_1, S_2) + \Psi(S_1, L_2) \tag{3}$$

which differs because of the presence of the last two terms, which corresponding to one photon passing the long arm and another passing the shorter arm of the interferometers. State (3) can not give any determination of the paths of the photon. It gives a maximum of 50% visibility, which can not be distinguished from a classical model. However, it will be seen in the next section, that according to quantum mechanics, the last two terms of (3) can be suppressed by using a coincidence time window which is shorter than the optical path difference of the interferometers.

3 Theoretical Discussion

Our version of the new type of E.P.R. experiment is illustrated in FIG. 1. The photon pair generated from parametric down conversion is sent through two independent Michelson interferometers I and II. The optical path differences $\Delta L_1 = L_1 - S_1$ and $\Delta L_2 = L_2 - S_2$ can be arranged to be shorter or longer then the coherence length of each beam of the down conversion field. The coincidence measurement is between the two output of the interferometers.

The two photon state of the parametric down conversion can be considered as,

$$\Psi = \int dk_1 \int dk_2 \delta(k_1 + k_2 - k_p) A(k_1) \mid k_1 \rangle \otimes \mid k_2 \rangle \tag{4}$$
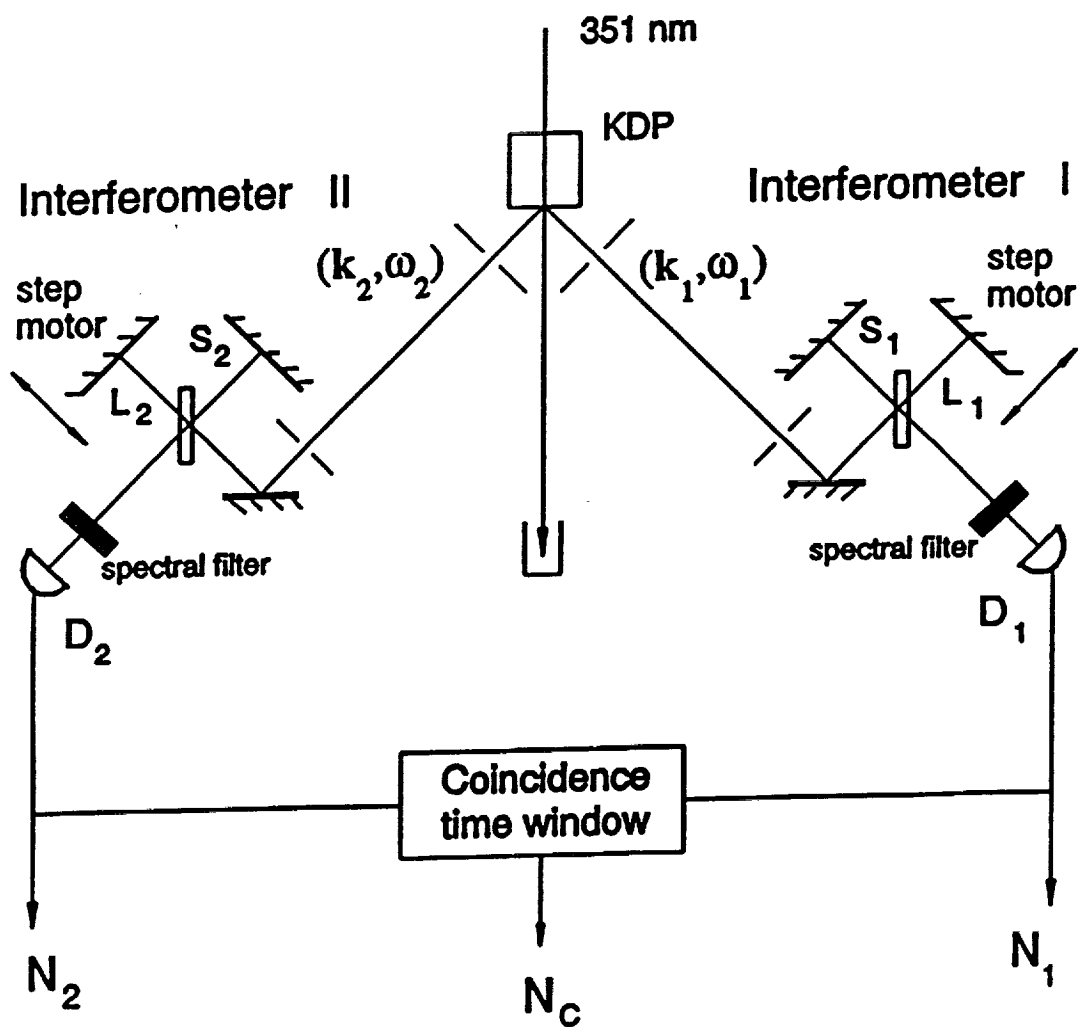
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FIG. 1: Schematic diagram of the experiment

where k_1 is the signal, k_2 is the idler and k_p is the pump wave number, the δ function comes from the perfect phase matching condition of the parametric down conversion, $A(k)$ is the wave packet distribution function and its width determines the coherence length of the wave packet. After leaving the interferometers, the wave function becomes,

$$\Psi = \tfrac{1}{4} \int dk_1 \int dk_2 \, \delta(k_1 + k_2 - k_p) \cdot A(k_1)$$

$$\cdot [|\,k_{1L}\rangle \,|\,k_{2L}\rangle + |\,k_{1S}\rangle \,|\,k_{2S}\rangle + |\,k_{1L}\rangle \,|\,k_{2S}\rangle + |\,k_{1S}\rangle \,|\,k_{2L}\rangle]$$

(5)

where $|\,k_{iM}\rangle = |\,k_i\rangle e^{i\varphi(Mi)}$, φ is the phase shift caused by passage of the wave through the system. The four terms of state (5) corresponding to the photons which have followed the long-long, short-short, long-short and short-long paths of the interferometers. State (5) is not an E.P.R. state, the coincidence rate can be estimated as, $R_c = R_{c0} \,|\,\Psi\,|^2$,

$$R_c = R_{c0} \int dk_1 F(k_1) \cdot \{1 + \cos k_1 \Delta L_1 + \cos(k_p - k_1)\Delta L_2$$

$$+ \tfrac{1}{2}\cos[k_1(\Delta L_1 + \Delta L_2) - k_p \Delta L_2] + \tfrac{1}{2}\cos[k_1(\Delta L_1 - \Delta L_2) + k_p \Delta L_2]\}$$

(6)

where $|\,A(k_1)\,|^2 \equiv F(k_1)$. Function $F(k_1)$ will generaly have about the same width as $|\,A(k)\,|^2$. If ΔL_1 and ΔL_2 are greater than the first order coherence length of the wave packets, the second, third, and fourth terms in (6) will vanish. The last term contains $\cos[k_1(\Delta L_1 - \Delta L_2) + k_p \Delta L_2]$; consequently, so long as $|\,\Delta L_1 - \Delta L_2\,|$ is less than the first order coherence length of the wave packet, this term gives rise to the interference fringes. If $|\,\Delta L_1 - \Delta L_2\,| \ll$ coherence length (equal optical path difference) then the visibility of these fringes attain their maximum value of 50%.

A similar result can be obtained from a classical model.[6][15] In the classical analog to the above experiment the electric field leaving the interferometer i will be

$$E_i = \frac{1}{\sqrt{2}} \int dk_i A(k_i) e^{i(k_i \cdot r - \omega_i t)} \cdot (e^{i\varphi(L_i)} + e^{i\varphi(S_i)})$$

(7)

where we neglect the polarization vector. The intensity is given by

$$I_i = \frac{1}{2} \int dk_i \,|\,A_i(k_i)\,|^2 \cdot (1 + \cos \delta_i)$$

(8)

where $\delta_i = k_i \Delta L_i = \varphi(L_i) - \varphi(S_i)$. The modulation as a function of the optical path difference ΔL_i is determined by the width of the function $|\,A_i(k_i)\,|^2$ and gives the first order interference coherence length of the field.

Now suppose the second order interference is measured, the coincidence counting rate $R_c \propto \langle I_1 I_2 \rangle$, where the bracket denotes an ensemble average,

$$\langle I_1 I_2 \rangle = \int dk_1 \int dk_2 \, \langle |\,A_1(k_1)\,|^2 |\,A_2(k_2)\,|^2 \rangle$$

$$\cdot \cos^2\left(\tfrac{k_1 \Delta L_1}{2}\right) \cos^2\left(\tfrac{k_2 \Delta L_2}{2}\right)$$

(9)

In order to model parametric down conversion it is necessary to account for the correlation in the two beams that is imposed by the phase matching condition. To do this assume perfect phase matching and take

$$\langle |\,A_1(k_1)\,|^2 |\,A_2(k_2)\,|^2 \rangle = \delta(k_1 + k_2 - k_p) \cdot G(k_1)$$

so that

$$R_c = R_{c0} \int dk_1 G(k_1)\{1 + \cos k_1 \Delta L_1 + \cos(k_p - k_1)\Delta L_2$$

$$+ \tfrac{1}{2}\cos[k_1(\Delta L_1 + \Delta L_2) - k_p \Delta L_2] + \tfrac{1}{2}\cos[k_1(\Delta L_1 - \Delta L_2) + k_p \Delta L_2]\} \tag{10}$$

It is the same as (6) which we have derived from the state (5).

It is not surprising that a classical model gives the same answer as that of quantum mechanics, because the above calculations have dealt with the wave nature of radiation for both the quantum and the classical models. However, if one can take advantage of the particle nature of the photon, the quantum prediction will be different. This idea has been demonstrated in the early polarization E.P.R. experiment using a coincidence measurement to produce an E.P.R. state.[13] For the two photon interference experiment a coincidence measurement is not enough to suppress the last two terms of (5) unless the coincidence time window is shorter than the optical path difference of the interferometers. Then the registration time difference in which the photons follow the long-short and short-long paths are outside the time window, i.e., the last two terms of (5) will not be registered by the coincidence counter.[16] This "cut off" effect will result in an E.P.R. state, which has no classical analog,

$$\Psi = \frac{1}{4}\int dk_1 \int dk_2 \delta(k_1 + k_2 - k_p)A(k_1) \cdot [|k_{1L}\rangle |k_{2L}\rangle + |k_{1S}\rangle |k_{2S}\rangle]\tag{11}$$

E.P.R. state (11) can provide 100% interference visibility,

$$R_c = R_{c0}\int dk_1 F(k_1) \cdot \{1 + \cos[k_1(\Delta L_1 - \Delta L_2) + k_p \Delta L_2]\}\tag{12}$$

To realize 100% visibility, besides equal optical path difference in the interferometers, a pump field with zero band width is required along with perfect phase matching for the parametric down conversion. One can easily arrange a narrow enough spectral band width of the pump field by means of a single mode laser as was done in this experiment, but, in principle, it is impossible to achieve perfect phase matching. When the finite size of the crystal and the finite interaction time of the down conversion is taken into account, the δ functions of $(k_1 + k_2 - k_p)$ and $(\omega_1 + \omega_2 - \omega_p)$ are replaced by functions with non-zero widths giving $k_1 + k_2 = k_p \pm \Delta k$ and $\omega_1 + \omega_2 = \omega_p \pm \Delta \omega$.[17] In this case (12) becomes,

$$R_c = R_{c0}\int dk_1 F(k_1) \cdot \{1 + \cos[k_1(\Delta L_1 - \Delta L_2) + k_p \Delta L_2 \pm \Delta k \Delta L_2]\}\tag{13}$$

The uncertainty Δk will reduce the interference visibility.

A detailed and careful study of the influence of the coincidence time window and the non-perfect phase matching can be found in reference (6). For a quasi monochromatic wave model, which is reasonable for parametric down conversion, the general solution of R_c may be written as

$$R_c = R_{c0}\{f_0 + f_1 \cos[k_1(\Delta L_1 - \Delta L_2) + k_p \Delta L_2]\}\tag{14}$$

where we assume that the optical path difference is much longer then the coherence length of the down conversion beams and ignore the trivial terms. The f's depend on the detail of the experiment, in particular the coincidence time window and the uncertainty Δk. For a large coincidence window, f_1/f_0 attains a maximum value of 50%. When the time window becomes shorter and shorter especially shorter than the optical path difference of the interferometers, f_1/f_0 reaches 100% for zero Δk.

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4 Experiment

The experimental arrangement is shown in FIG.1. A 351.1 nm single mode CW Argon laser beam was used to pump a 50 mm long potassium dihydrogen phosphate (KDP) nonlinear crystal for optical parametric down conversion. The coherence length of the 351.1 nm pump beam was measured to be longer than 5 meters. The KDP crystal was cut at TYPE I phase matching angle for generation of ω_1 and ω_2 photons. Both degenerate and nondegenerate (in frequency) photon pairs have been used in the experiment. In the degenerate case, $\lambda_1 = \lambda_2 = 702.2$ nm. The emission angles were about $2°$ relative to the pump. In the nondegenerate case,632.8 nm and 788.7 nm signal and idler pair were generated. The signal and idler photons were emitted at angles 1.8°and 2.3°relative to the pump beam, respectively. The signal and idler photons then were selected by pinholes and sent to two independent Michelson interferometers I and II. The interferometers are 5 m apart in order to have space-like separated detections. Two Geiger mode avalanche photodiodes D_1 and D_2 with 1 nm spectral filters (centered at 702.2 nm for degenerate case and 788.7 nm and 632.8 nm for nondegenerate case, respectively) were used for monitoring the first order and the second order interferences by means of direct counting and coincidence counting. The coincident circuit provides 20 psec, 50 psec and 3 nsec time window. N_1, N_2, N_c which corresponding to the number of counts from detector 1, detector 2 and from the coincidence time window were recorded simultaneously. The above measurements have taken advantage of the state-of-the-art millimeter lunar laser ranging high resolution timing diagnostic technique, which has been developed at the University of Maryland.

The optical path difference $\Delta L_1 = L_1 - L_2$ and $\Delta L_2 = L_2 - S_2$ of the two independent Michelson interferometers I and II can be changed by step motors continually from white light condition to about 7.2 mm which is longer then both the coherence length of the down converted fields and the 20 psec time window. It is also possible to move one of the mirrors discontinuously to a maximum $\Delta L = 12$ cm.

The experiment was performed in two steps. First, we used a 50 psec and a 3 nsec time window simultaneously for the coincidence measurement. By comparing the interference visibilities for $\Delta L > 1.5$ cm between the 50 psec and 3 nsec coincidence window, we expect to see the "cut off" effect.702.2 nm, photon pairs were used for the first step measurement.

1: $\Delta L_i <$ coherence length

We have measured the first order and the second order interference visibilities when both ΔL_1 and ΔL_2 were shorter than the coherence length of the field. We have also measured the first and second order interference visibilities when the optical path difference of one interferometer was shorter than the coherence length and that of the other was much longer than the coherence length. Fig. 2 (a,b) shows the second order and the first order interference visibilities with $\Delta L_2 = 5$ mm and ΔL_1 scanned starting from the white light condition. 97% second order and 82% first order interference visibilities were observed at the beginning of the scan. All reported values are directly measured without noise reduction and theoretical corrections.

2: $\Delta L_i >$ coherence length

Fig. 3(a,b,) reports two typical second order interference visibility measurements in which $\Delta L_2$was set to a value which was longer than the coherence length and ΔL_1 was scanned from white light condition. For each data point, the visibility was calculated from measurements similar to these shown in fig. 2. It is clear that the interference disappeared at about $\Delta L_1 = 500\mu m$ which
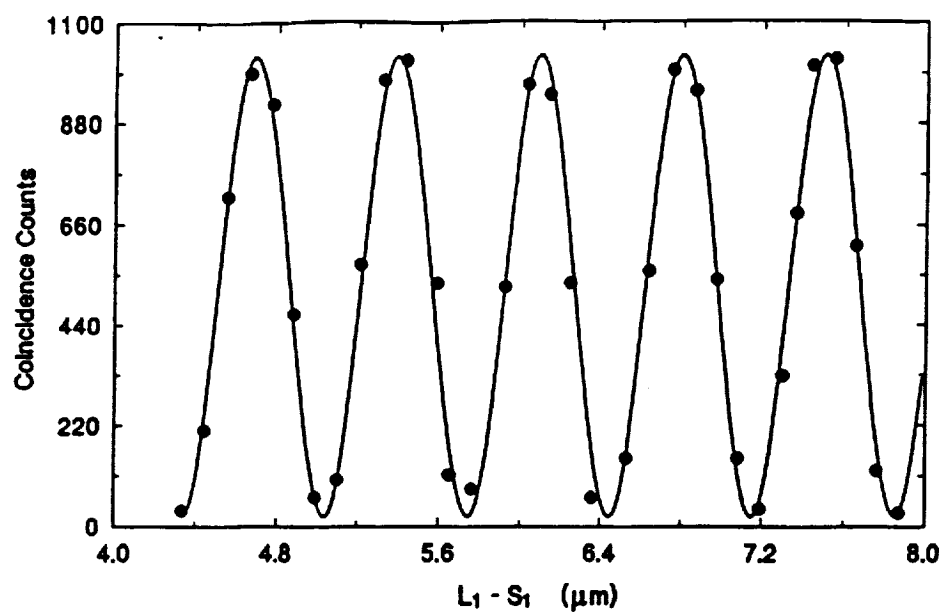
$C \cdot 3$

Fig. 2(a): Second order interference near white light condition, showing visibility near 100%
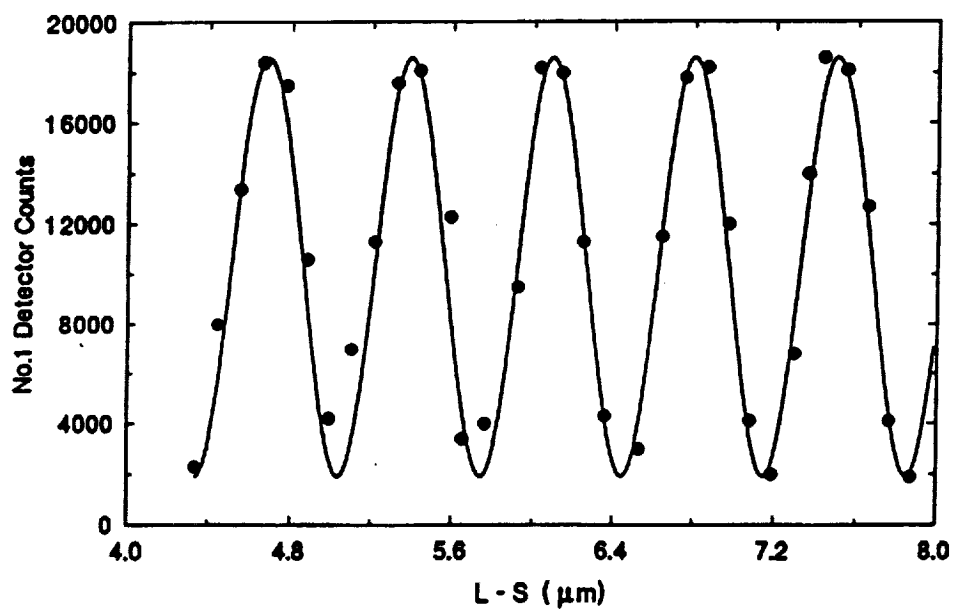

Fig. 2(b): First order interference near white light condition, showing visibility 82% (noise was not subtracted).

Fig. 3(a): Second order interference visibility with 50-psec coincidence window ($\Delta L_2 = 1.8$ mm, ΔL_1 scanned from white light condition).
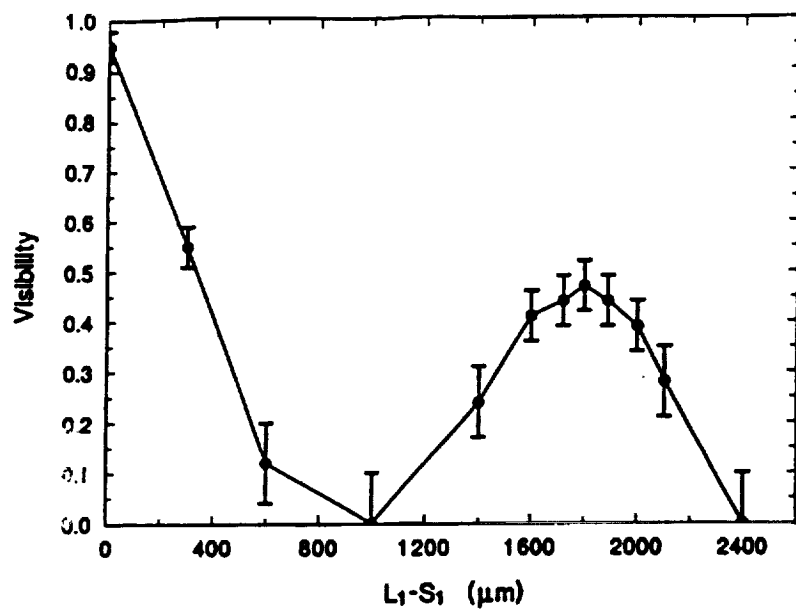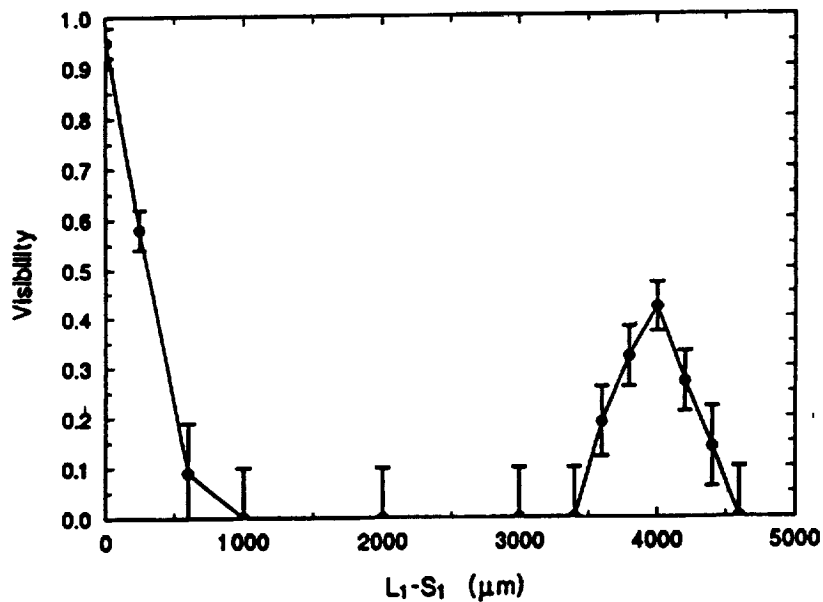

Fig. 3(b): Second order interference visibility with 50-psec coincidence window ($\Delta L_2 = 4$ mm, ΔL_1 scanned from white light condition).

corresponding to the first order coherence length of the field (determined by the band width of the spectral filter) and reappeared around $\Delta L_1 = \Delta L_2$. These measurements were repeated many times.

Fig. 4 and table 1 report the second order interference visibility measurement for $\Delta L_1 = \Delta L_2$ with 50 psec time window and 3 nsec time window. The interference visibilities were measured to be $(38 \pm 6)\%$ and $(21 \pm 7)\%$ for the 50 psec window and $(22 \pm 2)\%$ and $(7 \pm 3)\%$ for the 3 nsec window, when the optical path difference of the interferometers were 2 cm and 4 cm, respectively. The ratios are about 1.7 ± 0.3 for $\Delta L = 2$ cm and about 3.0 ± 1.6 for $\Delta L = 4$ cm, respectively. The "cut off" effect is clearly demonstrated. However, we still need a visibility more than 50% in order to have a unambiguous quantum result.

The second step of the experiment used a 20 psec coincidence time window. Higher interference visibility (>50%) was expected at $\Delta L > 6$ mm. In this experiment, 632.8 nm and 788.7 nm photon pairs were used for the measurement. The wavelength 632.8 nm was used for easy alignment. We used a CW He-Ne laser beam as input signal to match the 632.8 nm down conversion mode. Both 632.8 nm and 788.7 nm radiation have much longer coherence length due to the stimulated down conversion (or so called induced coherence). The parametric amplified signal and idler radiation were used for careful alignment. High visibility first order interference of the stimulated down conversion beams were observed before taking date.

Fig. 5, 6 and 7 report the experimental results. Fig. 5 (fig. 6) is a typical measurement in which $\Delta L_1(\Delta L_2)$ was fixed at 7 mm and $\Delta L_2(\Delta L_1)$ scanned around 7 mm. Fig. 7. reports the measurement in which both interferometers were scanned around 7 mm. The 7 mm optical path difference was much longer than the coherence length of the down conversion beam, no first order interference can be observed in N_1 or N_2, however, the coincidence measurement N_c showed clear interference fringes in the above measurements. The fringe visibilities are 59% with a period of 632.8 nm and 59% with a period of 788.7 nm for the type of measurements in fig. 5 nd fig. 6, respectively. When both ΔL_1 and ΔL_2 are changed together the visibility is 58% with a period of 351.1 nm. The solid curves in fig. 5, fig. 6 and fig. 7 are the fittings for 632.6 nm,788.7 nm and 351 nm, respectively. The standard deviation for these measurements is about 2%.

In summary:

1. The existence of E.P.R state has been observed by means of:

(1). the "cut off" effect, i.e., the interference visibility comparison between 50 psec and 3 nsec coincidence time window.

(2). direct measurement of more than 50% interference visibility for a 20 psec coincidence time window. This is the first observation of visibility greater than 50% for the two independent interferometers experiment.

2. The second order interference coherence (second order in intensity fourth order in field) is not limited by the coherence length of the pump beam only, but also by the non-perfect phase matching of the parametric down conversion. The uncertainty of the correlation in frequency determines the second order coherence length. We believe it is the non-perfect phase matching of the down conversion that reduced the visibility of the second order interference fringes in our experiment.

TABLE I: Second order interference visibility for equal optical path difference with 50-psec and 3-nsec coincidence time window.

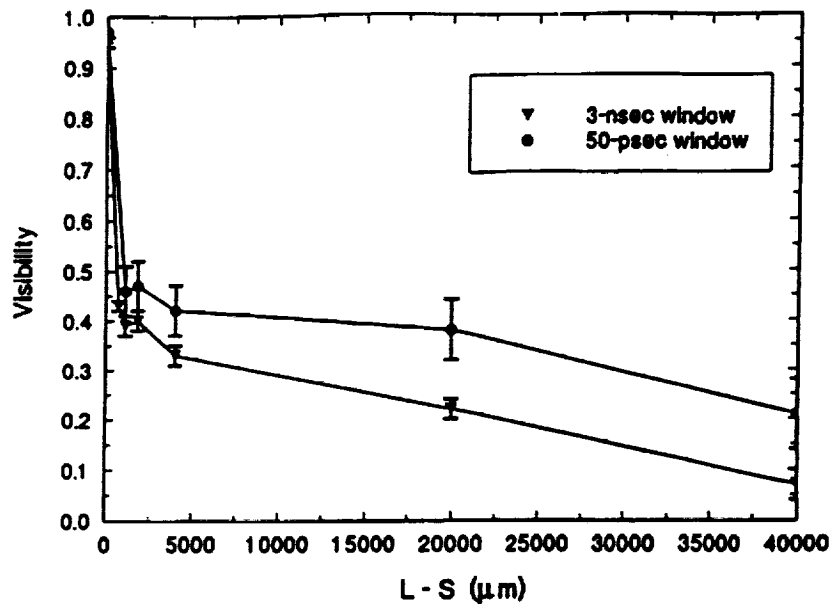| Second Order Interference Visibility Equal optical path difference $L_1 - S_1 = L_2 - S_2$			
$L_i - S_i$ (mm)	3-nsec window	50-psec window	Visibility ratio $(V_{50-psec}/V_{3-nsec})$
0	$(95 \pm 1)\%$	$(97 \pm 3)\%$	1.02 ± 0.03
1.1	$(39 \pm 2)\%$	$(46 \pm 5)\%$	1.18 ± 0.14
1.8	$(40 \pm 2)\%$	$(47 \pm 5)\%$	1.17 ± 0.14
4.0	$(33 \pm 2)\%$	$(42 \pm 5)\%$	1.27 ± 0.17
20.0	$(22 \pm 2)\%$	$(38 \pm 6)\%$	1.72 ± 0.32
40.0	$(7 \pm 3)\%$	$(21 \pm 7)\%$	3.00 ± 1.63

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Fig. 4: Second order interference visibility for equal optical path differences with 50-psec and 3-nsec coincidence time window.

Fig. 5: Second order interference fringes for 632.8 nm ($\Delta L_1 = 7$ mm, ΔL_2 scanned around 7 mm, 100 second for each point) with 20-psec coincidence time window.
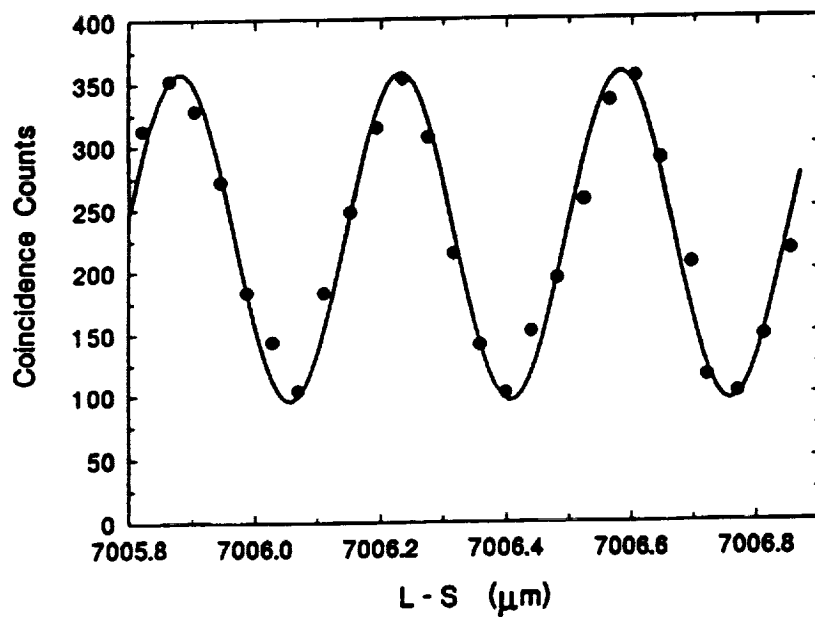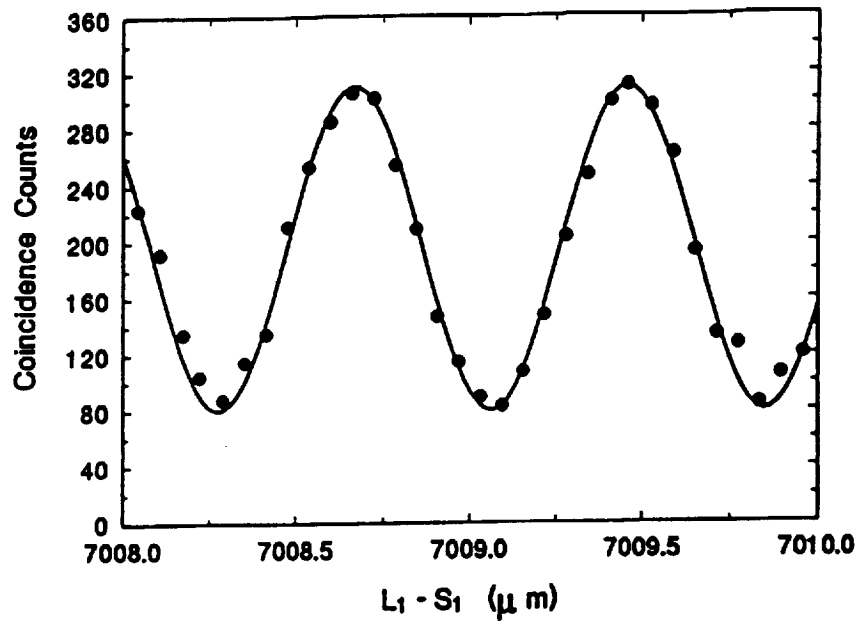
Fig. 6: Second order interference fringes for 788.7 nm (ΔL_2= 7 mm, ΔL_1 scanned around 7 mm,100 second for each point) with 20-psec coincidence time window.
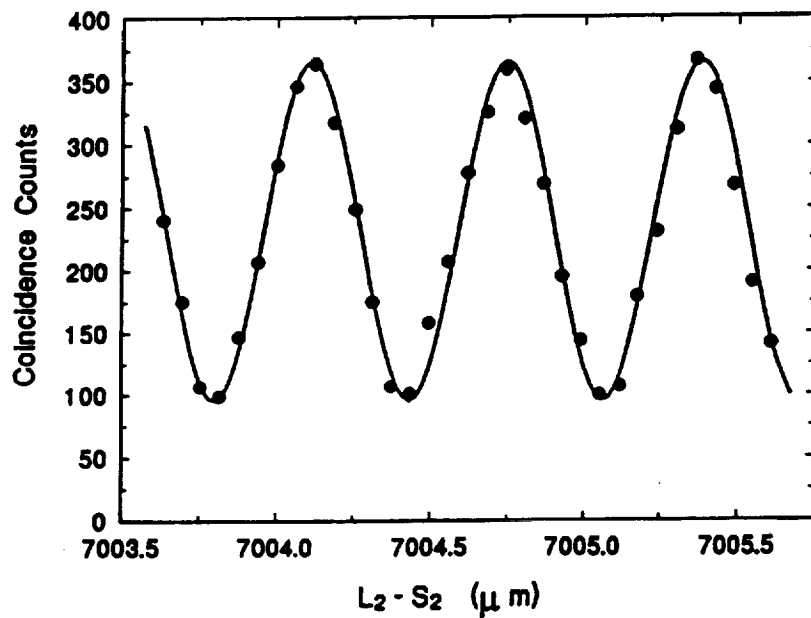

Fig. 7: Second order interference fringes for 351.1 nm($\Delta L_1 = \Delta L_2, \Delta L_1$ and ΔL_2 scanned together around 7 mm, 100 second for each point) with 20-psec coincidence time window.

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ON A LAGRANGE-HAMILTON FORMALISM
DESCRIBING POSITION AND MOMENTUM UNCERTAINTIES

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Abstract

According to Heisenberg's uncertainty relation, in quantum mechanics it is not possible to determine simultaneously exact values for position and momentum of a material system. Calculating the mean value of the Hamiltonian operator with the aid of exact analytic Gaussian wave packet solutions, these uncertainties cause an energy contribution additional to the classical energy of the system. For the harmonic oscillator, e.g., this nonclassical energy represents the ground state energy. It will be shown that this additional energy contribution can be considered as a Hamiltonian function, if it is written in appropriate variables. With the help of the usual Lagrange-Hamilton formalism known from classical particle mechanics, but now considering this new Hamiltonian function, it is possible to obtain the equations of motion for position and momentum uncertainties.

1 Introduction

According to quantum mechanics it is in principle impossible to simultaneously determine the exact values of two canonically conjugate variables like position and momentum. These values can be given only with a finite uncertainty, a mean square deviation or fluctuation $\langle \tilde{x}^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$ and $\langle \tilde{p}^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2$, where the brackets $\langle ... \rangle$ denote quantum mechanical mean values. The lower bound of these uncertainties, the minimum uncertainty product is defined by Heisenberg's uncertainty relation

$$U = \langle \tilde{x}^2 \rangle \langle \tilde{p}^2 \rangle \; \geqq \; \frac{\hbar^2}{4} \; . \tag{1}$$

In this paper the most simple but also most important one-dimensional problems, the free motion and the harmonic oscillator (HO) will be discussed in detail (the results for the free motion can be obtained in the limit $\omega \to 0$, where ω is the frequency of oscillation). The corresponding time-dependent Schrödinger equation (SE) (in position space),

$$i\hbar \, \frac{\partial}{\partial t} \, \Psi(x,t) = \; H_{op} \, \Psi(x,t) = \{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \} \, \Psi(x,t) \; , \tag{2}$$

has exact analytic Gaussian-shaped wave packet-type (WP) solutions $\Psi(x,t)$. The uncertainty of

position, reflecting the wave aspect, causes the finite width of this function, which can be time-dependent as it is known from the spreading of the "free-particle" WP. The particle aspect is expressed by the fact that the maximum of the WP follows the trajectory of the corresponding classical problem.

Calculating the mean value of the Hamiltonian operator H_{op} with the help of the Gaussian WPs to obtain the energy of the system,

$$
\begin{aligned}
\langle E \rangle &= \langle H_{op} \rangle = \frac{1}{2m}\langle p^2 \rangle + \frac{m}{2}\omega^2\langle x^2 \rangle \\
&= \left(\frac{1}{2m}\langle p \rangle^2 + \frac{m}{2}\omega^2\langle x \rangle^2\right) + \left(\frac{1}{2m}\langle \tilde{p}^2 \rangle + \frac{m}{2}\omega^2\langle \tilde{x}^2 \rangle\right) \\
&= E_{cl} + \tilde{E} .
\end{aligned}
\tag{3}
$$

the uncertainty of position and momentum causes, that in addition to the classical energy E_{cl}, a contribution $\tilde{E}$ occurs.

In classical mechanics, the (conserved) energy E_{cl} of the system is equivalent to the Hamiltonian function, $E_{cl} = H$, which also determines the dynamics of the system via the Hamiltonian equations of motion.

In this work, it will be shown that in analogy to classical particle mechanics, the additional contribution $\tilde{E}$ in (3) can be considered as Hamiltonian function for the position and momentum uncertainties. Therefore, the dynamics of these properties reflecting the (nonclassical), wave aspect, i.e. the equations of motion, can be obtained from this Hamiltonian function in exactly the same way as it is known from the formalism for classical particles.

For this purpose, $\tilde{E}$ has to be expressed in terms of appropriate variables and corresponding canonically conjugate momenta to provide the Hamiltonian $\tilde{\mathcal{H}}_L$.

2 Appropriate Variables for the Uncertainties

Using the Gaussian WP-solutions of the SE, exact analytic expressions for E_{cl} and $\tilde{E}$ can be obtained. In the case of the HO $\tilde{E}$ just represents the groundstate energy, usually given in the form $E_{GS} = \frac{1}{2}\hbar\omega$. However, there is much more information contained in $\tilde{E}$, especially connected with the dynamics of position and momentum uncertainties. In order to extract this information, the Gaussian WP used to calculate the mean values shall be given in the form

$$
\Psi_L(x,t) = N(t)exp\left\{i[y(t)\tilde{x}^2 + \frac{1}{\hbar}\langle p \rangle\tilde{x} + K(t)]\right\} ,
\tag{4}
$$

where $\tilde{x} = x - \langle x \rangle = x - \eta(t)$ (the explicit form of N(t) and K(t) is not relevant for the following discussion). The maximum of the WP at position x = $\langle x \rangle$ follows the classical trajectory $\eta(t)$. The WP width $\sqrt{\langle \tilde{x}^2 \rangle}$ is connected with the imaginary part y_I of the complex coefficient of $\tilde{x}^2$ in

the exponent, y(t), via

$$\frac{2\hbar}{m} y_I = \frac{\hbar}{2m\langle \tilde{x}^2 \rangle} = \frac{1}{\alpha^2(t)} \; . \tag{5}$$

Inserting the WP into the SE proves that $\langle x \rangle = \eta(t)$ obeys the classical Newtonian equation for a corresponding point particle,

$$\ddot{\eta} + \omega^2 \, \eta = 0 \; . \tag{6}$$

To determine the time dependence of the WP width, the complex (quadratically) nonlinear equation of Ricatti-type,

$$\frac{2\hbar}{m} \dot{y} + (\frac{2\hbar}{m} y)^2 + \omega^2 = 0 \; , \tag{7}$$

has to be solved. With the help of the new variable $\alpha(t)$ introduced in Eq.(5), the complex Ricatti equation can finally be transformed into the real (nonlinear) Newton-type equation

$$\ddot{\alpha} + \omega^2 \alpha = \frac{1}{\alpha^3} \; . \tag{8}$$

In contrast to the equation for the WP maximum, Eq.(6), the equation for the QP width, Eq.(8), contains an inverse cubic term on the rhs.

Additional insight into the dynamics of the investigated systems can be obtained by linearizing the Ricatti equation (7) with the help of

$$\frac{2\hbar}{m} y = \frac{\dot{\lambda}}{\lambda} \; , \tag{9}$$

introducing a new *complex* variable $\lambda = \hat{u} + i\hat{z} = \alpha e^{i\varphi}$, to provide the complex linear equation of motion

$$\ddot{\lambda} + \omega^2 \lambda = 0 \; , \tag{10}$$

which has exactly the same form as Eq. (6), but now for a complex variable.

It can be shown [1-3] that in cartesian coordinates, $\hat{z}$ is directly proportional to the classical trajectory,

$$\frac{\hat{z}\alpha_0 p_0}{m} = \langle x \rangle = \eta(t) , \tag{11}$$

and in polar coordinates, the absolute value α is identical with $\alpha(t) = (2m\langle \tilde{x}^2 \rangle / \hbar)^{\frac{1}{2}}$ from Eq. (8), and thus directly proportional to the WP width.

Furthermore, $\hat{u}$ and $\hat{z}$ (in cartesian coordinates), or α and φ (in polar coordinates), respectively, are not independent of each other, but coupled via the relation

$$\dot{\hat{z}}\hat{u} - \dot{\hat{u}}\hat{z} = \alpha^2 \dot{\varphi} = 1 . \tag{12}$$

The physical meaning of this relation is that $\lambda(t)$ moves in the *complex* plane like a particle in a real two-dimensional plane with *conserved angular momentum*. Therefore, the $1/\alpha^3$-term in Eq. (8) represents the "centrifugal force" for this motion in the *complex* plane.

3 Lagrange and Hamilton Functions for Uncertainties

In Eq. (5) it is shown how the mean square deviation of position, $\langle \tilde{x}^2 \rangle$, is connected with y_I or α (and thus λ), respectively. In a similar way the momentum uncertainty $\langle \tilde{p}^2 \rangle$ is connected with y_R and y_I or $\dot{\alpha}$ and $\dot{\varphi}$ (and thus $\dot{\lambda}$, respectively, via

$$\langle \tilde{p}^2 \rangle = \hbar^2 \frac{y_R^2 + y_I^2}{2\,y_I} = \frac{\hbar m}{2} (\dot{\lambda}\dot{\lambda}^*) = \frac{\hbar m}{2} (\dot{\alpha}^2 + \alpha^2 \dot{\varphi}^2) . \tag{13}$$

Therefore, the energy contribution $\tilde{E}$ can be written as

$$\tilde{E} = \frac{\hbar}{4}(\dot{\lambda}\dot{\lambda}^* + \omega^2 \lambda\lambda^*) = \frac{\hbar}{4} (\dot{\alpha}^2 + \alpha^2\dot{\varphi}^2 + \omega^2\alpha^2) . \tag{14}$$

Assuming that α and φ are the required appropriate generalized coordinates, still the canonically conjugate momenta have to be determined in order to express $\tilde{E}$ in a proper Hamiltonian form. In analogy to classical mechanics, a Lagrangian function for the position and momentum uncertainties can be obtained by simply changing the sign of the potential energy contribution into minus, leading to

$$\tilde{\mathcal{L}}_L(\alpha, \varphi, \dot{\alpha}, \dot{\varphi}) = \frac{\hbar}{4} (\dot{\alpha}^2 + \alpha^2\dot{\varphi}^2 - \omega^2\alpha^2) . \tag{15}$$

Thus, the generalized momenta are given by

$$\frac{\partial \tilde{\mathcal{L}}_L}{\partial \dot{\alpha}} = \frac{\hbar}{2}\, \dot{\alpha} \;=\; p_\alpha \tag{16}$$

$$\frac{\partial \tilde{\mathcal{L}}_L}{\partial \dot{\varphi}} = \frac{\hbar}{2}\, \alpha^2 \dot{\varphi} \;=\; p_\varphi \; . \tag{17}$$

With the help of these definitions, the energy fluctuation $\tilde{E}$ can be written in the correct Hamiltonian form

$$\tilde{\mathcal{H}}_L = \frac{p_\alpha^2}{\hbar} + \frac{p_\varphi^2}{\hbar \alpha^2} + \frac{\hbar}{4}\, \omega^2 \alpha^2 \; . \tag{18}$$

This Hamiltonian function $\tilde{\mathcal{H}}_L$ provides the equations of motion for the variables describing the *wave* aspect in exactly the same way as the classical Hamiltonian function of particle mechanics yields the equations of motion for the variables describing the *particle* aspect.

In addition, an interesting consequence follows from Eq. (17), defining the angular momentum p_φ. As mentioned in the previous section, this is an angular momentum property connected with the motion of λ in the *complex* plane under the additional condition, that the "conservation law" $\dot{\varphi} = \frac{1}{\alpha^2}$ is fulfilled.

However, inserting this into (17) shows that the conserved angular momentum-type quantity p_φ has the constant value

$$p_\varphi = \frac{\hbar}{2} \; , \tag{19}$$

a value that usually does not describe an orbital angular momentum but the nonclassical angular momentum-type property spin!

Furthermore, it should be mentioned that the uncertainty product (1), if it is written in terms of the new coordinates and momenta, takes the form

$$U(t) = p_\varphi^2 + (\alpha p_\alpha)^2 \; . \tag{20}$$

From Eq. (19) follows that $p_\varphi^2 = \hbar^2/4$, i.e. it is just the (constant) minimum uncertainty. The second term, however, represents the square at the position-momentum correlations, as

$$\langle [\tilde{x}, \tilde{p}]_+ \rangle \;=\; \langle \tilde{x}\tilde{p} + \tilde{p}\tilde{x} \rangle \;=\; \frac{\hbar}{2}\, \frac{\partial}{\partial t}\, (\lambda \lambda^*) = \hbar \dot{\alpha}\alpha \;=\; 2\,(\alpha p_\alpha) \tag{21}$$

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is valid.

For $p_\alpha = 0$ and thus $\dot{\alpha} = 0$, i.e. the WP width is constant and no correlations between position and momentum exist.

4 Conclusions

The information on the dynamics of the considered systems contained in the time-dependent SE can also be obtained from a corresponding Newtonian equation for these systems, if a complex variable is used, where the imaginary part of this variable is proportional to the classical trajectory and the real part is uniquely connected with the imaginary part. The connecting relation expresses a kind of conservation of angular momentum for the two-dimensional motion in the complex plane.

In polar coordinates, the absolute value of the complex variable, $\alpha(t)$, is directly proportional to the WP width $\sqrt{\langle \tilde{x}^2 \rangle}$, and thus to the uncertainty $\langle \tilde{x}^2 \rangle$.

It is possible to express the difference between the mean value of the Hamiltonian operator, $\langle H_{op} \rangle$, and the classical energy, E_{cl}, in terms of the coordinates α and φ and the corresponding canonically conjugate momenta. Thus, it is possible to write $\tilde{E}$ in the form of the Hamiltonian function $\mathcal{H}_L$, where from the correct equations of motion for the "wave properties" (uncertainties) can be obtained in exactly the same way as the equations of motion for the particle properties can be obtained from the classical energy, respectively Hamiltonian function.

5 Acknowledgements

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DISSIPATION IN A SQUEEZED-STATE ENVIRONMENT

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Abstract

The problem of a quantum particle coupled to a quantum-mechanical heat bath has a broad and general description in terms of a generalized quantum Langevin equation, as described in a series of papers by Ford, Lewis and O'Connell. Here we show how a squeezed-state environment may be incorporated in this general framework.

1 INTRODUCTION

In a paper entitled "Quantum Langevin Equation", Ford, Lewis and O'Connell [1] gave a broad and general description, in terms of a generalized quantum Langevin equation (GLE), of a quantum particle, moving in an arbitrarily external potential and coupled to a quantum-mechanical heat bath. Related papers included an extension incorporating the presence of an external time-dependent field [2]. In Ref. 1, we presented the general form of this equation consistent with fundamental physical requirements, in particular causality and the second law of thermodynamics. Next, we discussed an independent-oscillator (IO) model of the heat bath and we showed that, in addition to being a simple and convenient model with which to calculate, the most general GLE can be realized with an IO model. In addition, the IO model incorporates many other models that have appeared in the literature, in particular the blackbody radiation heat bath.

In the IO model, the quantum particle is surrounded by an infinitely large number of heat-bath particles, each attached to it by a spring. In Ref .1, the heat-bath is taken to be at temperature T. Here, we assume that the modes of the bath are squeezed and our purpose is to outline what aspects of Ref. 1 need

to be modified as a result. As it turns out, the only changes occur in expressions involving ensemble averages, specifically the autocorrelations of the random (noise) force $F(t)$ and the oscillator position $x(t)$.

2 DISCUSSION

As before, the Hamiltonian of the IO system is

$$H = \frac{p^2}{2m} + V(x) + \sum_j \left(\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2(q_j - x)^2 \right). \tag{1}$$

Here m is the mass of the quantum particle while m_j and ω_j refer to the mass and oscillator frequency of heat-bath oscillator j. In addition, x and p are the coordinate and momentum operators for the quantum particle and q_j and p_j are the corresponding quantities for the heat-bath oscillators. Also, $V(x)$ is a one-dimensional potential (but generalization to three dimensions is straightforward[1]). Use of the Heisenberg equations of motion lead to the GLE describing the time development of the particle motion:

$$m\ddot{x} + \int_{-\infty}^{t} dt' \, \mu(t - t')\dot{x}(t') + V'(x) = F(t), \tag{2}$$

where the dot and prime denote, respectively, the derivative with respect to t and x. In addition, $\mu(t)$ is the memory function:

$$\mu(t) = \sum_j m_j\omega_j^2 \cos(\omega_j t)\theta(t), \tag{3}$$

where $\theta(t)$ is the Heaviside step function. Also

$$F(t) = \sum_j m_j\omega_j^2 q_j^h(t) \tag{4}$$

is the random (fluctuation) force, where $q_j^h(t)$ denotes the general solution of the homogeneous equation (corresponding to no interaction). In Ref. 1, to find the expression for the (symmetric) autocorrelation of $F(t)$, we assumed that in the

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distant past the oscillators are in equilibrium at temperature T and with respect to the heat-bath Hamiltonian. This led to the result

$$\frac{1}{2}\langle F(t)\,F(t') + F(t')\,F(t)\rangle$$

$$= \frac{1}{\pi}\int_0^\infty d\omega\ \mathrm{Re}\big[\tilde{\mu}(\omega + i0^+)\big]\hbar\omega \tag{5}$$

$$\times \coth(\hbar\omega/2kT)\cos[\omega(t - t')],$$

where $\tilde{\mu}(\omega)$ is the Fourier transform of the memory function $\mu(t)$. To get the corresponding result in the case of a squeezed bath, we essentially have to generalize the expressions for $\langle q_j\,q_k\rangle$ etc. appearing in Eq.(4.12) of Ref. 1. To this end, it is convenient to use the familiar oscillator operators a, a^+ and a_j, a_j^+. As a result, using the procedure of Ref. 1, we obtain

$$\frac{1}{2}\langle F(t)\,F(t') + F(t')\,F(t)\rangle$$

$$= \sum_j \hbar m_j\,\omega_j^3\big\{(\langle a_j^+\,a_j\rangle + 1/2)\cos\omega_j(t - t')$$

$$+ \mathrm{Re}\,\langle a_j\,a_j\rangle\cos\omega_j(t + t')$$

$$+ \mathrm{Im}\,\langle a_j\,a_j\rangle\sin\omega_j(t - t')\big\}$$

$$= \frac{2}{\pi}\int_0^\infty d\omega\ \mathrm{Re}\,\tilde{\mu}(\omega)\,\hbar\omega\,\{\langle a^+(\omega)\,a(\omega) + \tfrac{1}{2}\rangle)\cos\omega(t - t')$$

$$+ \mathrm{Re}\,\langle a(\omega)\,a(\omega)\rangle\cos\omega(t + t')$$

$$+ \mathrm{Im}\,\langle a(\omega)\,a(\omega)\rangle\sin\omega(t + t')\}, \tag{6}$$

where the second equality follows from the use of the expession for $\mathrm{Re}\,\tilde{\mu}(\omega)$ given by Eq. (4.16) of Ref. 1.

In the particular case of the bath being in a thermal state, at temperature T, the last two terms on the right-side of Eq.(6) are zero and Eq.(6) reduces to Eq.(5). In the case of a squeezed bath, all of the terms in Eq.(6) are non-zero and detailed expressions for the various quantities may be found, for example, in the work of Gardiner et al. [3].

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As in the case of a thermal bath, the result for the symmetric position autocorrelation viz. 1/2 <x(t) x (t') + x(t') x(t)> is given by the right-side of Eq.(6) except that the integrand has an additional factor $|\alpha(\omega)|^2$, where $\alpha(\omega)$ is the generalized susceptibility. Such a relation is, in essence, a generalization of the fluctuation-dissipation theorem to the case of a non-thermal bath.

In conclusion, the results of Refs. 1 and 2, supplemented by Eq.(6) of the present paper, provide a general framework for discussing the problem of a quantum particle in a heat-bath whose modes are squeezed.

3 ACKNOWLEDGEMENTS

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Squeezed Spin States
— Squeezing the Spin Uncertainty Relations —

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Abstract

The notion of squeezing in spin systems is clarified and principle for *spin squeezing* is shown. Two *twisting* schemes are proposed as building blocks for spin squeezing and are shown to reduce the standard quantum noise, $\frac{S}{2}$, of the coherent S-spin state down to the order of $S^{1/3}$ and $\frac{1}{2}$. Applications to partition noise suppression are briefly discussed.

1 Introduction

First, we will review the uncertainty relations and coherent states of spin [1] compared to those of boson. Then we will define squeezing in spin systems and show the principle for *spin squeezing* [2]. Secondly, we will propose fundamental schemes for spin squeezing, namely, *one-axis twisting* and *two-axis counter-twisting*, and discuss their limits [2]. Finally, applications are briefly discussed.
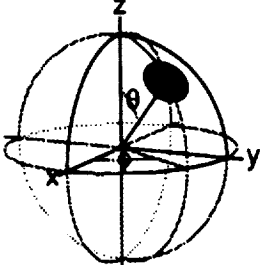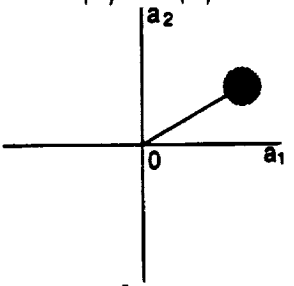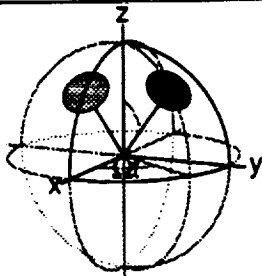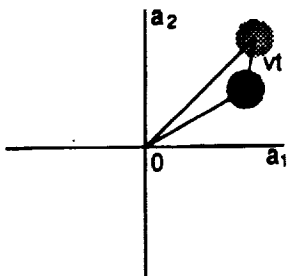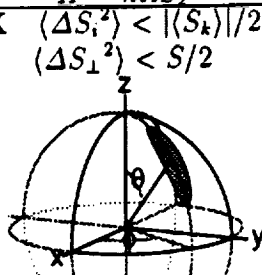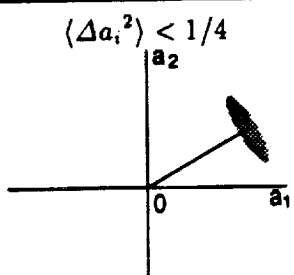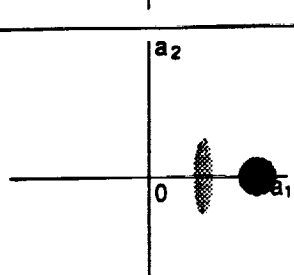
2 Uncertainty Relations — Spin vs. Boson —

Let us begin by comparing spins and bosons with respect to their uncertainty relations (TABLE I.) The spin commutation relation is $[S_i, S_j] = iS_k$, where $S_{i,j,k}$ are orthogonal spin components and the relation holds for any permutation of i, j, k. The same is true for associated uncertainty relations, $\langle \Delta S_i^2 \rangle \langle \Delta S_j^2 \rangle \geq |\langle S_k \rangle|^2/4$. This is quite different from the boson uncertainty relation since the right hand side (RHS) is state-dependent [3].

The coherent states can be defined as the minimum and equal uncertainty state; the state that minimizes the left hand side with the two uncertainties being equal. The eigenstate of the spin component of a certain direction (θ, ϕ), $S_{\theta,\phi} = S_x \sin\theta \cos\phi + S_y \sin\theta \sin\phi + S_z \cos\theta$, with eigenvalue S satisfies this condition if S_k is the eigen component (which is S) and S_i and S_j are normal components (whose variances are $S/2$). This state is called a coherent spin state (CSS), Bloch state, or directed angular momentum state [1].

Before talking about squeezing, let's look at the linear motions. A linear Hamiltonian proportional to an arbitrary spin component rotates the spin vector about an axis. This is a precession.

TABLE I. Spin vs. boson with respect to uncertainty relation, coherent state and squeezing

	Spin	Boson
Uncertainty Relations	$[S_i, S_j] = iS_k$ $\Downarrow$ $\langle \Delta S_i^2 \rangle \langle \Delta S_j^2 \rangle \geq \lvert \langle S_k \rangle \rvert^2 / 4$	$[a_i, a_j] = i/2$ $\Downarrow$ $\langle \Delta a_i^2 \rangle \langle \Delta a_j^2 \rangle \geq 1/16$
Coherent States	$S_{\theta,\phi}\lvert\theta,\phi\rangle = S\lvert\theta,\phi\rangle$ $\langle \Delta S_\perp^2 \rangle = S/2$	$a\lvert\alpha\rangle = \alpha\lvert\alpha\rangle$ $\langle \Delta a_i^2 \rangle = 1/4$
Translation	 $H = \hbar\Omega S_z$	 $H = \hbar i v a^\dagger + \text{H.c.}$
Squeezed States	X $\quad \langle \Delta S_i^2 \rangle < \lvert\langle S_k\rangle\rvert/2$ $\langle \Delta S_\perp^2 \rangle < S/2$ 	$\langle \Delta a_i^2 \rangle < 1/4$
Squeezing	Spin Squeezing ? X $\quad S_i(t){=}g_i S_i(0)$, $S_j(t){=}g_j S_j(0)$ $II =?$	 $a_1(t){=}G^{-1}a_1(0)$, $a_2(t){=}Ga_2(0)$ $H = \hbar i \chi (a^\dagger)^2 + \text{H.c.}$

It is regarded as a translation of the state on the spherical phase space of the spin. Although the rotation may change the uncertainties of the original spin components, the coherent spin state remains the minimum and equal uncertainty state as long as the component on the RHS is taken parallel to the mean spin vector.

Now let's discuss squeezing in a spin system. In a boson system, it is always regarded as squeezing if a certain quadrature amplitude has a variance smaller than the square root of the RHS of the uncertainty relation; that is 1/4. If we define the squeezing of spin likewise [4] — a certain spin component has a variance smaller than the square root of the RHS — we can squeeze the spin by just rotating it. If this were really squeezing, the experimentalists would be very happy since they could do this easily. Unfortunately, it doesn't offer any improvement beyond the standard quantum limit.

The quantum limit of spin systems can be attributed to the directional uncertainties of the spin vector. Therefore the uncertainties normal to the mean spin vector are the relevant quantities to be squeezed. To eliminate the superficial coordinate dependency, we write the criterion of the spin squeezing as $(\Delta S_\perp^2) < S/2$ (one of the component normal to the mean spin vector has a variance smaller than $S/2$) [2].

The next problem is how to squeeze the spin. Boson squeezing is regarded as attenuation of one quadrature amplitude and amplification of the other by the same factor. This can be done by a degenerate parametric amplifier described by a quadratic nonlinear Hamiltonian. Geometrically it is an area-preserving linear transformation on the boson phase space $\mathbf{R}^2$. The global shrinking/stretching is possible because boson phase space is an open plane.

In the spin case, permutative commutation relations obviously prohibit such a simple attenuation/amplification. In other words, global shrinking/stretching is impossible on the spherical phase space $\mathbf{S}^2$ of spin. The squeezing of spin is inevitably localized in phase space and, therefore, can be quite different from that of bosons.

3   Squeezed Spin States

Let's see how spin can, in principle, be squeezed. An S–spin system can be considered as a collection of a number, 2S, of 1/2-spins. In the coherent spin state pointing *up*, all spins are *up* (Fig.1 (a)). Therefore the z–component of the total system is S. However, whether the x–component of each spin takes 1/2 or -1/2 is completely independent and random. Therefore the variance is simply the sum of individual variances, 1/4, which is S/2. The same is true for the y–component. These uncertainties are the origin of the standard quantum noise of the CSS. The spin vector $\mathbf{S}$ is like a cone rather than an arrow. The diverging angle of the cone decreases with increasing S, since the base radius of the cone is proportional to $\sqrt{S}$.

In practical applications, it is desirable to reduce quantum noise for a given S. We have just seen that the origin of the standard quantum noise is a lack of quantum correlation among individual spins. If they are correlated, fluctuations of individual spins can cancel each other out (Fig.1 (b)). We refer to such a state as a squeezed spin state (SSS) [2]. Such a state can be conceived as an elliptical cone [5].

One way to establish the quantum correlations among individual spins is to let them interact with each other. This is a nonlinear interaction. Another way is to let them interact with an already correlated system such as squeezed light.
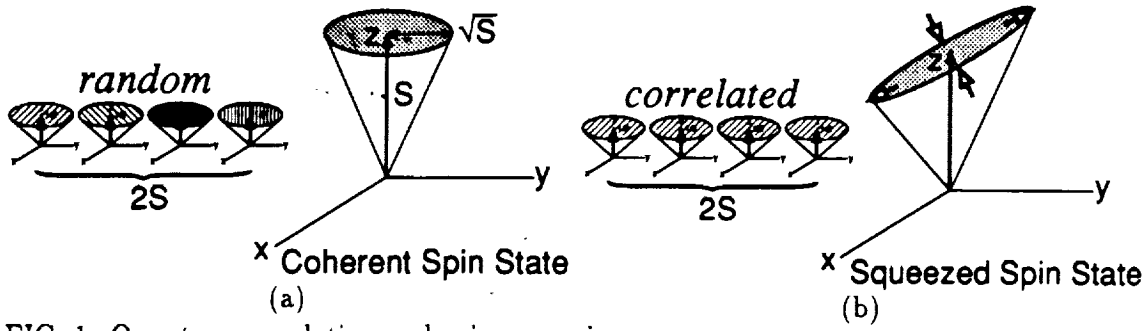
FIG. 1. Quantum correlation and spin squeezing

3.1 One-axis twisting

Let's consider the simplest nonlinear Hamiltonian, the square of a spin component, for example, $H = \hbar \chi S_z^2$. This interaction leads to $S_+(\mu) = S_+(0) \exp[i\mu(S_z + \frac{1}{2})]$, a rotation proportional to S_z, where μ is the strength of the interaction. If the initial state is on the equator of the sphere (Fig. 2 (a)), the interaction twists the noise distribution (Fig. 2 (b)).

(a) $\mu = 0$, CSS $|\frac{\pi}{2}, 0\rangle$ (b) $\mu = 0.2$ *(optimum)*   (c) $\mu = 0.4$ *(excessive)*
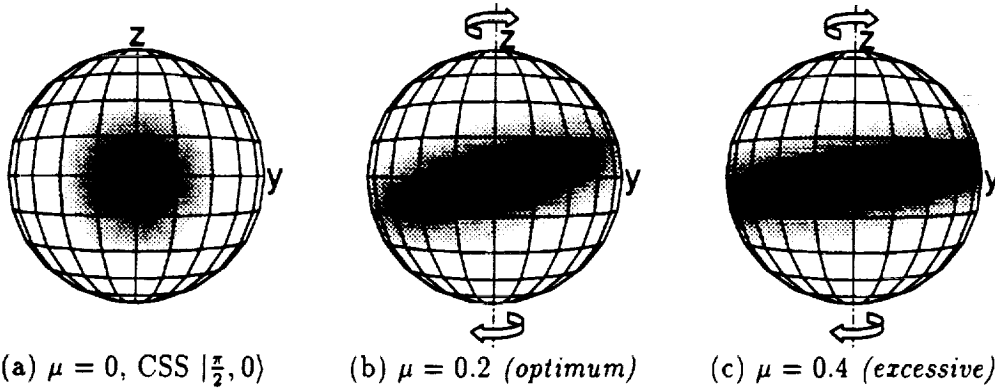
FIG. 2. Quasi-probability distribution $Q(\theta, \phi)$ [6] for one-axis twisting. ($S = 20$, $\mu = 2\chi t$).

The increased and decreased variances are,

$$V_+ \approx \frac{S}{2}(\mu S)^2$$

$$V_- \approx \frac{S}{2}\left[(\mu S)^{-2} + \frac{1}{24}(\mu^2 S)^2 \right] \geq 24^{-1/3} S^{1/3}$$

$$\text{squeezing} \quad \text{swirliness} \quad (\text{at } \mu = 24^{1/6} S^{-2/3})$$

where $\mu S > 1$ and $\mu^2 S \ll 1$ are assumed. The noise distribution is stretched by a factor of μS in a certain direction, while it is shrunk by the same factor in the orthogonal direction. This is nothing but squeezing. However, the stretching of the distribution is not exactly along a geodesic of the sphere, it is slightly S-shaped. The second term arises from this non-ideal effect, swirliness. The deviation from the geodesic becomes comparable to the reduced width of the distribution when μ is increased to the order of $S^{-2/3}$. Then the variance reaches its minimum of $S^{1/3}$. Because of the swirliness, it is impossible to further reduce the quantum noise by means of one-axis twisting.

3.2 Two-axis counter-twisting

The swirliness can be canceled out if we twist the noise distribution simultaneously clockwise and counterclockwise about two orthogonal axes both normal to the mean spin vector (Fig. 3 (b)). This can be done, for example, by the following Hamiltonian,

$$H = \hbar\chi(S_{\frac{\pi}{2},\frac{\pi}{4}}{}^2 - S_{\frac{\pi}{2},-\frac{\pi}{4}}{}^2) = \frac{\hbar\chi}{2i}(S_+^2 - S_-^2)$$

We refer to this as two-axis counter-twisting. The noise distribution is shrunk along a geodesic and stretched along the orthogonal geodesic until it spans almost half the sphere. If we twist the distribution more, it splits into two and no further improvement occurs.
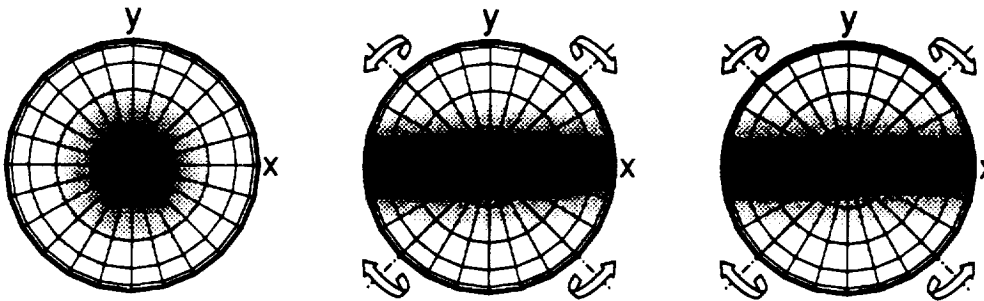
(a) $\mu = 0$, CSS $|0,0\rangle$ (b) $\mu = 0.203$ *(optimum)*   (c) $\mu = 0.248$ *(excessive)*

FIG. 3. Quasi-probability distribution $Q(\theta,\phi)$ for two-axis counter-twisting. ($S = 20$, $\mu = 4\chi t$).

3.3 Limits of noise reduction

Let's compare the minimum variances of two kinds of squeezed spin states. The dots show the exact minimum attainable variances calculated numerically (Fig. 4). The variance of the ordinary coherent spin state increases linearly with S. One-axis twisting can reduce it to the order of $S^{1/3}$. Two-axis counter-twisting can further reduce it to 1/2.
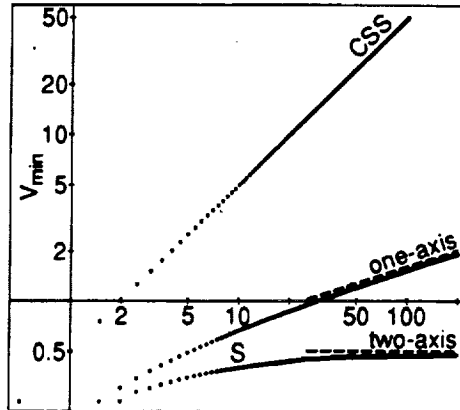
FIG. 4. Minimum variances vs. S

4 Applications to Partition Noise Suppression

There are many systems which can be described as a spin system. Spin squeezing described here offers better performance in these systems. For example, dispersion-less beamsplitters and interferometers for bosons and fermions can be described as a spin system with S being the

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half of the total particle number N passing through them [7, 5]. The operator S_z corresponds to the half of the particle number difference $N_A - N_B$ between two paths (A and B), and S_+ transfers a particle from one path (A) to the other (B). The outputs of 50% beamsplitters (i.e., $\langle S_z \rangle = 0$) have the number and phase partition noises $\delta N \equiv \langle [\Delta(N_A - N_B)]^2 \rangle^{1/2} = 2\langle \Delta S_z{}^2 \rangle^{1/2}$ and $\delta \phi \equiv \langle [\Delta(\phi_A - \phi_B)]^2 \rangle^{1/2} \approx \langle \Delta S_y{}^2 \rangle^{1/2}/|\langle S_x \rangle|$. For ordinary linear beamsplitters, they are $\delta N = \sqrt{N}$ and $\delta \phi \approx 1/\sqrt{N}$ since the output is in CSS $|\pi/2, 0\rangle$. Their ratio can be changed by *spin squeezing* without violating the uncertainty principle $\delta N \delta \phi \geq 1$. Physically, they can be realized as nonlinear interferometers. Both self-phase-modulation $H_I = \hbar \chi (N_A^2 + N_B^2) = 2\hbar \chi (N^2/4 + S_z^2)$ of particles in both paths and mutual-phase-modulation $H_I = \hbar \chi N_A N_B = \hbar \chi (N^2/4 - S_z^2)$ between particles in different paths lead to one-axis twisting. Optical Kerr effect and Coulomb interaction give these number-dependent phase modulations. These nonlinear beamsplitters can achieve either $\delta N \approx N^{1/6}$ or $\delta \phi \approx N^{-5/6}$ [8].

5 Summary

In summary, We have clarified the notion of squeezing in a spin system. Spin is squeezed if one of the components normal to the mean spin vector has a variance smaller than S/2. We have shown the principle for spin squeezing. The spin can be squeezed by establishing quantum correlations among elementary spins. We have proposed the fundamental schemes for spin squeezing. One-axis twisting can reduce the noise down to $S^{1/3}$ and two-axis counter-twisting can reduce it to 1/2. We have also discussed possible applications of spin squeezing to the sub-quantum-limit partition of quanta. Partition noise in either particle number or phase can be suppressed with a nonlinear beamsplitter which performs spin squeezing.

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QUANTUM ENTROPY AND UNCERTAINTY
FOR TWO-MODE SQUEEZED,
COHERENT AND INTELLIGENT SPIN STATES.

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Abstract

We compute the quantum entropy for monomode and two-mode systems set in squeezed states. Thereafter it is also calculated the quantum entropy for angular momentum algebra when the system is either in a coherent or in an intelligent spin state. These values are compared with the corresponding values of the respective uncertainties. In general, quantum entropies and uncertainties have the same minimun and maximun points. However for coherent and intelligent spin state it is found that some minima for the quantum entropy turn out to be uncertainty maxima. We feel that the quantum entropy we use provide the right answer since it is given in an essentially unique way.

1 INTRODUCTION

Some years ago Deutsch [1] proposed a new definition for the quantum uncertainty of a physical observable which immediately was taken up by Partovi [2] to carefully analyze the measurement of the system (x,p). Time ago, trying to understand the physical properties of supercoherent states [3] we started to call this new quantity $S(\Phi, |\psi>) \cong -|<\psi|\varphi>|^2 \ln |<\psi|\varphi>|^2$ the quantum entropy of the system Φ in the state $|\psi>$. In this article we keep using this notation which we feel is more appropriate. There will not be any sort of ambiguity with the standard use of the density operator for the statistical quantum entropy since in the following calculations we will only deal with pure states.

Our motivation is to go further with the quantum entropy and to calculate its values for physical systems less trivial than the monomode (x,p) one. We take the two-mode system when it is set on a two-mode squeezed state and the non-canonical, finite, angular momentum algebra of observables when it is either in a coherent (CSS) or in an intelligent spin state (ISS). Thereafter

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we estimate the (Heisenberg like) uncertainties I of these different systems and compare them with those previously obtained for the quantum entropy. Roughly speaking $S \cong \ln I$ implying coincidence of their extremals. It will be seen that, however, for ISS, it happens that states which minimize the quantum entropy are local maximuns for corresponding uncertainty.

In the next section we study the continuous, canonical cases of monomodal and two-modes systems. The third section deal with the three-dimensional angular momentum algebra. In the last one we discuss the results we have obtained.

The problem we are interested in is to compute the quantum entropy $S(\Phi, |\psi >)$ of different physical systems $\Phi = \{(x, p); (x_\pm, p_\pm); (J_i)\}$ and if possible to determine the states for which $S(\Phi, |\psi >)$ attains its minimum. In this work we do not solve this problem in its full generality. We calculate $S(\Phi, |\psi >)$ for some subspaces of $|\psi >$ and we find the states belonging to these subspaces for which S is extreme. We take the oportunity to compare with uncertainty functionals naturally related to these systems. It is worth pointing out what is the origin of the states we consider: all of them arise through the Heinsenberg relations, either by minimizing uncertainty functionals $I(A, B, |\psi >)$ or by introducing intelligent states, i.e. those states which satisfy the functional equation $I(A, B, |\psi >) = C([A, B], |\psi >)$ (C is given below).

2   ONE AND TWO MODE SQUEEZED STATES

This is the continous and canonical case. We start considering the monomodal case, where $\Phi_1 = \{x, p\}$ and the states $|\psi >$ for which we calculate $S(\Phi_1, |\psi >)$ are the squeezed states [4] (SS) (note the SS arise from the Heinserberg uncertainty relation). If we denote by $|z >$ the standard coherent states

$$|z > \equiv D(a, z)|0 >, \quad D(a, z) \equiv \exp\{za^\dagger - z^*a\}, \qquad (1a, 1b)$$

the SS are defined

$$|z, r\varphi > \equiv S_1(r, \varphi)|z >, \qquad (1c)$$

$$S_1(r, \varphi) \equiv \exp\{\frac{1}{2}r(e^{-2i\varphi}a^2 - e^{2i\varphi}(a^\dagger))^2\}. \qquad (1d)$$

If one introduces the squeezed annihilation operator $a(r, \varphi)$

$$a(r, \varphi) \equiv S_1(r, \varphi)aS_1^\dagger(r, \varphi) = \cosh r \; a + e^{2i\varphi} \sinh r \; a^\dagger, \qquad (2a)$$

the SS turn out to be their eigenvector with eigenvalues z,

$$a(r, \varphi)|z, r\varphi > = z|z, r\varphi > . \qquad (2b)$$

Following Deutsch the quantum entropy $S(\Phi_1, |\psi >)$ is defined by

$$S(\Phi_1, |\psi >) \equiv S(x, |z, r\varphi >) + S(p, |z, r\varphi >) \qquad (3a)$$

where

$$S(x, |z, r\varphi >) \equiv -\int_{-\infty}^{\infty} | < x|z, r\varphi > |^2 \ln | < x|z, r\varphi >)|^2 dx$$

$$= 2^{-1}(1 + \ln\{\pi (Re\gamma)^{-1}\}), \tag{3b}$$

$$S(p, |z, r\varphi >) \equiv -\int_{-\infty}^{\infty} | < p|z, r\varphi > |^2 \ln | < p|z, r\varphi >)|^2 dp$$

$$= 2^{-1}(1 + \ln\{\pi (Re\gamma)^{-1}\}), \tag{3c}$$

where

$$\gamma \equiv (\cosh r + e^{2i\varphi}\sinh r)(\cosh r - \sinh r e^{2i\varphi})^{-1}, \quad \gamma^{-1} = \gamma(-r, \varphi). \tag{3d}$$

Consequently S has the value

$$S(\Phi_1, |\psi >) = 1 + \ln\pi + 2^{-1}\ln\{1 + \sinh^2 2r \sin^2 2\varphi\}. \tag{4}$$

$S(\Phi_1, |z, r\varphi >)$ attains its minimun $1 + \ln\pi$ in two cases: i. if r=0, i.e. for pure coherent states since $|z, r\varphi >= |z >$ or ii. if $\varphi = n\pi/2$, $n \in \mathbf{Z}$ which corresponds to the proper SS. In both cases one gets what it has been shown [5] to be the minimun of the quantum entropy for this simple system.

To obtain the final results is eqs. (4a-b) we have used that [6]

$$\psi\{|z, r\varphi >\}(x) \equiv < x|z, r\varphi >=$$

$$= \pi^{-\frac{1}{4}}(Re\gamma)^{\frac{1}{4}} \exp(-i2^{-\frac{1}{2}}Im\ z)\exp(i2^{\frac{1}{2}}Im\ z)\exp\{-\gamma 2^{-1}(x - 2^{\frac{1}{2}}Rez)^2\} \tag{5a}$$

$$\bar{\psi}\{|z, r\varphi >\}(p) \equiv < p|z, r\varphi >$$

$$= \pi^{-\frac{1}{4}}(Re\gamma^{-1})^{\frac{1}{4}} \exp(i2^{-\frac{1}{2}}Im\ z)\exp(-i2^{\frac{1}{2}}Im\ z)\exp\{-(2\gamma)^{-1}(p - 2^{\frac{1}{2}}Imz)^2\}. \tag{5b}$$

The two-mode system $\Phi_2 = \{x_i, p_i, i \in (1, 2)\}$ has two annihilation and two creation operators $a_i, a_i^\dagger$,

$$[a_i, a_i^\dagger] = \delta_{ij}, \quad [a_i, a_j] = 0 \tag{6}$$

The two-mode coherent states are defined by

$$|z > \equiv \mathbf{D}(\mathbf{a}, \mathbf{z})|0 >_2 \equiv (D(a_1, z_1) \otimes D(a_2, z_2))|0 > \otimes|0 > \tag{7a}$$

where in an obvious two-dimensional vector notation

$$\mathbf{a}|z > = \mathbf{z}|z >, \quad \mathbf{a} \equiv (a_1 \otimes 1_{2\times 2}, 1_{2\times 2} \otimes a_2) \tag{7b}$$

The two-mode SS are given by

$$|z, r\varphi > \equiv S_2(r, \varphi)|z >, \tag{8}$$

$$S_2(r, \varphi) \equiv \exp\{r(e^{-2i\varphi}a_1 a_2 - e^{2i\varphi}a_1^\dagger a_2^\dagger)\}.$$

Observe that $S_2(r, \varphi)$ contains $(a_1, a_1^\dagger)$ corresponding with two modes we have now in the system. It is possible to generalize eqs. (2) to

$$\mathbf{a}(r, \varphi) \equiv S_2(r, \varphi)\mathbf{a}S_2^\dagger(r, \varphi) = \cosh r\ \mathbf{a} + e^{2i\varphi}\sinh r\sigma_1\mathbf{a}^\dagger, \tag{9a}$$

$$\mathbf{a}(r,\varphi)|\mathbf{z}, r\varphi >= \mathbf{z}|\mathbf{z} r\varphi > . \tag{9b}$$

these eqs. are based upon

$$S_2^\dagger(r,\varphi)\mathbf{D}(\mathbf{a},\mathbf{z})S_2(r,\varphi) = \mathbf{D}(\mathbf{a}, M(r,\varphi)\mathbf{z}) \equiv \mathbf{D}(\mathbf{a},\mathbf{w}), \tag{10a}$$

where $M(r,\varphi)$ is defined by

$$\mathbf{w} \equiv M(r,\varphi)\mathbf{z} \equiv \cosh r \ \mathbf{z} - e^{2i\varphi} \sinh r\sigma_1 \mathbf{z} = (\alpha, \beta)^T \tag{10b}$$

(σ_1 is the standard antidiagonal Pauli matrix). Computation of $S(\Phi_2, |\mathbf{z}, r\varphi >)$ entails the evaluation of $S(\mathbf{x}, |\mathbf{z}, r\varphi >)$ and $S(\mathbf{p}, |\mathbf{z}, r\varphi >)$,

$$S(\Phi_2, |\mathbf{z}, r\varphi >) = S(\mathbf{x}, |\mathbf{z}, r\varphi >) + S(\mathbf{p}, |\mathbf{z}, r\varphi >)$$

$$= -\int | < \mathbf{x}|\mathbf{z}, r\varphi > |^2 \ln | < \mathbf{x}|\mathbf{z}, r\varphi > |^2 d^2x$$

$$- \int | < \mathbf{p}|\mathbf{z}, r\varphi > |^2 \ln | < \mathbf{p}|\mathbf{z}, r\varphi > |^2 d^2p$$

$$= 2\{1 + \ln \pi + 2^{-1} \ln\{1 + \sinh^2 2r \sin^2 2\varphi\} = 2S(\Phi_1, |\mathbf{z}, r\varphi >). \tag{11}$$

As it happened for the monomodal case, there are two cases where the entropy has a minimun: i. for $r = 0$, wich corresponds to two-mode coherent states and ii. for $\varphi = n\pi/2$, these are the proper two-mode squeezed states. Calculation of $S(\phi_2, |\mathbf{z}, r\varphi >)$ (11) becomes straightforward after deducing the two (dual) representation of the wave functions $< \mathbf{x}|\mathbf{z}, r\varphi >$, $< \mathbf{p}|\mathbf{z}, r\varphi >$. It can be seen that

$$| < \mathbf{x}|\mathbf{z}, r\varphi > |^2 = \pi^{-1} Re\alpha\{1 - (Re\beta)^2(Re\alpha)^{-2}\}^{\frac{1}{2}}\cdot$$

$$\cdot \exp\{-Re\alpha(\mathbf{x} - 2^{\frac{1}{2}} Re\mathbf{w})^2 - Re\beta(\mathbf{x} - 2^{\frac{1}{2}} Re\mathbf{w})^T\sigma_1(\mathbf{x} - 2^{\frac{1}{2}} Re\mathbf{w})\} \tag{12a}$$

$$| < \mathbf{p}|\mathbf{z}, r\varphi > |^2 = \pi^{-1} Re\alpha\{1 - (Re\beta)^2(Re\alpha)^2\}^{\frac{1}{2}}\cdot$$

$$\cdot \exp\{-Re\alpha(\mathbf{p} - 2^{\frac{1}{2}} Im\mathbf{w})^2 - Re\beta(\mathbf{p} - 2^{\frac{1}{2}} Im\mathbf{w})^T\sigma_1(\mathbf{p} - 2^{\frac{1}{2}} Im\mathbf{w})\} \tag{12b}$$

The uncertainty $I(x, p, |z, r\varphi >)$ for the monomodal case is the standard quantity $(\Delta x)^2(|z, r\varphi >)$ $(\Delta p)^2(|z, r\varphi >)$. It turns out to be

$$I(\Phi_1, |z, r\varphi >) = 4^{-1}(1 + \sinh^2 2r \sin^2 \varphi)$$

Consequently, since $S(\Phi_2, |z, r\varphi >) \cong C_1 + 2^{-1} \ln I(\Phi_1, |r\varphi >)$ we observe that their minima coincide. For the two-mode system one has the uncertainty matrix [6] $I(\Phi_2, |z, r\varphi >)$ defined by

$$I(\Phi_2, |z, r\varphi >) \equiv (\Delta \mathbf{x})^2(\Delta \mathbf{p})^2 =$$

$$\begin{pmatrix} (\Delta x_1)^2 & (\Delta x_1)(\Delta x_2) \\ (\Delta x_1)(\Delta x_2) & (\Delta x_2)^2 \end{pmatrix} \begin{pmatrix} (\Delta p_1)^2 & (\Delta p_1)(\Delta p_2) \\ (\Delta p_1)(\Delta p_2) & (\Delta p_2)^2 \end{pmatrix} \tag{14a}$$

Minimum uncertainty states (MUS) are defined as those for which $I(\Phi_2, |\psi >) \cong 1_{2\times2}/4$. In the present case we have that

$$I(\Phi_2, |z, r\varphi >) = 4^{-1}(1 + \sinh^2 2r \sin^2 \varphi) \ 1_{2\times2}.$$

We have the qualitative situation already discussed for the monomodal case: minima of $S(\Phi_2)$ and $I(\Phi_2)$ coincide wuth either two-mode coherent or with proper squeezed states. Now we shift our interest to consider the less traditional, finite, non canonical system generated by 3-dimensional angular momentum algebra.

3 THE ANGULAR MOMENTUM ALGEBRA, COHERENT AND INTELLIGENT SPIN STATES.

The three-dimensional angular momentum algebra Φ_J provide a simple example of what one might think to be a general physical system. Its three generators J_i, $i \in (1, 2, 3)$ satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijl}J_l. \tag{15}$$

In general we will think of a physical system Φ_A to be a set of observables $\Phi_A = \{A_i, i \in \omega_p\}$ constituting some algebraic structure (i.e. very often this structure is a Lie algebra). The natural generalization of the quantum entropy definition initially given [1] for Φ_1 (eqs. (3)) is

$$S(\Phi_A, |\psi>) \equiv \sum_{i=0}^{i=p} S(A_i, |\psi>) = -\sum_{i=1}^{i=p} \sum_{\alpha_i} | <\alpha_i|\psi> |^2 \ln | <\alpha_i|\psi> |^2 \tag{16a}$$

where $|\alpha_1>$ are the eigenstates with eigenvalues α_1 of the observable A_i,

$$A_i|\alpha_i> = \alpha_i|\alpha_i> . \tag{16b}$$

Actually, a physical system Φ might be considered represented by different sets of observables $\{A_i\}$, $\{B_i\}$, $\cdots$ wich can be thought as equivalent quantum atlases which represent Φ.

In terms of field theory one is thinking in the possibility of $\{B_i\}$ being a redefinition of the initial observables $\{A_i\}$.

An already interesting, and non trivial example is whether, following this definitions of a physical system, $\Phi_1 = \{x, p\}$ can also be represented by $\{N \equiv a^\dagger a, \hat{\phi}\}$, the number and a convenient phase operator [7]. Of course, one expects that the quantum entropy of a physical system Φ must be independent of its quantum representation, $S(\Phi_A, |\psi>) = S(\Phi_B, |\psi>)$. We will not dwell on this interesting point in this article. Entropic calculations will be compared with uncertainties, which do not have a clearly cut, inambiguous definition, as we will comment below.

One of our main motivations of the present calculations is to better understand which are, for each specific given physical system, the states $|\psi>$ minimizing its quantum entropy, i.e. those states satisfying

$$\frac{\delta}{\delta|\psi>}S(\Phi, |\psi>) + \lambda|\psi> = 0, \quad <\psi|\psi> = 1. \tag{17a-b}$$

Instead of directly solving this problem, which we cannot do now, we study the behaviour of $S(\Phi, |\psi>)$ for subfamilies $|\psi>$ having a relevant physical origin, related to or stemming in uncertainty relations.

It is worth recalling what is the general situation concerning uncertainties functionals [8]. Given two physical observables A_1, A_2 Schwarz's inequality tell us that, for physical states $|\psi>$: $<\psi|\psi> = 1$,

$$I(A_1, A_2, |\psi>) \equiv$$
$$(\Delta A_1^2)(\Delta A_2^2) \geq 4^{-1}| <[A_1, A_2] > |^2 \equiv C([A_1, A_2], |\psi>). \tag{18}$$

MUS (minimum uncertainty states) are those for which I has a local minimum and IS (intelligent states) are states that satisfy the equality in eq. (18). The role of physical theories is to

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provide the value of commutator $[A_1, A_2]$. In principle one may find $|\psi_{MUS} > \neq IS$, $|\psi_I \neq MUS$ and $|\psi_{MUS,I} >$. It seems that IS constitute a very large set, being the states corresponding to intersection of two functionals.

Φ_J has two properties: i. is finite, i.e. it has irreducible unitary representations which are finite (due to compactness of $SO(3)$) and ii. is non canonical, i.e. there are not additional observables $K_j : [J_i, K_j] = i\delta_{ij} \cdot \Phi_J$ is one the simplest physical systems where there are IS which are not MUS [8]. The two kind of states that will be considered here are the coherent (CSS) [9] and the intelligent spin states (ISS) [8].

CSS are given by

$$|CSS > \equiv |\tau > \equiv R(\tau)| - j > = (1 + \tau\tau^*)^{-j} e^{\tau J}| - j >$$ (19a)

$$R(\tau) \cdot \equiv \exp\{\tau J_+\} \exp\{\ln(1 + \tau\tau^*) J_3\} \exp\{-\tau^* J_-\}$$

$$= \exp\{-i\theta J_n\}, \hat{n} = (\sin\varphi, -\cos\varphi, 0)$$ (19b)

where

$$\tau = e^{-i\varphi} \tan(\frac{\theta}{2}), \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi.$$ (19c)

ISS $|w_{j,n}(\tau) >$ have been defined as those for which $I(J_1, J_2, |w(\tau) >) = C(J_3, |w(\tau) >)$. They turn out to be

$$|w_{j,n}(\tau) > = a_n Y_1 \partial_y^n \{y^{2j} e^{\tau_y J_+}| - j >\}, \quad 0 \le n \le 2j$$ (20a)

where

$$a_n \equiv \{Z_1 Y_1 \partial_y^n \partial_z^n [yz + \tau\tau^*(y - 2)(z - 2)]^{2j}\}^{-\frac{1}{2}},$$ (20b)

$$Y_1 F(y, z) \equiv F(1, z), \tau_y \equiv \tau(1 - 2/y), \tau^2 = \tau^{*2}$$ (20c − d)

In particular $|w_{j,0}(\tau) > = | - \tau >$ and $|w_{j,2j}(\tau) > = |\tau >$ are CSS. We denote $|m >_i$ the respective eigenstates of J_i,

$$J_i|m >_i = m|m >_i$$ (21)

We first calculate $S(\Phi_J, |\tau >)$. According to eqs. (16)

$$S(\Phi_J, |\tau >) = -\sum_{i=1}^{i=3} \sum_{m=-j}^{m=j} |_i < m|\tau > |^2 \ln |_i < m|\tau > |^2.$$ (22a)

It is immediate to obtain the values of $_i < m|\tau >$ and its associated probabilities

$$|_{1,2} < m|\tau > |^2 =$$

$$\{2(1 + \tau\tau^*)\}^{-2j} \alpha(j, m)|1, i + \tau|^{2(j+m)}|1, i - \tau|^{2(j-m)},$$ (22b)

$$|_3 < m|\tau > |^2 = \{(1 + \tau\tau^*)\}^{-2j} \alpha(j, m)|\tau|^{2(j+m)},$$ (22c)

$$\alpha(j, m) \equiv 2j!(j + m)!/(j - m)!.$$

No closed expression has been obtained for eq. (22a). The same happens with the entropy for ISS. Its value is

$$S(\Phi_J, |w_n(\tau) >) = -\sum_{i=1}^{i=3} \sum_{m=-j}^{m=j} |_i < m|w_n(\tau) > |^2 \ln |_i < m|w_n(\tau) > |^2$$ (23)

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where

$$< w_n(\tau)|m >_{1,2} = a_n\alpha(j,m)^{\frac{1}{2}} \sum_{p=0}^{p=j-m} 2^{-m-p}(-1,-i)^p(j+m+p)!(p!(j-m-p)!)^{-1}$$

$$\sum_{q=0}^{p=j+m+p} (1,-i)^q(q!(j+m+p-q)!)^{-1}\tau^{j+m+p-q}\rho_n^{j+m+p-q}, \tag{24a}$$

$$| < w_n(\tau)|m >_3 |^2 = a_n\alpha(j,m)^{\frac{1}{2}}|\tau|^{2(j+m)}(\rho_n^{j+m})^2, \tag{24b}$$

$$\tau = \tan(\frac{\theta}{2})e^{-in\pi/2}, \quad n \in \mathbf{Z}, \quad \rho_n^k \equiv Y_1\partial_y^n\{y^{2j-k}(y-2)^k\}. \tag{24c}$$

Fig. 1 shows the structure of $S(\Phi_J, |\tau >)$ in terms of the θ, φ parametrization eq. (19c). S has local minimums at $\varphi = n\pi/2$, $\theta = \pi/2$. Details of the θ dependence for $\varphi = n\pi/2$ appear in fig. 2,3. It can be also observed that the minimum values of S increase with j.

Then we present in fig. 4 $S(\Phi_J|w_{j,n}(\tau) >)$ for the first proper ISS $|w_{1,1}(\tau) > (|w_{1,0}(\tau) >$ and $|w_{1,2}(\tau) >$ are CSS), and just to have a better feeling of it behaviour we show, in fig. 5, the shape of $S(\Phi_J, |w_{j,n}(\tau) >)$ for $j = 2$, $n = 1, 3$, proper non coherent intelligent spin states.

Then, fig. 6 shows that the minimum for S occur for the central ISS, i.e. in case of $j = 2$ for $|w_{2,2}(\tau) >$. In general it will occur for $n = (j, j \pm /2)$ according to whether j is integer or half-integer.

What can be said about the uncertainties?

In spite of arguments given [10] in favor of $\Delta J \equiv (\Delta J^2)^{1/2}$ as the right quantity one should take to define the uncertainty of Φ_j (ΔJ is a clear rotational invariant quantity), we will take partial and full quadratic uncertainties $I(J_1, J_2, |\psi >)$,

$$I(\Phi_J, |\psi >) = I(J_1, J_2), |\psi >) + I(J_3, J_3, |\psi >) + I(J_3, J_1, |\psi >)$$

as the physical relevant quantities which provide an additional insight concerning informational behaviour of Φ_J.

It seems to us that quadratic uncertainties are the typical elements of a quantum mechanically based definition.

As it is shown in figs. 7,8 there is a sharp qualitative difference in the behaviour of $I(\Phi_J, |\psi >)$ and $I(J_1, J_2, |\psi >)$. While $I(J_1, J_2, |w_n(\tau) >$ presents a local minimum at $\theta = \pi/2$ ($|\tau| \cong 1$); $I(\Phi_J|w_{j,n}(\tau) >)$ has a local maximum at this same point.

Since $S(\Phi_J, |w_{j,n}(\tau) >)$ exhibits a local minimum at $\theta = \pi/2$ and the full uncertainty shows a maximum, one cannot qualitatively relate anymore these two quantities through $S \cong \ln I$. these property is exhibit in fig. 7 where it is shown the anomalus behaviour of $I(\Phi_J, |w_{j,n}(\tau) >)$. Partial uncertainty for $I(J_1, J_2, |w_{j,n}(\tau) >)$ is shown in fig. 8. Its behaviour is completely different of the full uncertainty. Partial uncertainty minima coincide with entropy minimums.

4 CONCLUSIONS

We have estimated the values of the quantum entropy (not to be confused with the statistical quantum entropy due to the statistical mixture of quantum states) for monomode squeezed and
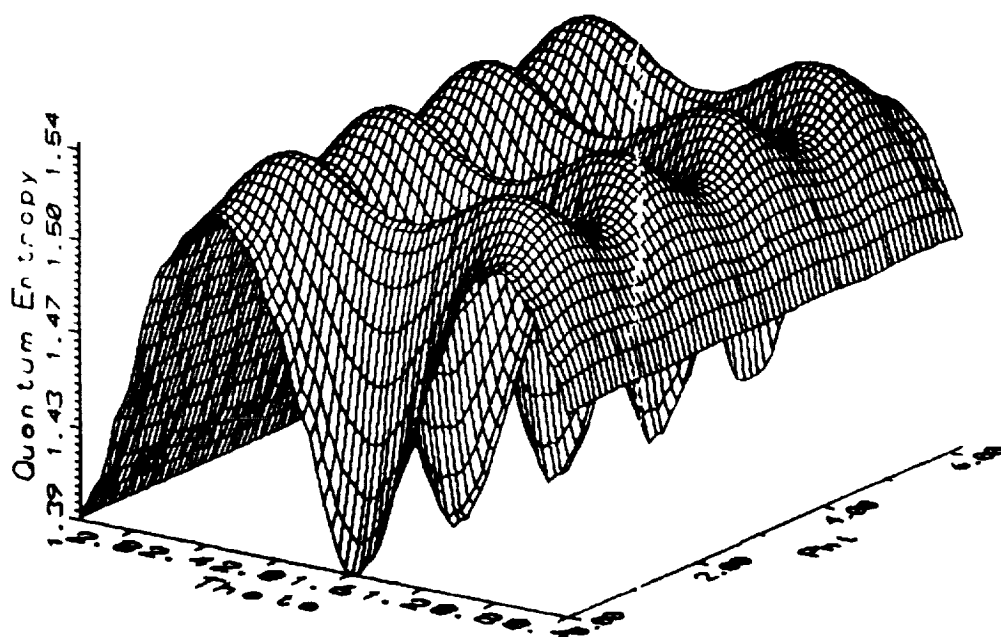
Fig. 1 Quantum entropy for coherent spin states
(CSS : j = 0.5)
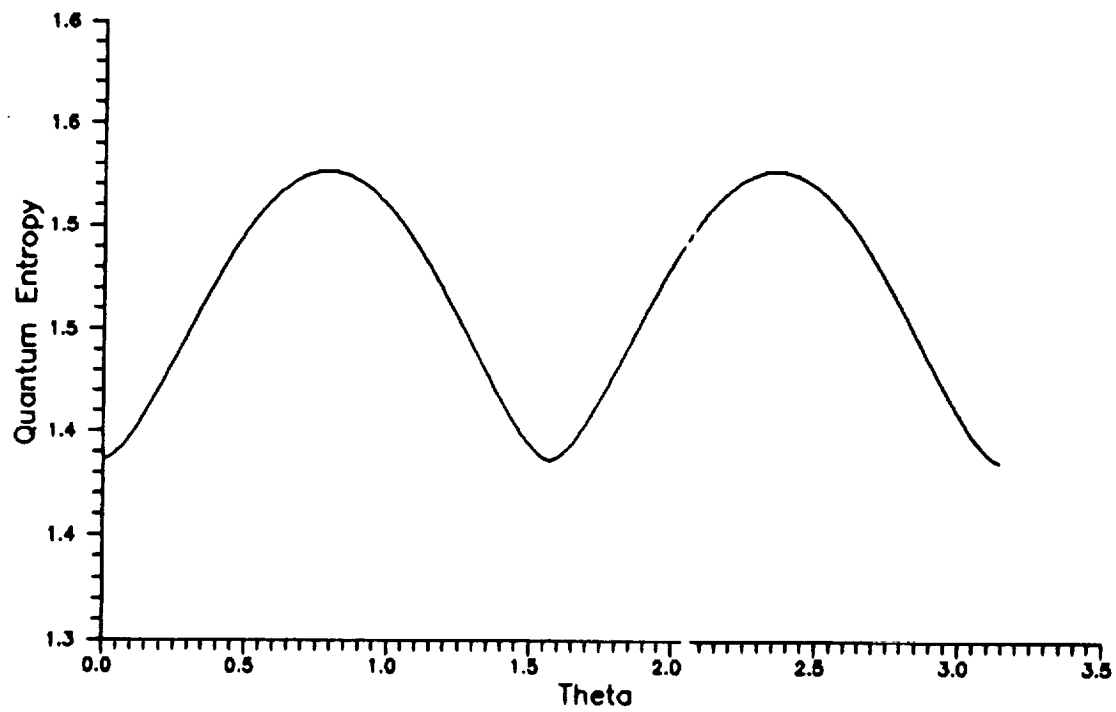
Fig.2 Quantum entropy for coherent spin
states (CSS : j = 1/2, φ = nπ /2).
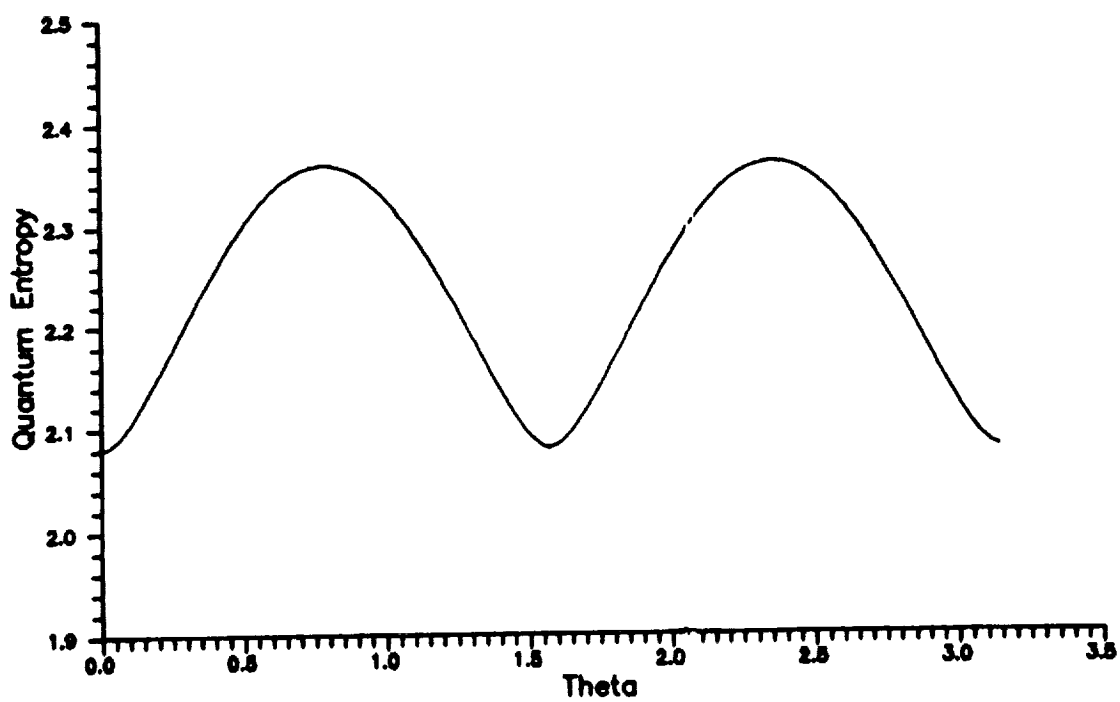
Fig.3 Quantum entropy for coherent spin states (CSS : j = 1, φ = n π /2).
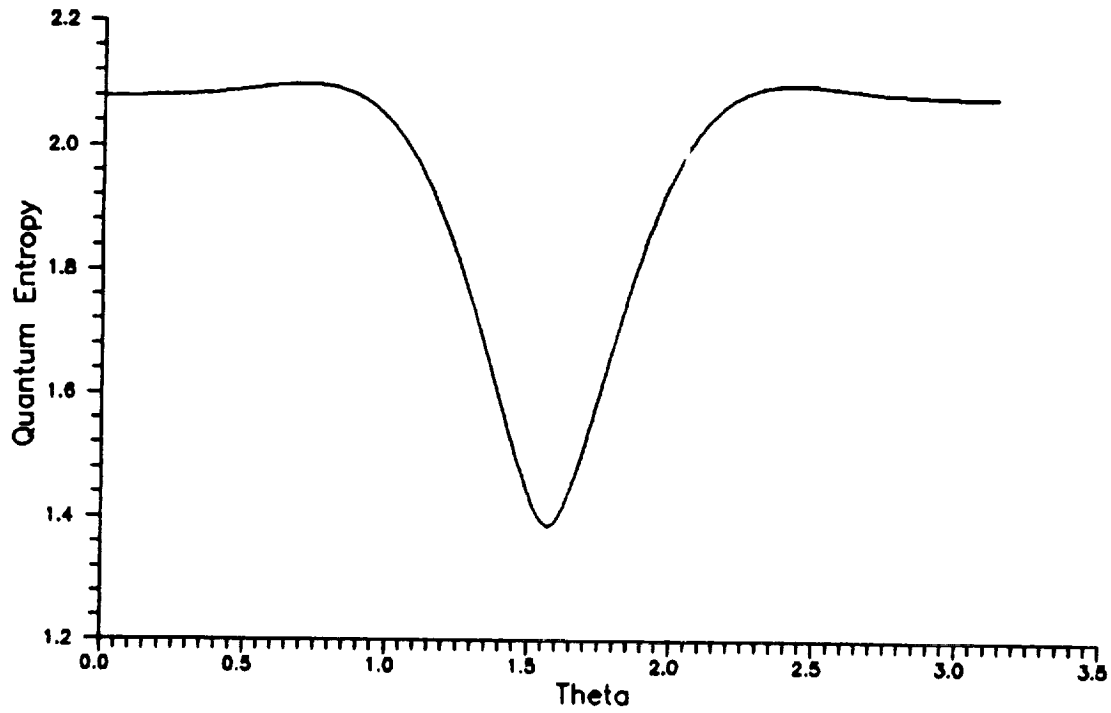
Fig.4 Quantum entropy for intelligent spin
states (ISS : j = 1, n = 1).
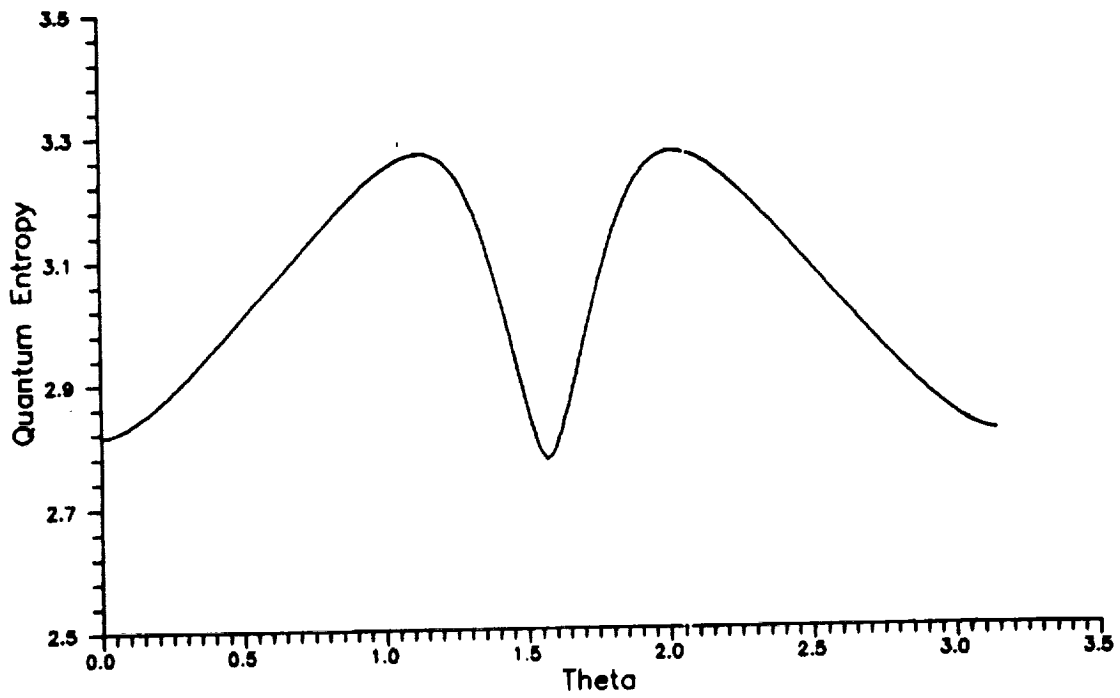
Fig.5 Quantum entropy for intelligent spin
states (ISS : j = 2, n = 1,3).
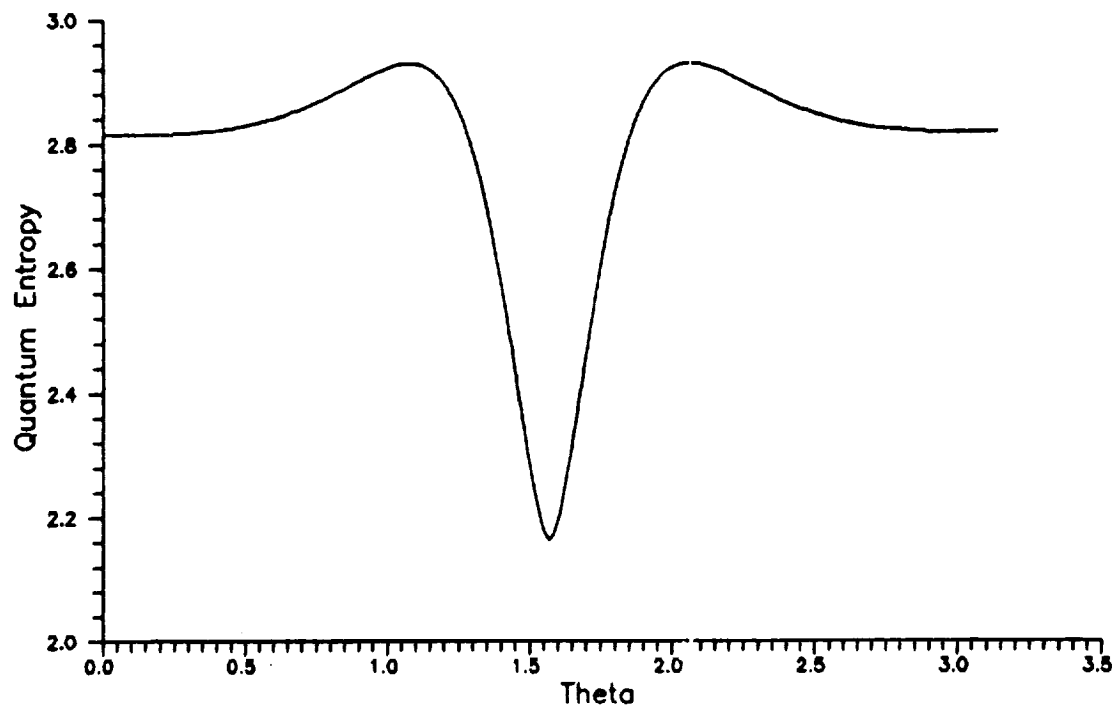
Fig.6 Quantum entropy for intelligent spin
states (ISS : j = 2, n = 2).
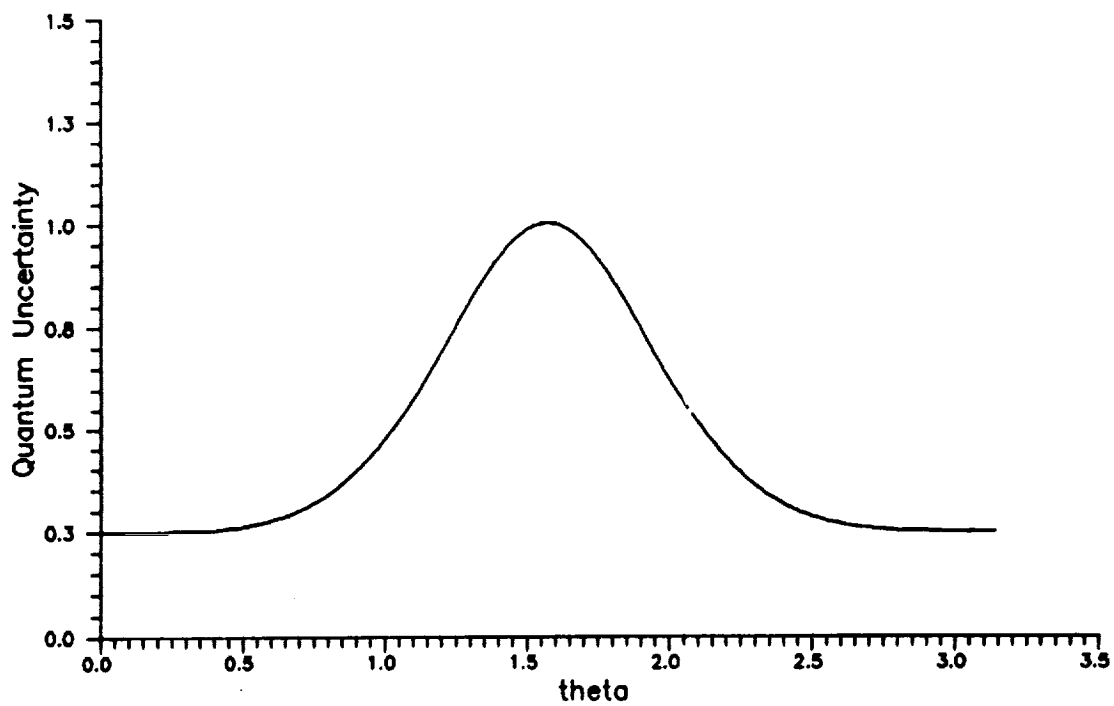
Fig.7 Full quantum uncertainty for intelligent spin state (ISS : j = 1, n = 1)
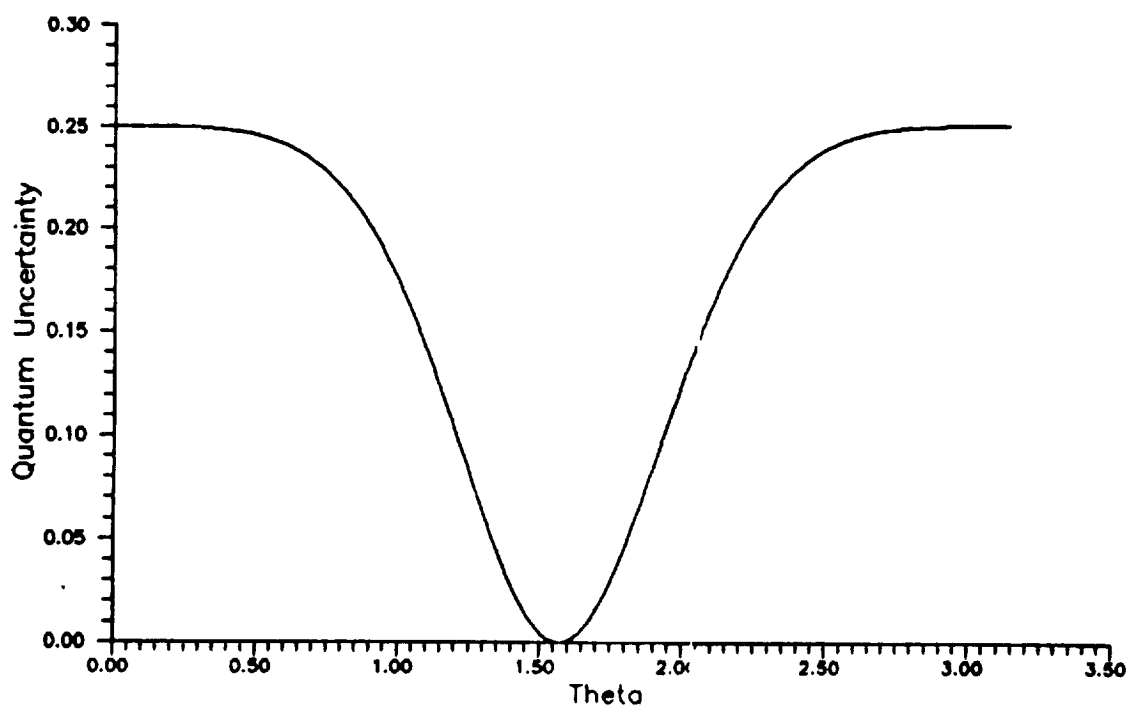
Fig.8   Partial quantum uncertainty for
intelligent spin states (ISS : j = 1, n = 1)

two mode squeezed states. Calculations were extended to the angular momentum system Φ_J where the states we used to probe $S(\Phi_J)$ came from the natural generalization of the standard coherent states or by imposing intelligence, i.e. states which satisfy the now operatorial Heisenberg equality.

In this case, the proper central IS were shown to be the best ones, i.e. they minimize $S(\Phi_J)$. We systematically compared the behaviour of $S(\phi_J)$ with that of $I(\Phi_J)$ just to understand why one must abandon the use of these latter quantities in favor of $S(\Phi_J)$. We observed the presence of anomalous behaviour in $I(\Phi_J)$ when one considers ISS, giving additional support to the choice of $S(\Phi_J)$ as the right physical quantity one has to consider for every physical system.

5 ACKNOWLEDGMENT

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THE UNCERTAINTY PRINCIPLE IN RESONANT GRAVITATIONAL WAVE ANTENNAE AND QUANTUM NON-DEMOLITION MEASUREMENT SCHEMES

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Abstract

A review on the current efforts to approach and to surpass the fundamental limit in the sensitivity of the Weber type gravitational wave antennae is reported. Applications of quantum non-demolition techniques to the concrete example of an antenna resonant with the transducer are discussed in detail. Analogies and differences from the framework of the squeezed states in quantum optics are finally discussed.

1 Introduction

The importance of detecting gravitational waves, as frequently pointed out, consists not only in verifying one of the most direct and astonishing predictions of the simplest metric theory of gravitation, i.e. General Relativity, but also in the possibility to open new windows on phenomena in the Universe in which only violent releases of gravitational energy occur [1]. Gravitational waves have not yet been directly measured because of the extreme smallness of the power emitted even by astronomical systems. The hypothetical sources which are strong candidates for emitting gravitational waves, according to our understanding of them actually only due to informations collected via the electromagnetic astronomy, are divided into two classes based upon the time evolution. Impulsive sources can be catastrofic events such as supernovae explosions and collapsing binary systems. The frequency spectrum of gravitational waves of this kind is flat up to 10^3 Hz, these impulsive phenomena having a characteristic duration of the order of milliseconds. One expects a perturbation of the metric tensor $h \approx 10^{-21} - 10^{-18}$ for events in our Galaxy and $h \approx 10^{-23} - 10^{-21}$ for events in the Virgo Cluster. Periodic sources can be pulsars if they deviate substantially from axial symmetry. The expected frequencies range is in this case between 10^{-2} and

10^2 Hz, while $h \approx 10^{-27} - 10^{-25}$. The efforts to detect gravitational waves have been concentrated from the very beginning on the impulsive events because of the larger expected perturbation to the metric tensor. It turns out that the modulation of the space-time induced by a gravitational wave on an extended body can also be seen as a production of a force field in it. Detecting the gravitational wave is therefore translated into the problem of detecting this small force of geometrical nature and the displacements produced by it in a test mass. The displacement induced in a body of reasonable sizes, $\approx 1\,m$, has therefore an amplitude of the order of 10^{-21} if the event is due to the a supernovae event in the Virgo Cluster. The accuracy required to measure such a small displacement is so high that the quantum nature of the detector has to be taken into account because the De Broglie wavelenght of a macroscopic test mass is of the same order of magnitude of the expected signal due to the gravitational waves. Here we report on the status of the art of the measurement techniques developed to allow monitoring of a class of gravitational wave detectors in a quantum regime. After a brief introduction for schematizing the detectors of gravitational waves and the sensitivity limit due to the fundamental noise in part 2, we introduce, in part 3, the quantum non-demolition measurement schemes for overcoming these limitations. The applications of stroboscopic and continuous quantum non-demolition schemes for a gravitational bar antenna resonant with the transducer are described respectively in part 4 and part 5. Conclusions deal also with the analogies and the differences from the quantum optics framework and the importance of this topic for understanding quantum mechanics applied to single macroscopic degrees of freedom repeatedly monitored.

2 Weber gravitational antennae:fundamental sensitivity limits

The gravitational wave detectors devised so far are based upon monitoring of the distance between two masses localized at different points. The equivalence principle requires a non-local, extended, structure of a gravitational wave detector because it is possible to nullify locally the effects of a gravitational field by means of a suitable choice of the reference frame.

Let us consider two masses in free fall: what is then measured is their variable distance which is supposed to be much smaller than the gravitational wavelength. The effect of a gravitational wave coming along z axis with proper polarization is to increase of $h/2$ the distance along y axis and to decrease by $h/2$ the distance along x axis. A classification of the gravitational wave detectors divides these into non resonant and resonant detectors if the two masses are respectively free or elastically coupled.

In non resonant detectors the distance between the two masses is measured by means of interferometric devices. The arms of the interferometer proposed so far are of the order of Km and use of multiple reflections allows to increase the physical paths by several orders of magnitude. In this contribution we will not be concerned with this kind of detectors but we shall instead consider the resonant detectors (Weber type gravitational wave antennae), the quantum limit in a interferometric antenna being enforced by the shot noise and the momentum fluctuations imparted by the photon flux to the central mirrow of the interferometer [2].

Resonant antennae are tipically cylindrical bars of materials having low internal dissipations. The materials used are silicon, sapphire, niobium or a particular aluminum alloy (Al 5056) and

the mass of the antennae is of few tons.

One can show that the motion of the ends of a cylindrical bar of mass M and length L oscillating in its fundamental longitudinal mode is equivalent to that of a harmonic oscillator of mass $M/2$ and equivalent length $4L/\pi^2$. If x is the displacement from equilibrium position the equation of motion of the Weber oscillator is

$$\ddot{x} + \frac{\dot{x}}{\tau_0} + \omega_0^2 x = \frac{2}{\pi^2} L \ddot{h}(t) \tag{1}$$

where τ_0 is the damping time, ω_0 is the proper frequency and $h(t)$ is the amplitude of the incoming gravitational wave. The forcing term due to the gravitational field is proportional to the distance between the two masses. From this formula one can calculate the cross section for the transfer of energy from the wave to the antenna and one finds that this is proportional to the mass of the antenna and to L^2. The proper frequency ω_0 is chosen to be tuned with the frequency of the expected wave ($10^3 Hz$) and the corresponding wavelenght is very large compared to the size of the antenna. To amplify the extremely small oscillations coupling of the bar with another oscillator of very small mass is used [3],[4]. In this case a system of two coupled harmonic oscillators is obtained in which the energy is continuosly transferred back and forth from M to m via beating. If the dissipations in the two oscillators are made negligible the amplitude of the oscillations in the second resonator is increased by a factor $1/\sqrt{\mu}$ with respect to the first resonator, where $\mu = m/M$, provided that the frequencies of the two uncoupled oscillators are made coincident. The motion of the transducer is transformed into an electric signal by means of a variable capacity and an amplifier schematizable as an ideal amplifier of gain A and two noise sources generators with current and voltage spectral densities respectively SI_n and SV_n. The sources of noise are the thermal noise, i.e. Brownian motion of antenna, which gives a contribution KT to the energy of the oscillator, being K the Boltzmann's constant and T the thermodynamical temperature of the antenna and the amplifier noise, which is expressed by means of the parameter $T_n = (SV_n SI_n)^{1/2}/K_B$, called noise temperature of the amplifier. This last noise has two effects: it contributes directly as an additive noise source at the output and it acts on the transducer leading to an increase of the temperature. In other words every transducer is at the same time an actuator and the amplifier noise gives rise to a back-action force acting on the mechanical oscillator.

If we define a noise temperature T_{eff} as the temperature which corresponds to the minimum detectable energy $E_{eff} = K T_{eff}$ transferred to the bar by an impulsive signal with an output signal/noise ratio equal to 1, we find, using a Wiener algorithm in the data analysis [5]

$$T_{eff} = 2 T_n \sqrt{\left(1 + \frac{1}{\lambda_0^2}\right)\left(1 + \frac{2T\lambda_0}{\beta Q T_n}\right)} \tag{2}$$

where $Q = \omega_0 \tau_0$ is the quality factor of the mechanical system, β is substantially the fraction of energy transferred to the electromagnetic circuit by the bar throug the capacitive coupling and λ_0 the impedance matching factor defined as

$$\lambda_0 = \frac{SV_n}{SI_n} \frac{1}{Z_0}. \tag{3}$$

For the antenna of the Rome group continously operating since one year at CERN one has a thermodynamical temperature of $\approx 4.2K$; the other parameters are $Q \simeq 5 \cdot 10^6$ and an amplifier

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noise temperature $T_n \simeq 10^{-7}$K [6]. It has been possible to achieve this last result making use of a SQUID amplifier. So one gets for T_{eff} a value of $\approx 10\mu K$, which is not far from the quantum limit temperature

$$T_{QL} = \frac{\hbar\omega}{K} \simeq 10^{-8}\text{K}. \qquad (4)$$

One expects that the force with which a gravitational wave acts on the antenna is by many orders of magnitude below the thermal noise even at thermodynamical temperatures as low as 10mK which is the temperature at which the third generation antennae will operate. Howewer, due to the particular features of the data analysis based on the variation of energy in the oscillator in the time, the quantum regime is reached earlier than as expected by (4). By writing the amount of energy which is exchanged during the measurement time Δt between the harmonic oscillator and the thermal reservoir and the quantized energy introduced by the measuring apparatus is easy to show that the quantum regime is obtained when the following condition is satisfied

$$\frac{KT\Delta t}{Q} \ll \hbar \qquad (5)$$

This can be also shown by reasoning in terms of displacements instead of energy. The variation of the length of the bar due to a gravitational wave with amplitude h is, according to (1)

$$\frac{\Delta l}{l} \simeq \frac{h}{2}. \qquad (6)$$

Because typical values for h are $h = 10^{-21}$ (which corresponds to a supernova explosion in the center of the Galaxy) taking $L = 1$ m, one gets from (5) a variation of the length of the bar $\Delta L \simeq 10^{-19}$ cm which coincides with the standard quantum limit (i.e. the root square mean of the position of a harmonic oscillator in his fundamental mode)

$$\sqrt{\langle \Delta l^2 \rangle} = \sqrt{\frac{\hbar}{2M\omega}}. \qquad (7)$$

It follows therefore that if we do not overcome this limit no information can be obtained on the evolution of the harmonic oscillator.

In these conditions one can find a method to measure the position of the quantum oscillator and to see if an external force has acted on it. However in doing this one must take into account that the position operator $\hat{x}(t)$ does not commute with itself at different times. Indeed with a measurement of $\hat{x}(t)$ at time t one put the oscillator into an eigenstate of $\hat{x}(t)$; if one repeats this measurement at the instant $t + \tau$ one puts the oscillator into another eigenstate. It turns out that it is not possible to know if the change in $\hat{x}(t)$ is caused by a very weak classical external force because of the quantum demolition of the state. What is needed is therefore a measurement which does not prevent the execution of the next measurements of the same observable avoiding the demolition of the projection of the state on that observable. This is possible in non-relativistic quantum mechanics as we will discuss in the following considerations, because this theory make limitations only on a simultaneous, perfect knowledge of two canonical observables.

3 Quantum non demolition measurements

The introduction of the quantum non-demolition measurements (QND) dates back to an article by Landau and Peierls [7] in 1931. However only recently, after understanding the role of quantum mechanics in the fundamental limits to the amplifier sensitivity [8],[9] and under the request to surpass the quantum limit in detectors of small displacements [10],[11], the problem has been studied in detail [12],[13]. The idea of a QND strategy is to perform a series of measurements of one observable of a single object in such a way that the act of the measurement itself does not affect the predictability of the result of the next measurements of the same observable. In order to do this the observable, the instants of time in which it is observed and the interaction Hamiltonian should be all carefully chosen for a given dynamical system. For instance, a first high precision measurement of the position of a free particle implies a large dispersion in the possible values of measurements of momentum. If a second measurement of position is made, due to the Heisenberg evolution, the result will have a large dispersion too. Instead, if a measurement of momentum in a free particle is made at a given instant of time, a second possible measurement will give the same result due to the constant value of the momentum between the two consecutive measurement, provided that the interaction due to the first measurement has not demolished the state. This simple example shows the route to define quantum non-demolition measurements. Only particular observables which satisfy a commutation relation at different times t_i and t_j are allowed to be monitored in a QND way, i.e. if

$$[\hat{x}(t_i), \hat{x}(t_j)] = 0. \tag{8}$$

Moreover, we must also take into account the perturbation on $\hat{x}(t)$ induced by the measuring apparatus which is coupled to the observed system by means of the operator Hamiltonian $\widehat{H}_i$. To avoid changes in the expected value of the observable during the measurement the following condition must be satisfied:

$$[\hat{x}(t), \widehat{H}_i] = 0. \tag{9}$$

This condition assures that the interaction Hamiltonian is simultaneously diagonalizable with the measured observable, no changes are induced in the measured observable during the measurement time in which only the interaction Hamiltonian will be responsible for the time evolution. A sequence of measurements performed under conditions (8) and (9) will give always the same result. This is a definition of a QND measurement. If the instants of time in which it is satisfied (8) are discrete the QND scheme is named stroboscopic or, in a realistic configuration with a duration of the measurement small with respect to the characteristic timescale of the motion of the observed system, quasi-stroboscopic [14],[15],[16]. Otherwise, having a continuous set of instants of time, the QND scheme is named continuous.

In the case of a single oscillator one introduces the two components of the complex amplitude

$$\begin{cases} \widehat{X}_1 = Re[(\hat{x} + i\frac{\hat{p}}{m\omega})e^{i\omega t}] \\ \\ \widehat{X}_2 = Im[(\hat{x} + i\frac{\hat{p}}{m\omega})e^{i\omega t}] \end{cases} \tag{10}$$

such that $\hat{x}(t) = \widehat{X}_1 \cos \omega t + \widehat{X}_2 \sin \omega t$. Their properties are

$$(a) \qquad \frac{d\widehat{X_1}}{dt} = \frac{d\widehat{X_2}}{dt} = 0 \Rightarrow [\widehat{X_1}(t), \widehat{X_1}(t+\tau)] = [\widehat{X_2}(t), \widehat{X_2}(t+\tau)] = 0 \qquad (11)$$

$$(b) \qquad [\widehat{X_1}(t), \widehat{X_2}(t)] = \frac{i\hbar}{m\omega}. \qquad (12)$$

By using (a) and (b) we get

$$[\hat{x}(t), \hat{x}(t+\tau)] = -[\widehat{X_1}, \widehat{X_2}] \{\cos \omega t \, \sin \omega(t+\tau) - \sin \omega t \, \cos \omega(t+\tau)\} = \frac{i\hbar}{m\omega} \sin \omega\tau. \qquad (13)$$

This means that to do a QND measurement of the operator $\hat{x}(t)$ in a single harmonic oscillator one needs the Hamiltonian (here $\hat{q}$ is the variable of the measuring apparatus which couples with the oscillator)

$$\widehat{H_i} = E_0 \, \delta(t - \frac{n\pi}{\omega})\hat{x}\hat{q} \qquad (14)$$

such that the interaction between the system and the measuring apparatus is turn on only when $\hat{x}(t)$ commutes with itself, that is why this kinds of measurements are called stroboscopic Q.N.D..

For a component of the complex amplitude, $\widehat{X_1}$, a QND interaction Hamiltonian should be [12]

$$\widehat{H_i} = E_0 \widehat{X_1} \hat{q} \qquad (15)$$

that is approximately obtained by using the interaction Hamiltonian

$$\widehat{H_i} = 2E_0 \cos \omega_m t \, \hat{x} \, \hat{q} \qquad (16)$$

provided a low-pass filter at $\omega_c << \omega_m$ is used. For practical reasons a different pumping is used, namely a up-conversion around an electrical frequency ω_e such that the interaction Hamiltonian is now

$$\widehat{H_i} = E_0 \cos \omega_e t \cos \omega_m t \, \hat{x} \, \hat{q} = \frac{E_0}{2} [\cos(\omega_e + \omega_m)t + \cos(\omega_e - \omega_m)t] \, \hat{x} \, \hat{q} \qquad (17)$$

which allows an approximate measurement of $\widehat{X_1}$ if a filtering around ω_e is performed with a selectivity such that the terms oscillating at $\omega_e \pm 2\omega_m$ are made negligible. It has been pointed out that the continous approximate QND measurement scheme of one component of the complex amplitude is obtained as a first order approximation of the corresponding stroboscopic scheme [17]. If we start from the interaction Hamiltonian of a stroboscopic measurement of $\widehat{X_1}$ expressed in terms of the physical observable $\hat{x}$

$$\widehat{H_i} = E_0 \cos \omega_e \, t \sum_n \delta(t - \frac{n\pi}{\omega_1})\widehat{X_1}\hat{q} = E_0 \cos \omega_e \, t \sum_n (-1)^n \delta(t - \frac{n\pi}{\omega_1}) \, \hat{x} \, \hat{q} \qquad (18)$$

we will see that, by Fourier expanding the Dirac-distribution, it is obtained

$$\widehat{H_i} = E_0 \cos \omega_e \, t \sum_n \cos(2n+1)\omega_1 t \hat{x} \, \hat{q} \qquad (19)$$

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that, at the first order, is

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \omega_1 t \hat{x} \, \hat{q} \tag{20}$$

i.e. the usual approximate scheme for monitoring of $\widehat{X}_1$. Thus knowing a QND stroboscopic strategy it is simple to write the corresponding QND approximate continuous strategy. This property will be particularly useful in the following considerations, where the more complicated but realistic case of two coupled harmonic oscillators will be treated.

It has been pointed out that also in the classical regime, i.e. when the amplifier is not quantum limited, the QND measurement schemes provide a better sensitivity because one phase of the signal is shielded by the back-action force of the amplifier. A quantitative model in the classical limit has been developed in [18]: it turns out that by writing the noise temperature as

$$T_b = \frac{\omega_m}{\omega_e} T_n \frac{1}{r} \tag{21}$$

for a standard 'amplitude and phase' monitoring is $r < 1$, and for a QND/BAE scheme r may be greater than unity. This is due to the squeezing of the electrical noise into one mechanical phase. A generalized uncertainty relation for the two classical conjugate observables due to the back-action of the amplifier noise is introduced as

$$\Delta X_1 \Delta X_2 \simeq \frac{K_B T_n}{2m\omega_m \omega_e} \tag{22}$$

which may be obtained through a replacement on the right hand side in the standard quantum uncertainty relationship

$$\Delta X_1 \Delta X_2 \simeq \frac{\hbar}{2m\omega_m} \tag{23}$$

of $\hbar$ with $K_B T_n/\omega_2$. If a squeezing factor ρ such that $\Delta X_1 = \rho \Delta X_2$ is introduced ($\rho \to 0$ means a noise-free measurement of $\widehat{X}_1$) the minimum burst noise temperature can be written as

$$T_b = \frac{m\omega_m^2 \Delta X_1^2}{2} \simeq \frac{1}{4} T_n \frac{\omega_1}{\omega_2} \rho \tag{24}$$

showing that the r figure of merit has a dynamical interpretation in terms of a squeezing factor. Recently, an interpretation of the back-action evasion strategies in which they are seen as an alternative to the usual impedance matching for maximizing the signal to noise ratio has been discussed [19].

The description of the QND measurement suggests how to measure small forces below the standard quantum limit. By means of a simple integration of the Heisenberg equation in presence of an external force $F(t)$, one gets for the QND operator $\widehat{X}_1$

$$\widehat{X}_1(t) = \widehat{X}_1(t_0) - \hat{I} \int_{t_0}^{t} \frac{F(t)}{m\omega} \sin \omega t' \, dt'. \tag{25}$$

A sequence of measurement of $\widehat{X}_1$ will then give as a result a sequence of eigenstates linked to the value of the external force

$$\xi(t,\tau) = \xi(t_0) - \int_{t_0}^t \frac{F(t)}{m\omega} \sin \omega t', \, dt'. \tag{26}$$

By means of successive measurements it is possible to study the form of $F(t)$ simply inverting (26)

$$F(\tau) = -\frac{m\omega}{\sin \omega t} \frac{d}{dt} \xi(t_0,t)\Big|_{t=\tau} \tag{27}$$

The singularities for $t = n\pi/\omega$ corresponds to a null information on the force acting on the harmonic oscillator on some instants of time. This can be compensated by using a second oscillator (i.e. a second antenna) with complex amplitude $\hat{Y}_1 + i\hat{Y}_2$ which has eigenvalues

$$\zeta(t,\tau) = \zeta(t_0) - \int_{t_0}^t \frac{F(t)}{m\omega} \sin \omega t', \, dt' \tag{28}$$

here obviously the singularities are in $t_n = (2n+1)\pi/2\omega$.

4 QND quasi-stroboscopic scheme for coupled harmonic oscillators

The current generation of gravitational wave antenna of the Weber type operates by means of an antenna coupled to a small mechanical resonator. In such a way the energy deposited in the antenna by a gravitational wave burst is transferred to the transducer. In the case of an ideal transfer of energy, i.e. with both a perfect tuning of the two uncoupled frequencies and negligible dissipations during the beating period, the amplitude of the oscillations in the transducer is larger than that in the antenna by a factor equal to the square root of the ratio of the equivalent masses of the two resonators. All the detectors operating in coincidence as described in [6] were equipped with a resonant transducer and the same is also planned for the third generation of gravitational wave antennas cooled at 50 mK now under development. It is therefore important to generalize the previous considerations on the QND schemes to this situation, as already outlined in [20]. As we have seen, it is possible to schematize the gravitational cryogenic antenna and the resonant transducer with two coupled harmonic oscillator having masses respectively m_x and m_y (with $\mu = \frac{m_y}{m_x} \ll 1$). The two coupled mechanicals oscillators are described by the Lagrangian

$$L = \frac{1}{2}m_x\dot{x}^2 + \frac{1}{2}m_y\dot{y}^2 - \frac{1}{2}m_x\omega_x^2 x^2 - \frac{1}{2}m_y\omega_y^2(y-x)^2 = \frac{1}{2}(\dot{\xi}\ \dot{\eta})\ T\begin{pmatrix}\dot{\xi}\\\dot{\eta}\end{pmatrix} - \frac{1}{2}(\xi,\eta)\ V\begin{pmatrix}\xi\\\eta\end{pmatrix} \tag{29}$$

where the normalized coordinates $\xi = \sqrt{m_x}x$ and $\eta = \sqrt{m_y}y$ have been introduced, together with the matrices T and V

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{30}$$

$$V = \begin{pmatrix} \omega_x^2 + \mu\omega_y^2 & -\sqrt{\mu}\omega_y^2 \\ -\sqrt{\mu}\omega_y^2 & \omega_y^2 \end{pmatrix}. \tag{31}$$

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As we have already cited to obtain the maximum coupling the two oscillators should have the same frequency $\omega_x = \omega_y = \omega$, i.e. they should be tuned. In this case one finds the solutions

$$\omega_\pm{}^2 = \omega_0{}^2\left(1 + \frac{\mu}{2} \pm \sqrt{\mu\left(1 + \frac{\mu}{4}\right)}\right) \tag{32}$$

which we can write more easily introducing $\alpha_\pm = \frac{\mu}{2} \pm \sqrt{\mu\left(1 + \frac{\mu}{4}\right)}$ obtaining $\omega_\pm^2 = \omega_0^2(1 + \alpha_\pm)$.

The normal coordinates $\Xi_\pm$ corresponding to the eigenfrequencies $\omega_\pm$ are linked to the physical coordinates by means of an orthogonal matrix

$$\begin{pmatrix} \hat{\Xi}_+ \\ \hat{\Xi}_- \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{\alpha_- + 2}} & \dfrac{\alpha_-}{\sqrt{\mu}\sqrt{\alpha_- + 2}} \\ \dfrac{1}{\sqrt{\alpha_+ + 2}} & \dfrac{\alpha}{\sqrt{\mu}\sqrt{\alpha_+ + 2}} \end{pmatrix} \begin{pmatrix} \sqrt{m_x}\hat{x} \\ \sqrt{m_y}\hat{y} \end{pmatrix}. \tag{33}$$

Let us introduce the complex amplitudes of the normal modes

$$\begin{cases} \widehat{X}_1^\pm = \hat{\Xi}_\pm \cos\omega_\pm t - \dfrac{\hat{P}_{\Xi\pm}}{\omega_\pm}\sin\omega_\pm t \\[3mm] \widehat{X}_2^\pm = \hat{\Xi}_\pm \sin\omega_\pm t + \dfrac{\hat{P}_{\Xi\pm}}{\omega_\pm}\cos\omega_\pm t \end{cases} \tag{34}$$

which satisfy the relations

$$[\widehat{X}_1^+, \widehat{X}_2^+] = \frac{i\hbar}{\omega_+} \qquad [\widehat{X}_1^-, \widehat{X}_2^-] = \frac{i\hbar}{\omega_-} \tag{35}$$

as well as

$$[\widehat{X}_{1,2}^+(t), \widehat{X}_{1,2}^+(t + \tau)] = [\widehat{X}_{1,2}^-(t), \widehat{X}_{1,2}^-(t + \tau)] = 0. \tag{36}$$

We can also rewrite the Hamiltonian $\widehat{H}$ of the system as

$$\widehat{H} = \frac{\omega_+^2}{2}[(\widehat{X}_1^+)^2 + (\widehat{X}_2^+)^2] + \frac{\omega_-^2}{2}[(\widehat{X}_1^-)^2 + (\widehat{X}_2^-)^2]. \tag{37}$$

The commutator $[\hat{y}(t) - \hat{x}(t), \hat{y}(t + \tau) - \hat{x}(t + \tau)]$ is calculated by writing $\hat{y}$ and $\hat{x}$ in terms of the complex amplitudes $\widehat{X}_{1,2}^\pm, \widehat{X}_{1,2}^\pm$ of the normal modes which are integral of the motion and by using the same computation procedure which led us to formula (13). Using (35),(36) we obtain, finally, the expression

$$[\hat{y}(t) - \hat{x}(t), \hat{y}(t + \tau) - \hat{x}(t + \tau)] = \frac{i\hbar}{M\omega\mu\sqrt{\mu + 4}}\left[\frac{\omega_+^2}{\omega^2}\sin\omega_+\tau + \frac{\omega_-^2}{\omega^2}\sin\omega_-\tau\right]. \tag{38}$$

This quantity becomes, in the limit $\mu \to 0$

$$[\hat{y}(t) - \hat{x}(t), \hat{y}(t + \tau) - \hat{x}(t + \tau)] = \frac{i\hbar}{2m_y\omega}\frac{\omega_+^2 + \omega_-^2}{\omega^2}\sin\bar{\omega}\tau \, \cos\omega_B\tau \tag{39}$$

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where $\bar{\omega} = \frac{\omega_+ + \omega_-}{2} = \frac{\omega}{2}\sqrt{\mu+4} \rightarrow \omega$ and $\omega_B = \frac{\omega_+ - \omega_-}{2} = \frac{\omega}{2}\sqrt{\mu}$. The (38) and (39) show that the commutator of the operator $\hat{y} - \hat{x}$ with itself at different times is time dependent and it has a characteristic beating behaviour. We have seen that in a quasi-stroboscopic scheme for a single harmonic oscillator the commutator is zero each half a period of the motion and the stroboscopicity is defined whenever measurements with a duration small compared to the period of the motion are performed. This implies a measurement time, a duty cycle, very small and a consequent small value of the effective electromechanical quality factor. In the case of a double harmonic oscillator this drawback is less pronounced because the commutativity is assured every half of a beating period for a time of the order of a period of oscillation. Thus quasi-stroboscopic QND schemes already proposed as a generalization of the conventional BAE scheme based upon a continuous monitoring [17] and already tested on a single oscillator system [21] can be adapted to this situation. In the case of a single harmonic oscillator the duration of the measurement must be small compared to the period of the harmonic oscillator T, in the case of two coupled harmonic oscillators this duration is of the order of some periods of the uncoupled oscillator, although the interaction must be turned on every quarter of a beating period. The interaction Hamiltonian for a two coupled harmonic oscillator system is therefore

$$\widehat{H}_i = \frac{E_0}{2} \sum_n [\theta(t - \frac{nT_B}{2} + \frac{\Delta T}{2}) + \theta(-t + \frac{nT_B}{2} + \frac{\Delta T}{2})](\hat{y} - \hat{x})\hat{q} \qquad (40)$$

where T_B is the beat period and ΔT is of the order of the period of a single harmonic oscillator. Practical values are $T_B \simeq 40\text{ms}$ and $\Delta T \simeq 2\text{ms}$. To calculate the error in a quasi stroboscopic measurement of the operator $\hat{y} - \hat{x}$ performed for instance in the interval $\frac{\pi}{2\omega_B} - \frac{2\pi}{\omega}, \frac{\pi}{2\omega_B} + \frac{2\pi}{\omega}$ we identify the conjugate observable of $\hat{y} - \hat{x}$ as the quantity $(\hat{p}_y - \hat{p}_x)/2$. This last can be expressed in terms of the components of the amplitudes of the normal modes and the commutator at different times of the two conjugate observables is obtained as

$$[\hat{y}(t+\tau) - \hat{x}(t+\tau), \frac{1}{2}(\hat{p}_y(t) - \hat{p}_x(t))] = \frac{i\hbar}{2}(\frac{\alpha_- - 1}{\alpha_-(\alpha_- + 2)} \cos\omega_+\tau + \frac{\alpha_+ - 1}{\alpha_+(\alpha_+ + 2)} \cos\omega_-\tau). \qquad (41)$$

When $\tau = 0$ the commutator relationship (41) is written as

$$[\hat{y}(t) - \hat{x}(t), \frac{1}{2}(\hat{p}_y(t) - \hat{p}_x(t))] = i\hbar \qquad (42)$$

which is exactly the quantity $[\hat{x}(t), \frac{1}{2}\hat{p}_x(t)] + [\hat{y}(t), \frac{1}{2}\hat{p}_y(t)]$.

By expressing ω_+ and ω_- in terms of the frequencies $\bar{\omega}$ and ω_B and substituting in $\alpha_\pm$ their expressions in terms of μ we get finally

$$[\hat{y}(t+\tau) - \hat{x}(t+\tau), \frac{1}{2}(\hat{p}_y(t) - \hat{p}_x(t))] = i\hbar(\cos\bar{\omega}\tau\cos\omega_B\tau - \frac{1+\mu}{\sqrt{\mu(\mu+4)}}\sin\bar{\omega}\tau\sin\omega_B\tau) \qquad (43)$$

If the measurement is performed in the interval $[\frac{\pi}{2\omega_B} - \frac{2\pi}{\omega}, \frac{\pi}{2\omega_B} + \frac{2\pi}{\omega}]$, we can approximate $\cos\omega_B\tau \simeq 1$ and $\sin\omega_B\tau \simeq \omega_B\tau - \frac{\pi}{2}$ and a measurement of infinitesimal duration t' performed in such interval and with a precision $\Delta[\hat{y}(t) - \hat{x}(t)]$ allows to evaluate the error introduced in the measurement process on the uncertainty product as

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$$\Delta[\hat{y}(t+t') - \hat{x}(t+t')] \cdot \frac{1}{2}\Delta[\hat{p}_y(t) - \hat{p}_x(t)] \approx \frac{\hbar}{2}|\cos\bar{\omega}t' - \frac{1+\mu}{\sqrt{\mu(\mu+4)}}\sin\bar{\omega}t'(\omega_B t' - \frac{\pi}{2})| \quad (44)$$

from which, under the approximation for the trigonometric functions, we obtain

$$\frac{1}{2}\Delta[\hat{p}_y(t) - \hat{p}_x(t)] \approx \frac{\hbar}{2\Delta[\hat{y}(t) - \hat{x}(t)]}. \quad (45)$$

The error due to a measurement of duration t' on the operator $\hat{y} - \hat{x}$ is calculated starting from $\Delta[\hat{y}(t) - \hat{x}(t)]$ because

$$\Delta[\hat{y}(t+t') - \hat{x}(t+t')] \approx \Delta[\hat{y}(t) - \hat{x}(t)]|\cos\bar{\omega}t' - \frac{1+\mu}{\sqrt{\mu(\mu+4)}}\sin\bar{\omega}t'(\omega_B t' - \frac{\pi}{2})| \quad (46)$$

If the notation now is changed defining $\Delta_t = \Delta[\hat{y}(t) - \hat{x}(t)]$ we have

$$\frac{\Delta_t}{t'}[\cos\bar{\omega}t' - \frac{1+\mu}{\sqrt{\mu(\mu+4)}}\sin\bar{\omega}t'|\omega_B t' - \frac{\pi}{2}|] \quad (47)$$

and in the limit of $t' \to 0$ we get

$$\frac{d\Delta_t}{dt} = \frac{1+\mu}{\sqrt{\mu(\mu+4)}}\frac{\pi\bar{\omega}}{2} \cdot \Delta_t \quad (48)$$

from which, by integrating, we obtain the error on a measurement performed around $t = \frac{\pi}{2\omega_B}$ as

$$\Delta[\hat{y}(t+\tau) - \hat{x}(t+\tau)] \approx \Delta[\hat{y}(t) - \hat{x}(t)]\exp[\frac{\pi(\mu+1)}{2\sqrt{\mu(\mu+4)}}\bar{\omega}\tau]. \quad (49)$$

For instance, for a choice $t = [\frac{\pi}{2\omega_B} - \frac{2\pi}{\bar{\omega}}]$ and $\tau = \frac{4\pi}{\bar{\omega}}$ we obtain

$$\Delta[\hat{y}(\frac{\pi}{2\omega_B} + \frac{2\pi}{\bar{\omega}}) - \hat{x}(\frac{\pi}{2\omega_B} + \frac{2\pi}{\omega})] \approx \Delta[\hat{y}(\frac{\pi}{2\omega_B} - \frac{2\pi}{\bar{\omega}}) - \hat{x}(\frac{\pi}{2\omega_B} - \frac{2\pi}{\bar{\omega}})]\exp[\frac{2\pi^2(\mu+1)}{\sqrt{\mu(\mu+4)}}] \quad (50)$$

A drawback of these measurement scheme appears when μ is very small and the frequency of the measurement is consequently very small too. To overcome this problem a multimode configuration can be used. In this case the commutator at different times more frequently approaches zero when compared to a two-mode configuration of the same final mass ratio. A more detailed description of this point can be found in [22].

5   QND continuous schemes for coupled harmonic oscillators

Also QND continuous schemes can be used for coupled harmonic oscillator. A first example is given by a monitoring of the complex amplitude of the physical modes $\hat{x}$ and $\hat{y}$ [23]. Introducing the complex amplitudes such that

$$\begin{cases} \hat{y} = Re[(\widehat{Y}_1 + i\widehat{Y}_2)e^{-i\omega_y t}] \\ \hat{p}_y/m_y\omega_y = Img[(\widehat{Y}_1 + i\widehat{Y}_2)e^{-i\omega_y t}] \end{cases} \tag{51}$$

we can rewrite the Hamiltonian in terms of $\widehat{Y}_1$ and $\widehat{Y}_2$ and, by writing the Heisenberg equations for the time evolution of $\widehat{Y}_1$, we obtain

$$\frac{d\widehat{Y}_1}{dt} = -\omega_y \hat{x}\sin\omega_y t. \tag{52}$$

The complex amplitude is not a constant of the motion. However it is easily proved that it is a QND observable. A relationship valid for an infinitesimal time τ is derived for the time evolution

$$\widehat{Y}_1(t + \tau) = \widehat{Y}_1(t) - \omega_y \hat{x}\sin\omega_y t\, \tau \tag{53}$$

and this implies the commutation rule for $\widehat{Y}_1$ at different times

$$[\widehat{Y}_1(t+\tau), \widehat{Y}_1(t)] = [\widehat{Y}_1(t) - \omega_y\,\hat{x}\sin\omega_y t, \widehat{Y}_1] = 0 \tag{54}$$

because of the commutativity between $\widehat{Y}_1$ and $\hat{x}$. Thus $\widehat{Y}_1$ (or $\widehat{Y}_2$, for which similar relationships hold) is a QND observable, although it is not conserved during the motion. From (52) the coordinate $\hat{x}$ is inferred as

$$\hat{x}(t) = -\frac{1}{\omega_y\,\sin\omega_y t}\frac{d\widehat{Y}_1}{dt} \tag{55}$$

apart from the singularities already discussed appearing when $\sin\omega_y t = 0$. When a classical force $F(t)$ acts on the system the Hamiltonian operator is modified and the added term is

$$\widehat{H}_f = -(\hat{x} + \hat{y})\,F(t) \tag{56}$$

obtaining, in this case, the following expression for the time evolution of $\widehat{Y}_1$

$$\frac{d\widehat{Y}_1}{dt} = -\omega_y \hat{x}\sin\omega_y t - \frac{\sin\omega_y t}{m_y\omega_y}F(t). \tag{57}$$

However the effect of the external force to be detected, in our case of geometrical nature, on the transducer is negligible compared to the effect on the antenna, due to the smaller size of the transducer. Thus $\widehat{H}_f \simeq -\hat{x}F(t)$ and the second term in (57) can be omitted. In this reasonable approximation, i.e. $F(t)$ acting only on the antenna, $\widehat{Y}_1$ is also QNDF, i.e. QND also in presence of an external force. To obtain a continuous monitoring of $\widehat{Y}_1$ we need a QND interaction Hamiltonian of the type

$$\widehat{H}_i = E_0\cos\omega_e t \cos\omega_y t\,(\hat{y} - \hat{x})\hat{q} \tag{58}$$

that is a coherent superposition of pumpings at frequencies $\omega_e \pm \omega_y$. Analogous considerations can be made for the monitoring of the real or the imaginary part of the complex amplitude of one normal mode expressed in terms of the physical modes through (33). The advantage in this case is that the quantity $\widehat{X}_1^+$ is a constant of the motion and its monitoring is the standard one

already discussed for a single harmonic oscillator. This is obtained by means of the interaction Hamiltonian

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \omega_+ t \, (\hat{y} - \hat{x})\hat{q} \tag{59}$$

and the analogous for monitoring a component of the complex amplitude $\widehat{X}_1^-$ by substituting ω_+ with ω_-. One drawback of monitoring one component of the complex amplitude of the normal modes is that the information on the other mode is lost, and it is crucial to have informations on both the modes to take full advantage of the resonant schemes.

An alternative scheme suggested by the time dependence of the commutator consists in a monitoring corresponding to the following Hamiltonian:

$$\widehat{H}_i = E_0 \cos \omega_e t \cos \bar{\omega} t \cos \omega_B t \, (\hat{y} - \hat{x})\hat{q}. \tag{60}$$

This coupling allows to infer informations on both the modes because, upon filtering around ω_e in such a way to neglect terms oscillating at $\omega_e \pm 2\omega_B$, $\omega_e \pm 2\bar{\omega}$, $\omega_e \pm 2(\bar{\omega} \pm \omega_B)$, it can be rewritten as

$$\widehat{H}_i = \frac{E_0}{4} \cos \omega_e t (\beta_+ \hat{X}_1^+ + \beta_- \hat{X}_1^-)\hat{q} \tag{61}$$

where $\beta_\pm$ are coefficients related to the coefficients of the matrix (33) and are expressed as

$$\beta_\pm = [m_x(2 + \frac{\mu}{2} \mp \sqrt{\mu(1 + \frac{\mu}{4})})]^{-1/2}(\frac{1}{2\sqrt{\mu}} \mp \sqrt{\frac{(1 + \frac{\mu}{4})}{\mu} - \frac{1}{\mu^{3/2}}}) \tag{62}$$

which, in the limit of $\mu \to 0$, goes to $\beta_\pm = \mp 1/\sqrt{2m_x\mu}$. In this limit the interaction Hamiltonian assumes a simple form

$$\widehat{H}_i = \frac{E_0}{\sqrt{32m_x\mu}} \cos \omega_e t (\hat{X}_1^+ - \hat{X}_1^-)\hat{q} \tag{63}$$

which contains informations on both the normal modes and in such a way that QND measurements can be performed on both the modes. In all the three cases here discussed the selectivity requirements on the electrical circuit are more stringent than in the case of a single harmonic oscillator, because now the electrical oscillator must have a quality factor $Q_e \gg \omega_e/\omega_B$ in order to avoid detection of sidebands contributions. The interaction Hamiltonian (60) can also be written as

$$\widehat{H}_i = \frac{E_0}{2} \cos \omega_e t (\cos \omega_+ t + \cos \omega_- t)(\hat{y} - \hat{x})\hat{q}. \tag{64}$$

With the analogy to the multipump scheme discussed for a single oscillator we can imagine a interaction Hamiltonian of which (64) is only the first order approximation

$$\widehat{H}_i \simeq \frac{E_0}{2} \cos \omega_e t [\sum_{n=0}^{+\infty} \cos (2n + 1)\omega_+ t + \sum_{m=0}^{+\infty} \cos (2m + 1)\omega_- t](\hat{y} - \hat{x})\hat{q} \tag{65}$$

which corresponds, in the limit of a stroboscopic pumping of the kind

$$\widehat{H}_i = E_0 [\sum_{n=0}^{+\infty}(-1)^n \delta(t - \frac{n\pi}{\omega}) + \sum_{m=0}^{+\infty}(-1)^m \delta(t - \frac{n\pi}{\omega})](\hat{y} - \hat{x})\hat{q} \tag{66}$$

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It is interesting to observe that after a time equal to $T_B/2$ both the trains of Dirac distributions will coincide, i.e. $T_B/2 = n\pi/\omega_+ = m\pi/\omega_-$ where $n = m + 2$ (the fact that n and m have the same parity assures the same sign of the corresponding Dirac pulses at those times). So each half a period the two trains are summed and the quasi-stroboscopic scheme discussed in the previous section can be considered as the first order approximation of the stroboscopic scheme resulting from (66). This completes the connection between the multipump continuous schemes and the quasi-stroboscopic scheme introduced in the previous section.

6   Conclusions

We have shown the scenario under which quantum non-demolition measurement schemes should be demanded for detecting gravitational waves in the generation of resonant gravitational wave antennae currently under development, particularly ultra-low temperature resonant bar antennae such as the Rome, Legnaro and Stanford ones which will work at a thermodynamical temperature of $\simeq 50$ mK. Both QND stroboscopic and continuous schemes have been discussed as well as their link and practical schemes to implement them. However the interest of quantum non-demolition measurement schemes goes beyond the only detectability of the gravitational radiation, involving also the quantum measurement theory and the predictions of it for repeated measurements on a single macroscopic oscillator. Feasibility of the generation of macroscopically distinguishable states using a QND scheme has been recently discussed in quantum optics [24], [25]. It has been pointed out that the generation of Schroedinger cats using micromechanical oscillators with quantum limited sensitivity is also feasible [26]. Unlike the optical case, in which the QND measurement is obtained with a frequency mixing due to non-linear susceptivity, the QND measurement for the mechanical case is obtained using an electric field which can be large as one wants. Dissipations in a mechanical oscillator also are quite low compared to electrical or optical oscillators. Moreover, analogies to the production and the detection of squeezed states in optics [27] have been shown. We want to point out a fundamental difference between the two topics: in the case of the optical squeezed states we deal with a quantized field in which its quantum nature is responsible for the limitation to the sensitivity, in the case of quantum non-demolition measurements on a harmonic oscillator the eventual force field which has to be monitored is considered classical and the fundamental limitations comes from the process of the measurement and the interaction of the meter with the external environment. What is squeezed in a QND measure is the back-action noise generated by the amplifier and the squeezing is made in a phase orthogonal to the one which is detected [21]. Despite of this conceptual difference the formalisms to deal with QND strategies are similar to the one used to deal with squeezed states. This analogy is so narrow that also multipump [28], [29] and quasi-stroboscopic [30], [31] schemes have been indipendently and successfully implemented for squeezing the light. Further thoughts on the analogies and the differences between quantum non-demolition measurements on a harmonic oscillator and the squeezing of the quantum noise can give rise to a better understanding on the same interpretation of Quantum Electrodynamics and the operative origin of the vacuum fluctuations of the field in terms of a measurement process [32], an aspect of this fascinating and successful theory which has been very little investigated until now.

7 Acknowledgments

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GEOMETRIC ASPECTS OF UNCERTAINTY AND CORRELATION

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Abstract

It is discussed that the metric induced on the quantum evolution submanifold of the projective Hilbert space describes the uncertainties and correlations of the operators generating the quantum-state evolution, and exhibits the inherently-quantized geometry.

1 Introduction

Berry's phase and its extensions [1-6] are the striking phenomena that show how the law of quantum-state evolution is geometric. It is determined by the evolution curve in the projective Hilbert space $\mathcal{P}$, and is independent of a specific choice of the Hamiltonian as long as it gives that projected curve in $\mathcal{P}$.

The phase difference due to the 1-parameter (λ) evolution is seen in the first-order term of $d\lambda$ in the transition amplitude $\langle \psi(\lambda)|\psi(\lambda + d\lambda)\rangle$. On the other hand, geometry of the evolution curve C in $\mathcal{P}$ is characterized by the Fubini-Study metric [7,8] induced on C: $ds^2 = 1 - |\langle\psi(\lambda)|\psi(\lambda+d\lambda)\rangle|^2$. (Here and hereafter, the state vectors are assumed to be normalized.)

Recently, Anandan and Aharonov [9,10] have obtained a remarkable result that if the 1-parameter evolution is generated by a Hermitian operator A, then the relation $ds = \Delta A d\lambda$ holds, where ΔA is the variance $(\Delta A)^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$. This means that the "velocity" of evolution along C is just equal to the uncertainty of the generator of that evolution.

The purpose of this paper is to report briefly the further results recently obtained in the study of geometric aspects of quantum evolution. More detailed discussions will be found in Ref. [11].

2 Geometry of Uncertainty and Correlation

There are a variety of 1-parameter evolutions for a generic quantum state. Each evolution gives each curve in the projective Hilbert space $\mathcal{P}$. It is preferable to consider the multi-dimensional submanifold $\mathcal{N}$ of $\mathcal{P}$, in which various evolution curves are embedded. $\mathcal{N}$ is properly called here the quantum evolution submanifold. If a given state is parametrized by a set of n real numbers $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$, then a local coordinate of $\mathcal{N}$ is identified with α. In this case, the metric induced on $\mathcal{N}$ is given by

$$ds^2 = 1 - |\langle \psi(\alpha)|\psi(\alpha + d\alpha)\rangle|^2. \tag{1}$$

If the evolution of the state $|\psi(\alpha)\rangle$ is assumed to be generated by n independent Hermitian operators $\{A_i(\alpha)\}_{i=1,2,\ldots,n}$, that is,

$$-i\partial_i|\psi(\alpha)\rangle = A_i(\alpha)|\psi(\alpha)\rangle \qquad (i = 1, 2, \ldots, n), \tag{2}$$

then Eq.(1) has the form $ds^2 = g_{ij}(\alpha)d\alpha^i d\alpha^j$, where

$$
\begin{aligned}
g_{ij}(\alpha) = \frac{1}{2}\langle\psi(\alpha)|A_i(\alpha)A_j(\alpha) &+ A_j(\alpha)A_i(\alpha)|\psi(\alpha)\rangle \\
&- \langle\psi(\alpha)|A_i(\alpha)|\psi(\alpha)\rangle\langle\psi(\alpha)|A_j(\alpha)|\psi(\alpha)\rangle,
\end{aligned} \tag{3}
$$

provided that $\partial_i \equiv \partial/\partial\alpha^i$ and the summation convention is understood for the repeated upper and lower indices. Thus, one can see that the diagonal g_{ii} and off-diagonal g_{ij} ($i \neq j$) components are respectively equal to the uncertainties and correlations of the operators generating the evolution [11,12].

The metric (3) defines the Riemannian structure of $\mathcal{N}$. The metric-compatible connection can be expressed as a simple quantum expectation value [11]:

$$
\begin{aligned}
\Gamma^{\bullet}_{kij} = \frac{1}{4}\langle\psi|[\partial_i B_j &+ \partial_j B_i - i(B_i B_j + B_j B_i)]B_k \\
&+ B_k[\partial_i B_j + \partial_j B_i + i(B_i B_j + B_j B_i)]|\psi\rangle,
\end{aligned} \tag{4}
$$

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where the operators B_i are given by $B_i(\alpha) = A_i(\alpha) - \langle\psi(\alpha)|A_i(\alpha)|\psi(\alpha)\rangle$. With this expression, it is straightforward to ascertain the Riemannian parallelism: $\nabla_k g_{ij} = \partial_k g_{ij} - \Gamma^h{}_{ik}g_{hj} - \Gamma^h{}_{jk}g_{ih} = 0$.

Geometric aspects of the uncertainties and correlations can be seen best in the squeezed state example. The single-mode two-photon squeezed state [13] is given by $|z\rangle_\xi = D(z)S(\xi)|0\rangle = \exp(za^\dagger - z^*a)\exp[\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)]|0\rangle$. $a^\dagger$ and a are the usual bosonic creation and annihilation operators. $|0\rangle$ is the vacuum state annihilated by a. $D(z)$ and $S(\xi)$ are called Glauber's displacement operator and the squeeze operator, respectively. The displacement operator gives a correspondence relation between relevant operators and their classical counterparts in the phase space (x,p) with the parametrization $z = (x + ip)/\sqrt{2}$. x and p are respectively equal to the expectation values of the position $X = (a + a^\dagger)/\sqrt{2}$ and momentum $P = (a - a^\dagger)/i\sqrt{2}$ operators in the squeezed state.

Consider the translational evolution: $|z(x,p)\rangle_\xi \longrightarrow |(z + dz)(x + dx, p + dp)\rangle_\xi$, where the squeeze parameter is fixed. From the transition amplitude, the metric $ds^2 = 1 - |_\xi \langle z|z + dz\rangle_\xi|^2$ is directly calculated as

$$ds^2 = \frac{1}{2}(\cosh 2r - \sinh 2r \cos 2\phi)dx^2 + \frac{1}{2}(\cosh 2r + \sinh 2r \cos 2\phi)dp^2$$

$$+ 2 \times \frac{1}{2}\sinh 2r \sin 2\phi \, dx\, dp, \quad (5)$$

provided the parametrization $\xi = re^{-2i\phi}$ $(0 \leq r, 0 \leq \phi < 2\pi)$ has been used. This is the Euclidean metric in a non-Cartesian coordinate. On the other hand, the above translational evolution is generated by the following operators:

$$-i\frac{\partial}{\partial x}|z\rangle_\xi = A_x|z\rangle_\xi, \qquad A_x = -P + \frac{p}{2}, \quad (6a)$$

$$-i\frac{\partial}{\partial p}|z\rangle_\xi = A_p|z\rangle_\xi, \qquad A_p = X - \frac{x}{2}. \quad (6b)$$

The uncertainties and correlations in the squeezed state are the familiar ones:

$$(\Delta A_x)^2 = (\Delta P)^2 = \frac{1}{2}(\cosh 2r - \sinh 2r \cos 2\phi), \quad (7a)$$

$$(\Delta A_p)^2 = (\Delta X)^2 = \frac{1}{2}(\cosh 2r + \sinh 2r \cos 2\phi), \quad (7b)$$

$$C(A_x, A_p) = -C(X, P) = \frac{1}{2}\sinh 2r \sin 2\phi, \quad (7c)$$

where $C(A, B) = C(B, A) = \frac{1}{2}\langle\psi|AB + BA|\psi\rangle - \langle\psi|A|\psi\rangle\langle\psi|B|\psi\rangle$. These quantities in fact give the components of the metric (5).

The effects of squeezing as the expansion, contraction, and rotation in the phase space has been explored geometrically by the methods of phase-space representations of quantum theory in the literature [14,15]. The metric (5) describes those effects in a peculiar *representation-free* manner.

Since the metric is given in terms of a reference state, it carries some of quantum numbers characterizing that state. Accordingly, $\mathcal{N}$ possesses the quantized structure, in general. In what follows, such examples are given.

The first example is the displaced number state [16]: $|z\rangle_n = D(z)|n\rangle$, where $|n\rangle \equiv (n!)^{-1/2}(a^\dagger)^n|0\rangle$ $(n = 0, 1, 2, \ldots)$. Consider the translational evolution $|z(x, p)\rangle_n \longrightarrow |z(x + dx, p + dp)\rangle_n$. The metric is calculated as

$$ds^2 = (n + \frac{1}{2})(dx^2 + dp^2). \tag{8}$$

Therefore, the phase space locally identified with $\mathcal{N}$ associated with the evolution of the displaced number state has a Euclidean metric with a quantized conformal factor.

Another example is the squeezed number state [17]: $|\xi\rangle_n = S(\xi)|n\rangle$ $(n = 0, 1, 2, \ldots)$. The squeeze parameter is again parametrized as $\xi = re^{-2i\phi}$. Consider the evolution $|\xi(r, \phi)\rangle_n \longrightarrow |(\xi + d\xi)(r + dr, \phi + d\phi)\rangle_n$. $\mathcal{N}$ is locally labelled by (r, ϕ). The metric is then found to be

$$ds^2 = \frac{1}{2}(n^2 + n + 1)(dr^2 + \sinh^2 2r d\phi^2). \tag{9}$$

This is the metric of the Lobachevsky space [18] with a quantized conformal factor. Its Gaussian curvature [18] is also quantized as $K = -8/(n^2 + n + 1)$. It is interesting to see that the curvature vanishes in the "classical limit" $n \to \infty$.

3 Conclusions

It has been demonstrated that the Fubini-Study metric induced on the quantum evolution submanifold $\mathcal{N}$ is completely given by the uncertainties and correlations of the operators generating various evolutions, and $\mathcal{N}$ admits the quantized Riemannian structure.

In the above simple examples, only the conformal factors of the metrics are quantized. This may be partially due to the mathematical fact [18] that all two-dimensional spaces are conformally equivalent to the Euclidean space. In general, each component of the metric is individually quantized.

4 Acknowledgements

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OBSERVABLES, MEASUREMENTS AND PHASE OPERATORS FROM A BOHMIAN PERSPECTIVE

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Abstract

Bohmian mechanics is a deterministic theory of point particles in motion. While avoiding all the paradoxes of nonrelativistic quantum mechanics, it yields the quantum formalism itself—especially the role of self-adjoint operators—as a macroscopic measurement formalism. As an "application" it is shown that much of the confusion connected with the phase operator for the electromagnetic field arises from a misunderstanding of the role of operators in quantum theory.

1 Introduction

We would like to apologize for the bad title: we will try to explain why the casual use of the words "observables" and "measurements," which are on John Bell's list of bad words in his article "Against Measurement"[1], "measurement" being the worst of all, leads to much unnecessary confusion concerning the meaning of the quantum formalism. But first we introduce an even worse word: following Bell we will use the abbreviation "FAPP" for "for all practical purposes."

Quantum mechanics suffers from its irreducible reference to "observers" and "measurements": We have, for example, the fundamental rule that $|\psi(q)|^2 dq$ is the *probability* of *observing* a particle in dq about q in a position *measurement*. This rule entails 1) indeterminism, because it deals with probabilities on a fundamental level; 2) subjectivity, because it refers to an observer and 3) vagueness, because the notion of measurement is vague. It has repeatedly been emphasized, however, that these are inescapable components of modern physics. The following reasons are frequently cited:

- It is *meaningless* to talk about trajectories of particles, because the uncertainty principle doesn't allow for a simultaneous measurement of position and velocity (Heisenberg).

- It leads to *contradictions* even to think that a particle might have a well-defined position and velocity at the same time.

- It is mathematically *impossible* to add "hidden variables" (e.g., actual positions) as a further specification of the quantum state (von Neumann [2]).

This is wrong! In fact it is almost trivially wrong: A counterexample has existed for more than four decades, namely Bohm's quantum theory [3], which we prefer to call "Bohmian mechanics." By trying the obvious, namely by seeking a motion of particles in space compatible

with Schrödinger's equation, one is led directly to Bohmian mechanics. This theory is clear, objective and deterministic. The entire quantum formalism—operators as observables, randomness, etc.—emerges as a measurement formalism, or more precisely, as a phenomenological formalism for describing measurement-like experiments. Thus one arrives at an explanation for the quantum formalism rather than at an alternative theory which might give rise to "new predictions." We will argue, however, that Bohmian mechanics nonetheless refutes most of the approaches to the problem of the phase operator in quantum optics. It turns out, in fact, that there is no problem! But let us first give a brief review of nonrelativistic quantum mechanics.

2 The Quantum Formalism

- State: The state of an N-particle system is given by a vector $\psi \in \mathcal{H} = L_2(\mathbb{R}^{3N})$.

- Dynamics: The time evolution is given by the unitary evolution $\psi_t := e^{-\frac{i}{\hbar}Ht}\psi_0$, which is equivalent to Schrödingers equation $i\hbar\frac{\partial}{\partial t}\psi = H\psi$.

- Observables: The observables of the system are given by self-adjoint operators on $\mathcal{H}$. To find operators corresponding to classical observables one replaces the classical Poisson brackets by the commutator: $\{\ ,\ \} \mapsto \frac{1}{i\hbar}[\ ,\ ]$.

- Measurements: In a measurement of an operator $A = \sum \lambda_i |a_i\rangle\langle a_i|$ on a system in the state ψ one may find only one of its eigenvalues λ_i, with probability $\text{prob}(i) = |\langle a_i|\psi\rangle|^2$. After the measurement the system is in the corresponding eigenstate $|a_i\rangle$ (collapse rule).

3 The Fundamental Ambiguity

There can be no doubt that the predictions of quantum mechanics are of an amazing accuracy. But neither this nor the mathematical simplicity and beauty of unitary evolution in Hilbert space should hide the fact that a fundamental ambiguity enters at the very point where mathematics makes contact with reality: Measurements! Measurements of what—if the wave function ψ is really the complete state? And as J.S Bell has said [1]:

> It would seem that the theory is exclusively concerned about "results of measurement", and has nothing to say about anything else. What exactly qualifies some physical systems to play the role of "measurer"? Was the wave function of the world waiting to jump for thousands of millions of years until a single-celled living creature appeared? Or did it have to wait a little longer, for some better qualified system ... with a Ph.D.?

This fundamental ambiguity, connected with "measurement" and collapse is also responsible for the familiar paradoxes associated with orthodox quantum mechanics such as Schrödinger's cat paradox or the measurement problem. In the following we shall show that these difficulties simply evaporate by giving up the unquestioned assumption that ψ alone provides a complete description of the state of a system. Bohmian mechanics will permit an understanding of quantum phenomena in a language everybody is using anyway: a theory of particles moving in space.

4 Bohmian Mechanics

- State: $(q, \psi), q \in \mathbb{R}^{3N}, \psi \in L_2(\mathbb{R}^{3N})$, i.e., the state of an N-particle system is given by its wave function *and* the actual positions $q = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$ of the particles *which the theory is about*.

- Dynamics: The time evolution is given by a first-order differential equation for the positions of the particles, with ψ evolving in the usual way:

$$\frac{d}{dt}\mathbf{q}_k(t) = \mathbf{v}_k^{\psi_t}(q(t)) = \frac{\hbar}{m_k}\mathrm{Im}\frac{\nabla_k\psi_t}{\psi_t}(q(t)) \tag{1}$$

$$i\hbar\frac{\partial}{\partial t}\psi(x, t) = (-\sum_{k=1}^{N}\frac{\hbar^2}{2m_k}\Delta_k + V(x))\psi(x, t) \tag{2}$$

(Note that the role of ψ is to generate a Galilean covariant vector field on configuration space which guides the motion, and this leads directly to (1).)

This is all we need! It is a crucial property of this dynamical system that it conserves the distribution $\rho = |\psi|^2$, which we call the equivariant measure. The quantum formalism, randomness, Born's rule—"If a system has wave function ψ then its configuration has distribution $|\psi|^2$"—and all the rest emerges from a detailed analysis of these equations. No further axioms about measurements are necessary nor is there room for any such axioms. That this is so was already sketched by David Bohm in his 1952 paper [3]; a more detailed analysis can be found in [4]. Let us give a summary of the main crucial features of Bohmian mechanics: In addition to being clear, objective and deterministic it also agrees with experiment. There is, however no need for collapse, no measurement paradox and no need to split the world into system and observer.

Let us look at some simple examples.

4.1 Example: the motion of a Gaussian wave packet

Consider the time dependent one-particle wave function $\psi_t(x)$ of a freely evolving Gaussian, which starts at the origin with velocity v_0 and width σ. From (1) one obtains the velocity vector field and easily solves the differential equation for the positions, obtaining the solution flow $\Phi_{q_0}(t) := q(t) = v_0 t + q_0\sqrt{1 + t^2/\sigma^4}$. Note that the motion is clearly non-Newtonian. Only in the limit of large times, the particles move with constant velocity $v_\infty(q_0) := v_0 + q_0/\sigma^2$, which means v_∞ is a random variable with a Gaussian distribution, centered around v_0. Now let us define the momentum as the random variable $p := mv_\infty$, which can be approximately determined by measuring the position $q(T)$ at a large time T: $p \approx mq(T)/T$. Clearly the probability distribution for p is exactly the same as the one obtained by projecting the initial state on the eigenstates of the momentum operator. It can in fact be shown quite generally that for an arbitrary freely evolving wave function $\psi_t(x)$, p is well defined, with distribution given in the usual way by the Fourier transform $|\hat{\psi}_0(p)|^2$. Note that p is not at all the same as the "classical momentum" given by m times the actual velocity.

4.2 Example: the two-slit experiment in Bohmian mechanics

The particle passes through either the upper or the lower slit. The interference pattern occurs because the wave function guiding the particle develops this pattern. Closing one slit will lead to a different wave function and therefore to different paths and a different—or no—pattern. The randomness observed in the experiment is due to uncertainty in the initial conditions, as in classical chaotic systems.

5 Measurements/Experiments

Let us sketch an analysis of measurement-like experiments; for a much more detailed analysis see [5]. We describe the combined evolution of a composite system consisting of System $\otimes$ Apparatus. Let the initial state of the apparatus be ϕ_0 and let ϕ_i denote the orthogonal apparatus wave functions corresponding to the possible outcomes. (Think of separated wave packets corresponding to possible pointer positions or patterns of ink spots on a computer printout—which may, for example, register detection by a photocounter.) We assign the values λ_i to the "pointer states" ϕ_i. It turns out [5] that if an experiment is repeatable then in the simplest case there exists a basis $\{|\psi_i\rangle\}$ of the system Hilbert space such that under the interaction with the apparatus

$$|\psi_i\rangle \otimes |\phi_0\rangle \rightarrow |\psi_i\rangle \otimes |\phi_i\rangle. \tag{3}$$

(Note that the unitarity of the time evolution together with the orthogonality of the $|\phi_i\rangle$ forces the orthogonality of the $|\psi_i\rangle$.)

An arbitrary state $|\psi\rangle = \sum c_i |\psi_i\rangle$ may be expressed in this basis, with $c_i = \langle \psi_i | \psi \rangle$. The linearity of the time evolution implies that

$$|\psi\rangle \otimes |\phi_0\rangle \rightarrow \sum c_i |\psi_i\rangle \otimes |\phi_i\rangle. \tag{4}$$

Thus using Born's rule—which we remind you is a *consequence* of Bohmian mechanics—we find that $|\langle \psi_i | \psi \rangle|^2$ is the probability to find the outcome λ_i.

Let us make a remark on the "measurement problem": Certainly the wave function is in a superposition after interaction with a superposition of eigenstates, but the complete state is given by the wave function *and* the actual configuration. The trajectory will end up in but one of the different disjoint wave packets, and thus the dynamics does not lead to a macroscopic superposition of outcomes, as would be the case if we had only a Schrödinger wave function. Moreover, for the further evolution the influence of the other wave packets turns out to be FAPP negligible. In this way collapse is merely a matter of convenience.

Now let us make contact with the usual operator formalism. Define the self-adjoint operator

$$A := \sum \lambda_i |\psi_i\rangle\langle\psi_i|. \tag{5}$$

With this operator we can calculate the statistics for the outcome in the usual way.

The fact that a self-adjoint operator on the system Hilbert space *alone* suffices to describe the full statistics for the outcome of the experiment supports the misleading idea that some preexisting properties of the system have actually been "measured," the apparatus playing a purely passive

role. That this is not generally the case, that we rather have to regard the result as being the joint product of the system *and* the apparatus, has been emphasized by Bohr.[1]

For the analysis of more general experiments it is convenient to introduce the following notation. The map $\Delta \mapsto P(\Delta) := \sum_{\lambda_i \in \Delta} |\psi_i\rangle\langle\psi_i|$, from subsets of $\mathbb{R}$ to projectors on $\mathcal{H}$, is what mathematicians call a projection-valued measure (PV).[2] With this notation $\langle\psi|P(\Delta)|\psi\rangle$ is the probability to find the result in the set Δ. Note that $A = \int \lambda P(d\lambda)$.

It turns out [5] that if one doesn't assume repeatability a positive-operator-valued measure (POV) $O(\Delta)$ plays the role of $P(\Delta)$. These operators need not be projectors, i.e., it may be that $O(\Delta)^2 \neq O(\Delta)$. The probability of finding the result in the set Δ is given by $\langle\psi|O(\Delta)|\psi\rangle$. Define the self-adjoint operator $B := \sum \lambda_i O(\lambda_i)$ $(= \int \lambda O(d\lambda))$. Thus the expected value of the outcome is given by $\langle\psi|B|\psi\rangle$. Note that knowledge of B alone does not provide complete information about the statistics of the outcome, as it does for repeatable experiments, because in general $B^n \neq \sum \lambda_i^n O(\lambda_i)$. Thus for nonrepeatable measurements it is not possible to cast the information about the entire statistics into a bilinear form involving a single self-adjoint operator.

POV's have been proposed as a means of providing a generalized description for "fuzzy measurements"[6]. Note, however, that POV's arise naturally from a measurement analysis in Bohmian mechanics, in which there is no "intrinsic fuzziness."

6 The Phase Problem in Quantum Optics

6.1 A brief history of the phase operator

For the following discussion it will be sufficient to focus on a single mode of the electromagnetic field, which is well-known to be equivalent to an one-dimensional harmonic oscillator. We will use the standard notation $a, a^\dagger$ for the annihilation and creation operators, and $N := a^\dagger a$ for the number operator.

For a classical harmonic oscillator the phase is a respectable observable. What is its quantum mechanical counterpart? We give a short sketch of some of the main approaches to the "phase problem." A detailed discussion can be found in [7].

- 1927 Dirac [8]: A polar decomposition of the creation and annihilation operator into $e^{i\Phi}\sqrt{N} := a$, which seems to imply $\sqrt{N}e^{-i\Phi} = a^\dagger$, "yields" $[\Phi, N] = -i$. Dirac noticed himself that this definition leads to contradictions, e.g., if one takes the expectation value of the commutator for an energy eigenstate.

- 1964: Susskind and Glogower [9] prove that there is no way to define an unitary operator U with the property $U\sqrt{N} = a$. Therefore there can be no self-adjoint operator Φ such that $U = e^{i\Phi}$, which explains the flaw in Dirac's ansatz. They conclude that a self-adjoint phase operator doesn't exist.

- 1968: Loudon defines nonorthogonal "phase eigenstates" [10]: $|\phi\rangle := \sqrt{\frac{1}{2\pi}} \sum_{n=0}^{\infty} e^{in\phi}|n\rangle$.

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[1] Position measurements are exceptions. Position plays a distinguished role in Bohmian mechanics, as it does in the real world.

[2] $P^2 = P = P^\dagger; P(\emptyset) = 0, P(\Omega) = 1; P(\cup\Delta_i) = \sum P(\Delta_i)$ for mutually disjoint sets Δ_i.

- 1976: Lévy-Leblond, using the Loudon states, constructs a POV [11]: $\Delta \to \int_\Delta d\phi |\phi\rangle\langle\phi|$, for Δ any subset of $[-\pi, \pi]$.

- 1986: Barnett and Pegg introduce "negative-photon-number" states and define the unitary operator [12]: $e^{i\Phi} := \sum_{-\infty}^{\infty} |n\rangle\langle n+1|$.

- 1988: Barnett and Pegg suggest a limiting procedure, based on the definition of phase eigenstates in a finite-dimensional Hilbert space [13]. $|\phi_n\rangle := \frac{1}{\sqrt{s+1}} \sum_{m=0}^{s} e^{im\phi_n} |m\rangle$, $\phi_n := \phi_0 + 2\pi/(s+1)n, n = 0 \ldots s$, $\Phi_s := \sum_{n=0}^{s} \phi_n |\phi_n\rangle\langle\phi_n|$. The limit $s \to \infty$ is then taken at the *end* of any calculation.

- 1991: Mandel proposes an operational approach to the quantum phase [14]. He suggests an experiment, together with some procedure to derive quantities he calls the "cosine and sine of the phase difference." He finds disagreement with the predictions based on the (second) Barnett-Pegg operator or the Susskind-Glogower operator.

6.2 Discussion of the different approaches

Let us first address two questions which might now be irritating the reader: 1) How can it be the case that we have a nonexistence proof and several explicit constructions of self-adjoint phase operators at the same time? 2) What exactly is going on in this peculiar (hi)story?

The answer to the first question is easy. The nonexistence proof of Susskind and Glogower tells us that there is no polar decomposition of the annihilation operator into a positive and a unitary operator. None of the "phase operators" suggested by Barnett and Pegg provide such a decomposition (if they serve any purpose at all, it is certainly not for this). But how can one decide who is right? And, perhaps more to the point, what is the physical relevance of all these operators?

This leads us to the second question. We are often told that for every classical observable there exists a corresponding self-adjoint operator. Recipes such as "replace the classical Poisson brackets by the commutator" are used as a guide to postulate the correct commutation relations. This seems to work perfectly well for position and momentum but not for the phase. But so what? Why should it?

We have sketched in (4.1) how to describe "momentum measurements" without invoking postulated commutation relations. The analysis of the experiment shows that the momentum operator as a multiplication operator in Fourier space yields the correct statistics. Note, however, that it can be shown that for the actual velocity—certainly a classical observable—there is neither a corresponding operator nor a POV! This simply means that there is no experiment which measures the actual velocity *in the sense of section (5)*.

The POV proposed by Lévy-Leblond is an explicit example how to describe an *abstract* phase "measurement" without a self-adjoint operator. In order to decide which is the "right" description for the phase one would have to ask for the experiment which an operator or POV is supposed to describe.[3] But what is the physical relevance of pursuing the question as to which experiments are described by a given operator? Note, however, that for a given experiment, say Mandel's

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[3]This has been emphasized by Lévy-Leblond. His focus, however, was more on advertising a more general formalism for describing experiments than on applying it to a special example.

experiment, it is well-known how to calculate the photocount statistics, which is all that is relevant. There is no room left for *postulating* operators or eigenstates. An analysis of the experiment at hand shows what quantities are actually "measured" and which mathematical objects, be they operators or POV's or what have you, simplify the description of the predictions. And, as is also stressed by Mandel, different experiments yield different operators. There is no unique phase operator, nor do we need one. In other words: There is no problem!

7 Conclusion

We end by quoting Bell one last time [15]:

> ..... in physics the only observations we must consider are position observations, if only the positions of instrument pointers. It is a great merit of the de Broglie-Bohm picture to force us to consider this fact. If you make axioms, rather than definitions and theorems, about the "measurement" of anything else, then you commit redundancy and risk inconsistency.

8 Acknowledgments

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III. Theoretical Development

UNCERTAINTY RELATIONS, ZERO POINT ENERGY AND THE LINEAR CANONICAL GROUP

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Abstract

The close relationship between the zero point energy, the uncertainty relations, coherent states, squeezed states and correlated states for one mode is investigated. This group-theoretic perspective enables the parametrization and identification of their multimode generalization. In particular the generalized Schrödinger-Robertson uncertainty relations are analyzed. An elementary method of determining the canonical structure of the generalized correlated states is presented.

1 Introduction

Advances in atomic physics and quantum optics have made it possible to examine and verify many of the immediate predictions of quantum mechanics. The most celebrated of these is the Heisenberg [?] uncertainty relation

$$(\Delta q)^2 \ (\Delta p)^2 \geq \left(\frac{\hbar}{2}\right)^2 \tag{1}$$

where

$$(\Delta q)^2 = \langle q^2 \rangle - \langle q \rangle^2, \tag{2}$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \tag{3}$$

are the dispersons in the coordinate and momentum variable. The Heisenberg uncertainty relation in the form

$$\Delta q \cdot \Delta p \geq \frac{\hbar}{2} \tag{4}$$

has been verified in gedanken experiments like the Heisenberg microscope and in the simple pictures of de Broglie waves.

Since Δq and Δp have different dimensions their individual magnitudes cannot be compared without choosing units for length and momentum. By a suitable scale change we could scale them inversely as long as the unit of action is fixed; in this case the change is in the unit of $\{mass^2/(time)^2\}$ or equally well in the unit of length since action has the dimensions of

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{mass × (length))²/time}. Having fixed any such choice we can talk of the numerical values of Δp and Δq. Another and earlier result of quantum theory is the existence of zero point energy [?]. If p and q are canonical operators satisfying the commutation relations

$$q\,p - p\,q = i\hbar \tag{5}$$

then the "energy" $\frac{1}{2}(p^2 + \omega^2 q^2)$ has a nonzero minimum value:

$$\frac{1}{2}(p^2 + \omega^2 q^2) = \omega\left\{\frac{\omega q - ip}{\sqrt{2\omega}} \cdot \frac{\omega q + ip}{\sqrt{2\omega}}\right\} + \frac{\hbar\omega}{2} \geq \hbar\omega/2. \tag{6}$$

Since the first term is non negative, $\omega\, a^\dagger a$, there is the zeropoint energy $\hbar\omega/2$ for the ground state which is annihilated by the operator

$$a = (\omega q + ip)/\sqrt{2\omega}. \tag{7}$$

While the notation is new, the zeropoint energy is as old as quantum theory!

It is well known that there is an immediate connection between the two relations. For every ω, $-\infty < \omega < \infty$

$$E(\omega) = (\omega q - ip)(\omega q + ip) \geq 0 \tag{8}$$

but this implies

$$\omega^2\langle q^2\rangle + \langle p^2\rangle + i\omega\langle qp - pq\rangle \tag{9}$$

$$= \omega^2\langle q^2\rangle - \omega\hbar + \langle p^2\rangle \geq 0. \tag{10}$$

Hence the discriminant of this quadratic form should be negative: that is,

$$4\,\langle q^2\rangle\langle p^2\rangle \geq \hbar^2. \tag{11}$$

Noting that the deviations from the mean

$$Q = q - \langle q\rangle, \qquad P = p - \langle p\rangle \tag{12}$$

also satisfy the canonical commutation relations we, derive

$$\langle Q^2\rangle\langle P^2\rangle \geq \frac{1}{4}\hbar^2 \tag{13}$$

which is Heisenberg's uncertainty relation.

We may therefore say that the zeropoint energy relation (6) was not invariant under the linear canonical transformation

$$q \longrightarrow Q = q - \langle q\rangle \tag{14}$$

$$p \longrightarrow P = p - \langle p\rangle \tag{15}$$

nor under

$$q \longrightarrow Q = \omega^{\frac{1}{2}} q \tag{16}$$

$$p \longrightarrow P = \omega^{-\frac{1}{2}} p. \tag{17}$$

Imposition of these canonical transformations on the Planck zeropoint energy inequality (6) gives the Heisenberg uncertainty relation.

But there are yet other linear canonical transformations: the simplest one is

$$q \longrightarrow q \cos \theta - \omega^{-1} p \sin \theta \tag{18}$$

$$p \longrightarrow \omega q \sin \theta + p \cos \theta. \tag{19}$$

While the Planck zeropoint inequality is invariant under this transformation, the Heisenberg uncertainty relation is not. We get, for any θ,

$$\left\{ \langle q^2 \rangle \cos^2 \theta + \langle p^2 \rangle \sin^2 \theta - \langle qp + pq \rangle \cos \theta \sin \theta \right\}. \tag{20}$$

$$\left\{ \langle q^2 \rangle \sin^2 \theta + \langle p^2 \rangle \cos^2 \theta + \langle qp + pq \rangle \cos \theta \sin \theta \right\} \geq \frac{\hbar^2}{4}. \tag{21}$$

By an elementary rearrangement this gives

$$\left\{ \langle q^2 \rangle + \langle p^2 \rangle \right\}^2 - \left\{ \left(\langle q^2 \rangle - \langle p^2 \rangle \right) \cos 2\theta - \langle qp + pq \rangle \sin 2\theta \right\}^2 \geq \hbar^2. \tag{22}$$

By choosing

$$\tan 2\theta = - \langle qp + pq \rangle \Big/ \left\{ \langle q^2 \rangle - \langle p^2 \rangle \right\} \tag{23}$$

we get the inequality

$$\langle q^2 \rangle \langle p^2 \rangle - \frac{\langle qp + pq \rangle^2}{4} \geq \frac{\hbar^2}{4}. \tag{24}$$

This is the Schrödinger uncertainty relation provided we replace q and p by $q - \langle q \rangle$ and $p - \langle p \rangle$. It was derived by Schrödinger and by Robertson[?]. It is stronger than the Heisenberg uncertainty relations and reduces to it in the special case of "uncorrelated states" for which

$$\langle (q - \langle q \rangle) (p - \langle p \rangle) + pq \rangle = 0 \tag{25}$$

or equivalently

$$\langle qp + pq \rangle = \langle q \rangle \langle p \rangle + \langle p \rangle \langle q \rangle. \tag{26}$$

Even for a harmonic oscillator of frequency ν this is not in general true and the correlation oscillates with twice the frequency. So a Heisenberg minimum uncertainty state is not canonically invariant. For the harmonic oscillator this has been known for decades. Dodunov and Mańko [?] have given a general systematics of such a derivation. The clue to the Schrödinger-Robertson generalization of the Heisenberg uncertainty relations is the requirement of invariance under the group of linear canonical transformations. The state of the minimum energy for the harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}(p^2 + q^2) = \left(a^\dagger a + \frac{\hbar}{2} \right) \tag{27}$$

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is the vacuum state $|\Psi\rangle$ satisfying

$$a|\Psi\rangle = 0 \tag{28}$$

with the associated wave function

$$\psi(x) = (\pi)^{-1/4}\exp(-x^2/2). \tag{29}$$

This is a state of the minimum uncertainty. But the minimum uncertainty class is wider, among these are

$$a|z\rangle = z\,|z\rangle\,,\, z \text{ complex number} \tag{30}$$

with wave function

$$\psi(x) = (\pi)^{-1/4}\exp\left\{-(x-z)^2/2\right\}. \tag{31}$$

These are the "coherent states" introduced by Schrödinger [?] and rediscovered decades later in the context of quantum optics by Glauber [?] and by Sudarshan [?]. They constitute an overcomplete family of states in terms of which every state can be expressed in infinitely many ways; further in terms of them every density matrix can be exhibited as a sum of projectors $|z\rangle\langle z|$ to the coherent states with distribution valued weight [?] and [?].

But the coherent states are not a canonically invariant set. The scale transformation ("squeezing")

$$q \longrightarrow \exp\left(\omega^{\frac{1}{2}}\right)q\,, \qquad p \longrightarrow \exp\left(\omega^{-\frac{1}{2}}\right)p \tag{32}$$

takes a coherent state into a new class of [?] states which are now called squeezed states. In terms of $a, a^\dagger$ these are the Bogoliubov - Valatin transformations [?]. The unitary transformation

$$V = \exp\left\{-i\omega^{\frac{1}{2}}(qp+pq)/2\right\} \tag{33}$$

accomplishes the squeezing: and thus the one parameter family of overcomplete sets of squeezed coherent states with wave functions.

$$\psi(x) = (\pi)^{-\frac{1}{4}}\exp\left\{-\omega(x-\Gamma_2 z)^2/2\right\} \tag{34}$$

labelled by 3 parameters ω, $Re\ z$, $Im\ z$. For each ω we have an overcomplete family of states.

This is still not general enough. There are still more canonical transformations that can be performed which will make the state no longer a minimum uncertainty state in the Heisenberg sense but which would be minimum Schrödinger uncertainty states. These are the correlated states whose wave functions have been obtained by Dodunov, Kurmyshev and Mańko [?]. A simpler version of this is as a complex Gaussian:

$$\psi(x) = (\pi)^{-\frac{1}{4}}\exp\left[-\frac{1}{2}\left(\alpha x^2 - 2\beta x + \gamma\right)\right] \tag{35}$$

where α, β, γ are complex parameters satisfying $(\beta + \beta^*)^2\big/(\alpha + \alpha^*) = \gamma + \gamma^*$. The imaginary part of γ is arbitrary. These therefore contain two complex parameters

$$(\Delta q)^2 = \frac{1}{2\alpha_1}$$

$$(\Delta p)^2 = \frac{\alpha_1}{2} + \left(\frac{\alpha_2}{\alpha_1}\right)^2$$

$$\langle qp + pq\rangle - \langle q\rangle\langle p\rangle - \langle p\rangle\langle q\rangle = -\frac{2\alpha_2}{\alpha_1}. \tag{36}$$

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Making use of the appealing phase space picture introduced by Planck [?] for the quantum oscillator, the ground state with the zeropoint energy (for $\omega = 1$) has a phase space patch which is a circle with unit radius and an area π which is (2π) times the uncertainty. The mean value of $\frac{1}{2}(p^2 + q^2)$ within this circular disc is $\frac{1}{2}$ which satisfied Planck. So his picture of the ground state is a circle of unit radius centered at the orgin. By
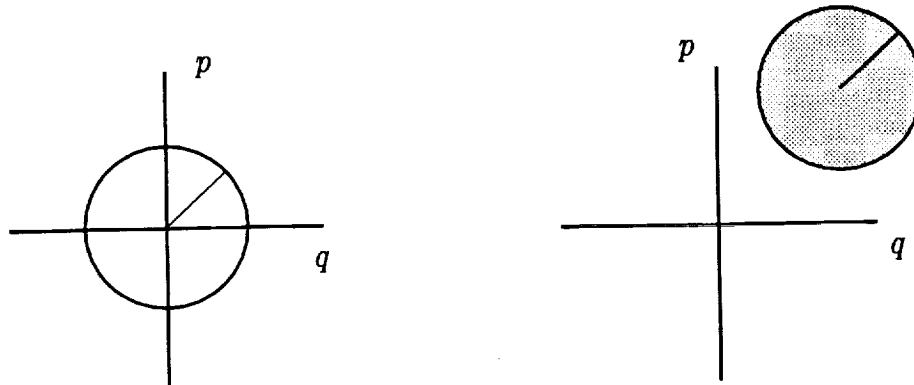
Fig.1. Planck's picture of the minimum energy state and the coherent states. The coherent states are centered at the point $\left(\frac{z+z^*}{\sqrt{2}}, \frac{z-z^*}{i\sqrt{2}}\right)$.

displacing the origin to $\sqrt{2}\,z$ we get the two parameter (one complex parameter) family of coherent states.

Squeezed states are obtained by area preserving deformations of the circles into ellipses with major (minor) axis along the coordinate directions.
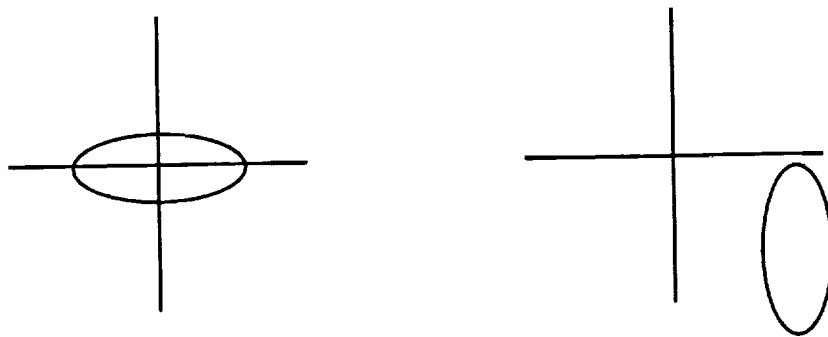
Fig.2. Planck pictures for squeezed states.

When the ellipse is tilted we get the more general family of correlated states discussed by Dodunov, Kurmyshev and Mańko. Of course this tilting alters things only for the squeezed states but not for the coherent states.

Fig.3. Planck pictures for correlated states.

2   The Group Theoretic Significance of the States Which Have Minimum Schrödinger Uncertainty.

The linear canonical transformations on a pair of canonical variables form a group $SL(2, R) \,\text{[}\sim\text{]}\, T(2)$, the semidirect product of the special linear group with translations. The minimum uncertainty s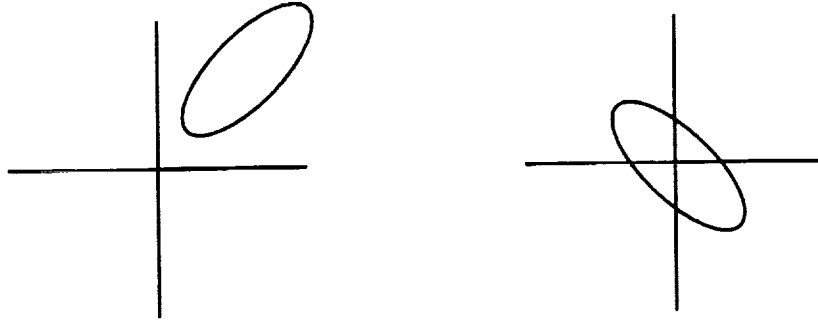tate of Planck are invariant under the harmonic SO(2) subgroup of this group; this is its stability group. So the quotient of the canonical group by the harmonic stability group the correlated states are in one-to-one correspondence with the elements of the coset of dimension $5 - 1 = 4$.

These states are realized by single mode lasers and states with substantial squeezing and/or correlation have been generated and identified.

It is a natural question to ask whether these notions and correspondences can be generalized to n-degrees of freedom and multimode laser beams. Group theory can be invoked to get a general answer to the problem.

3   Multimode Correlated States and Their Group-Theoretic Relevance

Consider a system of n canonical pairs $\{q_r, p_r\}$, $1 \leq r, s \leq n$. The homogeneous linear transformations are $Sp(2n, R)$ and the translations are $T(2n)$. So the linear canonical group is the semidirect product $Sp(2n, R) \,\text{[}\sim\text{]}\, T(2n)$ with $n(2n + 1) + 2n(2n + 3)$ parameters. We seek canonical invariants bilinear in the $2n$ canonical variables and look for the appropriate conditions to get the minimum generalized Schrödinger uncertainty. We expect this to come from the ground

state $|\Omega\rangle$ annililated by all annililation operators $(q_r + ip_r)/\sqrt{2}$ and states obtained from $|\Omega\rangle$ by the action of the linear canonical group. Since these involve individual harmonic SO(2) elements for each degree of freedom and any $O(n)$ rotation between the various degrees of freedom the stability group of $|\Omega\rangle$ has $n + \frac{n(n-1)}{2} = \frac{1}{2}n(n+1)$ parameters, we expect a family with $\frac{1}{2}n(3n+5)$ parameters corresponding to the dimension of the coset space.

Even for small values of n this dimension grows rapidly; we adopt a more elementary method to obtain the generalized correlated states. We describe in detail the case for $n = 2$ and remark that the method generalizes for arbitrary n. The multimode coherent states are $2n$ parameter states obtained by $T(2n)$ acting on $|\Omega\rangle$. Let us consider the group $Sp(4, R)$ which is a double covering of SO(3,2) and has the same Lie algebra of dimension ten. This algebra can be obtained by the three $(p_r p_s)$, the three $(q_r q_s)$ and the four $\frac{1}{2}(q_r p_s + p_s q_r)$ which close under commutation. The generic SO(3,2) algebra has two invariants, one of the second order and one of the fourth order. If we consider the expectation values of the ten quantities $(p_r p_s)$, $(q_r q_s)$, $\frac{1}{2}(q_r p_s + p_s q_r)$ they furnish a 4×4 symmetric non negative matrix which is bounded below by the zero point energy

1. Let this matrix be denoted by:

$$T_{\mu\nu} = \begin{pmatrix} e_{11} & e_{12} & a & b \\ e_{12} & e_{22} & c & d \\ a & c & f_{11} & f_{12} \\ b & d & f_{12} & f_{22} \end{pmatrix}. \tag{37}$$

By suitable harmonic SO(2) transformations in (q_1, p_1) and in (q_2, p_2) this can be reduced to the form

$$\begin{pmatrix} e_1 & 0 & a' & b' \\ 0 & e_2 & c' & d' \\ a' & e' & f_1 & 0 \\ b' & a' & 0 & f_2 \end{pmatrix}. \tag{38}$$

By scale transformations independently for the two degrees of freedom we can reduce this to the form

$$\begin{pmatrix} e & 0 & a'' & b'' \\ 0 & e & c'' & d''' \\ a'' & c'' & f & 0 \\ b'' & d' & 0 & f \end{pmatrix}. \tag{39}$$

Now harmonic SO(2) transformations in (q_1, p_1) and in (q_2, p_2) can be used to diagonalize the other diagonal blocks to get

$$\begin{pmatrix} e & 0 & a' & 0 \\ 0 & e & 0 & d' \\ a' & 0 & f & 0 \\ 0 & d' & 0 & f \end{pmatrix}. \tag{40}$$

Now the SO(2) rotation between the two degrees of freedom can be used to transform this into

$$\begin{pmatrix} e + a' & 0 & 0 & 0 \\ 0 & e + d' & 0 & 0 \\ 0 & 0 & f + a' & 0 \\ 0 & 0 & 0 & f + d' \end{pmatrix}. \tag{41}$$

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Further scale transformations in the two degrees of freedom can render this to the final form

$$\begin{pmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_1 & 0 & 0 \\ 0 & 0 & g_2 & 0 \\ 0 & 0 & 0 & g_2 \end{pmatrix}. \tag{42}$$

Thus there are two invariant quantities g_1, g_2 which maybe recognized as the uncertainties in the two natural modes. Note that g_1, g_2 are both positive and not less than $\frac{1}{2}\hbar$.

Naturally the minimum uncertainty state must have degenerate structure with

$$g_1 = g_2 = \frac{1}{2}\hbar. \tag{43}$$

This is the vacuum state $|\Omega\rangle$ in the natural modes. The correlated states are obtained by the action of the group $Sp(4, R) \boxed{\sim} T(4)$. The $T(4)$ action demands that we replace q, p by $q - \langle q \rangle$, $p - \langle p \rangle$, after which we may ignore them. Since the state $|\Omega\rangle$ has a 3-parameter stability group we may restrict attention to the quotient manifold of cosets.

This construction can be immediately generalized. We take the 4×4 diagonal block of the $2n \times 2n$ matrix and carry out the transformations outlined in the previous scheme and then take the bordering 4×2, 2×4 and 2×2 blocks. Now make orthogonal transformations between the modes to make the 6×6 block diagonal with possibly unequal diagonal elelments. Scale transformations independently in the three modes will make them diagonal with pairs of values equal. Now the process can be repeated with the bordering 2×6, 6×2 and 2×2 blocks; and repeating the procedure we can diagonalize the 8×8 matrix with

$$\langle p_1^2 \rangle = \langle q_1^2 \rangle \ , \ \langle p_2^2 \rangle = \langle q_2^2 \rangle, \ldots, \langle p_4^2 \rangle = \langle q_4^2 \rangle \ . \tag{44}$$

This can be done with the $2n \times 2n$ has matrix is fully diagonalized with adjacent pairs of diagonal elements equal; that is the eigenvalues are

$$g_1, g_1, g_2, g_2, g_3, g_3, \ldots, g_n, g_n \ . \tag{45}$$

This is the canonical form with n invariants $g_1, g_2, \ldots, g_n$ with each $g_r \geq \frac{1}{2}\hbar$. The distinguished generalized correlated states have degenerate eigenvalues

$$g_1 = g_2 = \cdots = g_n = \frac{1}{2}\hbar. \tag{46}$$

This is the multimode vaccum state! We can get the multimode coherent states by displacements which are the real and imaginary parts of $z_1, z_2, \ldots, z_n$. Squeezed states are obtained by scale transformations in each mode independently so that the diagonal eigenvalues became

$$\lambda_1 g_1, \lambda_1^{-1} g_1, \ldots, \lambda_n g_n, \lambda_n^{-1} g_n \ . \tag{47}$$

The displacements and squeezings introduce $2n + n = 3n$ parameters. But the generalized correlated state is obtained by the full coset of the linear canonical group $Sp(2n, R) \boxed{\sim} T(2n)$ by the stability group of the N-mode vacuum state $|\Omega\rangle$.

These correlated states maybe displayed explicitly but are too cumbersome. The multimode correlated states have wave functions which are displaced Gaussians with phase factors. Depending upon the experimental requirements we may obtain intensity correlations, photocount statistics etc. directly. The number of parameters describing such correlated states are enormous and would be restricted by the method of generation of such states.

4 Discussion

Some remarks are in order about the correlated states in quantum field theory. As long as the number of excited modes is finite, however many, there exists a unitary transformation from the multimode vacuum state to the multimode correlated state. These unitary transformations are generated by a quantity bilinear in the canonical variables. These operators are unbounded but do generate unitary transformations. When the number of modes became infinite, the generic correlated state cannot be obtained form the vacuum state they would be in a different Hilbert space from the Fock vacuum. [?]

It was the purpose of this paper to demonstrate the close relation between the correlated states and the linear canonical group; and to show that the correlated states which minimize the Schrödinger uncertainties is related to the canonical multimode vacuum which is invariant under linear unitary transformations of the modes. The generic wave functions are Gaussians with a determined number of independent parameters.

The one and two-mode analysis is equally applicable to the propogation of the Gaussian Schell mode paraxial wave fronts through a system of thin lenses which are, respectively, isotropic and nonisotropic. This has been carried out elsewhere [?].

Correlated states are the generic family which include squeezed states and coherent states as special cases. For each value of the complex parameter α, we have an overcomplete family of states in the case of one degree of freedom. For the multimode case the parameter defining the generic form (37) from the canonical form (42) are such labelling parameters.

5 Acknowledgments

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SUPERSQUEEZED STATES FROM SQUEEZED STATES

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Abstract

Using super-Baker-Campbell-Hausdorff relations on the elements of the supergroup OSP(2/2), we derive the supersqueeze operator and the supersqueezed states, which are the supersymmetric generalization of the squeezed states of the harmonic oscillator.

1   Introduction

The concept of supersymmetry became of wide interest to physicists because of attempts to obtain a grand unified theory of the fundamental interactions. In particular, such supersymmetric theories predict that there are fermion partners to fundamental bosons, and *vice versa*. However, searches for this fundamental supersymmetry have so far proven fruitless, and something of a, "Trust me, we'll find it at the next accelerator"-attitude has emerged.

On the other hand, phenomenological manifestations of supersymmetry have been found at low energies, e.g., in the contexts of nuclear physics [1], atomic physics [2], and WKB-theory [3]. The supersymmetry and atomic physics interests [2, 4] of Alan Kostelecký, Rod Truax, and myself, combined with our interest in coherent states [5, 6], led us to develop super-BCH relations [7, 8] as a precursor to deriving a complete supercoherent states formalism. With Alan's graduate student, Beata Fatyga [9], we gave supercoherent states for three distinct systems: (i) the super Heisenberg-Weyl algebra, which defines the supersymmetric harmonic oscillator; (ii) an electron in a constant magnetic field, which is a supersymmetric quantum-mechanical system with a Heisenberg-Weyl algebra plus another bosonic degree of freedom, and (iii) the electron-monopole system, which has an OSP(1/2) supersymmetry. (I also want to mention that Alan, Rod, and I have joined forces with Man'ko to obtain time-dependent supercoherent states [10].)

At the first International Workshop on Squeezed States [11], Alan reported on our supercoherent states [12]. In the question and answer session of Alan's talk, he was asked if we were trying to extend our results to the supersqueezed states of the harmonic oscillator. (Honest! That was not a set-up question.) Alan replied that we were, but that it was a harder problem. (If that response had come from me, instead of Alan, you might now suspect that it was a set-up answer.) Anyway, having committed ourselves, we hoped to do it before this Second International Workshop on Squeezed States. And we did–by the skins of our teeth. The last calculation (although not the last check) was finished on May 18.

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In Sec. 2, I will give a quick review of coherent states and squeezed states. (See, also, Ref. [5]). Then, I go on to show how, given the superdisplacement operator for coherent states [9, 12], one can obtain supersqueezed states if one can first obtain the supersqueeze operator. This supersqueeze operator is derived in the following section. I conclude with a description of the supersqueezed states. Further details and results will appear elsewhere [13].

2   Coherent states and squeezed states

In the Schrödinger formalism, those states which minimize the $x - p$ uncertainty relation are

$$\psi(x) = [2\pi\sigma^2]^{-1/4} \exp\left[-\left(\frac{x - x_0}{2\sigma}\right)^2 + ip_0x\right], \tag{1}$$

$$\sigma = \mathcal{S}\sigma_0 = \mathcal{S}/[2m\omega]^{1/2}. \tag{2}$$

When $\mathcal{S} = 1$, these Gaussians have the width of the ground state of the harmonic oscillator with natural frequency $\nu = \omega/(2\pi)$, and are the coherent states. When $\mathcal{S} \neq 1$, they are the "squeezed states" of the harmonic oscillator. Their uncertainty product evolves with time as

$$[\Delta x(t)]^2[\Delta p(t)]^2 = \frac{1}{4}\left[1 + \frac{1}{4}\left(\mathcal{S}^2 - \frac{1}{\mathcal{S}^2}\right)^2 \sin^2(2\omega t)\right]. \tag{3}$$

In the (displacement) operator formalism, the coherent states are given by

$$D(\alpha)|0\rangle = \exp[\alpha a^\dagger - \alpha^* a]|0\rangle = \exp\left[-\frac{1}{2}|\alpha|^2\right]\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \equiv |\alpha\rangle, \tag{4}$$

where $|n\rangle$ are the number states. The displacement operator, $D(\alpha)$, is the unitary exponentiation of the elements of the factor algebra, spanned by a and $a^\dagger$:

$$D(\alpha) = \exp[\alpha a^\dagger - \alpha^* a] = \exp\left[-\frac{1}{2}|\alpha|^2\right]\exp[\alpha a^\dagger]\exp[-\alpha^* a], \tag{5}$$

where the last equality comes from using a BCH relation. With the identifications $Re(\alpha) = [m\omega/2]^{1/2}x_0$ and $Im(\alpha) = p_0/[2m\omega]^{1/2}$, these are the same as the minimum-uncertainty coherent states, up to an irrelevant phase factor.

Obtaining the squeezed states from the displacement operator coherent states is more complicated than from the minimum-uncertainty coherent states. One starts with the "unitary squeeze operator"

$$S(z) = \exp\left[z\frac{a^\dagger a^\dagger}{2} - z^*\frac{aa}{2}\right] \tag{6}$$

$$\equiv \exp\left[G_+\frac{a^\dagger a^\dagger}{2}\right]\exp\left[G_0\frac{\left(a^\dagger a + \frac{1}{2}\right)}{2}\right]\exp\left[G_-\frac{aa}{2}\right] \tag{7}$$

$$= \exp\left[e^{i\phi}(\tanh r)\frac{a^\dagger a^\dagger}{2}\right]\left(\frac{1}{\cosh r}\right)^{(\frac{1}{2}+a^\dagger a)}\exp\left[-e^{-i\phi}(\tanh r)\frac{aa}{2}\right], \tag{8}$$

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where $z \equiv re^{i\phi}$ and Eq. (8) is obtained from a BCH relation. A normal-ordered form for the second term in Eq. (7) is

$$\left(\frac{1}{\cosh r}\right)^{(\frac{1}{2}+a^\dagger a)} = \left(\frac{1}{\cosh r}\right)^{\frac{1}{2}} \left[\sum_{n=0}^{\infty} \frac{(\operatorname{sech} r - 1)^n}{n!}(a^\dagger)^n(a)^n\right]. \tag{9}$$

Note that $S(z)$ by itself can be considered to be the displacement operator for the group SU(1,1) defined by

$$K_+ = \frac{1}{2}a^\dagger a^\dagger, \qquad K_- = \frac{1}{2}aa, \qquad K_0 = \frac{1}{2}(a^\dagger a + \frac{1}{2}). \tag{10}$$

The squeezed states equivalent to the ψ of Eqs. (1-2) are obtained by operating on the ground state by

$$T(\alpha, z)|0\rangle = D(\alpha)S(z)|0\rangle \equiv |(\alpha, z)\rangle, \tag{11}$$

$$z \equiv re^{i\phi}, \qquad r = \ln S. \tag{12}$$

[ϕ is a phase which defines the starting time, $t_0 = (\phi/2\omega)$, and S is the wave-function squeeze of Eq. (2).]

Although the operator method appears, at first sight, to be more complicated, it has a distinct advantage when one wants to consider supersymmetry. The operator method has a direct supersymmetric generalization. The mathematics is clear, and so one does not have to solve the problem of how to include the fermionic sector in the wave-function formalism. That answer will come out in the end.

3   How to obtain supersqueezed states

Recently, we used the operator method to find supercoherent states [9]. Among the examples in this study, the supercoherent states of the harmonic oscillator were obtained. From the super Heisenberg-Weyl algebra defined by

$$[a, a^\dagger] = I, \quad \{b, b^\dagger\} = I, \tag{13}$$

the superdisplacement operator was obtained:

$$\mathbf{D}(A, \theta) = \exp[Aa^\dagger - \overline{A}a + \theta b^\dagger + \overline{\theta}b] \tag{14}$$

$$= \left(\exp[-\frac{1}{2}|A|^2]\exp[Aa^\dagger]\exp[-\overline{A}a]\right)\left(\exp[-\frac{1}{2}\overline{\theta}\theta]\exp[\theta b^\dagger]\exp[\overline{\theta}b]\right). \tag{15}$$

θ and $\overline{\theta}$ are odd Grassmann numbers. They are nilpotent and they satisfy anticommutation relations among themselves and with the fermion operators b and $b^\dagger$. A and $\overline{A}$ are complex, even, Grassmann numbers. Explicit calculation yields

$$\mathbf{D}(A, \theta)|0, 0\rangle = [1 - (1/2)\overline{\theta}\theta]|A, 0\rangle + \theta|A, 1\rangle. \tag{16}$$

The two labels of $|0, 0\rangle$ in Eq. (16) represent the even (bosonic) and odd (fermionic) spaces. The bosonic space contains an ordinary coherent state $|A\rangle$ and the fermionic space has zero or one fermions. (See Ref. [9] for further details.)

From the above it is clear that the supersymmetric generalization of the SU(1,1) squeeze operator of Eqs. (6-8) is what is needed to obtain the supersqueeze operator and, hence, the supersqueezed states. The group involved is the supergroup OSP(2/2). In addition to the su(1,1) algebra elements of Eq. (10), it has five more:

$$
\begin{aligned}
M_0 &= \frac{1}{2}(b^\dagger b - \frac{1}{2}), \\
Q_1 &= \frac{1}{2}a^\dagger b^\dagger, \qquad Q_2 = \frac{1}{2}ab, \qquad Q_3 = \frac{1}{2}a^\dagger b, \qquad Q_4 = \frac{1}{2}ab^\dagger.
\end{aligned}
\tag{17}
$$

4 The supersqueeze operator

To obtain the supersqueeze operator as a product, one solves the t-dependent equation

$$
\begin{aligned}
\mathbf{S}(Z,\theta_j,t) &= \exp[t(ZK_+ - \overline{Z}K_- + \theta_1 Q_1 + \overline{\theta_1}Q_2 + \overline{\theta_2}Q_3 + \theta_2 Q_4)] \\
&= e^{\gamma_+ K_+} e^{\gamma_0 K_0} e^{\gamma_- K_-} e^{\beta_1 Q_1} e^{\mu M_0} e^{\beta_4 Q_4} e^{\beta_3 Q_3} e^{\beta_2 Q_2} \\
&\equiv \mathbf{S}_1(\mu, \gamma_i, \beta_k, t).
\end{aligned}
\tag{18}
$$

By construction, μ, the γ_i, and the β_k's are functions of t. Thus, taking the derivative of Eq. (18) with respect to t and then multiplying on the right by $\mathbf{S}^{-1}$ yields

$$
\left[\frac{d}{dt}\mathbf{S}\right]\mathbf{S}^{-1} = \left[\frac{d}{dt}\mathbf{S}_1\right]\mathbf{S}_1^{-1}.
\tag{19}
$$

This can explicitly be written as ("dot" signifies $\frac{d}{dt}$)

$$
\begin{aligned}
[ZK_+ - \overline{Z}K_- &+ \theta_1 Q_1 + \overline{\theta_1}Q_2 + \overline{\theta_2}Q_3 + \theta_2 Q_4] \\
&= \dot{\gamma}_+ K_+ \\
&\quad + [e^{\gamma_+ K_+}]\dot{\gamma}_0 K_0 [e^{-\gamma_+ K_+}] \\
&\quad + [e^{\gamma_+ K_+} e^{\gamma_0 K_0}]\dot{\gamma}_- K_- [e^{-\gamma_0 K_0} e^{-\gamma_+ K_+}] \\
&\quad + S_B \dot{\beta}_1 Q_1 S_B^{-1} \\
&\quad + S_B [e^{\beta_1 Q_1}]\dot{\mu} M_0 [e^{-\beta_1 Q_1}] S_B^{-1} \\
&\quad + S_B [e^{\beta_1 Q_1} e^{\mu M_0}]\dot{\beta}_4 Q_4 [e^{-\mu M_0} e^{-\beta_1 Q_1}] S_B^{-1} \\
&\quad + S_B [e^{\beta_1 Q_1} e^{\mu M_0} e^{\beta_4 Q_4}]\dot{\beta}_3 Q_3 [e^{-\beta_4 Q_4} e^{-\mu M_0} e^{-\beta_1 Q_1}] S_B^{-1} \\
&\quad + S_B [e^{\beta_1 Q_1} e^{\mu M_0} e^{\beta_4 Q_4} e^{\beta_3 Q_3}]\dot{\beta}_2 Q_2 [e^{-\beta_3 Q_3} e^{-\beta_4 Q_4} e^{-\mu M_0} e^{-\beta_1 Q_1}] S_B^{-1},
\end{aligned}
\tag{20}
$$

where

$$
S_B = [e^{\gamma_+ K_+} e^{\gamma_0 K_0} e^{\gamma_- K_-}].
\tag{21}
$$

Note that S_B is the form of the ordinary squeeze operator defined in Eq. (7).

All the terms on the right hand side of Eq. (20) can be written in nonexponential form by using super-BCH formulas and the graded commutation relations. When this is done, there are really eight equations, one for each of the factors multiplying the eight elements of the algebra

osp(2/2); i.e., an equation for each of the factors multiplying K_+, K_0, etc. With some algebra, each of the eight equations can be changed to a set of equations having only one time-differential in each.

These eight equations can actually be solved as twenty separate, coupled, differential equations, of simpler form. This is because the four even group parameters $\{\mu, \gamma_+, \gamma_0, \gamma_-\}$ can each be written as having three terms, containing products of zero, two, or four of the θ_j, respectively, and the four odd group parameters $\{\beta_k\}$ can be written as having two terms, containing products of one or three of the θ_j, respectively. (We wll use a presubscript to denote this; e.g., $\beta_1 = (_1\beta_1) + (_3\beta_1)$.) One takes the eight equations and expands all of the expressions in powers of the θ_j. The order-zero, -two, and -four pieces of the even equations are separated and, similarly, the order-one and -three pieces of the odd equations are separated. One places the lower-order solutions into the higher-order equations. (Note that the boundary conditions needed are that the solutions must all be zero when $t = 0$. Then the supersqueeze operator will be obtained when we set $t = 1$.)

One can do this in a well-defined manner. In particular, the solutions shown below were obtained by finding, in order: $(_0\mu)$, $(_0\gamma_+)$, $(_0\gamma_0)$, $(_0\gamma_-)$, $(_1\beta_1)$, $(_1\beta_2)$, $(_1\beta_3)$, $(_1\beta_4)$, $(_2\mu)$, $(_2\gamma_+)$, $(_2\gamma_0)$, $(_2\gamma_-)$, $(_3\beta_1)$, $(_3\beta_2)$, $(_3\beta_3)$, $(_3\beta_4)$, $(_4\mu)$, $(_4\gamma_+)$, $(_4\gamma_0)$ and $(_4\gamma_-)$.

In the solutions we will use the suggestive notation

$$ r \equiv [Z\overline{Z}]^{1/2}, \qquad e^{i\phi} \equiv [Z/\overline{Z}]^{1/2}, \tag{22} $$

where r and $e^{i\phi}$ are now understood to represent Grassmann-valued quantities. Then, one can make the replacements

$$ Z \to re^{i\phi}, \qquad \overline{Z} \to re^{-i\phi}. \tag{23} $$

Some care is needed because the quantity $e^{i\phi}$ is strictly defined only for $|\overline{Z}| \neq 0$ and $\overline{z} \neq 0$, where $\overline{z}$ is the body of $\overline{Z}$. However, the solutions given below are not affected by this. Even so, the physical meaning of Grassmann numbers remains an open question [14].

We also define

$$ c \equiv \cosh y, \qquad s \equiv \sinh y, \qquad y \equiv rt, \tag{24} $$

$$ \Phi \equiv \overline{\theta_2}\theta_2\overline{\theta_1}\theta_1 = \overline{\theta_2\theta_1}\theta_1\theta_2. \tag{25} $$

With this, the complete solutions to the group parameters are:

$$
\begin{aligned}
\mu \;=\; & 0 \\
& + \frac{1}{2r^2}\{[\overline{\theta}_1\theta_1 - \overline{\theta}_2\theta_2](c-1) + [\overline{\theta}_2\theta_1 e^{-i\phi} - \overline{\theta}_1\theta_2 e^{i\phi}](s-y)\} \\
& + \frac{\Phi}{r^4}\left[c - 1 - \frac{1}{2}sy\right],
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\gamma_+ \;=\; & \left[e^{i\phi}\frac{s}{c}\right] \\
& - \frac{e^{i\phi}}{4r^2c^2}[\overline{\theta}_1\theta_1(sc - y) + e^{i\phi}\overline{\theta}_1\theta_2(c-1)^2 \\
& \qquad + e^{-i\phi}\overline{\theta}_2\theta_1 s^2 + \overline{\theta}_2\theta_2(sc + y - 2s)] \\
& + \frac{\Phi e^{i\phi}}{8r^4c^3}\left[(2y + sy^2 - s) + c(\frac{11}{8}y - 2s) + \left(-\frac{5}{8}sc^2 + \frac{1}{4}sc^4\right)\right],
\end{aligned} \tag{27}
$$

$$ \gamma_0 \;=\; [-2\ln c] $$

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$$+\frac{1}{2r^2}[\overline{\theta}_1\theta_1(\frac{-ys}{c}+c-1)+e^{i\phi}\overline{\theta}_1\theta_2(\frac{s}{c}-s)$$

$$+e^{-i\phi}\overline{\theta}_2\theta_1(-\frac{s}{c}+s)+\overline{\theta}_2\theta_2(\frac{2+ys}{c}-c-1)]$$

$$+\frac{\Phi}{8r^4c^2}[(y^2-1-2ys)-c(\frac{11}{4}ys+4)$$

$$+c^2(2\ln c+8c-3-4ys-\frac{1}{4}s^2)], \tag{28}$$

$$\gamma_- = \left[-e^{-i\phi}\frac{s}{c}\right]$$

$$+(\frac{e^{-i\phi}}{4r^2c^2})[\overline{\theta}_1\theta_1(sc-y)-e^{i\phi}\overline{\theta}_1\theta_2s^2$$

$$-e^{-i\phi}\overline{\theta}_2\theta_1(c-1)^2+\overline{\theta}_2\theta_2(sc+y-2s)]$$

$$-\frac{\Phi e^{-i\phi}}{8r^4c^3}\left[(2y+sy^2-s)+c(\frac{11}{8}y-2s)+\left(sc^2(\frac{15}{8}+2\ln c)-\frac{9}{4}c^3y\right)\right], \tag{29}$$

$$\beta_1 = \frac{1}{r}[s\theta_1+(c-1)e^{i\phi}\theta_2]$$

$$+\frac{1}{4r^3}[\overline{\theta}_2\theta_1\theta_2(y-2cs+yc)+e^{i\phi}\overline{\theta}_1\theta_1\theta_2(2c(1-c)+ys)], \tag{30}$$

$$\beta_2 = \frac{1}{r}[s\overline{\theta}_1+(c-1)e^{-i\phi}\overline{\theta}_2]$$

$$+\frac{1}{4r^3}[\overline{\theta}_2\theta_1\theta_2(yc-s+\frac{1}{2}(sc-y))+\overline{\theta}_2\overline{\theta}_1\theta_1e^{-i\phi}(ys-3(c-1)-\frac{1}{2}s^2)], \tag{31}$$

$$\beta_3 = \frac{1}{r}[(c-1)e^{i\phi}\overline{\theta}_1+s\overline{\theta}_2]$$

$$+\frac{1}{4r^3}[e^{i\phi}\overline{\theta}_2\overline{\theta}_1\theta_12(ys-2(c-1))+\overline{\theta}_2\overline{\theta}_1\theta_12(yc-s)], \tag{32}$$

$$\beta_4 = \frac{1}{r}[(c-1)e^{-i\phi}\theta_1+s\theta_2]$$

$$+\frac{1}{4r^3}[e^{-i\phi}\overline{\theta}_2\theta_1\theta_2(-4c^2+4c+2ys)+\overline{\theta}_1\theta_1\theta_2(-4sc+2s+2yc)]. \tag{33}$$

Setting $t=1$ yields the general supersqueeze group parameters.

5   The supersqueezed states

Then, using the above group parameters, the graded commutation relations among the generators, and the properties of Grassmann algebra, the supersqueezed states can be found to be

$$\mathbf{T}(A,\theta,Z,\theta_i)|0,0\rangle = \mathbf{D}(A,\theta)\mathbf{S}(Z,\theta_i)|0,0\rangle = |A,\theta;Z,\theta_i\rangle$$

$$= \hat{\mu}\Gamma_-h_1(a^\dagger)\left[(1-\frac{1}{2}\overline{\theta}\theta)|(A,Z),0\rangle+\theta|(A,Z),1\rangle\right]$$

$$+\hat{\mu}\Gamma_+\frac{\beta_1}{2}h_2(a^+)\left[\overline{\theta}|(A,Z),0\rangle+(1+\frac{1}{2}\overline{\theta}\theta)|(A,Z),1\rangle\right], \tag{34}$$

where

$$\hat{\mu} = 1 - \frac{1}{4}[(_2\mu) + (_4\mu)] + \frac{1}{32}(_2\mu)^2, \tag{35}$$

$$\Gamma_\pm = 1 + \frac{(2 \pm 1)}{4}[(_2\gamma_0) + (_4\gamma_0)] + \frac{(2 \pm 1)^2}{32}(_2\gamma_0)^2, \tag{36}$$

$$h_1(a^\dagger) = 1 + \frac{1}{2}[(_2\gamma_+) + (_4\gamma_+)](a^\dagger - \overline{A})^2 + \frac{1}{8}(_2\gamma_+)^2(a^\dagger - \overline{A})^4, \tag{37}$$

$$h_2(a^\dagger) = \frac{(a^\dagger - \overline{A})}{c}\left[1 + \frac{1}{2}(_2\gamma_+)(a^\dagger - \overline{A})^2\right]. \tag{38}$$

As with the supercoherent states, we find that the supersqueezed states are a linear combination of squeezed states in the bosonic sector with zero or one fermion in the odd sector. What is different, however, is that the squeezed states are multiplied by a linear combination of boson raising operators up to order four.

In the limits $A \to 0$ and $Z \to 0$, the supersqueezed states reduce to the "fermisqueezed states"

$$
\begin{aligned}
\mathbf{D}(0,\theta)\mathbf{S}(0,\theta_i)|0,0\rangle = {}& \left[1 - \frac{1}{2}\left(\frac{\overline{\theta_1}\theta_1}{4}\right) - \frac{1}{12}\left(\frac{\Phi}{16}\right)\right]\left[\left(1 - \frac{1}{2}\overline{\theta}\theta\right)|0,0\rangle + \theta|0,1\rangle\right] \\
& + \left[\frac{\theta_1}{2} - \frac{1}{3}\left(\frac{\overline{\theta_2}\theta_1\theta_2}{8}\right)\right]\left[\left(1 + \frac{1}{2}\overline{\theta}\theta\right)|1,1\rangle + \overline{\theta}|1,0\rangle\right] \\
& + \left[-\frac{1}{\sqrt{2}}\left(\frac{\overline{\theta_2}\theta_1}{4}\right)\right]\left[\left(1 - \frac{1}{2}\overline{\theta}\theta\right)|2,0\rangle + \theta|2,1\rangle\right].
\end{aligned} \tag{39}
$$

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INFORMATION ENTROPY VIA
GLAUBER'S Q-REPRESENTATION

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Abstract

We present a convenient way to evaluate the information entropy of a quantum mechanical state via the Glauber Q-representation. As an example we discuss the information entropy of a thermally relaxing squeezed state in terms of its Q-representation and show the validity of the corresponding entropic uncertainty- and Araki-Lieb inequalities.

1   The information entropy

Shannon and Wehrl were the first to describe the information of a quantum mechanical state in terms of its probability distributions [1]. Later, there has also been a substantial amount of work on this topic from the quantum optics point of view [2]. The question of comparability of the information entropy with the Heisenberg uncertainty has been treated as well. The Heisenberg uncertainty has turned out to be of enormous significance because of its experimental measurability. However, it only takes the second moments into account whereas the information entropy is supposed to be an exact measure of the information and thus of the uncertainty or non-information. In comparison to the significant Heisenberg uncertainty inequality, there is a similarly meaningful entropic uncertainty relation. Bialynicki-Birula et al., derived such an inequality more than 15 years ago [3].

In this paper we would like to put forward a possibility to evaluate the information entropy as a function of the Q-representation since this representation is well-known for many interesting quantum mechanical states and completely describes the state. In particular we here would like to investigate the information entropy for the squeezed state which evolves to a thermal state via an appropriate Fokker-Planck equation. Special interest is devoted to the entropic uncertainty relation. As a major result we show that a squeezed state also obeys the minimum entropic uncertainty relation. However, it turns out that the evolution of the squeezed state via the Fokker-Planck equation, does lead to a change of the information entropy and the marginal contributions but surprisingly does not influence the minimality of the uncertainty relation. This even means

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that a thermal state fulfills the entropic uncertainty relation with an equal sign. We moreover investigate the Araki-Lieb inequality [4] for information entropies and find agreement with the well-known results of von Neumann entropies for the thermally relaxing squeezed state.

We start off the paper with some basic facts on entropies and develop an expression for the information entropy in terms of the Glauber Q-representation. The definition of the quantum mechanical entropy is given by:

$$S = -Tr\{\hat{\rho}\ln\hat{\rho}\} \tag{1}$$

with $\hat{\rho}$ being the density matrix operator and assuming the Boltzmann constant to be 1. This often called von Neumann entropy is zero for a pure state and non-zero for a mixed state. It is moreover known to be constant for a closed system which arises from the fact that a unitary time evolution does not change the eigenvalues of $\hat{\rho}$.

Thus, normally, the evolution of the entropy of subsystems of a closed system is of greater interest. Considering two disjunct interacting systems that form together the whole system being described by $\hat{\rho}$, we can introduce the reduced density operators $\hat{\rho}_A = Tr_B\{\hat{\rho}\}$ and $\hat{\rho}_B = Tr_A\{\hat{\rho}\}$, where Tr_A and Tr_B abbreviate the tracing over the variables of the subsystems A and B, respectively. This leads to the definition of the entropy of the subsystem A: $S(\hat{\rho}_A) = -Tr_A\{\hat{\rho}_A\ln\hat{\rho}_A\}$ and to the analogous expression for subsystem B by replacing A by B in the above formula.

These reduced or here called marginal entropies describe information or more directly disorder and uncertainty of A and B and are not necessarily time independent like the entropy of the whole system S of Eq.(1). Information about the interaction of A and B is neither included in $S(\hat{\rho}_A)$ nor in $S(\hat{\rho}_B)$ so that we expect the sum of $S(\hat{\rho}_A)$ and $S(\hat{\rho}_B)$ not to be smaller than S. And, in fact, Araki and Lieb [4] proofed the following triangle inequality:

$$|S(\hat{\rho}_A) - S(\hat{\rho}_B)| \le S \le S(\hat{\rho}_A) + S(\hat{\rho}_B). \tag{2}$$

Because of the close relation of entropy and uncertainty and moreover the existence of a lower bound of S, the second inequality can be interpreted as uncertainty relation. The calculation of the above entropies requires the diagonalization of the reduced density operators. Since this is often difficult, the information entropy or Shannon-Wehrl-entropy was introduced according to:

$$\tilde{S}(\hat{\rho}; \hat{O}) = -\sum_e \langle e|\hat{\rho}|e\rangle \ln\langle e|\hat{\rho}|e\rangle, \tag{3}$$

with

$$\hat{O}|e\rangle = e|e\rangle. \tag{4}$$

The corresponding expressions for the subsystems can be obtained by exchanging $\hat{\rho}$ by $\hat{\rho}_A$ or $\hat{\rho}_B$, where the so far arbitrary operator $\hat{O}$ may be chosen differently. If we are dealing with operators that can be expressed in terms of the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ of a boson field, it is reasonable to consider the information entropy

$$\tilde{S}(\hat{\rho}, \hat{a}) = -\int d^2\alpha \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \ln\left(\frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle\right) \tag{5}$$

$$= -\int d^2\alpha Q(\alpha)\ln Q(\alpha), \tag{6}$$

where

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \tag{7}$$

is the well-known Glauber representation of a state $\hat{\rho}$, and where $|\alpha\rangle$ is the boson coherent state with the decomposition of unity $\int d^2\alpha|\alpha\rangle\langle\alpha| = \pi$.

According to following calculations, the information entropy of a squeezed state is not zero as opposed to the von Neumann entropy which is always zero for a pure state. This obviously makes the information entropy more interesting than the von Neumann entropy. The form of Eq.(5) as well as Shannon's early work [1] suggest to define an information entropy for any phase space distribution. The Q-representation, however, has turned out to be appropriate for realitic measurements as shown in the analysis in terms of phase propensities [5] and heterodyne measurements. [6] For the investigation of the entropic uncertainty principle for information entropies, the marginal entropies are evaluated by inserting the marginal Q-representation in the above expressions, instead. Letting α_1 and α_2 be arbitrary coordinates in the complex plane of α, we thus define

$$Q_i(\alpha_i) = \int d\alpha_j Q(\alpha_i, \alpha_j) \tag{8}$$

and

$$S_i = -\int d\alpha_i Q_i(\alpha_i)\ln Q_i(\alpha_i) \tag{9}$$

for $i, j \epsilon \{1, 2\}, i$ unequal j. This leads to the entropic uncertainty relation for information entropies, which, as a major result of this study, will turn out to hold for the squeezed state and its thermally relaxing state.

In the following, we put forward the time independent information entropy of a squeezed state and its evolution to a thermal state via the Fokker-Planck equation and evaluate the corresponding information entropies.

2 Information entropy and entropic uncertainty relations of a squeezed state

2.1 Statics

In this section the squeezed state is described by the time independent Q-representation

$$Q(\alpha, \alpha^*) = \frac{1}{\pi\mathrm{ch}(s)}\exp\left[-|\alpha - \alpha_0|^2 + \frac{\mathrm{th}(s)}{2}\{(\alpha - \alpha_0)^2 + (\alpha^* - \alpha_0^*)^2\}\right], \tag{10}$$

with the corresponding information entropy:

$$S = -\int d^2\alpha Q(\alpha, \alpha^*)\ln Q(\alpha, \alpha^*) = 1 + \ln\frac{\pi}{2} + \ln(e^s + e^{-s}). \tag{11}$$

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The letter s here denotes the squeezing parameters and α_0 describes the coherent state which has been squeezed. We now want to compare S with the information entropies obtained out of the marginal Q-representations. Those marginal information entropies are obviously dependent on the choice of coordinates, where our considerations in the following will concentraate on the most interesting Cartesian coordinates.

The Q-representation in Cartesian coordinates ($\alpha_x = \mathrm{Re}\alpha, \alpha_y = \mathrm{Im}\alpha$) has the form

$$Q(\alpha_x, \alpha_y) = \frac{1}{\pi \mathrm{ch}(s)} \exp\left[-\frac{2}{1+e^{2s}}(\alpha_x - (\alpha_0)_x)^2\right] \times \exp\left[-\frac{2}{1+e^{-2s}}(\alpha_y - (\alpha_0)_y)^2\right], \qquad (12)$$

leading to the marginal quasi-probability distributions,e.g.:

$$Q_x(\alpha_x) = \int_{-\infty}^{+\infty} d\alpha_y Q(\alpha_x, \alpha_y) = \frac{1}{\pi \mathrm{ch}(s)} \left(\frac{\pi(1+e^{-2s})}{2}\right)^{1/2} \exp\left[-\frac{2}{1+e^{2s}}(\alpha_x - (\alpha_0)_x)^2\right], \quad (13)$$

and thus to the marginal information entropies, e.g.:

$$S_x = -\int Q_x(\alpha_x)\ln Q_x(\alpha_x)d\alpha_x = \frac{1}{2} + \frac{1}{2}\ln\frac{\pi}{2} + \frac{1}{2}\ln(1 + e^{2s}), \qquad (14)$$

and correspondingly $S_y = \frac{1}{2} + \frac{1}{2}\ln\frac{\pi}{2} + \frac{1}{2}\ln(1 + e^{-2s})$. Considering above equations, it is now easy to see that squeezed states fulfill minimum entropic uncertainty

$$S = S_x + S_y, \qquad (15)$$

and that the Araki-Lieb inequality is valid as well: $|S_x - S_y| < S$.

A similar consideration can be done for polar coordinates with $\alpha = re^{i\phi}$ and $\alpha_0 = r_0 e^{i\phi_0}$. The integrals here are not as straight forward as in the Cartesian case. For special cases as the weakly squeezed vacuum, however, it was possible to show the validity of the uncertainty and Araki-Lieb relation [7].

2.2 Dynamics

Our interest now turns to the time evolution of the information entropy, its marginal information entropies and its influence on the inequalities investigated in the preceeding section. The time evolution of the Q-representation is governed by the Fokker-Planck equation

$$\partial_t Q = \left[\frac{\gamma}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right) + \eta\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]Q. \qquad (16)$$

This equation follows from the well-known Fokker-Planck equation for the P-representation with $Q(\alpha, t) = \int \frac{d^2\beta}{\pi}\exp[-|\alpha - \beta|^2]P(\beta, t)$, $\bar{n}$ is the mean number of photons and η turns out to be $\gamma(\bar{n} + 1)$.

We now move on with the solution of the Fokker-Planck equation for Q, assuming the squeezed state to be the initial state at time t equal to 0. Following reference [7] this turns out to be:

$$Q(\alpha, t) = \int d^2\beta \frac{1}{\pi n(t)} \exp\left(-\frac{|\alpha - \beta e^{-(\gamma/2)t}|^2}{n(t)}\right) \tag{17}$$

$$\times \frac{1}{\pi \text{ch}(s)} \exp\left[-|\beta - \alpha_0|^2 + \frac{1}{2}\text{th}(s)\{(\beta - \alpha_0)^2 + (\beta^* - \alpha_0^*)^2\}\right] \tag{18}$$

$$= \exp[a(t)|\alpha|^2 + b(t)(\alpha^2 + \alpha^{*2}) + c(t)\alpha + c(t)^*\alpha^* + N(t)], \tag{19}$$

with $n(t) = \frac{n}{\gamma}(1 - e^{-\gamma t})$. For the rather long analytic expressions of $a(t), b(t), c(t)$ and $N(t)$ we refer to reference [7]. At this point it will only be of interest that the Fokker-Planck equation has preserved the Gaussian character of the initial state.

In the following we would like to point out that also the time dependent Q representation due to the last equation leads to minimum uncertainty. Even more it turns out that every normalized Gaussian function of the following form has this property:

$$Q(\alpha, t) = \exp[a(t)|\alpha|^2 + b(t)(\alpha^2 + \alpha^{*2}) + c(t)\alpha + c^*(t)\alpha^* + N(t)], \tag{20}$$

where the real coefficients $a(t), b(t)$ and the complex $c(t)$ are now arbitrary with the only restrictions to fulfill: $a(t) < 0$, $2|b(t)| < |a(t)|$ and $N(t)$ is determined such that $Q(\alpha, t)$ is normalized. Using the same notation as in the static case, some algebra gives rise to the corresponding information entropy and marginal entropies:

$$S(t) = 1 + \ln\pi - \frac{1}{2}\ln(a^2(t) - 4b^2(t)), \tag{21}$$

$$S_x(t) = \frac{1}{2} + \frac{1}{2}\ln\pi - \frac{1}{2}\ln(-a(t) - 2b(t)), \tag{22}$$

$$S_y(t) = \frac{1}{2} + \frac{1}{2}\ln\pi - \frac{1}{2}\ln(-a(t) + 2b(t)), \tag{23}$$

yielding immediately for all times t

$$S(t) = S_x(t) + S_y(t). \tag{24}$$

At $t = 0$ this is in agreement with the minimum Heisenberg uncertainty relation of a Gaussian wavepacket of the above form because the product of uncertainties in space and momentum is exactly one. Since the Fokker-Planck equation does conserve the Gaussian character of the wave function and does moreover give not rise to any phase factor, the Cartesian entropic uncertainty relation is even fulfilled with the equal sign for all times. Thus we have also a minimum uncertainty relation for the thermal state, what is not expected from the Heisenberg uncertainty inequality.

Moreover one finds for any Gaussian distribution that the Araki-Lieb inequality is equivalent to:

$$\frac{1}{2}\left|\ln\frac{-a - 2b}{-a + 2b}\right| \leq -\frac{1}{2}\ln(a^2 - 4b^2) + 1 + \ln\pi \tag{25}$$

In the case of a thermally squeezed with its particular values for the coefficients $a(t), b(t), c(t)$ and $N(t)$ an even stronger inequalility can be derived:

$$|S_x(t) - S_y(t)| \le S(t) - (1 + \ln\frac{\pi}{2}).\tag{26}$$

which , however, does not mean that all phase space distributions fulfill the Araki-Lieb or even the improved inequality [7].

In conclusion, we introduced a way to evaluate the information entropy in terms of the Glauber Q-representation. Taking advantage of these entropies, we approached the question of comparability of the Heisenberg uncertainty and the Shannon-Wehrl-entropy like description of information for the example of a thermally relaxing squeezed state. The first just considers second moments and is therefore a – though very important – approximation whereas the other is exact but academic. We find full accordance concerning the validity of the Heisenberg and entropic uncertainty inequalities for the thermally relaxing squeezed state but as expected also observe disagreement in the case when the equal sign holds.

3   Acknowledgments

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Squeezed States of Electrons
and
Transitions of the Density of States

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Abstract

Electron systems which have low dimensional properties have been constructed by squeezing the motion in zero, one or two-direction. An isolated quantum dot is modelled by a potential box with delta-profiled, penetrable potential walls embedded in a large outer box with infinitely high potential walls which represent the work function with respect to vacuum. We show the smooth crossover of the density of states from the three-dimension to quasi-zero-dimensional electron gas.

1 Introduction

Quantum wires and quantum boxes with three-dimensionally confined electrons constitute a considerable part of recent semiconductor research [1, 2]. To study the optical properties of these systems, one should investigate the density of states (DOS) carefully, because the change in the density of states affects directly the optical properties of these structures as a result of reduced dimensionality.

The DOS of a low dimensional electron gas(LDEG) in the presence of magnetic field has been discussed in many literatures measuring the magnetocapacitance [3, 4]. Furthermore an electrical confinement which is usually controlled by (alternate) gate voltage [5] and, so called the illumination method [6] are used to get a LDEG. The etched silicon filaments also discussed recently as quantum wires or quantum dots [7]. But the DOS of a LDEG of confined electrons in small space which is constructed by reducing the size of the confinement is not discussed frequently, see ref. [1]. A typical example of an ideal system having Q0D charactor is that of electron confined in a quantum box with impenetrable potential barriers. Despite of the large number of studies on quantum wire and quantum box structures up to date, we have not found research on the crossover of the DOS from a three dimensional DOS to a quasi-zero dimensional DOS. In Section 2, to illustrate the formation of a quasi-one-dimensional electron gas(Q1DEG) using the classical electrostatic method, a simple metal-insulator-semiconductor(MIS) structure

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with very many parallel gate electrodes has been treated by making use of the conformal mapping method. In section 3, we consider a rather artificial quantum box structure, so called the three directional double-barrier resonant-tunneling structures(DBRTS) to study quantum mechanically, and have calculated the local density of states and the global density of states. In section 4, the crossovers of the DOS is calculated. Especially, we reveal the crossovers of the DOS from 3D to Q0D.

2   Construction of the very many parallel quantum wires

To study electrical properties of a quantum wire, we first start with a quasi-two-dimensional electron gas(Q2DEG) at simple metal-insulator- semiconductor(MIS) structure. A Q2DEG system with many parallel gate electrodes is shown in Fig.1 in which an electron gas is confined to the x-z plane. We actually try to confine the electrons in the z-direction as well to form a Q1DEG system.
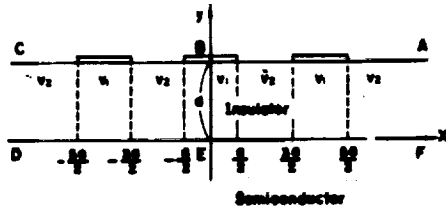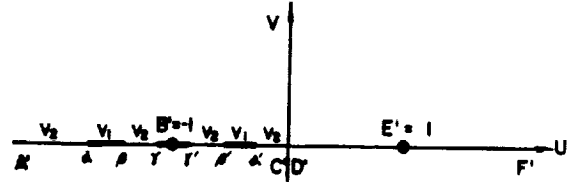

Fig.1

Fig.2

Fig.1 A Structure of symetric gate arrays. Fig.2 The boundary condition in w-plane.

To calculate the charge(density) distribution at the MIS interface (y=0 plane) to see the formation of a Q1DEG, we will use the conformal mapping method which is useful especially for two-dimensional problems and we assume that significant changes in the electrode potential (and thus is density in the channel) cause only a slight change in the near junction band bending. This type of approximation has been used by Shik [8] to calculate various properties of the MIS structure.

The problem is solving the Laplace equation in insulator region.

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \tag{1}$$

with the boundary condition;

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At $y = d$, $\Psi |_{y=d}$ is alternate gate potential V_1 and V_2 and at $y = 0$, $\Psi |_{y=0}$ is constant, i.e. equipotential surface. The next step is getting the distribution of the surface carrier density.

$$n_s(x) = \frac{K_d}{4\pi e} \frac{\partial \Psi}{\partial y} |_{y=0} .$$ (2)

We solve the problem by taking the following conformal transformation [9].

$$W = U + iV = e^{\pi z/d}, z = x + iy$$

where d is the thickness of the insulator, Now the insulator region is mapped on the upper half plane and the boundary condition is given as in Fig.2.

After getting the potential which satisfy the boundary condition, we now get the electron density distribution n(x) analytically from Eq.(2):

$$n(x) = \frac{K_d V_2}{4\pi e}[1 + (V_1/V_2)^{-1} \sum_{n=1}^{m}(-1)^{n+1} < \sinh((2n-1)\pi a/2d)$$
$$[\cosh(\pi x/d) + \cosh((2n-1)/2d)] >]$$ (3)

where a is the gate interval and m is the number of gates.
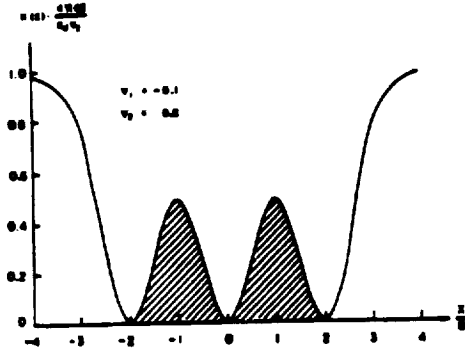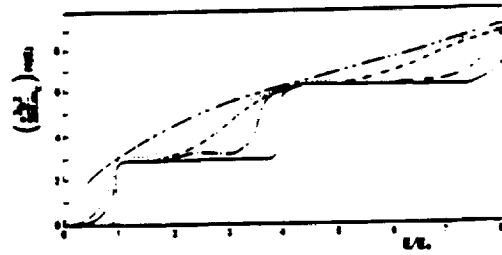

Fig.3   Fig.4

Fig.3 The surface charge density vs. position.
Fig.4 Crossover of the global DOS from 3D to 2D in the range $U_1 = 0$ to $U_1 = 20$,
as a function of E/E_0. Here U_1 takes the values 0, 2, 12, 16, 20.

A typical density distribution is illustrated in Fig.3 where we can see immediately the many parallel Q1DEG (eventhough we show here only two wires). The one dimensional electron density of the order of $10^6/cm$ is obtained for the typical operating gate voltage when a is 1000Å. We also investigated the case of anti-symmetric gate voltage. Similar results have been obtained but in symmetric case it is easy to construct one dimensional electron channels especially for smaller number of gates. In our calculation we took $a/d = 1$, which satisfy the first approximation, $y = 0$ plane is equipotential.

3 The quantum box model and the DOS

Now we come back to quantum system with a delta-profiled quantum box. Usually a quantum dot is an element of the array of quantum dots. But the interaction among quantum dots decreases rapidly with increasing dot separation [10] and is unimpotant for the usual experimental situation [11].

Therefore a separated single quantum box is taken for our study. For a rectangular wire($L_{y(x)} \ll \lambda_D$), Arora and others[12] used the impenetrable potential walls, but in this paper we consider three sets of penetrable barriers. We start our calculations with the model, i.e., the typical three directional DBRTS, which consists of two thin($\sim 50\AA$)$Al_xGa_{1-x}As$ layers, separated by a thin GaAs layer along all three directions. The potential is expressed by

$$V(x,y,z) = V_1\delta(x+a) + \delta(x-a) + V_2\delta(y+b) + \delta(y-b) + V_3\delta(z+c) + \delta(z-c). \qquad (4)$$

In this potential, the six $Al_xGa_{1-x}As$ potential barriers have been replaced by δ-functions with strengthes V_1, V_2 and V_3 in the x, y, and z direction, respectively. The parameters $V_i(i = 1, 2, 3)$ are given by

$$V_i = d_i\Delta V_{ci} \qquad (5)$$

where d_i are the barrier widths and ΔV_{ci} are the conduction-band discontinuities. In order to deal with finite density of states,[13], we must place our structure in a large impenetrable rigid box extending from $-L/2$ to $L/2$. With proper boundary conditions[14], the Schrödinger equation is separable. we can write the wave function in the product form

$$\Psi(r) = \psi(x)\psi(y)\psi(z) = \prod_{i=1}^{3} \Psi_i \qquad (6)$$

The separated wave functions, Ψ_i, satisfy the reduced equations.

$$\Psi_i'' + [2m_c/\hbar^2][E_i - V_i]\Psi_i = 0 \qquad (7)$$

with

$$E = \sum_{i=1}^{3} E_i \qquad (8)$$

Here E is the total energy corresponding to the Hamiltonian H and $E_i(i = x, y, z)$ is the energy eigenvalue of Ψ_i.

The local density of states in the DBRTS has been obtained in various cases [15]. It is defined as a function of $\mathbf{r} = (x, y, z)$ and E by

$$\begin{aligned}
N(x,y,z;E) &= -(2/\pi)ImG(\mathbf{r},\mathbf{r}';E) \\
&= 2\sum_{\alpha\beta\gamma}\sum_{k_x}\sum_{k_y}\sum_{k_z} | \Psi_{\alpha k_x}(x) |^2| \Psi_{\beta k_y}(y) |^2| \Psi_{\gamma k_z}(z) |^2 \delta(E - E_k), \qquad (9)
\end{aligned}$$

where the factor of 2 implies spin degeneracy, $G(\mathbf{r}, \mathbf{r}'; E)$ is the single particle Green's function, and α, β, and $\gamma(= e$ or $0)$ label state parity. Next we consider the global DOS $N(E)$. It can be calculated by taking the integration over the box volume,

$$N(E) = 8\int_0^a dx \int_0^b dy \int_0^c dz N(x,y,z;E). \qquad (10)$$

The amplitude of the wave function inside the well for both even and odd parities of x, y, and z componenets are given elsewhere[14]. $N(E)$ can be rewritten as follows:

$$N(E) = (2/\pi^3) \int_0^\infty dp_1 [G_e(p_1) + G_o(p_1) + (G_e(p_1) - G_o(p_1)) \sin(2p_1)/2p_1]$$

$$\times \int_0^\infty dp_2 [G_e(p_2) + G_o(p_2) + (G_e(p_2) - G_o(p_2)) \sin(2p_2)/2p_2]$$

$$\times \int_0^\infty dp_3 [G_e(p_3) + G_o(p_3) + (G_e(p_3) - G_o(p_3)) \sin(2p_3)/2p_3] \delta(E - E_k). \quad (11)$$

The properties of functions $G_e(p_i)$ and $G_o(p_i)$ are already revealed in Ref.14.

When we take appropriate limiting cases, the Eq.(11) recovers all the well-known expressions of the DOS of 3D, 2D, 1D, and 0D. since the calculations are straightforward, we haven't repeated here.

4   Crossovers of the density of states

Now we consider crossovers of the global DOS from a high dimension to a low dimension.

4.1   From 3D to 2D

This case may happen when two of three potentials $U_i's$(see the reference 15) approach zero, while the remainder varies from zero, i.e., 3D case, to infinity, i.e., 2D case. The Eq.(11) can be modified as

$$\frac{2\pi}{4bc} \frac{\hbar^2}{2m_e} N(E) = \int_0^{\pi/2(E-E_o)^{1/2}} dp_1 [G_e(p_1) + G_o(p_1) + (G_e(p_1) - G_o(p_1)) \sin(2p_1)/2p_1] \quad (12)$$

The result of the numerical behavior of Eq.(12) is shown in Fig.4 and indicates the transition of the DOS from 3D to 2D. In this case we take $U_2 = U_3 = 0$, U_1 changes from 0 to 20, and E/E_o varies from 0 to 8. Higher values of U_1 correspond to a staircase-like 2D behavior which shows steps at $E/E_o = l^2$ with $l = 1, 2, 3, \cdots$.

4.2   From 2D to 1D

This corresponds to the case of U_3 going to zero, U_1 to infinity, and U_2 varying from zero, i.e., 2D case, to infinity, 1D case. So $G(p_3) = 1, G_e(p_1) = \pi \sum \delta(p_1 - (l + 1/2)\pi), G_o(p_1) = \pi \sum \delta(p_1 - (l + 1)\pi)$. Then we can get the modified equation of N(E) as follows ;

$$\frac{\pi^2 \hbar^2}{4cm_e} N(E) = \sum_{m=0}^\infty \int_0^{\pi/2(E-E_o)} dp_2 [G_e(p_2) + G_o(p_2) + (G_e(p_2) - G_o(p_2)) \sin(2p_2)/2p_2]$$

$$\times [1/[(\pi/2a)^2 E/E_o - ((m + 1/2)\pi/a)^2 - (p_2/b)^2]^{1/2}$$

$$+ 1/[(\pi/2a)^2 E/E_o - (m + 1)\pi/a^2 - (p_2/b)^2]^{1/2}]. \quad (13)$$

The Fig.5 shows the graphical result, that is, the crossover of the global DOS from 2D to 1D. In this case, for the sake of convenience, we take $a = b, U_3 = 0$ and $U_1 \to \infty, U_2$ takes the values

of 0, 2, 8, 16, 20, and E/E_o is taken from 0.0 to 8.0. One can see that the steps at $E/E_o = 1$, 4, 9, $\cdots$, that is, the two dimensional band edges, are shifted to peaks at $E/E_o = 2, 5, 8, 10$, $\cdots$, i.e., the one dimensional band edges. As U_2 increases, the motion of confined electrons along the y axis starts to shrink and is guided only along the z axis. This weak additional confinement shifts the 2D band edges towards higher energies and finally the typical 1D characteristics of the DOS comes to bevisualized. Higher values of U_2 correspond to increased sharp peaks of the DOS of the 1-D quantum wire case, which are in good agreement with those of Arakawa and Sakaki [13] and of Tsang [16]. The values at $E/E_o = 5$ and 10 are roughly twice those at $E/E_o = 2$ and 8, respectively, which comes from the double degeneracy of the eigenstates. Similar discussions were treated by Berggrent and Newson [17] in the case of the 2D electrons in the presence of a magnetic field.
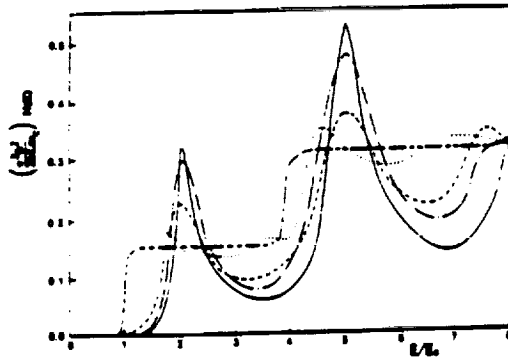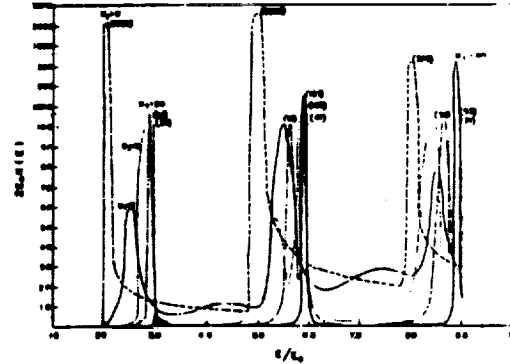

Fig.5

Fig.6

Fig.5 Crossover of the global DOS from 2D to 1D. Here we take $U_1 = \infty$ and $U_2 = 0, 2, 8, 16, 20$. Higher values of U_2 correspond to a sawtoothlike 1D behavior.

Fig.6 Crossover of the global DOS from 1D to 0D. Here we take $U_1 = U_2 = \infty$ and $U_3 = 0, 2, 10, 20, 80$. Higher values of U_3 correspond to a sharp line shape 0D behavior.

4.3 From 1D to 0D

In this case, we take both U_1 and U_2 to be infinity, and U_3 vary from zero, 1D case, to infinity, 0D case. Then Eq.(11) becomes

$$N(E) = (2/\pi) \sum_{l,m=1}^{\infty} \int_0^{\pi/2(E/E_o)^{1/2}} dp_3 [G_e(p_3) + G_o(p_3) + (G_e(p_3) - G_o(p_3)) \sin(2p_3)/2p_3]$$

$$\times \quad \delta[E - (\hbar^2/2m_c)(l\pi/2a)^2 + (m\pi/2b)^2 + (p_3/c)^2 \tag{14}$$

$$(2E_o)N(E) = \sum_{l,m=1}^{\infty} \frac{G_e(t) + G_o(t) + (G_e(t) - G_o(t)) \sin(2t)/2t}{[E/E_o - l^2 - m^2]^{1/2}} \tag{15}$$

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where $t = (\pi/2)(E/E_o - l^2 - m^2)^{1/2}$.

Fig.6 shows the transition of the global DOS from 1D to 0D. For the sake of convenience, we put $a = b = c$. The Eq.(15) recovers the well-known DOS of a quantum dot [16], when we take U_3 to be infinity. Sawtooth type maximums at $E/E_o = 2, 5, 8, 10, \cdots$, are now moved to the positions at $E/E_o = 3, 6, 9, 11, \cdots$, as the strength of U_3 increases. When the confining potential increase, both primary peaks and secondary ones appear, which reflect the coexistance of 1D and 0D behavior. The secondary peak with the lower energy is a reminescence of the 1D DOS shifted towards higher energy due to the additional confinement, and the primary peak (higher energetic peak) arises from quasi 0D states. Bacause the differences between peaks are so high, we used different scales for the DOS axis ranging from 1.0 to over 2000. The DOS clearly shows the potential strength (U_i) dependence of the spatial quantization through E_o. This kind of secondary peaks are also shown in many experimental data of a transport measurement [18]. We know that the electron systems used in above experiments are in an intermediate state between 1D and 0D, because the potential strengths are not infinitely high.

We believe that this kind of DOS transition which shows intermediate states will also occur in real systems where, for example, the barriers have finite widths. For barriers with finite thickness, the effective mass of the electron changes in passing from the quantum-well region (GaAs) to the barrier regions (AlGaAs) of the structure. Bruno and Bahder [15] have considered this for the one directional DBRTS case and showed that the DOS at the low-energy subband edges is higher than the DOS at the same energies in the absences of barriers (for delta-profiled barriers). In our case, we can estimate that our result for the DOS will be increased a bit upward at the same energies because of the additive form of the potential which we have taken.

In this paper, first we showed as an example the formation of many electron wires using the conformal mapping method. Next, considering a penetrable quantum box, with a volume of a $axbxc$, in a very large rigid box of volume L^3, we calculated the general form of the local and global DOS.

The merit of this model is as follows :

1) the model is simple to handle and easy to calculate analytically,

2) in this model, one can recover the results of all the limiting cases of the 3D, 2D, 1D, and 0D,

3) starting from one equation we can discuss all three crossover cases.

5 Acknowledgments

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QUANTUM MECHANICAL EFFECTS OF TOPOLOGICAL ORIGIN

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Abstract

Following a brief review of the original Casimir and Aharonov-Bohm effects, some other effects of similar natures are mentioned. A Casimir interaction between AB fluxes is presented. Possible realizations of the Casimir effects for massive charged fields in solid state structures; and a new AB effect for photons are suggested.

1 Casimir and Aharonov-Bohm Effects

There are two types of quantum mechanical effects which can be attributed to the non-trivial topologies of the configuration or phase spaces.

First kinds of these effects are named after Casimir [1]. When the space is bounded the vacuum expectation value of quantized fields acquires non-zero values and becomes space dependent, which then creates a force on the boundaries. The attractive force between the parallel conductive plates is the first example of this kind [1]: the vacuum fluctuations of the electromagnetic field produces an attractive force on the unit area of the plates given by $F = -(\pi^2/240)(\hbar c/a^4)$ where a is the seperation of the plates. This force is already observed in experiments [2]. The topological nature of this effect is in the fact that the field momentum perpendicular to the plates is discretized; i.e., the effective topology is not $\mathbf{R}^3$ but $\mathbf{S}^1 \times \mathbf{R}^2$.

The second kind of effects are known as the Aharonov-Bohm (AB) effects, which involve the electron field: when a confined flux is placed in the space, the electrons moving in te outside region pick up a phase which is observable in the interferance experiments [3]. In this effect the topology of the plane perpendicular to the flux line is multiply connected.

These exist some other examples which are similar to the above mentioned effects:

Several calculations have been made for Casimir effects involving boundaries of different shapes [2]. Examples with moving boundaries are also studied which are used to obtain squezzed states of light [4].

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[1] Mailing address.

Some of the physical effects which have been proposed to be similar to the AB effect are the followings:

Introducing an impenetrable charged line in place of the magnetic flux and replacing the electrons with neutral magnetic dipoles one obtains a system equivalent to the AB effect [5].

AB experiments involving the correlated charged particles are also proposed. Ideas involving electron-positron pairs extended to obtain AB effect for the effective photon field [6].

2 An Interaction Connecting Two Effects

Consider two parallel, tightly wound solenoids, confining fluxes Φ_1 and Φ_2 in them, which are seperated by the distance a. The vacuum expectation value of the energy for the massive, charged field in the region outside the flux lines have a finite "interaction" term depending on $\Phi_1\overline{\Phi_2}$ and on a. For a scalar charged field the interaction energy in the slice of space with unit thickness having its normal parallel to the fluxes is

$$E_{\Phi_1\Phi_2} = -\frac{\hbar c}{2\pi^2}\frac{\Lambda_1^2\Lambda_2^2}{a^2}, \quad \Lambda \equiv \frac{e\Phi}{2\pi}. \tag{1}$$

Note that the mass of the field does not contribute to the interaction term which only appears in the self energy terms involving Φ_1^2 and Φ_2^2 seperately. The energy (1) leads to an attractive force on the unit length of the flux lines given by

$$F_{\Phi_1\Phi_2} = -\frac{\partial}{\partial a}E_{\Phi_1\Phi_2} = \frac{\hbar c}{\pi^2}\frac{\Lambda_1^2\Lambda_2^2}{a^3}. \tag{2}$$

The above force is derived for a hypotetical scalar, charged field. For charged fields with spin, for each spin degree of freedom we expect to thave a force equivalent to (2). For example for the electron field the force should be multiplied by two.

If the fluxes are quantized, for integer fluxes, fields with integer charges, that is $e^\pm$, $\mu^\pm$, $\tau^\pm$ particles contribute to the Casimir interaction. On the other hand the quark fields can contribute only to the interaction of the fluxes quantized to the one or two-thirds of integers. In conclusion we can say that the Casimir force between the quantized AB fluxes may count the number of families [7].

3 Comments on Possible New Realizations

The recent developments in solid state physics enables one to create two-dimensional and one-dimensional structures (=quantum dots) in which we can trap charged particles. These structures my raise the hope of observing new Casimir effects involving massive fields.

Finally we like to suggest an experimental set up which may realize an AB geometry for the photon field: In the double slit experiment, if we place an infinitely long, thin, neutral and perfectly conductive wire perpendicular to the incoming light beam, we expect to observe an AB type effect for the photon field. This, unlike the one suggested in Ref.[6], would be a purely neutral AB effect.

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NONUNITARY AND UNITARY APPROACH TO EIGENVALUE PROBLEM OF BOSON OPERATORS AND SQUEEZED COHERENT STATES

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Abstract

The eigenvalue problem of the operator $a + \zeta a^\dagger$ is solved for arbitrary complex ζ by applying a nonunitary operator to the vacuum state. This nonunitary approach is compared with the unitary approach leading for $|\zeta| < 1$ to squeezed coherent states.

1 Introduction

The eigenvalue problem to linear combinations of boson operators in the standardized form $a + \zeta a^\dagger$ can be solved with squeezed coherent states only in the case $|\zeta| < 1$ when it is equivalent to the eigenvalue problem of an operator $\kappa a + \mu a^\dagger$ under the condition $|\kappa|^2 - |\mu|^2 = 1$ with the substitution $\zeta \equiv \frac{\mu}{\kappa}$, ($\zeta, \kappa, \mu$ arbitrary complex numbers) [1]. This corresponds to the unitary approach because the squeezed coherent states can be obtained by applying unitary squeezing operators to coherent states [2]. However, this eigenvalue problem can also be solved for arbitrary complex ζ with a nonunitary approach providing in the limiting case $\zeta \to \infty$ even the solution of the eigenvalue problem for the boson creation operator $a^\dagger$. The corresponding eigenstates are not normalizable for $|\zeta| \geq 1$ and are not states of the usual Hilbert space H (Fock space) in this case but they are states of a rigged Hilbert space K' in Gelfand triplets of spaces $K \subset H \subset K'$ [3]. Such states that do not give finite expectation values for relevant operators as, for example, for the number operator could be, therefore, considered as pathological ones. However, they play an important auxiliary role for the formulation of a new kind of orthogonality and completeness relations on paths through the complex plane of eigenvalues, where at once two dual states belonging to the parameters ζ and $\zeta' = \frac{1}{\zeta^*}$ or $\zeta \zeta'^* = 1$ are involved [1].

2 Nonunitary approach to the eigenvalue problem

The solution of the eigenvalue problem

$$(a + \zeta a^\dagger)|\alpha; \zeta> = \alpha|\alpha; \zeta>, \tag{1}$$

can be represented in the number-state basis $|n>$ in the following nonnormalized form

$$|\alpha;\zeta>=\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}H_n\left(\frac{\alpha}{\sqrt{2\zeta}}\right)\left(\frac{\sqrt{2\zeta}}{2}\right)^n|n>$$

$$=\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{k!(n-2k)!}\left(\frac{\zeta}{2\alpha^2}\right)^k|n>, \tag{2}$$

or by derivatives of a Gaussian function in the form

$$|\alpha;\zeta>=\exp\left(\frac{\alpha^2}{2\zeta}\right)\sum_{n=0}^{\infty}\frac{(-\zeta)^n}{\sqrt{n!}}\frac{\partial^n}{\partial\alpha^n}\exp\left(-\frac{\alpha^2}{2\zeta}\right)|n>. \tag{3}$$

Substituting in (2) the number states by the generation from the vacuum state, one obtains by means of the generating function of the Hermite polynomials $H_n(z)$

$$|\alpha;\zeta>=\exp\left(\alpha a^\dagger-\frac{\zeta}{2}a^{\dagger 2}\right)|0>. \tag{4}$$

Two special cases are easily obtained from these formula, the coherent states $|\alpha;0>$

$$|\alpha;0>=\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n>\equiv\exp\left(\frac{\alpha\alpha^*}{2}\right)|\alpha>, \tag{5}$$

and the squeezed vacuum states $|0;\zeta>$

$$|0;\zeta>=\sum_{m=0}^{\infty}\sqrt{\frac{(2m-1)!!}{2^m m!}}(-\zeta)^m|2m>. \tag{6}$$

The nonunitary operator $\exp\left(\alpha a^\dagger-\frac{\zeta}{2}a^{\dagger 2}\right)$ does not preserve the normalization of the states. The corresponding normalized states

$$|\alpha;\zeta>_{norm}=(1-\zeta\zeta^*)^{\frac{1}{4}}\exp\left\{-\frac{2\alpha\alpha^*-(\zeta^*\alpha^2+\zeta\alpha^{*2})}{4(1-\zeta\zeta^*)}\right\}|\alpha;\zeta>, \tag{7}$$

are only possible for $|\zeta|<1$.

The expectation values of the canonical operators

$$Q(\varphi)\equiv\sqrt{\frac{\hbar}{2}}(ae^{i\varphi}+a^\dagger e^{-i\varphi}),\quad P(\varphi)\equiv-i\sqrt{\frac{\hbar}{2}}(ae^{i\varphi}-a^\dagger e^{-i\varphi}), \tag{8}$$

denoted by cross-lines are

$$\overline{Q(\varphi)}=\sqrt{\frac{\hbar}{2}}\frac{\alpha e^{i\varphi}(1-\zeta^* e^{-i2\varphi})+\alpha^* e^{-i\varphi}(1-\zeta e^{i2\varphi})}{1-\zeta\zeta^*},$$

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$$\overline{P(\varphi)} = -i\sqrt{\frac{\hbar}{2}} \frac{\alpha e^{i\varphi}(1 + \zeta^* e^{-i2\varphi}) - \alpha^* e^{-i\varphi}(1 + \zeta e^{i2\varphi})}{1 - \zeta\zeta^*}, \tag{9}$$

and their variances are

$$\overline{(\Delta Q(\varphi))^2} = \frac{\hbar}{2} \frac{(1 - \zeta e^{i2\varphi})(1 - \zeta^* e^{-i2\varphi})}{1 - \zeta\zeta^*},$$

$$\overline{(\Delta P(\varphi))^2} = \frac{\hbar}{2} \frac{(1 + \zeta e^{i2\varphi})(1 + \zeta^* e^{-i2\varphi})}{1 - \zeta\zeta^*}, \tag{10}$$

The uncertainty product

$$\overline{(\Delta Q(\varphi))^2} \; \overline{(\Delta P(\varphi))^2} = \frac{\hbar^2}{4} \left\{ 1 - \frac{(\zeta e^{i2\varphi} - \zeta^* e^{-i2\varphi})^2}{(1 - \zeta\zeta^*)^2} \right\}, \tag{11}$$

is the minimal possible one for 4 angles φ_{ext} according to

$$e^{i4\varphi_{ext}} = \frac{\zeta^*}{\zeta}, \quad \overline{(\Delta Q(\varphi_{ext}))^2} \; \overline{(\Delta P(\varphi_{ext}))^2} = \frac{\hbar^2}{4}, \tag{12}$$

corresponding to the extreme values of the variances

$$\overline{(\Delta Q(\varphi_{ext}))^2} = \frac{\hbar}{2} \frac{1 \mp |\zeta|}{1 \pm |\zeta|}, \overline{(\Delta P(\varphi_{ext}))^2} = \frac{\hbar}{2} \frac{1 \pm |\zeta|}{1 \mp |\zeta|}. \tag{13}$$

One has pure amplitude (phase) squeezing if the minimal (maximal) value of $\overline{(\Delta Q(\varphi_{ext}))^2}$ corresponds to $\overline{P(\varphi_{ext})} = 0$. This leads to the following coordinate–invariant conditions for the arguments of the Hermite polynomials in (2):

1. amplitude squeezing, $\dfrac{\alpha}{\sqrt{2\zeta}}$ real numbers,

2. phase squeezing, $\dfrac{\alpha}{\sqrt{2\zeta}}$ imaginary numbers.

In the more general case, $\dfrac{\alpha}{\sqrt{2\zeta}}$ is a complex number. The expectation value of the number operator is

$$\overline{N} = \left(\frac{\alpha - \zeta\alpha^*}{1 - \zeta\zeta^*} \right) \left(\frac{\alpha^* - \zeta^*\alpha}{1 - \zeta\zeta^*} \right) + \frac{\zeta\zeta^*}{1 - \zeta\zeta^*}. \tag{14}$$

and its variance

$$\overline{(\Delta N)^2} = \frac{((1 + \zeta\zeta^*)\alpha - 2\zeta\alpha^*)\left((1 + \zeta\zeta^*)\alpha^* - 2\zeta^*\alpha\right)}{(1 - \zeta\zeta^*)^3} + \frac{2\zeta\zeta^*}{(1 - \zeta\zeta^*)^2}. \tag{15}$$

The nonunitary approach provides a new convenient parametrization of the squeezed coherent states.

3 Unitary approach to the eigenvalue problem

The unitary squeezing operators

$$S(\xi, \eta = \eta^*, \xi^*) = \exp\left\{\xi\frac{1}{2}a^2 - \xi^*\frac{1}{2}a^{\dagger 2} + i\eta\frac{1}{2}(aa^\dagger + a^\dagger a)\right\} \tag{16}$$

transform the basis operators a and $a^\dagger$ according to

$$S(\xi, \eta, \xi^*)(a, a^\dagger)(S(\xi, \eta, \xi^*))^\dagger = (a, a^\dagger)\begin{pmatrix} \kappa, \mu^* \\ \mu, \kappa^* \end{pmatrix}$$

$$\kappa = ch\epsilon - i\eta\frac{sh\epsilon}{\epsilon}, \mu = \xi^*\frac{sh\epsilon}{\epsilon}, \epsilon \equiv \sqrt{|\xi|^2 - \eta^2}, |\kappa|^2 - |\mu|^2 = 1. \tag{17}$$

The solution of the eigenvalue problem of the operator $a + \zeta a^\dagger$ is obtained by the following application of the unitary squeezing operators to coherent states $|\gamma>$

$$\frac{e^{i\chi}}{\sqrt{1 - |\zeta|^2}}(a + \zeta a^\dagger)S(\xi, \eta, \xi^*)\left|\frac{e^{i\chi}\alpha}{\sqrt{1 - |\zeta|^2}}\right> = (\kappa a + \mu a^\dagger)S(\xi, \eta, \xi^*)|\gamma>$$

$$= \frac{e^{i\chi}\alpha}{\sqrt{1 - |\zeta|^2}}S(\xi, \eta, \xi^*)\left|\frac{e^{i\chi}\alpha}{\sqrt{1 - |\zeta|^2}}\right> \tag{18}$$

where χ is an arbitrary angle and ζ and η are given by

$$\xi = \frac{e^{-i\chi}\zeta^*}{\sqrt{1 - |\zeta|^2}}\theta, \eta = -\frac{\sin\chi}{\sqrt{1 - |\zeta|^2}}\theta,$$

$$\theta \equiv \sqrt{\frac{1 - |\zeta|^2}{|\zeta|^2 - \sin^2\chi}}\text{Arsh}\sqrt{\frac{|\zeta|^2 - \sin^2\chi}{1 - |\zeta|^2}}. \tag{19}$$

By choosing $\chi = 0$ one finds

$$|\alpha; \zeta>_{norm} = \exp\left\{-\frac{\zeta^*\alpha^2 - \zeta\alpha^{*2}}{4(1 - |\zeta|^2)}\right\}$$

$$\cdot \exp\left\{\frac{1}{2|\zeta|}\text{Arsh}\left(\frac{|\zeta|}{\sqrt{1 - |\zeta|^2}}\right)\left(\zeta^*a^2 - \zeta a^{\dagger 2}\right)\right\}\left|\frac{\alpha}{\sqrt{1 - |\zeta|^2}}\right> . \tag{20}$$

The unitary approach is restricted to $|\zeta| < 1$.

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4 Dual states and eigenstates of the creation operator

The states $< \frac{\beta^*}{\zeta^*}; \frac{1}{\zeta^*}|$ are left eigenstates to the operator $a + \zeta a^\dagger$ according to

$$< \frac{\beta^*}{\zeta^*}; \frac{1}{\zeta^*}|(a + \zeta a^\dagger) = \beta < \frac{\beta^*}{\zeta^*}; \frac{1}{\zeta^*}|, \tag{21}$$

and they are dual to the states $|\alpha; \zeta >$ in the sense of the orthogonality relation

$$< \frac{\beta^*}{\zeta^*}; \frac{1}{\zeta^*}| \alpha; \zeta >= \sqrt{2\pi\zeta} \exp\left(\frac{\alpha^2}{2\zeta}\right) \delta(\alpha - \beta), \tag{22}$$

and of the completeness relation

$$\frac{1}{\sqrt{2\pi\zeta}} \int_C d\alpha \exp\left(-\frac{\alpha^2}{2\zeta}\right) |\alpha; \zeta >< \frac{\alpha^*}{\zeta^*}; \frac{1}{\zeta^*}| = I. \tag{23}$$

The integration path C through the complex plane is widely arbitrary with the only restriction that it it must begin in one sector and end in the other sector where $\exp\left(-\frac{\alpha^2}{2\zeta}\right)$ vanishes in the infinity for fixed values of ζ.

The eigenstates of the creation operator $a^\dagger$ according to

$$a^\dagger|\tilde{\beta}; \infty >= \beta|\tilde{\beta}; \infty > \tag{24}$$

are

$$|\tilde{\beta}; \infty >= \exp\left(-a^\dagger \frac{\partial}{\partial\beta}\right) \delta(\beta)|0 >= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}} \frac{\partial^n}{\partial\beta^n} \delta(\beta)|n > \tag{25}$$

where $\delta(\beta)$ is the one-dimensional delta function of complex argument (analytic functional). They are orthogonal to the coherent states $|\alpha; 0 >$

$$< \alpha^*; 0|\tilde{\beta}; \infty >= \delta(\alpha - \beta). \tag{26}$$

This relation shows also that the coherent states are already complete on paths through the complex plane.

More details and references can be found in [1].

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NON-GAUGE PHASE TRANSFORMATIONS
IN QUANTUM TRANSITION AMPLITUDES

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Abstract

The prescription for introducing a gauge transformation into a quantum transition amplitude, nominally well known, contains an ambiguous feature. It is presumed by some authors that an appropriate transformation of the phase of a wave function will generate the associated gauge transformation. It is shown that this is a necessary but not sufficient step. Examples from the literature are cited to show the consequences of the failure of this procedure. One must distinguish between true gauge transformations and unitary transformations within a fixed gauge.

1. Introduction

The necessary procedure to introduce a change of gauge in quantum mechanics is quite standard [1,2]. (We adopt the terminology that the phrase "gauge transformation" implies the so-called "gauge transformation of the second kind" [1].) This quantum-mechanical procedure begins with a change in the potentials employed to represent an electromagnetic field, and then associates with these altered potentials a changed interaction Hamiltonian and a particular phase transformation of the wave function. Some practitioners presume the inverse: that the phase transformation of the wave function will always imply that a gauge transformation has been done. It is the aim of this paper to show that this inverse procedure does not necessarily produce a gauge transformation, and that significant misinterpretations can occur thereby.

When a non-gauge-changing unitary transformation (a "phase transformation") is presumed to actually produce a gauge transformation, it may not have practical ill consequences. In some cases, it simply induces an identity transformation in the

transition amplitude. The outcome is less benign when the non-gauge phase transformation is interpreted as a gauge transformation, and used to infer further physical conclusions. For example, this has led to the concept that one particular gauge is more fundamental than others. These difficulties are discussed in Sec. 3 after a review of basic information in Sec. 2.

2. Formal Background

The approach taken here is that of semi-classical electrodynamics. Quantization of the field is not necessary for present purposes. Both relativistic and non-relativistic formulations will be used; relativistic because matters are clearer in that context, and non-relativistic because that is where the difficulties have actually occurred. It is presumed throughout that the fields and the gauge-transformation functions are explicitly time dependent.

A gauge transformation of the electromagnetic four-vector potential A^μ by the real, scalar generating function Λ is accomplished by

$$A^\mu \to A^{G\mu} = A^\mu - \partial^\mu \Lambda \tag{2.1}$$

or the non-relativistic equivalent

$$\phi \to \phi^G = \phi - \partial\Lambda/\partial(ct) \tag{2.2}$$

$$\vec{A} \to \vec{A}^G = \vec{A} + \vec{\nabla}\Lambda, \tag{2.3}$$

where $A^\mu = (\phi, \vec{A})$. This is accompanied in quantum mechanics by a change in the phase of the wave function induced by the unitary transformation

$$\Psi \to \Psi^G = U\Psi, \tag{2.4}$$

with

$$U = exp(ie\Lambda). \tag{2.5}$$

When one wishes to change the gauge in which a transition amplitude is expressed, it is necessary to know how the Hamiltonian is transformed. It follows directly from the Schrödinger equation that this transformed Hamiltonian operator is given by

$$H^G - i\hbar\partial/\partial t = U (H - i\hbar\partial/\partial t) U^\dagger, \tag{2.6}$$

or, equivalently, by [3,4]

$$H^G = UHU^\dagger - i\hbar U\dot{U}^\dagger = UHU^\dagger + i\hbar\dot{U}U^\dagger, \tag{2.7}$$

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where the dot on the U indicates the time derivative. The Dirac analog of this Schrödinger result is instructively simple. From the Dirac equation

$$(i\partial - e\cancel{A} - m)\Psi = 0,\qquad(2.8)$$

one obtains

$$U(i\partial - e\cancel{A} - m)U^{\dagger}U\Psi = (i\partial - e\cancel{A}^{G} - m)\Psi^{G} = 0,\qquad(2.9)$$

where $\cancel{A} = \gamma^{\mu}A_{\mu}$, and A_{μ}^{G} is given by Eq.(2.1).

The transition amplitude employed will be the generic form

$$(S-1)_{fi} = -(i/\hbar) \int dt\ (\Phi_{f}, H_{I}\Psi_{i}),\qquad(2.10)$$

which is commonplace in scattering theory, but is useful also in bound-state problems. It represents a physical situation in which the transition-inducing electromagnetic field is not present at asymptotic times, i.e., there is no field present at large negative times when the initial state is prepared and at large positive times when final measurement of the transition products is made. The state Φ is one with no electromagnetic field present. Its Hamiltonian will be called H_0. The state Ψ satisfies the Schrödinger equation with full interaction. In other words,

$$(i\hbar\partial_{t} - H_{0})\Phi = 0,\qquad(2.11)$$

$$(i\hbar\partial_{t} - H)\Psi = 0,\qquad(2.12)$$

$$H = H_{0} + H_{I}.\qquad(2.13)$$

For the usual problem in which an atomic or molecular potential V is present at asymptotic times, as distinct from the electromagnetic field whose application causes transitions, one can state

$$H_{0} = (p^{2}/2m) + V,\quad H_{I} = (1/2m)(e\vec{p}\cdot\vec{A}/c + e\vec{A}\cdot\vec{p}/c + e^{2}A^{2}) + e\phi\qquad(2.14)$$

in an arbitrary gauge, where no stipulation has been made as to how the field is to be represented by scalar or vector potentials, or a combination of both. To be as straightforward as possible in this formalism, it is required that the field is to be turned on and off adiabatically, so that one can require the vector potential at both positive and negative asymptotic times to be the same (nominally zero). This restriction is known [3-8] to assure that the same physical result will arise from the transition amplitude in Eq.(2.10) in different gauges, but with the use of exactly the same non-interacting wave function Φ_{f}, regardless of the choice of gauge for H_{I} and Ψ_{i}.

Finally, the relativistic transition amplitude analogous to

Eq. (2.10) is stated [9]

$$(S-1)_{fi} = -(i/\hbar) \int d^4x \; \bar{\Phi}_f e\!\!\not{A}\Psi_i, \qquad (2.15)$$

where $\bar{\Phi}$ is the Dirac adjoint $\bar{\Phi} \equiv \Phi^\dagger \gamma^0$, and a static binding potential is singled out, so that the non-interacting and interacting Dirac equations are, respectively,

$$(i\partial\!\!\!/ - \gamma^0 V - m)\Psi = 0 \qquad (2.16)$$

$$(i\partial\!\!\!/ - e\!\!\not{A} - \gamma^0 V - m)\Psi = 0. \qquad (2.17)$$

3. Statement of the Problem

Whereas there is really no ambiguity in the information reviewed in Sec. 2, the way in which it is employed in the literature is not uniform. A simple unifying concept which serves to characterize the inconsistencies which arise is to note that they all stem from the improper notion that a gauge-change-like unitary transformation applied to the wave function is a guarantee that a gauge change has actually occurred.

Possibly the simplest example of this problem occurred in connection with the demonstration [10,11] that the substitution $\Psi \approx U\Phi$ in Eq. (2.10) (for a particular choice of U) can give a good approximation for certain classes of transitions in which dressing by a low frequency field is present. The result of this approximation is that Eq. (2.10) becomes

$$(S-1)_{fi} = -(i/\hbar) \int dt \; (\Phi_f, H_I U \Phi_i). \qquad (3.1)$$

This has, however, been characterized as a gauge transformation [12] solely on the grounds of the presence of the unitary factor U, even though there is no transformation at all of the interaction Hamiltonian H_I.

Another example is a procedure intended to change the gauge in which a transition amplitude is expressed in a fashion which is purported to be "manifestly gauge invariant". The device employed is simply to insert a unit operator into the transition amplitude in the form of $U^\dagger U$. Then the U factors are attached to the wave functions, and a gauge transformation is presumed to be accomplished. (A clear example of this is in Ref. 13.) Equation (2.10) would then become

$$(S-1)_{fi} = -(i/\hbar) \int dt \; (\Phi_f, H_I U^\dagger U \Psi_i)$$
$$= -(i/\hbar) \int dt \; ((U\Phi_f), (UH_I U^\dagger)(U\Psi_i)). \qquad (3.2)$$

Since the wave functions now bear the unitary transformation factors U as in Eq. (2.4), they are regarded by some authors as being in a new gauge.

There are several defects with the above procedure. One is

the fact that the non-interacting wave function Φ is transformed as well as the interacting wave function Ψ. This fact has been noted by some authors, and concluded to be necessary [13-18]. A corollary of this procedure is that there then exists a preferred gauge, since only in one gauge is it possible to have the non-interacting wave function appear without its unitary transformation factor. The preferred gauge normally selected is the so-called "length gauge", or "EF" gauge, where the dipole-approximation interaction Hamiltonian is $H_I = -e\vec{E} \cdot \vec{r}$. For example, the statement is made that [14] "... the textbook wave functions can, in general, only be applied in the $\vec{E} \cdot \vec{r}$ formalism ...". The presumed necessity to apply a field-dependent gauge transformation factor to represent a non-interacting state in any gauge other than the length gauge has been termed an oxymoron [19].

Another problem with the procedure expressed in Eq.(3.2) is that the interaction Hamiltonian is not properly stated. The true gauge-transformed interaction Hamiltonian follows from Eq.(2.6) or (2.7), taken together with Eq.(2.13). By contrast, the form

$$H_I^P = U H_I U^\dagger \qquad (3.3)$$

is simply a unitary (or phase) transformation of the operator H_I. It is not the gauge-transformed interaction Hamiltonian. The actual gauge-transformed interaction Hamiltonian is given by

$$H_I^G = U H_I U^\dagger + (U H_0 U^\dagger - H_0) + i\hbar \dot{U} U^\dagger. \qquad (3.4)$$

The clearest way to see the true meaning of Eq.(3.2) is to employ the relativistic form given in Eq.(2.15). The lack of second order differential operators in the Dirac equation and the simple form $e\mathcal{A}$ for the interaction term makes the relativistic form especially clear for formal purposes. The procedure analogous to Eq.(3.2) employed in Eq.(2.15) gives

$$(S-1)_{fi} = -(i/\hbar) \int d^4x \ \overline{\Psi}_f e\mathcal{A} U^\dagger U \Psi_i$$
$$= -(i/\hbar) \int d^4x \ (\overline{U\Phi}_f) e\mathcal{A} (U\Psi_i), \qquad (3.5)$$

since U always commutes with $e\mathcal{A}$. Equation (3.5) shows plainly that there is no gauge transformation at all. The interaction term remains identically the same as the original, and does not transform to the new gauge as would follow from Eq.(2.1).

The procedure in Eq.(3.5), as in Eq.(3.2), is simply a unitary transformation within a fixed gauge.

4. Resolution of the Problem

The resolution of the ambiguities discussed above is straightforward. One simply states a transition amplitude in an unspecified gauge, containing all four components of the electromagnetic potential function, as given in, for example,

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Eq.(2.15). In a particular gauge, designated by the superscript (a), this is

$$(S-1)_{fi} = -(i/\hbar) \int d^4x \; \overline{\Phi}_f e\Lambda^{(a)} \Psi_i^{(a)}. \tag{4.1}$$

In gauge (b), it is

$$(S-1)_{fi} = -(i/\hbar) \int d^4x \; \overline{\Phi}_f e\Lambda^{(b)} \Psi_i^{(b)}. \tag{4.2}$$

The non-interacting state Φ_f is the same in both instances since it is independent of the field. This is the type of transparent gauge invariance that has also been given the name "manifest gauge invariance" [19], although that description is risky, since the same phrase means different things to different researchers. A better name would be "strong gauge invariance", since it so strongly stresses the complete equivalence of all gauges.

There is no clear algebraic transformation that connects Eq.(4.1) with (4.2). Nevertheless, they must be equivalent if all gauges are equally valid. This has been shown by calculation of practical examples [6,7] as well as by the demonstration [3] that the formal difference between the expressions which are the non-relativistic analogs of Eqs.(4.1) and (4.2) has a null result.

The mis-identification of the simple phase transformation in Eq.(3.2) or (3.4) as a gauge transformation follows from an attempt to achieve algebraic identity between transition amplitudes in different gauges. What is achieved instead is simply a unitary transformation within a fixed gauge.

Another motivation for employing Eq.(3.3) as a gauge-transformed interaction Hamiltonian in place of Eq.(3.4) makes use of arguments [14,15] involving dependence on the dipole approximation and on the preferred use of the $\vec{E}\cdot\vec{r}$ interaction. Such arguments are inherently risky. One cannot view as fundamental a formalism which depends critically on an interaction which cannot extend to very strong fields or to the presence of significant magnetic influences.

5. Acknowledgment

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COLLISION-INDUCED SQUEEZING IN A HARMONIC OSCILLATOR

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The concept of squeezing has so far been applied mainly to light, as is evidenced by numerous research works on the subject of squeezed light. Since in quantum mechanics both light and the simple harmonic oscillator are described within the same mathematical framework, there is of course no difficulty in applying the concept to the simple harmonic oscillator as well. In fact, theoretical development of squeezed states and squeezed light owes much to physical insights one obtains as the analogy between light and the harmonic oscillator is exploited [1, 2]. There, however, exist only a few publications [3, 4, 5] that deal directly with generation of squeezing in a harmonic oscillator. Since the two quadrature operators for a simple harmonic oscillator carry the physical meaning of position and momentum operators apart from constants, a squeezed oscillator, i.e., a simple harmonic oscillator in a squeezed state, exhibits squeezing in actual position or momentum. Thus, a squeezed oscillator once generated can play an important role in atomic or molecular experiments that require precise initial determination of the position or momentum of the particles involved.

In our previous work [5], it was shown that squeezing can be generated in a harmonic oscillator by subjecting it to collisional interaction. The model chosen for this study is one-dimensional collision between a helium atom, taken as a structureless particle, and a hydrogen molecule, approximated as a simple harmonic oscillator. The harmonic oscillator was assumed to be prepared in its ground state before the collision. Thus,

$$|\psi(t=0)> = |0>, \tag{1}$$

and the initial quadrature variances are given by

$$(\Delta X_1)^2 = (\Delta X_2)^2 = \frac{1}{4}. \tag{2}$$

As the collision proceeds, the oscillator develops into a superposition state,

$$|\psi(t)> = \sum_n a_n(t)|n> = \sum_n |a_n(t)|e^{i\phi_n(t)}|n>. \tag{3}$$

The quadrature variances at time t are then given by [6]

$$(\Delta X_1)^2 = \frac{1}{4} + \sum_n n|a_n|^2 + \frac{1}{2}\sum_n \sqrt{n+2}\sqrt{n+1}|a_n||a_{n+2}|cos(\phi_{n+2} - \phi_n)$$
$$- [\sum_n \sqrt{n+1}|a_n||a_{n+1}|cos(\phi_{n+1} - \phi_n)]^2, \tag{4}$$

and a similar expression for $(\Delta X_2)^2$. Wodkiewicz et. al. [7] have shown that a superposition state consisting of a finite number of eigenstates $|n >$ can exhibit squeezing for appropriate values of the magnitudes $|a_n|$ and phases ϕ_n of probability amplitudes, and thus there is a possibility of squeezing in the collision state given by Eq.(3). Our calculations, as reported earlier [5], show that there occurs a relatively strong squeezing near the time of minimum separation and a weak squeezing altanately in position and momentum after the collision is over.

It should be noted that, in most of the collision studies in the past, attention was focused on the magnitudes $|a_n|$ of the probability amplitudes as they yield the transition probabilities. For our study of collision-induced squeezing, however, the question of how the phases develop in time as the collision proceeds is also an important issue, because the variances $(\Delta X_1)^2$ and $(\Delta X_2)^2$ depend not only on the magnitudes $|a_n|$ but also on the phases ϕ_n, as can be seen from Eq.(4). Even if the magnitudes $|a_n|$ are fixed, the variances can take on different values for different phases ϕ_n.

In order to emphasize the importance of the phases, we show below that squeezing can be achieved from a coherent state simply by changing the phases alone. Let us consider a harmonic oscillator in a coherent state $|\alpha >$ at time $t = 0$. If we let the oscillator develop freely in time, its state at time t is given by

$$|\psi(t) >= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-in\omega t} |n > .$$
(5)

The variances $(\Delta X_1)^2$ and $(\Delta X_2)^2$ remain $\frac{1}{4}$ throughout. Let us now assume that the phases of the coherent state are changed at time $t = 0$ so that the oscillator develops in time according to

$$|\psi(t) >= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{i\theta_n} e^{-in\omega t} |n > .$$
(6)

As compared with the coherent state, Eq(5), the state represented by Eq.(6) has additional constant phase factors θ_n. Althogh this state is not identical with the coherent state, it has the same Poissonian state distribution as the coherent state and may thus be called a "quasi-coherent state". It is our purpose to show that, with appropriate values of θ_n, the quasi-coherent state can show squeezing in X_1 or X_2. To illustrate this, let

$$\theta_n = \begin{cases} 0, & \text{if n is even,} \\ -\frac{\pi}{2} & \text{if n is odd.} \end{cases}$$
(7)

The state represented by Eqs. (6) and (7) are a linear combination of even and odd coherent states [8] with the relative phase between the even and odd states given by $\frac{\pi}{2}$. The variance $(\Delta X_1)^2$ for this state can easily be computed using Eq.(4), and similarly $(\Delta X_2)^2$. The result of the calculation is

$$(\Delta X_1)^2 = \frac{1}{4} + |\alpha|^2 - |\alpha|^2 \sin^2(\phi - \omega t) - |\alpha|^2 e^{-4|\alpha|^2} \sin^2(\phi - \omega t),$$
(8)

$$(\Delta X_2)^2 = \frac{1}{4} + |\alpha|^2 - |\alpha|^2 \cos^2(\phi - \omega t) - |\alpha|^2 e^{-4|\alpha|^2} \cos^2(\phi - \omega t).$$
(9)

The variances oscillate between v_{max} and v_{min} where,

$$v_{max} = \frac{1}{4} + |\alpha|^2, \quad v_{min} = \frac{1}{4} - |\alpha|^2 e^{-4|\alpha|^2}$$
(10)

It is evident that the quasi-coherent state with the phases given by Eq.(7) exhibits squeezing because $v_{min} < \frac{1}{4}$.

The example presented above shows clearly that two states with different phases in general have different degrees of squeezing, even if they have the same state distribution. This means that, even if one considers collision processes that produce the same state distribution, the degree of squeezing obtained during and after the collisions can be quite different, depending on how the phases ϕ_n of the probability amplitudes develop in time as the collisions proceed. It is therefore evident that, for a detailed study of collision-induced squeezing, further study on the time development of the phases in collisions and its relation to collision parameters such as potential energy surfaces and collision energy is needed.

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POSITIVE PHASE SPACE DISTRIBUTIONS AND UNCERTAINTY RELATIONS

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Abstract

In contradistinction to a widespread belief, Wigner's theorem allows the construction of true joint probabilities in phase space for distributions describing the object system as well as for distributions depending on the measurement apparatus. The fundamental role of Heisenberg's uncertainty relations in *Schrödinger* form (including correlations) is pointed out for these two possible interpretations of joint probability distributions. E.g., in order that a multivariate normal probability distribution in phase space may correspond to a Wigner distribution of a pure or a mixed state, it is necessary and *sufficient* that Heisenberg's uncertainty relation in Schrödinger form should be satisfied.

1 Introduction

Joint measurements of conjugate variables q and p are realized in many optical devices. This implies that one can think in this domain of a representation of quantum mechanics by means of joint probability distributions (j.p.d.) in the phase space of conjugate variables q and p [1]. This is perhaps the most convenient way to a realistic underpinning of quantum mechanics. A major advantage is that the incompatible variables q and p are c-numbers. The Wigner distribution function, which is widely used in optics, is the simplest language for coherent and squeezed states [2]. For these states the Wigner function is nonnegative. However, it is well known that the Wigner distribution cannot be considered as a true (nonnegative) probability distribution in general [3]. The aim of this paper is twofold: in the first part (sections 2 and 3) we present an analysis of the central question to consider phase space representations of quantum mechanics as *true* (nonnegative) probability distributions [4, 5] ; in the second part (sections 4 and 5) we emphasize the fundamental role of *Heisenberg's* uncertainty relations in *Schrödinger* form for Gaussian Wigner distributions and compare this with j.p.d. depending on the measurement arrangement (positive operator valued measures).

2 Wigner's theorem

On account of the commutation relations between the operators $\hat{q}$ and $\hat{p}$, there is *no* unique operator corresponding to the monomial $q^n p^m$. As a consequence there is *no* unique construction

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of the j.p.d.. In general, a j.p.d. is completely determined by a given correspondence rule. Notwithstanding this arbitrareness, the existence of true probabilities in phase space is severely restricted by Wigner's theorem [3], which considers the following five requirements:

(1) The j.p.d. is the mean value of an hermitian operator $\hat{K}(q,p)$ depending on the c-numbers q and p: $f(q,p) = \text{tr}[\hat{K}(q,p)\hat{\rho}]$.

(2) The j.p.d. is a linear functional of the density matrix (sesquilinear in the wavefunction): this means that $\hat{K}(q,p)$ is independent of $\hat{\rho}$.

(3) The j.p.d. is a true probability function: $f(q,p) \geq 0$.

(4) When integrating over momentum space, the marginal distributions coincide with the proper quantummechanical probabilities in q: $\int f(q,p)\mathrm{d}p = < q \mid \hat{\rho} \mid q >$.

(5) When integrating over position, the marginal distributions coincide with the proper quantummechanical probabilities in p: $\int f(q,p)\mathrm{d}q = < p \mid \hat{\rho} \mid p >$.

Theorem 1 *The five requirements (1)-(5) are incompatible.*

The requirement (2) is not explicitly present in the original version of Wigner's theorem; the necessity of this requirement was emphasized by Mügür-Schlächter [6], who observed that in the absence of the arbitrary restriction (2) Wigner's theorem cannot be realized. In the stronger version of Kruszynski and de Muynck [7] the requirement on *one* marginal distribution suffices.

3 Realisation of positive phase space distributions

For our purpose, it is sufficient to consider two different interpretations of j.p.d. as functionals of the density matrix.

(1) The j.p.d. $f(q,p)$ is interpreted as the probability that the variables q and p have certain values, the variable considered as a property possessed by the object system. In this case, two possibilities are left open for the construction of true j.p.d.:

(1.1) $f(q,p)$ *is a linear functional of* $\hat{\rho}$.

In this case the requirements (1)-(5) are only compatible with a restricted class of functions. E.g. for the Weyl correspondence rule, the restricted class of functions are Gaussons (see section 4). The Wigner distribution cannot be considered as a true probability distribution in general, because e.g. it takes necessarily negative values for pure states that are not Gaussons. However, one can easily construct positive non-Gaussian Wigner j.p.d. corresponding to *mixed* states. For a representation of quantum mechanics by means of true Wigner j.p.d. one can add the new requirement that only nonnegative j.p.d. are physical states . This means e.g. that a one photon state is represented by a mixed state [10]. This idea is made plausible by the experimental fact that it is impossible to prepare a pure state with 100 % efficiency.

(1.2) $f(q,p)$ *is a nonlinear functional of* $\hat{\rho}$.

J.p.d. which are a nonlinear functional of the density matrix are not restricted by Wigner's theorem. The j.p.d. which is the product of the proper quantum mechanical marginal distributions is a trivial example: $f(q,p) = < q \mid \hat{\rho} \mid q >< p \mid \hat{\rho} \mid p >$. Non-trivial examples with correlations exist also in the literature [11].In this case the j.p.d. is a *multilinear* functional of the density matrix. We have considered a complete analysis of true distributions which are *quadratic* functional

of the density matrix [5]. This results in a new concept of j.p.d. which is based on a consistent phase space interpretation of the energy eigenstates of the wave function.

(2) The j.p.d. $f(q,p)$ is not function of the object system alone, but may also depend on the measurement arrangement of two incompatible observables Q and P. The measurements mutually influence each other in such a way that the singly measured quantum probability functions cannot be reproduced from the measurement results. In this case it is no longer desirable that the marginal probability distributions equal the single measured ones, hence Wigner's theorem does not restrict this class of j.p.d. and $f(q,p)$ may be a linear function of the density matrix. The optimal stochastic phase-space representations introduced by Prugovečki [12] are an example of this class. In general the distributions of class (2) can be considered in the framework of positive operator valued measures [13].

4 Heisenberg's uncertainty relation in Schrödinger form and coherent and squeezed Wigner distributions

We consider case (1.1) for the Weyl correspondence rule. In this case the construction of true j.p.d. for pure states is restricted by the remarkable and important theorem which was proven by Hudson [8] for one-dimensional systems and generalized by Soto and Claverie [9] for systems with an arbitrary number of degrees of freedom.

Theorem 2 *The necessary and sufficient condition for the Wigner distribution function of a pure state to be nonnegative is that the corresponding wave function $< q|\psi >$ is the exponential of a quadratic form.*

As a consequence the wave function represents a coherent or a squeezed state and the j.p.d. is a bivariate or a multivariate normal (Gaussian) distribution in phase space. Conversely, in two-dimensional phase space of the conjugate random variables q and p the most general normalised bivariate normal probability distribution with mean values $\bar{q}$ and $\bar{p}$ can be put in the standard form

$$f(q,p) = \frac{1}{2\pi\sqrt{\Delta}} \exp\left\{ -\frac{1}{2\Delta} \left[\sigma_p(q-\bar{q})^2 - 2\sigma_{q,p}(q-\bar{q})(p-\bar{p}) + \sigma_q(p-\bar{p})^2 \right] \right\}, \qquad (1)$$

where σ_q and σ_p and $\sigma_{q,p}$ represent respectively the variances and the covariance $\sigma_q = E[(q-\bar{q})^2]$, etc.; E denotes the expectation value and Δ is the determinant of the covariance matrix: $\Delta = \sigma_q\sigma_p - \sigma_{q,p}^2 \geq 0$. Schrödinger derived a more general and stronger form of Heisenberg's uncertainty relation including the correlation $\sigma_{q,p}$:

$$\sigma_q\sigma_p - \sigma_{q,p}^2 \geq \hbar^2/4, \qquad (2)$$

which we call "Heisenberg's uncertainty relation in *Schrödinger* form". It is easy to derive and to diagonalise the corresponding density matrix. $f(q,p)$ may now represent a pure *or* a mixed state. The eigenfunctions $< q|\psi >$ are oscillator eigenfunctions functions multiplied by a common q-dependent phase factor which is characteristic for the correlation. We can show explicitly that there is a close connection between a Gaussian distribution in phase space, quadratic Hamiltonians and temperature dependent oscillator states. This implies a connection between physical and

statistical parameters. The eigenvalues of the corresponding density matrix are $(1 - z)z^n$ with $z = (\Delta - \hbar/2)/(\Delta + \hbar/2)$, which leads to a *sufficient* condition for a bivariate normal probability distribution to be a quantum state:

Theorem 3 *In order that a bivariate normal probability distribution in phase space with variances σ_q, σ_p and covariance $\sigma_{q,p}$ may correspond to a Wigner distribution of a pure or a mixed state, it is necessary and sufficient that Heisenberg's uncertainty relation in Schrödinger form $\sigma_q \sigma_p - \sigma_{q,p}^2 \geq \hbar^2/4$ should be satisfied [4, 14].*

It is very remarkable that the Schrödinger form of Heisenberg's uncertainty relation, which is a necessary condition to be fulfilled for every Wigner distribution function, is also a *sufficient* condition in the case of a bivariate normal probability distribution. Indeed, to be a Wigner distribution function, $f(q,p)$ must satisfy an infinite set of KLM [15] or equivalent conditions in general, but for the two-dimensional Gaussian distribution the infinite set reduces to *one* simple *necessary and sufficient* physical condition. In this respect, the uncertainty relation in Schrödinger form is *more fundamental* than Heisenberg's relation in the usual, less stronger form $\sigma_q \sigma_p \geq \hbar^2/4$. Moreover, the Schrödinger form is *invariant* for linear canonical transformations (in general $Sp(2n, R)$ invariant transformations), while the usual form is not. Finally, for quadratic Hamiltonians, which are closely related to the Gaussian Wigner distribution, the Schrödinger form remains *invariant* during the motion if the variances and the covariance are dependent on time. Indeed, in this case the quantum Liouville equation is equivalent to the classical Liouville equation and therefore $\bar{q}$, $\bar{p}, \sigma_q, \sigma_p$ and $\sigma_{q,p}$ have the same time dependence as in the classical case. These are further reasons why the uncertainty relation in Schrödinger form is *more relevant* than Heisenberg's relation in the usual form.

For systems with an arbitrary number of degrees of freedom the strong form of Heisenberg's uncertainty relation is derived from the inequality $\mathrm{tr}(a^\dagger \hat{\rho} a) \geq 0$ where the vector a is given by $a = A(\hat{q} - \bar{q}) + B(\hat{p} - \bar{p})$, A and B being arbitrary matrices, and which takes the form:

$$\begin{vmatrix} A^\dagger \\ B^\dagger \end{vmatrix} \begin{vmatrix} \sigma_{q,q} & \sigma_{q,p} - \imath\hbar/2 \\ \sigma_{p,q} - \imath\hbar/2 & \sigma_{p,p} \end{vmatrix} \begin{vmatrix} A \\ B \end{vmatrix} \geq 0. \tag{3}$$

Therefore Heisenberg's uncertainty relation in Schrödinger form takes now the matrix form:

$$\sigma - \imath\hbar\beta/2 \geq 0. \tag{4}$$

where σ is the covariance matrix $\begin{vmatrix} \sigma_{q,q} & \sigma_{q,p} \\ \sigma_{p,q} & \sigma_{p,p} \end{vmatrix}$ and β the fundamental symplectic matrix $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$.

Theorem 4 *The necessary and sufficient conditions for a Gaussian phase space function to be a Wigner distribution is that the covariance matrix σ satisfies Heisenberg's uncertainty relation in Schrödinger form: $\sigma - \imath\hbar\beta/2 \geq 0$ [4].*

Analogous remarks as for the bivariate j.p.d. are valid for the multivariate j.p.d., the eq. 4 is now $Sp(2n, R)$ invariant. The theorem entails a considerable simplification with respect to the theorem of Simon, Sudarshan and Mukanda [17], where $Sp(2n, R)$ invariant powers of $\beta\sigma^{-1}$ satisfy n complicated inequalities. The difference between a pure and a mixed state is given by a theorem of Littlejohn [16]:

Theorem 5 *The necessary and sufficient condition for a Gaussian Wigner distribution to be a pure state is that the matrix $2\sigma/\hbar$ is a symplectic matrix: $\sigma\beta\sigma = (\hbar^2/4)\beta$.*

In two dimensions the matrix relation reduces to $\sigma_q \sigma_p - \sigma_{q,p}^2 = \hbar^2/4$.

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5 Heisenberg's uncertainty relation in Schrödinger form for j.p.d. depending on the measurement arrangement

It was argued in section 3 that the construction of j.p.d. of class (2) is not restricted by Wigner's theorem. Requiring Galilei invariance, linearity and positivity for any density matrix describing the object system, we have for the most general form of the j.p.d.:

$$f(q,p) = \hbar^{-n} \text{tr}(\hat{D}_{q,p}^{\dagger} \hat{\rho}_{meas} \hat{D}_{q,p} \hat{\rho}_{obj})$$ (5)

where $\hat{\rho}_{meas}$ and $\hat{\rho}_{obj}$ are the density matrices describing exhaustively the measurement apparatus and the object system and $\hat{D}_{q,p}$ represents the displacement operator. If both $\hat{\rho}_{meas}$ and $\hat{\rho}_{obj}$ are pure states then $f(q,p)$ reduces to the transition probability $f(q,p) = \hbar^{-n} \text{tr}(\psi_{meas}, \hat{D}_{q,p} \psi_{obj})$. The marginal distributions are always given by the convolution of two true probability densities:

$$\int f(q,p) \mathrm{d}p = <q \mid \hat{\rho}_{meas} \mid q> \star <q \mid \hat{\rho}_{obj} \mid q>,$$ (6)

$$\int f(q,p) \mathrm{d}q = <p \mid \hat{\rho}_{meas} \mid p> \star <p \mid \hat{\rho}_{obj} \mid p>,$$ (7)

which can be seen as accuracy calibrations given by the confidence functions $<q \mid \hat{\rho}_{meas} \mid q>$ and $<p \mid \hat{\rho}_{meas} \mid p>$. The couple $q, <q \mid \hat{\rho}_{meas} \mid q>$ together with $p, <p \mid \hat{\rho}_{meas} \mid p>$ can also be interpreted as a fuzzy sample point in phase space [12]. Remark also that, for these j.p.d. the ordening of operators is equivalent with a measuring procedure. One can also write the j.p.d. as a convolution of two Wigner distributions:

$$f(q,p) = f_{meas}(q,p) \star f_{obj}(q,p),$$ (8)

the first one representing the measurement procedure and the second one describing the object system. This "smoothing" or "coarse graining" of the Wigner distribution eleminates fast oscillations in $\hbar$ and gives therefore a better representation in the classical limit [18]. Another consequence of the last formula is that the covariance matrix σ is the sum of the covariance matrix σ_{obj} of the object system and the σ_{meas} of the measurement procedure. Hence we obtain the "operational" uncertainty relation

$$\sigma - i\hbar\beta \geq 0.$$ (9)

which reduces in one dimension to $\sigma_q \sigma_p - \sigma_{q,p}^2 \geq \hbar^2$. This operational uncertainty relation is in accordance with the experimental uncertainty relation $(\Delta q)_{ex} (\Delta p)_{ex} \sim \hbar$ [19]. Comparing this with the uncertainty relations for the j.p.d. of the preceding section, we observe that the inequalities are the same, except for the essential difference that $\hbar$ replaced by $2\hbar$, expressing the presence of extra noise due to the measurement procedure.

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SQUEEZED COLOUR STATES IN GLUON JET

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Abstract

The possibility of formation of squeezed states of gluon fields in quantum chromodynamics due to nonlinear nonperturbative selfinteraction during jet evolution in the process of e^+e^- annihilation into hadrons, which are analogous to the quantum foton squeezed states in quantum electrodynamics is demonstrated and the squeezing parameters are calculated.

1 Evolution equation for gluon field

The gluon part of the quantum chromodynamics Hamiltonian has the form [1]

$$H_g = H^0 + H_{int} = \int \{ \tfrac{1}{2}(\vec{E}_a\vec{E}_a + \vec{B}_a\vec{B}_a) - \vec{E}_a g C_{abc}\vec{A}_b\vec{A}_c +$$

$$\tfrac{1}{2} g \vec{B}_a C_{abc}\vec{A}_b \times \vec{A}_c + \tfrac{1}{2}g^2(C_{abc}\vec{A}_b\vec{A}_c^0)^2 + \tfrac{1}{2}g^2(\tfrac{1}{2}C_{abc}\vec{A}_b \times \vec{A}_c)^2 \} d^3x \quad (1)$$

where $\vec{E}_a = -\vec{\nabla}A_a^0 - \partial_t\vec{A}_a, \vec{B}_a = \vec{\nabla} \times \vec{A}_a, \vec{A}_a$—vector potential of gluon field, C_{abc}—structure constants of the SU(3), $a, b, c, = 1, \ldots, 8$ are colour indices; i, j, k, l—indices of 3-vectors.

The field of gluons appears in the form of gluon jet or cascade, which is produced by the quark with large transfered momentum. Due to the cubic and quadratic nonlinearities in (1) bremstrahlung gluons divide and at the end of perturbative cascade we have a jet of gluons with approximately equal energies and momenta [2].

At the end of cascade multiplicity distribution of gluons is close to negative binomial distribution [3, 4] which can be considered as a set of Poisson (coherent) distributions.

The importance of nonperturbative hadronisation stage is connected with confinement, subpoisson multiplicity distributions at this stage [5, 6], connection with intermittency [7], pairing of partons during colour loosing, nonlinearities of (1) hint on the possibilities of squeesed gluon states.

Let us take for simplicity that all gluons in jet have equal energies and momenta. Choose such the system of coordinates that has axe x_1 coinciding with the direction of gluon momentum. Then in the momentum representation the operator of gluon selfinteraction takes the form

$$V = \frac{1}{2}g^2 C_{abc}C_{adb}\left[\left(2 - \frac{m_f^2}{k_0^2}\right) A_{1212}^{body} + \left(2 - \frac{m_f^2}{k_0^2}\right) A_{1313}^{body} + A_{2323}^{body}\right], \quad (2)$$

$$A_{i,j,i,j}^{b,c,d,f} = A_i^{+b}A_j^{+c}A_i^d A_j^f + A_i^{+b}A_j^c A_i^{+d}A_j^f + A_i^{+b}A_j^c A_i^d A_j^{+f} + (c.c.), \tag{3}$$

where $A_i^b(A_i^{+b})$ are annihilation (production) operators of gluons with b—colour and i—vector component, k_0 and m_0 are gluon energy and mass: $k_0^2 - \vec{k}^2 = m_g^2$.

Evolution equation for gluon operator with indices a and k

$$i\partial_t A_k^a = [A_k^a, H] \tag{4}$$

then takes the form

$$i\partial_t A_k^a = f_1 A_m^a + f_2 A_k^{+a} + f \tag{5}$$

The function f_1, f_2, f do not contain explicitly A_k^a and A_k^{+a}, $f_1 = f_1^{a+}$ [9].

2 Squeezing of the gluon field in jet

Let us solve the equation (5) for small time $\Delta t << 1/E, E = \sqrt{f_1^2 - |f_2|^2}$. Then the solution of (5), is written in the matrix form

$$\partial_t \begin{pmatrix} A_k^a(t) \\ A_k^{+a}(t) \end{pmatrix} = \frac{1}{i}\begin{pmatrix} f_1 & f_2 \\ -f_2^+ & f_1 \end{pmatrix}\begin{pmatrix} A_k^a(t) \\ A_k^{+a}(t) \end{pmatrix} + \frac{1}{i}\begin{pmatrix} f \\ f^+ \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} A_k^a(t) \\ A_k^{+a}(t) \end{pmatrix} = \exp\left[-i\int_0^{\Delta t}\begin{pmatrix} f_1 & f_2 \\ -f_2^+ & f_1 \end{pmatrix} dt'\right] \times$$

$$\left\{\begin{pmatrix} A_k^a(0) \\ A_k^{+a}(0) \end{pmatrix} + \frac{1}{i}\int_0^{\Delta t}\exp\left[i\begin{pmatrix} f_1 & f_2 \\ -f_2^+ & f_1 \end{pmatrix} dt''\right]\begin{pmatrix} f \\ -f^+ \end{pmatrix} dt'\right\} \tag{7}$$

Let us take at some moment $t_0 = 0$ the conditions: $f = 0, \Im f_2 = 0$, and that f_1 and f_2 vary slowly. Then the solution takes the form

$$A_k^a(t) = A_k^a(0) - iA_k^a(0)f_1\Delta t - iA_k^{+a}(0)f_2\Delta t. \tag{8}$$

This expression coincides with the expression for ideal squeezed state [8]

$$A_k^a = A_k^A \mathrm{chr} + e^{2i\theta}A_k^{+a}\mathrm{shr} \tag{9}$$

$$\mathrm{chr} = 1 - if_1 t, \qquad \mathrm{shr} = f_2, \qquad e^{2i\theta} = -i \tag{10}$$

where r and θ are squeezing parameters. Thus the selfsqueezing is possible for the gluon field with fixed colour and Lorentz component.

In quantum optics such states are named as pure quantum stated and operators $z_1 = (A - A^+)/2$ and $z_2 = (a = A - A^+)/2$ can have average fluctuations smaller then $1/4$.

3 Evolution of gluon multiplicity distribution in jet

Take vector of state $|n_1, n_2, \ldots, n_a >$ where n_a— the number of gluons with definite indices i and a. The operator of full gluon number $\hat{N}$ acts on the vector as

$$\hat{N}|n_1, n_2, \ldots, n_a >= (n_1 + n_2 + \ldots + n_a)|n_1, n_2, \ldots, n_a > \tag{11}$$

It is clear that any part of H_{int} acts on the vector as

$$|New> = A_i^{+a} A_j^{+b} A_k^c A_l^d |n_1, n_2, \ldots, n_a> = |n_1, n_2, \ldots, n_\xi+1, n_\beta+2, \ldots, n_\sigma-1, n_\gamma-1, \ldots, n_a> \quad (12)$$

It does not change the number of particle

$$\hat{N}|New> = (n_1 + n_2 + \ldots + n_a)|New> \quad (13)$$

and then

$$[\hat{N}, H] = 0. \quad (14)$$

Thus the total number of gluon is jet under made conditions ($\vec{k} = \text{const}, k_0 = \text{const}$) does not change with the time and it is not difficult to see that gluon multiplicity distribution does not change with the time.

It can be also shown that the value squeezing shr for every mode is limited [9]. Foton multiplicity of squeezing states distributions have been used earlier for phenomenological description of some properties of hadron multiplicity distribution [10, 11].

Here we obtain for model gluon jet that the squeezed states of colour gluon field can appear due selfinteraction and nonperturbative mechanism of gluon selfinteraction and can be particularly important at nonperturbative stage of jet evolution.

Due to nonperturbativeness, pairing of gluon and subpoisson multiplicity distributions squeezing states can be responsible partly for hadronisation of colour partons (confinment) and intermittency (fractal dimension) phenomenon in multiparticle processes.

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EXACT AND QUASI-CLASSICAL DENSITY MATRIX AND WIGNER
FUNCTIONS FOR A PARTICLE IN THE BOX AND HALF SPACE.

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The exact expressions for density matrix and Wigner
functions of quantum systems are known only in special cases
and ,practically, all of them and their references are
described in [1-3]. Correspondending Hamiltonians are
quadratic forms of Euclidean coordinates and momenta. In
this paper we consider the problem of one-dimensional free
particle movement in the bounded region $0 < x < a$ (including
the case $a = \infty$). For this problem the solutions of Schrödinger
equation are well known:

$$\psi_n = (2/a)^{1/2} \sin(\pi n x/a), \quad E_n = (\pi n \hbar)^2/(2ma^2), \quad n = 1, 2, 3, \ldots \quad (1)$$

Then the equilibrium density matrix can be calculated by
formula

$$\rho(x, x', \beta) = \sum \psi_n(x) \psi^*(x') \exp(-\beta E_n) \quad (2)$$

Indroducing the expression (1) to (2) and mading some
simple transformations we obtain two series, each of which

is, practicalliy the definition of teta-function [4]

$$\Theta_3(z|\tau)=1+2\sum_{n=1}^{\infty}\cos(2\pi nz)\exp(i\pi\tau n^2) \qquad (3)$$

As result we have the expression

$$\rho(x,x',\beta)=\frac{1}{2}\left[\Theta_3\left(\frac{x-x'}{2a}\middle|\frac{i\pi\hbar^2\beta}{2ma^2}\right)-\Theta_3\left(\frac{x+x'}{2a}\middle|\frac{i\pi\hbar^2\beta}{2ma^2}\right)\right] \qquad (4)$$

The replacement $\beta=it/\hbar$ transforms the expression (4) into the propogator of Schrödinger equation for the particle in a box obtained earlier by various methods in [3,5,7].

Evidently in the limit $\beta/a^2 \to \infty$ (i.e. low temperature and small size of a box) the density matrix can be good approximated only by the first order term of the expansion series (2). The question is in obtaining from the exact (but not very obvious) formula (4) the asymptotics of the density matrix in quasi - classical limit β/a^2 (high temperature and wide box).

The qualitive behaviour of the probability density $\rho(x,x,\beta)$ in this case is clear from physical consideration. It must be almoust constant at all points inside the box except very small region near the wall corresponding to de Brogle wave length. In this region the density matrix must leads to zero. However, it is interesting to obtain this result from the formula (4). More over we would like to know the character of the derlexion uniform distribution inside the box caused by quantum corrections. This problem can be solved using the equality [4] for teta-function

$$\Theta_3(z|\tau)\equiv(i/\tau)^{1/2}\exp(-\pi iz^2/\tau)\Theta_3(-z/\tau|-1/\tau) \qquad (5)$$

Due to the fact that in our case the parameter τ is pure

complex and restricting by the first term of expansion series(3) of the function $\Theta_3(-z/\tau|-1/\tau)$ when $\tau \to 0$ we obtain the following formula describing the quasi-classical behaviour of the density matrix

$$\rho(x,x',\beta) = (m/2\pi\beta\hbar^2)^{1/2}\left\{\exp\left[-m(x-x')^2/2\beta\hbar^2\right]\,(1+\right.$$
$$+\,2\exp\left[-ma^2/\beta\hbar^2\right]ch\left[-2m(x-x')/\beta\hbar^2\right]\}-\exp\left[-m(x+x')^2/2\beta\hbar^2\right]\,(1+$$
$$\left.+\,2\exp\left[-ma^2/\beta\hbar^2\right]ch\left[-2m(x+x')/\beta\hbar^2\right]\right)\right\} \tag{6}$$

This formula is correct in the region $|x \mp x'| \le a$ (i.e. at the left half of the box). For the points outside of this region one have to use the properties folloing from (3) and (4)

$$\rho(x,x') = \rho(x',x), \quad \rho(a-x,a-x') = \rho(x,x')$$

For the diagonal elements of probability density we have following expression

$$\rho(x,x',\beta) = (m/2\pi\beta\hbar^2)^{1/2}\left\{1-\exp\left[-2mx^2/\beta\hbar^2\right]+2\exp\left[-ma^2/\beta\hbar^2\right]\,[1-\right.$$
$$\left.-\,\exp\left[-2mx^2/\beta\hbar^2\right]ch\left[4mx/\beta\hbar^2\right]\right)\right\}, \quad x\le a/2, \quad ma^2/\beta\hbar^2 \gg 1 \tag{8}$$

The first two terms in figure brackets describe the probability density of particle position in the infinite half space right from the wall placed to the point $x=0$. The other terms give corrections caused by the-presence of the second wall. Note, that this corrections don't oscillate as it can be seemed from formulas (3) and (4).

In the centre of the box the density matrix is equal to

$$\rho(a/2,a/2,\beta) = const\,(1-2\exp(-ma^2/2\beta\hbar^2)$$

and the half space case on the same distance from the coordinate centre we have an analogous expression but without two in front of exponent. The exact expression of

statistical sum have the form

$$z(\beta)=\frac{1}{2}\left[\Theta_3\left(0\left|\frac{i\pi\hbar\,\beta^2}{2ma^2}\right.\right)-1\right]$$ (9)

It's quasi-classical expansion is

$$z(\beta)=a(m/2\pi\beta\hbar^2)^{1/2}(1+2\exp\ [-2ma^2/\beta\hbar^2])-1/2, \quad ma^2/\beta\hbar^2 >> 1$$ (10)

From (4) one can obtained the Wigner function

$$W(p,q,\beta)=\int\rho(q+\xi/2,q-\xi/2,\beta)\ \exp(-ip\xi/\hbar)\ d\xi$$ (11)

Taking into consideration that the integration region is bounded by the interval $-2q \leq \xi \leq 2q$ under the condition $0 \leq q \leq a/2$ we have [7]

$$W(p,q,\beta)=\frac{2}{a}\int_0^q\cos\ (2py/\hbar)\ \Theta_3\left(\frac{y}{a}\left|\frac{i\pi\hbar\,\beta^2}{2ma^2}\right.\right)dy-$$

$$-(\hbar/ap)\sin(2pq/\hbar)\Theta_3\left(\frac{q}{a}\left|\frac{i\pi\hbar\,\beta^2}{2ma^2}\right.\right)$$ (12)

but when $a/2 \leq q \leq a$ one have to use the equality
$$W(p,q,\beta) = W(p,a-q,\beta)$$

The Wigner for a free particle in half space was exactly expressed by the error-function for the first time in [8]

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NEW SQUEEZED LANDAU STATES

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Abstract

We introduce a new set of squeezed sates through the coupled two-mode squeezed operator. It is shown their behaviour is simpler than the correlated coherent states introduced by Dodonov, Kurmyshev and Man'ko in order to quantum mechanically describe the Landau system, i.e. a planar charged particle in a uniform magnetic field. We compare results for both sets of squeezed states.

A planar charged particle moving in a uniform magnetic field is a very interesting quantum mechanical system. It is not trivial, needs the two spatial dimensions to describe it, it has some reminiscence of the two dimensional oscillator, but requires in addition the peculiar presence of the angular momentum operator which play a role as important as the hamiltonian. As recently it has been pointed out [1], the system has an Osc(1) dynamical degeneracy group. It seemed to us the system has a physics rich enough and mathematically particularly well understood in terms of the holomorphic (and antiholomorphic) coordinates that deserved to be revisited.

A planar particle of charge e, mass m, moving in a uniform magnetic field $\vec{B} = B\hat{k}$ can be described by the classical first order action

$$S = <\vec{p} \cdot \dot{\vec{r}} - (2m)^{-1}[\vec{p} - 2^{-1}eB(i\vec{r})]^2 > \leqq < \vec{p} \cdot \dot{\vec{r}} - H >. \tag{1}$$

$\vec{r}$ is the two-dimensional vector position of e, $\vec{p}$ its canonical momenta (which in the presence of the vector potential $\vec{A} = 2^{-1}B(i\vec{r})$ does not coincide with $m\dot{\vec{r}}$), and the linear operator i indicates a positive $\pi/2$ rotation, i.e. $(i\vec{v})_j = -\epsilon_{ji}v_l$. We choose B such that $eB \leqq m\omega$ is always positive, without losing generality.

The Landau system $\Phi_L \leqq \{\vec{r}, \vec{p}, H, \Lambda \leqq -(i\vec{r}) \cdot \vec{p}\}$ is quantized by imposing

$$[r_i, p_j] = i\hbar\delta_{ij} \quad i,j = (1,2). \tag{2}$$

As shown in ref. [1] it is convenient to introduce two sets of additional, momentum-like variables

$$\vec{\pi} \leqq \vec{p} - 2^{-1}m\omega(i\vec{r}), \quad \vec{\omega} \leqq \vec{p} + 2^{-1}m\omega(i\vec{r}). \tag{3}$$

$\vec{\pi}$ is the q-operator representing the observable $m\dot{\vec{r}}$. In terms of these quantities the hamiltonian and the angular momentum take the form

$$H = (2m)^{-1}\{\vec{p}^2 + 4^{-1}m^2\omega^2\vec{r}^2 + m\omega\Lambda\} = (2m)^{-1}\vec{\pi}^2, \tag{4}$$

$$\Lambda = (-i\vec{r}) \cdot \vec{p} = (2m\omega)^{-1}\{\vec{\pi}^2 - \vec{\omega}^2\}. \tag{5}$$

Observe the interesting chiral aspect of Λ in terms of $\vec{\pi}$ and $\vec{\omega}$.

It is immediate to notice that ω_i commutes with π_j,

$$[\omega_i, \pi_j] = 0. \tag{6}$$

Consequently $\vec{\omega}$ and Λ commute with H. Since

$$[\Lambda, \omega_i] = -i\hbar\epsilon_{ij}\omega_j = i\hbar(i\vec{\omega})_i \tag{7a}$$

$$[\omega_i, \omega_j] = -i\hbar m\omega\epsilon_{ij}. \tag{7b}$$

we see that $\{1, \vec{\omega}, \Lambda\}$ constitute a dynamical symmetric group (which will be easily recognized, when represented by its holomorphic components ω_z, $\omega_{\bar{z}}$ to be Osc (1)), i.e. commutes with H.

It is convenient to introduce holomorphic dimensionless variables $z, \bar{z},\ p_z, p_{\bar{z}},\ \pi_z, \pi_{\bar{z}},\ \omega_z, \omega_{\bar{z}}$ to analyze the system,

$$z \rightleftharpoons (2^{-1}\hbar^{-1}m\omega)^{1/2}(x+iy), \quad p_z = (2\hbar m\omega)^{1/2}(p_x - ip_y) = -i\partial_z \ + c.c. \tag{8}$$

The two momentum-like set of variables take the form

$$\pi_z = p_z + 2^{-1}i\bar{z} \quad , \quad \pi_{\bar{z}} = p_{\bar{z}} - 2^{-1}iz \tag{9ab}$$

$$\omega_z = p_z - 2^{-1}i\bar{z} \quad , \quad \omega_{\bar{z}} = p_{\bar{z}} + 2^{-1}iz \tag{10ab}$$

while H and Λ become

$$H = \hbar\omega\{p_z p_{\bar{z}} + 4^{-1}z\bar{z} + 2^{-1}\lambda\} \rightleftharpoons \hbar\omega h, \tag{11}$$

$$\Lambda = i\hbar\{\bar{z}p_{\bar{z}} - zp_z\} \rightleftharpoons \hbar\lambda = \hbar[\bar{z}\partial_{\bar{z}} - z\partial_z]. \tag{12}$$

Heinsenberg commutation relations eqs. (2) change to

$$[z, p_z] = i = [z, \pi_z] = [z, \omega_z] \quad + c.c. \ . \tag{13}$$

The two main physical observables h, λ have a very simple structure

$$h = \pi_z \pi_{\bar{z}} + 2^{-1} \rightleftharpoons n_1 + 2^{-1}, \quad \lambda = \pi_z \pi_{\bar{z}} - \omega_{\bar{z}}\omega_z \rightleftharpoons n_1 - n_2 \tag{14ab}$$

where $\pi_{\bar{z}}, \omega_z,\ \pi_z, \omega_{\bar{z}}$ can be regarded as two sets of decoupled annihilation and creation operators

$$[\pi_{\bar{z}}, \pi_z] = 1 = [\omega_z, \omega_{\bar{z}}], \tag{15}$$

since $[\omega_{z,\bar{z}}, \pi_{z,\bar{z}}] = 0$. We emphasize the fundamental role of the both $h, \lambda(H, \Lambda)$ in determining the two-mode quantum structure of the system, The energy degeneracy is broken by the presence of n_2, the second fundamental quantum number. These two series of discretes numbers will become the origin of the two couplex parameters labelling the coherent Landau states dicovered long time

ago [2] by Mal'kin and Man'ko. (Incidentally our $\pi_{\overline{z}}$ coincides with a of ref. [3] and our ω_z equals $-ia_0$. To introduce coherent Landau states we introduce the state $|0,0>= \psi_{00}$

$$\psi_{00}(z\overline{z}) = \pi^{-\frac{1}{2}}e^{-\frac{1}{2}z\overline{z}}. \tag{16}$$

ψ_{00} belongs to the ground subspace, i.e. $\pi_{\overline{z}}|0,0>= 0$ and is unitary (using the natural measure $2^{-1}idzd\overline{z} = dxdy$. The ground subspace is determined by the orthonormal set $\psi_{0p} = (p!)^{-1/2}\omega_{\overline{z}}^p|0,0>= |0,p>$

$$|0,p>= (p!)^{-\frac{1}{2}}(iz)^p|0,0>. \tag{17}$$

Each level-n energy eigenspace has the discrete orthonormal basis

$$\psi_{np} = (n!)^{-\frac{1}{2}}(p!)^{-\frac{1}{2}}\pi_z^n\omega_{\overline{z}}^p|0,0>. \tag{18}$$

Equations (14) tell us $H\psi_{np} = \hbar\omega(n + 2^{-1})\psi_{np}$ and $\Lambda\psi_{np} = \hbar(n - p)$.
We define the coherent Landau states [2] by

$$|w,s> \stackrel{=}{=} e^{w\pi_z - \overline{w}\pi_{\overline{z}} + s\omega_{\overline{z}} - \overline{s}\omega_z}|0,0> \tag{19a}$$

$w,s \in \mathbb{C}$. They constitute an over complete unitary system of the Hilbert space $\{\psi_{np}, n; p \in 0,1,\cdots\}$ in the usual sense (for coherent states)

$$< w_1s_1|w_2s_2 >= e^{-\frac{1}{2}|w_2-w_1|^2-|s_2-s_1|^2+i|w_2||w_1|\sin(\varphi_2-\varphi_1)+i|s_2||s_1|\sin(\phi_3-\phi_1)} \tag{19b}$$

$w = |w|e^{i\varphi}$, $s = |s|e^{i\phi}$.
They have three basic properties: i. They are $\pi_{\overline{z}}$ eigenstates with eigenvalue w, ii. they also are eigenstates of ω_z with proper value s

$$\pi_{\overline{z}}|ws >= w|w,s > \quad , \quad \omega_z|ws >= s|ws >, \tag{20}$$

and iii. they propagate remaining in the family. If one starts on $|ws >$ leaving the system to evolve, at time t Φ_L will be described by

$$e^{-i\hbar\omega t}|ws >= |we^{-i\omega t}, s >. \tag{21}$$

Eqs. (20) suggest a way to compute q-mechanical expected values for physical observables $F(p,\overline{p}, z,\overline{z})$. One has to transform them to their representation in terms of the new variables $(\pi,\overline{\pi},\omega,\overline{\omega})$, then normal ordering in both types of variables and finally taking into account eqs. (20).
In this way we obtain:

$$< z >_{CL} =< ws|z|ws >=< ws|(i\pi_{\overline{z}} - i\omega_{\overline{z}})|w,s >= i(w - \overline{s}) \tag{22a}$$

$$< z^2 >_{CL}= -(w - \overline{s})^2 \quad , \quad < z\overline{z} >= (w - \overline{s})(\overline{w} - s) + 1 \tag{22b,c}$$

plus their respective complex (hermitian) conjugates. We also obtain

$$< p_z >_{CL}= 2^{-1} < \pi_z + \omega_z >_{CL}= 2^{-1}(\overline{w} + s) \quad + c.c. \tag{23a}$$

$$< p_z^2 >_{CL} = 4^{-1} < (\pi_z + \omega_z)^2 > = 4^{-1}(\overline{w} + s)^2 \quad + h.c. \,, \tag{23b}$$

$$< p_z p_{\overline{z}} >_{CL} = 4^{-1} + 4^{-1}(w + \overline{s})(\overline{w} + s) \tag{23c}$$

$$< h >_{CL} = < ws|h|ws > = w\overline{w} + 2^{-1} \,, \quad < \lambda >_{CL} = w\overline{w} - s\overline{s}, \tag{24a,b}$$

$$< h^2 >_{CL} = (w\overline{w} + 2^{-1})^2 + w\overline{w} \,, \quad < \lambda^2 >_{CL} = < \lambda >_{CL}^2 + w\overline{w} + s\overline{s}. \tag{25a,b}$$

Recalling definitions (8) relating $z, \overline{z}$ and real dimensionless variables x, y we can calculate physical uncertanties, which are defined for canonical sets of variables in terms of holomorphic variances $\Delta z, \Delta z\overline{z} = < z\overline{z} > - < z >< \overline{z} >, \Delta p_z, \Delta p_z p_{\overline{z}}$. they turn out to be

$$(\Delta x)_{CL}^2 = 4^{-1}(\Delta z)_{CL}^2 + 4^{-1}(\Delta \overline{z})_{CL}^2 + 2^{-1}(\Delta z\overline{z})_{CL} = 2^{-1} = (\Delta y)_{CL}^2, (\Delta xy)_{CL} = 0. \tag{26abc}$$

In a similar way, we find for the physical momenta

$$(\Delta p_x)_{CL}^2 = 2^{-1} = (\Delta p_x p_y)_{CL} = 0. \tag{27abc}$$

Consequently both uncertanties attain lowest bound

$$(\Delta x)_{CL}(\Delta p_x)_{CL} = 2^{-1} = (\Delta y)_{CL}(\Delta p_y)_{CL}. \tag{28}$$

Coherent Landau states are minimun uncertanty states (MUS).

Squeezing can be now analysed, since the standard procedure to consider this type of states involves the squeezing of associated coherents states. Complexive decoupled squeezed Landau states have been introduced in ref. [3], where they have been called correlated coherent states.

Since squeezing is not that intuitive we face in principle four different types of squeezing: partial squeezing in $\pi_{\overline{z}}\pi_z$, partial squeezing in $\omega_z\omega_{\overline{z}}$ or full, complexive squeezing in both sets of variables.

The complexive squeezing might be either decoupled or coupled in both set of variables. One might think that it could be enough to squeeze just in the dynamical constituents of the hamiltonian $\pi_{\overline{z}}\pi_z$ in order to obtain physically appealing results. This primary type of "squeezing" can be shown to lead to states which are irrelevant, since they are neither minimun uncertainty states nor the variances of any canonical variable can tend to zero.

We are obliged to turn our interest to more radical way of squeezing. As we said above, we must try complexive squeezing, i.e. to introduce squeeze operators which squeeze both type of quanta, the π and the ω-ones.

Let us first consider what we call "decoupled" squeezing, as it has been done in ref. [3]. The squeezing operator is defined as

$$S(q_1, q_2) = e^{\frac{1}{2}q_1^2\pi_z^2 - \frac{1}{2}\overline{q}_1^2\pi_{\overline{z}}^2 + \frac{1}{2}q_2^2\omega_{\overline{z}}^2 - \frac{1}{2}\overline{q}_2^2\omega_z^2} = S^{\pi}(q_1)S^{\omega}(q_2). \tag{29}$$

We consider the squeezed states

$$|ws, q_1, q_2 > \equiv S(q_1, q_2)|w, s > . \tag{30}$$

where both w and s are distorted.

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Both the π and ω variables transform non trivially here,

$$(\pi_{\bar{z}})_{q_1} \equiv S_{q_1}^+ \pi_{\bar{z}} S_{q_1} = \pi_{\bar{z}} chr_1 + e^{2i\varphi_1} shr_1 \pi_z \tag{31}$$

$$(\omega_{\bar{z}})_{q_2} \equiv S_{q_2}^+ \omega_z S_{q_2} = \omega_z chr_2 + e^{2i\varphi_2} shr_2 \omega_{\bar{z}}. \quad + h.c. \ . \tag{32}$$

The squeezed transformed of the Heinserberg canonical variable z $z_{q_1 q_2} \equiv S_{q_1 q_2}^+ z S_{q_1 q_2}$ becomes in the present case

$$z_{q_1 q_2} = i(\pi_{\bar{z}} chr_1 + e^{2i\varphi_1} shr_1 \pi_z - \omega_{\bar{z}} chr_2 - e^{-2i\varphi_2} shr_2 \omega_z). \tag{33}$$

The complexive squeezed expectation values of z and p_z are therefore

$$< z >_{q_1 q_2} = < z_{q_1 q_2} >_{CL} = i[s_{q_1}(w) - \bar{s}_{q_2}(s)], \tag{34a}$$

$$< p_z >_{q_1 q_2} = < (p_z)_{q_1 q_2} >_{CL} = 2^{-1}[\bar{s}_{q_1}(w) + s_{q_2}(s)], \tag{34b}$$

where subindex CL indicates the coherent Landau state $|w, s>$ and $s_q(u) \equiv chru + e^{2i\varphi} shr\bar{u}$. Quadratic complexive squeezed expectation values become

$$< z^2 >_{q_1 q_2} = -[s_{q_1}(w) - \bar{s}_{q_2}(s)]^2 - shr_1 chr_1 e^{2i\varphi_1} - shr_2 chr_2 e^{-2i\varphi_2} \tag{35a}$$

$$< z\bar{z} >_{q_1 q_2} = < \bar{z} >_{q_1 q_2} < z >_{q_1 q_2} + chr_1^2 + shr_2^2, \tag{35b}$$

$$< p_z^2 >_{q_1 q_2} = 4^{-1}(\bar{s}_{q_1}(w) + s_{q_2}(s))^2 + 4^{-1} shr_1 chr_1 e^{-2i\varphi_1} + 4^{-1} shr_2 chr_2 e^{2i\varphi_2}, \tag{36a}$$

$$< p_z p_{\bar{z}} >_{q_1 q_2} = 4^{-1}[\bar{s}_{q_1}(w) + s_{q_2}(s)][s_{q_1}(w) + \bar{s}_{q_2}(s)] 4^{-1} ch^2 r_1 + 4^{-1} sh^2 r_2. \tag{36b}$$

From this expressions for the holomorphic variables we can evaluate physical uncertanties to see how they behave for complexive decoupled squeezing. They are

$$(\Delta p_y)_{q_1 q_2}^2 = (\Delta x)_{q_1 q_2}^2 = 2^{-1} chr_1(chr_1 - shr_1 cos 2\varphi_1) + 2^{-1} chr_2(chr_2 - shr_2 cos 2\varphi_2) - 2^{-1}, \tag{37}$$

$$(\Delta y)_{q_1 q_2}^2 = (\Delta p_x)_{q_1 q_2}^2 = 2^{-1} chr_1(chr_1 + shr_1 cos 2\varphi_1) + 2^{-1} chr_2(chr_2 + shr_2 cos 2\varphi_2) - 2^{-1}. \tag{38}$$

For $\varphi_1 = 0 = \varphi_2$ Δx and Δp_y are squeezed since:

$$(\Delta p_y)_{\varphi_1=0=\varphi_2}^2 = (\Delta x)_{\varphi_1=0=\varphi_2}^2 = 4^{-1} e^{-2r_1} + 4^{-1} e^{-2r_2} \to 0^+ \ , \ r_1, r_2 \to \infty \tag{39}$$

while, of course Δp_x and Δy increase according to eq. (38). The partial uncertanties get closer to their lowest bound,

$$(\Delta y)_{q_1 q_2}^2 = (\Delta p_x)_{q_1 q_2}^2 \Big|_{\varphi_1=0=\varphi_2} = 8^{-1}[1 + chr_2(r_2 - r_1)] = (\Delta y)_{q_1 q_2}^2 = (\Delta p_y)_{q_1 q_2}^2 \Big|_{\varphi_1=0=\varphi_2} \tag{40}$$

This result indicates that physical squeezing, in the sense that the squeezed states are also minimun uncertanty states, is obtained just for $r_2 = r_1$. Complexive decoupled squeezing leads to physical squeezing modes, but the two independent "a priori" parameters q_1 and q_2 have to coincide.

A nicer solution to finding squeezed states of Φ_l arises by considering the fact that we have two modes in the system. For this situation a more natural squeezed operator can be defined,

similarly to what has been done for the two photon case in ref. [4]. The "coupled" squeezing operator we postulate is given by

$$S_q \equiv e^{\frac{1}{2}q^2\pi_s\omega_{\overline{s}} - \frac{1}{2}\overline{q}^2\overline{\pi}_s\omega_s}. \tag{41}$$

It naturally depends upon only one parameter. It is straighforward to show that the squeezed values of $\pi_{\overline{s}}$ and ω_s respectively are

$$S_q^+\pi_{\overline{s}}S_q = \pi_{\overline{s}}ch\frac{r}{2} + e^{2i\varphi}sh\frac{r}{2}\omega_{\overline{s}}, \tag{42}$$

$$S_q^+\omega_s S_q = \omega_s ch\frac{r}{2} + e^{2i\varphi}sh\frac{r}{2}\pi_s. \tag{43}$$

As expected this type of squeezing makes π-variables to have ω-components and viceversa.

The new associates squeezed states are defined by

$$|ws, q> \equiv S_q|w, s> \tag{44}$$

where S_q has been introduced in eq. (41), It is inmediate to perform in this case similar calculations to what has already been done for the previous case. Results turn out to be mathematically simpler and physically interesting. We get

$$< z >_q = < z_q >_{CL} = i(w - \overline{s})ch\frac{r}{2} + ish\frac{r}{2}(\overline{s}e^{2i\varphi} - we^{-2i\varphi}), \tag{45}$$

$$< p_z >_q = 2^{-1}(\overline{w} + s)ch\frac{r}{2} + 2^{-1}sh\frac{r}{2}(\overline{w}e^{2i\varphi} + se^{-2i\varphi}). \tag{46}$$

In addition one finds that $(\Delta_z)_q^2 = (\Delta_{p_z})_q^2 = 0$. Finally the variances of the canonical variables attain the respective forms.

$$(\Delta x)_q^2 = 4^{-1}e^r(1 - cos2\varphi) + 4^{-1}e^{-r}(1 + cos2\varphi) = (\Delta p_y)_q^2, \tag{47a}$$

$$(\Delta p_x)_q^2 = 4^{-1}e^r(1 + cos2\varphi) + 4^{-1}e^{-r}(1 - cos2\varphi) = (\Delta y)_q^2, \tag{47b}$$

Both uncertanties coincide, their value being

$$(\Delta x)_q^2(\Delta p_x)_q^2 = (\Delta y)_q^2(\Delta p_y)_q^2 = 4^{-1}(chr^2 - shr^2cos^22\varphi). \tag{48}$$

For $\varphi = k\pi/2$ we obtain squeezing and minimun uncertainty.

In conclusion we feel these coupled squeezed states (44) are the natural ones for introducing squeezing in the Landau system. We have shown they behave in a simpler way then those defined in ref. [3] while they also lead to physical squeezing.

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THE UNCERTAINTY PRINCIPLE AND QUANTUM CHAOS

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Abstract

The conception of quantum chaos is described in some detail. The most striking feature of this novel phenomenon is in that all the properties of classical dynamical chaos persist here but, typically, on the finite and different time scales only. The ultimate origin of such a universal quantum stability is in the fundamental uncertainty principle which makes discrete the phase space and, hence, the spectrum of bounded quantum motion. Reformulation of the ergodic theory, as a part of the general theory of dynamical systems, is briefly discussed.

1 Introduction

The main purpose of this talk is to explain new physical ideas in the so-called *quantum chaos* which since recently attracts ever growing interest of many researchers [1-5, 10]. In appendix I also briefly discuss the concept of coherent/squeezed states in nonlinear, particularly, chaotic systems in a more close relation to the topic of this Workshop.

The recent breakthrough in understanding of the quantum chaos has been achieved, particularly, due to a new philosophy accepted, explicitly or more often implicitly, in most studies of quantum chaos. Namely, the whole physical problem of quantum dynamics was separated into two different parts:

- The *proper quantum dynamics* as described by a specific dynamical variable, the wavefunction $\psi(t)$, and by some deterministic equation, for example the Schrödinger equation. This part naturally belongs to the general theory of dynamical systems and is essentially mathematical; the problem is well-posed and this allows for extensive studies.

- The *quantum measurement* including the registration of the result and, hence, the collapse of the ψ function. This part still remains very vague to the extent that there is no common agreement even on the question whether this is a real physical problem or an ill-posed one so that the Copenhagen interpretation of (or convention in) quantum mechanics answers all the admissible questions. In any event, there exists as yet no dynamical description of the quantum measurement including the ψ collapse.

In this way one can single out a very difficult problem of the fundamental randomness in quantum mechanics which is related to the second part only, and which is foreign, in a sense, to the proper quantum system. On the other hand, there is a close relation of this separate problem to the quantum chaos itself (see Section 4 below and Ref.[4]).

The importance of quantum chaos is not only in that it represents a new unexplored field of nonintegrable quantum dynamics with many applications but also, and this is most interesting for the fundamental science, in reconciling the two seemingly different dynamical mechanisms for the statistical laws in physics.

Historically, the first mechanism was related to the *thermodynamic limit* $N \to \infty$ in which the completely integrable system becomes chaotic for typical (random) initial conditions (see, e.g.,Ref.[6]). A natural question – what happens for a large but finite number of freedoms N – has still no rigorous answer but the new phenomenon of quantum chaos, at least, presents an insight into this problem too. This mechanism, which is equally applicable in both classical and quantum mechanics, may be called the *traditional statistical mechanics* (TSM).

The second (new) mechanism is based upon the strong (exponential) local instability of motion characterized by positive Lyapunov's exponents $\Lambda > 0$ [6, 7]. It is not at all restricted to large N, and is possible, e.g., for $N > 1$ in a Hamiltonian system. However, this mechanism has been operative, until recently, in the classical mechanics only. This phenomenon is called *dynamical chaos* as it does not require any random parameters or any noise in the equations of motion. Notice that in a Hamiltonian (time–reversible) system the motion is unstable in both directions of time because for each positive Λ there is the equal negative one, and for almost all trajectories the instability depends on positive (in a given direction of time) exponents only. Hence, the dynamical chaos is also time–reversible, and no 'time arrow' exists or is required in the theory.

The quantum system bounded in phase space has a discrete energy (frequency) spectrum and is similar, in this respect, to the finite–N TSM. In both cases the motion is almost periodic. Moreover, such quantum systems are even completely integrable in the Hilbert space (see, e.g., Ref.[3]). Yet, the fundamental correspondence principle requires the transition to the classical mechanics, including the dynamical chaos, in the *classical limit* $q \to \infty$, where q is some (big) quasi-classical parameter, e.g., the quantum number n (the action variable, $\hbar = 1$). Again, a natural physical conjecture is that for finite but large q there must be some chaos similar to the finite–N TSM. Yet, in a chaotic quantum system the number of freedoms N does not need to be large as well as in the classical chaos. The quantum counterpart of N is q, both quantities determining the number of frequencies which control the motion. Thus, mathematically, the problem of quantum chaos is similar to that for the finite–N TSM.

Some researchers believe that the only way out of the above apparent contradiction is the failure of the correspondence principle [37]. If it were so the quantum chaos would be, indeed, a great discovery. 'Unfortunately', there exists a less radical (but also interesting and important) resolution of this difficulty to be discussed below.

The main difficulty here is in that the both problems suggest some chaos in the discrete spectrum which is completely contrary to the existing theory of dynamical systems and to the ergodic theory where such dynamics corresponds to the opposite limit of regular motion.

The ultimate origin of the quantum integrability is discreteness of the phase space (but not, as yet, of the space–time!) or, in the modern mathematical language, the noncommutative geometry of the former. This is the very basis of the whole quantum physics directly related to the

fundamental uncertainty principle which implies a finite size of an elementary phase–space cell: $\Delta x \cdot \Delta p \gtrsim \hbar$ (per freedom).

As an illustration I will make use of the simple model described classically by the *standard map* (SM) [7, S]:

$$\bar{n} = n + k \cdot \sin\theta; \qquad \bar{\theta} = \theta + T \cdot \bar{n} \tag{1}$$

with action–angle variables n, θ, and perturbation parameters k, T. The quantized standard map (QSM) is given by [9, 10]

$$\bar{\psi} = \exp(-ik \cdot \cos\hat{\theta}) \cdot \exp\left(-i\frac{T}{2}\hat{n}^2\right)\psi, \tag{2}$$

where momentum operator $\hat{n} = -i\partial/\partial\theta$. To provide the complete boundedness of the motion consider SM on a torus of circumference (in n)

$$L = \frac{2\pi m}{T} \tag{3}$$

with integer m to avoid discontinuities. The quasi–classical transition corresponds to quantum parameters $k \to \infty$, $T \to 0$, $L \to \infty$ while classical parameters $K = kT = $ const, and $m = LT/2\pi = $ const remain unchanged.

The QSM models the *energy shell* of a conservative system which is the quantum counterpart of the classical energy surface.

In the studies of dynamical systems, both classical and quantal, most problems unreachable for rigorous mathematical analysis are treated "numerically" using computer as a universal model With all obvious drawbacks and limitations such "numerical experiments" have very important advantage as compared to the laboratory experiments, namely, they provide the complete information about the system under study. In quantum mechanics this advantage becomes crucial as in laboratory one cannot observe (measure) the quantum system without a radical change of its dynamics.

2   The definition of quantum chaos

The common definition of the classical chaos in physical literature is the *strongly unstable motion*, that is one with positive Lyapunov's exponents $\Lambda > 0$. The Alekseev – Brudno theorem then implies that almost all trajectories of such a motion are unpredictable, or random (see Ref.[11]). A similar definition of quantum chaos, which still has adherents among both mathematicians as well as a few physicists, fails because, for the bounded systems, the set of such motions is empty due to the discreteness of the phase space and, hence, of the spectrum.

The common definition of quantum chaos is *quantum dynamics of classically chaotic systems* whatever it could happen to be. Logically, this is most simple and clear definition. Yet, in my opinion, it is completely inadequate (and even somewhat helpless) from the physical viewpoint just because such a chaos may turn out to be a perfectly regular motion as, for example, in case of the *perturbative localization* [12]. In QSM the latter corresponds to $k \lesssim 1$ when all quantum transitions are suppressed independent of classical parameter K which controls the chaos.

I would like to define the quantum chaos in such a way to include some essential part of the classical chaos. The best definition I have managed to imagine so far reads:

the quantum chaos is finite-time (transient) dynamical chaos in discrete spectrum

In other words this new phenomenon reveals an intrinsic complexity and richness of the motion with discrete spectrum which has been considered since long ago as the most simple and regular. This is certainly in contradiction with the existing ergodic theory. In what follows I will try to explain a new approach to the ergodic theory which is necessary to describe the peculiar phenomenon of quantum chaos.

3   The time scales of quantum dynamics

Already the first numerical experiments with QSM revealed the quantum diffusion in n close to the classical one under conditions $K \gtrsim 1$ (classical stability border) and $k \gtrsim 1$ (quantum stability border) [9]. Futher studies confirmed this conclusion and showed that the former followed the latter in all details but on a *finite time interval* only [10, 13]. This observation was the clue to understanding the dynamical mechanism of the diffusion, which is apparently an aperiodic process, in discrete spectrum. Indeed, the fundamental uncertainty principle implies that the discreteness of the spectrum is not resolved on a sufficiently short time interval. Whence, the estimate for the *diffusion (relaxation) time scale* :

$$t_R \sim \varrho_0 \leq \varrho. \tag{4}$$

Here ϱ is the density of (quasi)energy levels, and ϱ_0 is the same for the *operative eigenstates* which are actually present in the initial quantum state $\psi(0)$ and, thus, do actually control the dynamics. In QSM the quasi-energies are determined mod $2\pi/T$ and, surprisingly, $\varrho = LT/2\pi = m$ is a classical parameter (3). As to ϱ_0, it depends on the dynamics and is given by the estimate [10, 13]:

$$\frac{\varrho_0}{T} \sim \frac{t_R}{T} \equiv \tau_R \sim D \equiv \frac{\langle (\Delta n)^2 \rangle}{\tau} \leq \frac{m}{T} \tag{5}$$

Here τ is discrete map's time (the number of iterations), and D is the classical diffusion rate. This remarkable expression relates an essentially quantum characteristic (τ_R) to the classical one (D). The latter inequality in Eq.(5) follows from that in Eq.(4), and it is explained by the boundedness of QSM on a torus. In the quasi-classical region $\tau_R \sim k^2 \to \infty$ (see Eq.(1)) in accordance with the correspondence principle.

Besides relatively long time scale (5) there is another one given by the estimate [14, 10]

$$t_r \sim \frac{\ln q}{\Lambda} \to \frac{\ln k}{\ln(K/2)} \tag{6}$$

where q is some (large) quasi-classical parameter, and where the latter expression holds for QSM. It may be termed the *random time scale* since here the quantum motion of a narrow wave packet is as random as classical trajectories according to the Ehrenfest theorem. This was well confirmed in a number of numerical experiments [15]. The physical meaning of scale t_r is in fast spreading of a wave packet due to the strong local instability of classical motion.

Even though the random time scale t_r is very short it grows indefinitely in the quasi–classical region (q, $k \to \infty$), again in agreement with the correspondence principle.

Big ratio t_R/t_r imlies another peculiarity of quantum diffusion: it is dynamically stable as was demonstrated in striking numerical experiments [16] with time reversal.

Thus, the quantum chaos possesses all the finite–time properties of the 'true' (classical–like) chaos on the corresponding time scales in spite of the discrete spectrum. To put it another way, the phenomenon of quantum chaos demonstrates that the limiting case of the regular motion in the general theory of dynamical systems, which appears to be fairly simple and transparent, reveals, in the quantum chaos, its internal complexity and richness to the extent of approaching its opposite, the 'true' classical chaos, or deterministic randomness.

I think that the conception of characteristic time scales of quantum dynamics is a satisfactory resolution of the apparent contradiction between the correspondence principle and the quantum transient (finite–time) pseudochaos. Some physicists, however, feel that such an explanation is, at least, ambiguous because it includes the two limits which do not commute:

$$\lim_{|t| \to \infty} \lim_{q \to \infty} \neq \lim_{q \to \infty} \lim_{|t| \to \infty}$$

While the first order leads to the classical chaos, the second one results in an essentially quantum behavior with no chaos at all. To relax these doubts I notice that in physics one does not need any limits at all, and can describe, principally, anything quantum–mechanically. If, nevertheless, we would like to make use of the much simpler classical mechanics (for practical purposes) the only one limit ($q \to \infty$) is quite sufficient as the physical time is certainly finite. At last, even if it would be helpful for some reason to formally take the limit $|t| \to \infty$ this should be *conditional* that is one for a fixed ratio $|t|/t_R(q)$, for example. The limit $|t| \to \infty$ is related to the existing ergodic theory which is asymptotic in t. Meanwhile the new phenomenon of the quantum chaos requires the modification of the theory to a finite time which is a difficult mathematical problem still to be solved. The main difficulty is in that even the distinction between the two opposite limits in the ergodic theory – discrete and continuous spectra – is asymptotic only.

In any event, since quantum mechanics is commonly accepted as the universal theory, the phenomenon of the 'true' (classical–like) dynamical chaos strictly speaking does not exist in nature. Nevertheless, it is very important in the theory as the limiting pattern to compare with the real quantum chaos. On the other hand, the practical importance of statistical laws even for a finite time interval is in that they provide a relatively simple description of the *essential* behavior for a very complicated dynamics.

4   The quantum steady state

As a result of quantum diffusion and relaxation some steady state is formed whose nature depends on the *ergodicity parameter*

$$\lambda = \frac{l_s}{L} \approx \frac{D}{L}. \tag{7}$$

where l_s is the so-called *localization length* (see Eq.(10) below). If $\lambda \gg 1$ the quantum steady state is close (at average) to the classical statistical equilibrium which is described by ergodic phase

density $g_{cl}(n) =$const (for SM on a torus) where n is a continuous variable. In quantum mechanics n is integer, and the quantum phase density $g_q(n,\tau)$ in the steady state fluctuates [17, 5], the ergodicity meaning

$$g_q(n) = \overline{|\psi_s(n,\tau)|^2} = \frac{1}{L} \tag{8}$$

where the bar denotes time averaging.

According to numerical experiments the ergodicity does not depend on the initial state which implies that all eigenfunctions $\phi_m(n)$ are also ergodic, at average, with Gaussian fluctuations [17, 5] and the dispersion

$$\langle |\phi_m(n)|^2 \rangle = \frac{1}{L}. \tag{9}$$

This is always the case sufficiently far in the quasi-classical region as $\lambda \sim k^2/L \sim Kk/m \to \infty$ with $k \to \infty$ ($K = kT$ and $m = LT/2\pi$ remain constant) in accordance with old Shnirelman's theorem [18].

An interesting unsolved problem is the microstructure of ergodic eigenfunctions, particularly, the so-called 'scars' [29] which reveal the set of classical (unstable) periodic trajectories (see Ref.[30] for the theory of scars).

Finite fluctuations (9) show that a single chaotic quantum system in a pure state described by $\psi_s(n,\tau)$ represents, in a sense, a finite statistical ensemble of $M \sim L$ "particles". Moreover, Eq.(9) shows that all the basis states in a chaotic quantum system are statistically independent as if the system were in a mixed state and not in a pure one as it actually is. This means that the quantum chaos provides the *dynamical* loss of quantum coherence which is of principal importance in many problems, for example, in the theory of quantum measurement. The fluctuations result, particularly, in partial recurrences toward the initial state but the recurrence time is much longer as compared to the relaxation time scale τ_R and sharply depends on the recurrence domain.

If $\lambda \ll 1$ the quantum steady state is qualitatively different from the classical one. Namely, it is localized in n within the region of size l_s around the initial state if the size of the latter $l_0 \ll l_s$. Numerical experiments show that the phase space density, or the *quantum statistical measure* , is approximately exponential [10, 13]

$$g_s(n) \approx \frac{1}{l_s} \exp\left(-\frac{2|n|}{l_s}\right); \qquad l_s \approx D \tag{10}$$

for initial $g(n,0) = \delta(n)$. The quantum ensemble is now characterized by $M \sim l_s \sim k^2$ "particles".

The relaxation to this steady state is called *diffusion localization* , and it is described approximately by the diffusion equation [19, 28]

$$\frac{\partial g}{\partial \tau'} = \frac{1}{2} \frac{\partial}{\partial n} D \frac{\partial g}{\partial n} + \frac{\partial g}{\partial n} \cdot \frac{n}{|n|} \tag{11}$$

for initial $g(n,0) = \delta(n)$ where new time

$$\tau' = \tau_R \ln\left(1 + \frac{\tau}{\tau_R}\right) \tag{12}$$

accounts for the discrete motion spectrum [20]. The last term in Eq.(11) describes the "backscattering" of ψ wave propagating in n which eventually results in the diffusion localization. The fitting parameter $\tau_R \approx 2D$ was derived from the best numerical data available (see Ref.[21] where a different theory of diffusion localization was also developed).

5 Concluding remarks

In conclusion I would like to briefly mention a few important results for unbounded quantum motion. In SM it corresponds to $L \to \infty$. First, there is an interesting analogy between dynamical localization in momentum space and the celebrated Anderson localization in disordered solids which is a statistical theory. The analogy was discovered in Ref.[22] and essentially developed in Ref.[23]. It is based upon (and restricted by) the equations for eigenfunctions. The most striking (and less known) difference between the two problems is in the absence of any diffusion regime in $1D$ solids [24]. This is because the energy level density of the operative eigenfunctions in solids

$$\rho_0 \sim \frac{l\,dp}{dE} \sim \frac{l}{u} \sim t_R \tag{13}$$

which is the relaxation time scale, is always of the order of the time interval for a free spreading of the initial wave packet at a characteristic velocity u. In other words, the localization length l is of the order of the electron scattering free path. On the contrary, in momentum space, for instance in the standard map, each scattering (one map's iteration) couples $\sim k$ unperturbed states, so that $\sim k^2 \gg 1$ scatterings are required to reach the localization $l \sim k^2$. Another (qualitative) explanation of this surprising difference is in that the density of quasienergy levels for driven systems is always much higher as compared to that of energy levels. The same is true for a conservative 2D system as compared with 1D motion in solids. Thus, the Anderson localization is the spreading, rather than diffusion, localization.

Another similarity between the two problems is in that the Bloch extended states in periodic potential correspond to a pecular quantum resonance in QSM for rational $T/4\pi$ [9, 10].

An interesting open question is the dynamics for irrational Liouville's (transcendental) $T/4\pi$. As was proved in Ref.[25] the motion can be unbounded in this case unlike that for a typical irrational value. The latter is the result of numerical experiments, no rigorous proof of localization for $k \gg 1$ has been found as yet. In Ref.[2S] the conjecture is put forward, supported by some semiqualitative considerations, that depending on a particular Liouville's number the broad range of motions is possible, from purely resonant one ($|n| \sim \tau$) down to complete localization ($|n| \lesssim l$).

If the quantum motion is not only unbounded but the growth of unbounded variables is exponential, the "true" chaos (not restricted to a finite time scale) can occur. A few exotic examples together with considerations from different viewpoints can be found in Refs.[10, 26]. One particular model is $3D$ linear oscillator with phase–dependent frequencies described by the Hamiltonian

$$\hat{H} = \frac{1}{2} \sum_{k=1}^{3} \left(\omega_k(\theta_1, \theta_2, \theta_3)\hat{n}_k + \hat{n}_k \omega_k(\theta_1, \theta_2, \theta_3) \right); \qquad \hat{n}_k = -i\frac{\partial}{\partial \theta_k}$$

However, such chaos does not seem to be a typical quantum dynamics.

The final remark is that the quantum chaos, as defined in Section 2, comprises not only quantum systems but also any linear, particularly classical, waves [27]. So, it is essentially the

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linear wave chaos. Moreover, a similar mechanism works also in completely integrable nonlinear systems like Toda lattice, for example [31]. From mathematical point of view all these new ideas require some *perestroika* in the existing ergodic theory. Perhaps, better to say that a new ergodic theory is wanted which, instead of benefiting from the asymptotic approximation ($|t| \to \infty$ or $N \to \infty$), could analyze the finite-time statistical properties of dynamical systems. In my opinion, this is the most important conclusion from the first attempts to comprehend the quantum chaos.

6   Acknowledgments

Appendix: universal squeezing of coherent states

The coherent states have been introduced and are widely used as the special most narrow wave packets for the linear harmonic oscillator. In this and only this case the packets do not spread, and it allows, particularly, for the unambiguous distinction between the coherent and squeezed states which attract recently much attention [32]. The generalization of both onto nonlinear oscillations remains unclear as many different attempts do attest [33]. The main obstacle here is a universal phenomenon of the stretching/squeezing for any narrow wave packet in nonlinear dynamics. Even in a completely integrable system the linear (in time) local instability of motion always occurs just as a result of nonlinearity which makes the frequencies $\omega(n)$ dependent on initial conditions. In quantum mechanics it corresponds to unequal energy level spacings. As a result the squeezing parameter

$$s(t) \equiv \frac{d_{max}}{d_{min}} \sim n\,(\Delta\theta)^2 \sim (\Delta n)^2\, t^2 \sim (\nu_1\omega t)^2\, \frac{(\Delta n)^2}{n}$$

permanently grows with time. Here $d_{max} \sim \sqrt{n}\Delta\theta \sim (\nu_1\omega t)\Delta n/\sqrt{n}$ and $d_{min} \sim 1/d_{max}$ are the maximal and minimal dimensions, respectively, of an initially 'round' (coherent) wave packet ($\Delta n/\sqrt{n} \sim \sqrt{n}\Delta\theta \sim 1$) on the action–angle phase plane in polar coordinates $\sqrt{n}$, θ; $\nu_1 = (n/\omega)|d\omega/dn|$ is dimensionless nonlinearity, and the minimum–uncertainty relation [34] $d_{max} \cdot d_{min} \sim 1$ used is the quantum counterpart of the classical phase–space area conservation. The former is not exact [35]

$$\frac{dW}{dt} \approx \frac{1}{24}\frac{d^2\omega}{dn^2}\frac{\partial^3 W}{\partial\theta^3} \sim W\,\omega\,\nu_2\,\frac{(\Delta n)^3}{n^2}$$

where $W(n,\theta,t)$ is the Wigner function, and $\nu_2 = (n^2/\omega)d^2\omega/dn^2$. This estimate determines the *inflation time scale* t_{if} when the phase–space area A, occupied by a quantum state, substantially increases ($\Delta A \sim A$):

$$\nu_2\omega t_{if} \sim \frac{n^2}{(\Delta n)^3} \to \sqrt{n} \quad (s_0 \sim 1)$$

The latter estimate holds for the coherent initial state $(s_0 = (\Delta n)^2/n = 1)$.

It is instructive to compare t_{if} with the two other characteristic time scales of the packet dynamics, namely

- *squeezing time scale* $(\Delta s \sim 1)$: $\nu_1 \omega t_{sq} \sim \sqrt{n}/\Delta n \to 1$ $(s_0 \sim 1)$

- *stretching time scale* $(\Delta \theta \sim 1)$: $\nu_1 \omega t_{st} \sim n/\Delta n \to \sqrt{n}$ $(s_0 \sim 1)$

In quasiclassics $(n \gg 1)$ $t_{sq} \ll t_{st} \sim t_{if}$ $(s_0 \sim 1)$. If $\Delta n \ll \sqrt{n}$ (initial squeezing parameter $s_0 \gg 1$) the discreteness (quantization) of action n comes into play and destroys the wave packet. Apparently, it happens when $\Delta n_c \sim 1$ at the packet center, or $\Delta \theta \sim 1$. Hence, beyond the stretching time scale t_{st} a single packet does no longer exist. In a sense, t_{st} is the packet life time.

The ultimate origin of the packet inflation is in that the uncertainty relations are generally inequality. An attempt [36] to reformulate them as the equality, using the universal relation

$$\int W^2 \, dp \, dq = \frac{1}{2\pi}$$

for any pure state, is very restrictive as W may be negative. Particularly, this is just the case during inflation when W oscillates around zero.

Recently another version of 'phase-space density' (also called Husimi distribution)

$$S(p, q, t) = \frac{1}{2\pi} |< \alpha | \psi >|^2$$

became very popular. Even though this function has a clear physical meaning as the expansion in the basis of the coherent states at points $\alpha = (q + ip)/\sqrt{2}$ and, moreover, is never negative it may substantially distort the picture of quantum evolution owing to the inherent restriction of resolution in both p and q separately. Particularly, for a classically unstable and, hence, chaotic motion the squeezing of a narrow wave packet is almost completely hidden, the stretching only showing up [15].

In the latter case the squeezing (as well as stretching) is most fast ($s(t) \sim \exp(2\Lambda t)$ where the instability rate Λ is the Lyapunov exponent), and it explains a very short random time scale (6). This scale essentially depends on the initial wave packet, estimate (6) corresponding to the special, least squeezing, packet with $\Delta n \sim (\Delta \theta)^{-1} \sim \sqrt{k}$. This is also a sort of coherent state but very unusual one which depends not on the action n but on perturbation parameter k $(\Delta n/\sqrt{k} \sim \sqrt{k}\Delta \theta \sim 1)$. The squeezing due to the local instability is terminated at time (6) by the distruction of the packet which disintegrates into many scattered pieces [15] when $\Delta \theta \sim \Delta n \sim 1$ as explained above. However, if the packet resides on a classical (unstable) periodic trajectory of period $P \lesssim t_r$ the squeezing is restricted, due to periodicity, by the time $P/2$, and a quasistationary structure may exist. This phenomenon manifests itself in the so-called 'scars' on the chaotic eigenfunctions [29, 30]. The set of such almost 'frozen' packets may form a natural coherent basis for chaotic quantum systems [19].

In conclusion I would like to emphasize again that even though the distinction between coherent and squeezed states remaines, as yet, ambiguous the squeezing itself is generic.

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WIGNER FUNCTIONS FOR NONCLASSICAL STATES
OF A COLLECTION OF TWO-LEVEL ATOMS

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Abstract

The general theory of atomic angular momentum states is used to derive the Wigner distribution function for atomic angular momentum number states, coherent states, and squeezed states. These Wigner functions $W(\theta, \varphi)$ are represented as a pseudo-probability distribution in spherical coordinates θ and φ on the surface of a sphere of radius $\sqrt{j(j+1)}$ where j is the total angular momentum.

1 Introduction

The phase space description of electromagnetic fields has had great success in leading to an understanding of the relationship between semiclassical and quantum theories of light. It was Sudarshan [1] who proved the optical equivalence theorem, i.e., he derived the relationship between the quantities measured by a photodetector and the mean values of the corresponding operators. He showed that the function appearing in the diagonal coherent state representation, that is calculated from the density matrix, provides a link between the semiclassical and quantum descriptions. This function, now denoted by $P(\alpha)$, is generally singular for nonclassical states [2]. In such cases the Wigner function [3,4] has proved to be especially attractive as an alternative. The Wigner function has also proved to be quite useful in discussing related topics [5] such as the photon number distribution and the phase operator distribution. In these problems, the concept of the area of overlap in phase space has been especially useful.

The nonclassical characteristics of the atomic systems, particularly a collection of two-level atoms, has been a subject of much investigation [6,7]. Much of the work has concentrated on the direct calculation of the variances in the atomic operators such as J_x, J^+, and J^-. Very little has been done on the relationship between the nonclassical

aspects and the phase space distributions for atomic operators. For general angular momentum systems, Arecchi, et al. [8] introduced the analog of the diagonal coherent state representation

$$\rho = \int P(\alpha,\beta) \, |\alpha,\beta\rangle \langle\alpha,\beta| \, \sin\alpha \, d\alpha \, d\beta, \tag{1}$$

where $|\alpha,\beta\rangle$ represents the atomic coherent state

$$|\alpha,\beta\rangle = \sum_{m=-j}^{+j} \binom{2j}{m+j}^{1/2} \left(\sin\frac{\alpha}{2}\right)^{j+m} \left(\cos\frac{\alpha}{2}\right)^{j-m} e^{-i(j+m)\beta} |jm\rangle, \tag{2}$$

and where $|jm\rangle$ is the eigenstate of J^2 and J_z. The parameters α and β correspond to θ and φ except that α is measured off the south pole. The coherent state obtains the minimum of the angular momentum uncertainty relation $\langle \Delta J_{x'}^2 \rangle \langle \Delta J_{y'}^2 \rangle \geq |\langle J_{z'} \rangle|^2/4$, where $x', y',$ and z' form an orthogonal coordinate system with z' in the $\hat{r}$ direction with angular coordinates (α,β). The coherent state is just a rotation of the ground Fock state $|j,-j\rangle$ away from the south pole. Arecchi and co-workers discussed the utility of the function $P(\alpha,\beta)$ in atomic problems, and Scully and co-workers have discussed the Wigner function for spin-$\frac{1}{2}$ particles [9]. Using the general theory of multipole operators [10], Agarwal [11] introduced the Wigner function for systems of arbitrary angular momentum. To arrive at this distribution, we first expand the atomic angular momentum operators as

$$G = \sum_{K=0}^{2j} \sum_{Q=-K}^{+K} G_{KQ} T_{KQ}, \tag{3}$$

where T_{KQ} is the multipole operator defined by

$$T_{KQ} = \sum_{m=-j}^{j} \sum_{m'=-j}^{+j} (-1)^{j-m} \sqrt{2k+1} \begin{pmatrix} j & K & j \\ -m & Q & m' \end{pmatrix} |jm\rangle \langle jm'|, \tag{4}$$

where $\begin{pmatrix} j & K & j \\ -m & Q & m' \end{pmatrix}$ is the usual Wigner $3j$ symbol. The expansion coefficients in Eq. (3) are obtained from the orthogonality of the multipole operators, namely

$$G_{KQ} = \text{Tr}\left(GT_{KQ}^{\dagger}\right). \tag{5}$$

The Wigner function associated with G is then defined by [11]

$$W(\theta,\varphi) = \sum_{K=0}^{2j} \sum_{Q=-K}^{+K} Y_{KQ}(\theta,\varphi) G_{KQ}, \tag{6}$$

where Y_{KQ} are the usual spherical harmonics. Note that

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$$\text{Tr } G = \sqrt{\frac{2j+1}{4\pi}} \int W(\theta,\varphi) \sin\theta \, d\theta \, d\varphi = 1 \,, \tag{7}$$

a general property desired of any distribution function. Note further that if two operators $G^{(1)}$ and $G^{(2)}$ are represented respectively by the Wigner functions $W^{(1)}$ and $W^{(2)}$, then

$$\text{Tr } (G_1 G_2) = \int W^{(1)}(\theta,\varphi) \, W^{(2)}(\theta,\varphi) \sin\theta \, d\theta \, d\varphi \,, \tag{8}$$

a defining property of the Wigner distribution. In fact these two features, Eqs. (7) and (8), can be used to derive the form, Eq. (6), of the Wigner function. Thus, unlike the P function, all expectation values can be obtained in terms of the Wigner functions alone.

In this paper we shall consider the structure of the Wigner function associated with the important states like (i) Fock states $|j,m\rangle$, (ii) coherent states $|\alpha,\beta\rangle$, and (iii) squeezed states $|\zeta,m\rangle$ associated with a collection of two-level atoms interacting with a squeezed photon bath. We examine how the quantum character of the state is reflected in the properties of the Wigner function.

2   Angular Momentum States $|jm\rangle$

We first obtain the Wigner function for the state $|jm\rangle$. The density matrix can be written in the form

$$\rho = |jm\rangle \langle jm| \,. \tag{9}$$

Upon using Eqs. (4) through (6), that are used in defining the Wigner function W, we find that

$$W_{jm}(\theta,\varphi) = \sum_{K=0}^{2j} Y_{K0}(\theta,\varphi) (-1)^{j-m} \sqrt{2K+1} \begin{pmatrix} j & K & j \\ -m & 0 & m \end{pmatrix}^* \tag{10}$$

As expected W_{jm} is independent of φ.

This function is plotted in Fig. 1 as a function of $\theta \in (0,\pi)$ and $\varphi \in (-\pi,\pi)$ for $j=5$ and $m=0, -1, \ldots -5$. We plot the distribution both as planar and spherical surfaces. If we suppose that $|jm\rangle$ is an orbital angular momentum state, then quantum-mechanically we would expect the angular momentum vector of length $\sqrt{j(j+1)} \, \hbar$ to be oriented inside a sphere of radius $\sqrt{j(j+1)} \, \hbar$ such that its z component is $m\hbar$ where $m = -5, \ldots, 5$. This situation is depicted in Fig. 2 [12]. The Wigner function $W(\theta,\varphi)$, when integrated over the domain of spherical angle, $\theta \in (0,\pi)$ and $\varphi \in (-\pi,\pi)$, contributes the most positive probability at precisely these locations in θ. At these θ values there is always one peak on the "wavy sea" that is not cancelled by a trough and so contributes a large amount of probability. In Fig. 1 we plot the function $W(\theta,\varphi)$ as a two-dimensional surface, and also the normalized function $\tilde{W} = W/\sqrt{j(j+1)}$ in spherical coordinates so that the oscillations can be viewed as variations in the surface of a sphere of radius one.
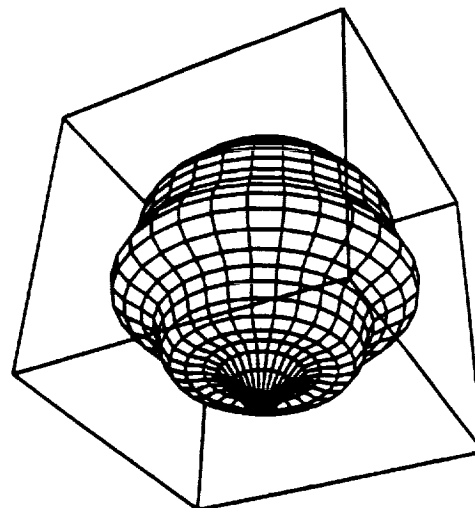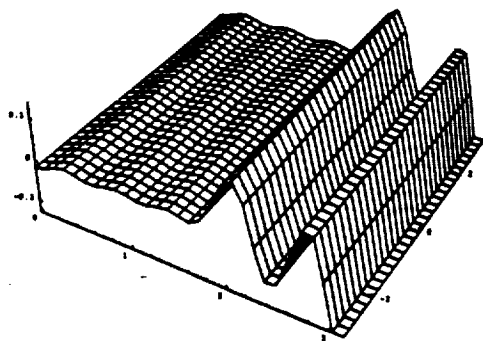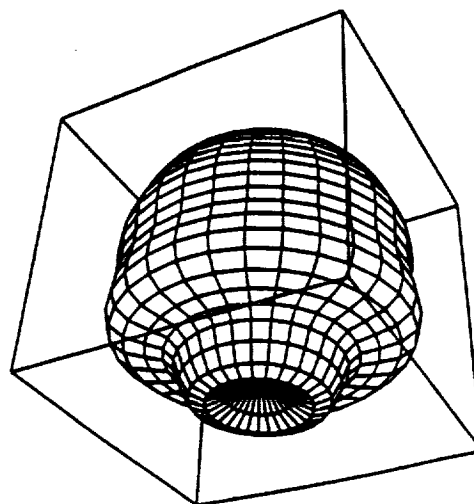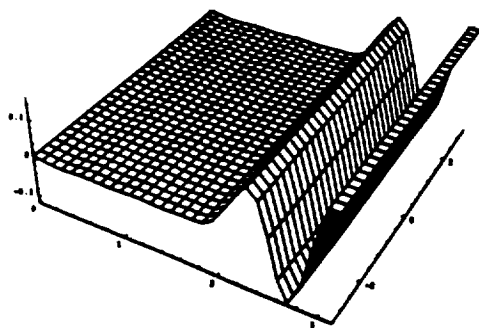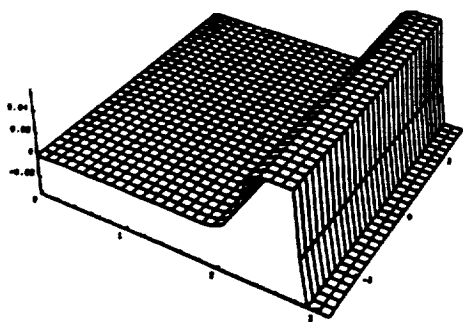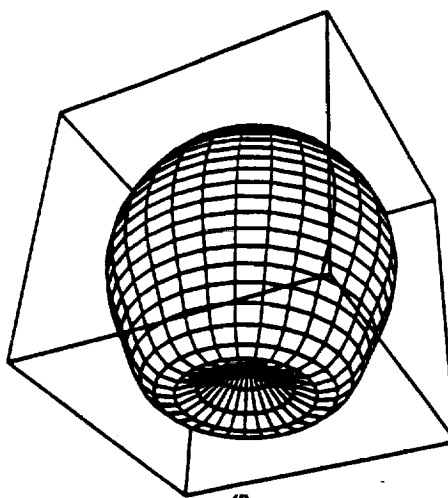
(a)

(b)

(c)

(d)

(e)

(f)

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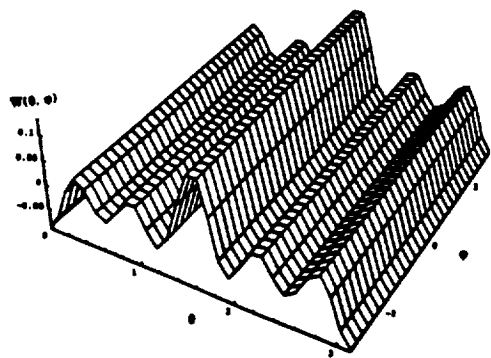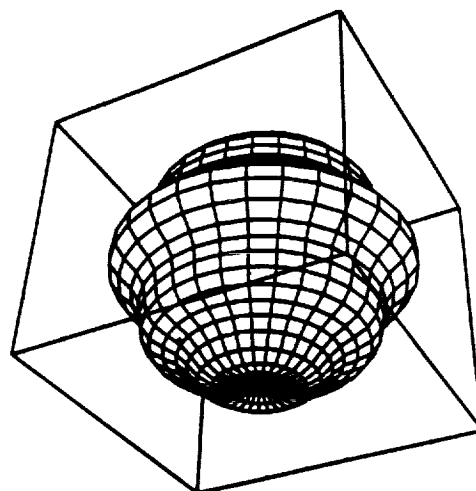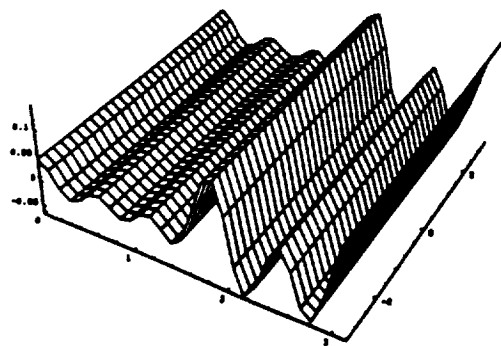
(g)
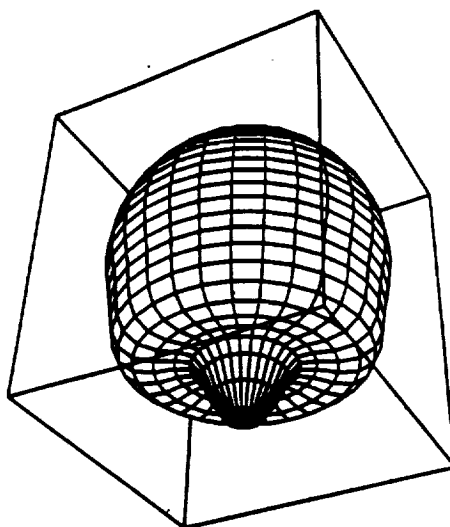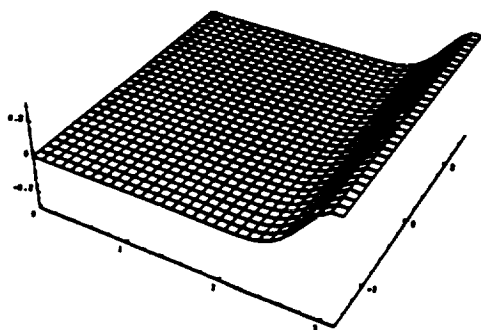

(h)
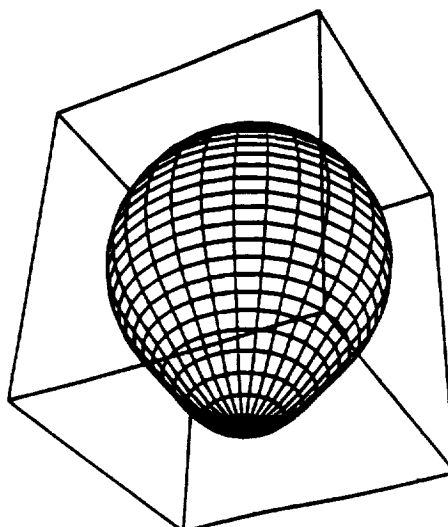

(i)

(j)

(k)

(l)

FIG. 1. Here we plot for $\theta \in (0,\pi)$ and $\varphi \in (-\pi,\pi)$, the normalized Wigner function $\widetilde{W}^{Fock} = W^{Fock}/\sqrt{j(j+1)}$, where $W^{Fock}(\theta,\varphi)$ is given by Eq. (10). The angular momentum Fock states represented here are $|jm\rangle = |5,m\rangle$ where $m = 0,$ $-1, -2, -3, -4, -5$. When integrated over θ and φ , the Wigner function contributes the most positive probability precisely at the locations where the angular momentum vector for $|jm\rangle$ of length $\sqrt{j(j+1)}\,\hbar$ has z component $m\hbar$ (see Fig. 2). These contributions occur where the dominant positive crest of the Wigner function — the peak that is not cancelled by any troughs — contributes. To bring out all the features of $W(\theta,\varphi)$ we plot it first as a two-dimensional surface function of θ and φ in (a), (c), (e), (g), (i), and (k). This method of presentation brings out the scale of the local positive and negative variations of W with respect to the plane $f(\theta,\varphi) = 0$. Then in (b), (d), (f), (h), (j), and (l), we take a global view by plotting $\widetilde{W}(\theta,\varphi) = W(\theta,\varphi)/\sqrt{j(j+1)}$ on a sphere of radius one.
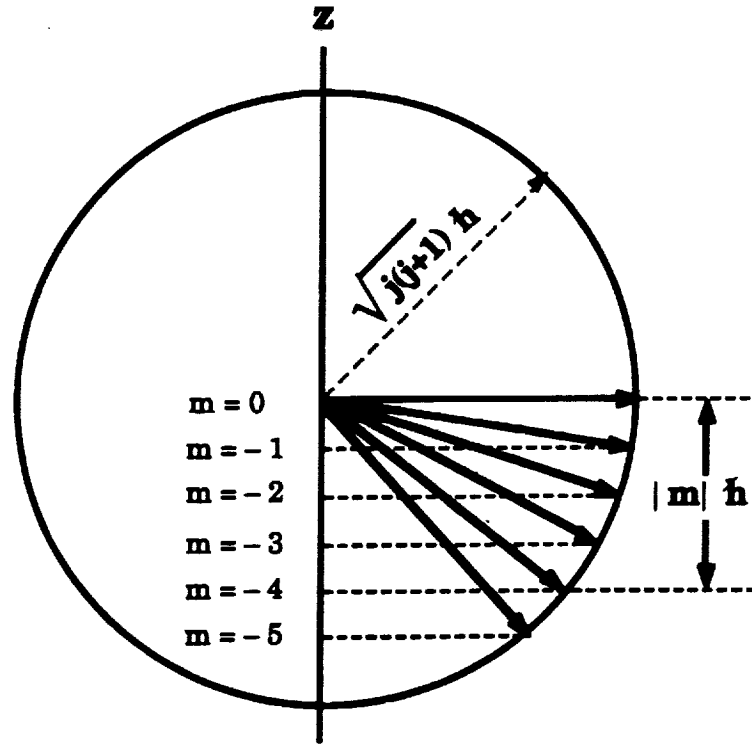

FIG. 2. Here we show a schematic diagram of the angular momentum vector for the Fock states inside a sphere of radius $\sqrt{j(j+1)}\,\hbar$. The vectors all have length $\sqrt{j(j+1)}\,\hbar$ but z component $m\hbar$. These vector locations correspond to the maximal contributions from the Wigner functions shown in Fig. 1. In particular, the Wigner function always has an uncancelled dominant peak at precisely these locations in the angle θ.

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3 Atomic Coherent State $|\alpha,\beta\rangle$

We next consider the Wigner function for the atomic coherent state, Eq. (2),

$$\rho = |\alpha,\beta\rangle \langle\alpha,\beta| . \qquad (11)$$

Using Eqs. (2), (4), and (5), the coefficients of the operator G for the density matrix, Eq. (11), are found to be

$$G_{KQ}^{coherent} = e^{-iQ\beta} (\tan \alpha/2)^Q \sum_{m=-j}^{j} \binom{2j}{j+m}^{1/2} \binom{2j}{j+m+j}^{1/2} (\sin \alpha/2)^{2j+2m}$$

$$\times (\cos \alpha/2)^{2j-2m} (-1)^{j-m-Q} (2K+1)^{1/2} \begin{pmatrix} j & j & K \\ m & -m-Q & Q \end{pmatrix} . \qquad (12)$$

The Wigner function $W^{coherent}(\theta,\varphi)$ is then given by Eq. (6) and is plotted in Fig. 3 for $\alpha = \beta = \pi/4$, recalling that α is measured at the south pole. (Again we have normalized $\widetilde{W} = W/\sqrt{j(j+1)}$.) The coherent state appears as a positive perturbation on the surface of a unit sphere. It is a Gaussian-like distribution located on the sphere's surface at $\theta = 3\pi/4$, $\varphi = \pi/4$; the "Wigner toothache" state. It is just a rotation of the ground Fock state Wigner function from section 2. The Gaussian shape is analogous to that found for the Wigner distribution for coherent states of the single mode radiation field.
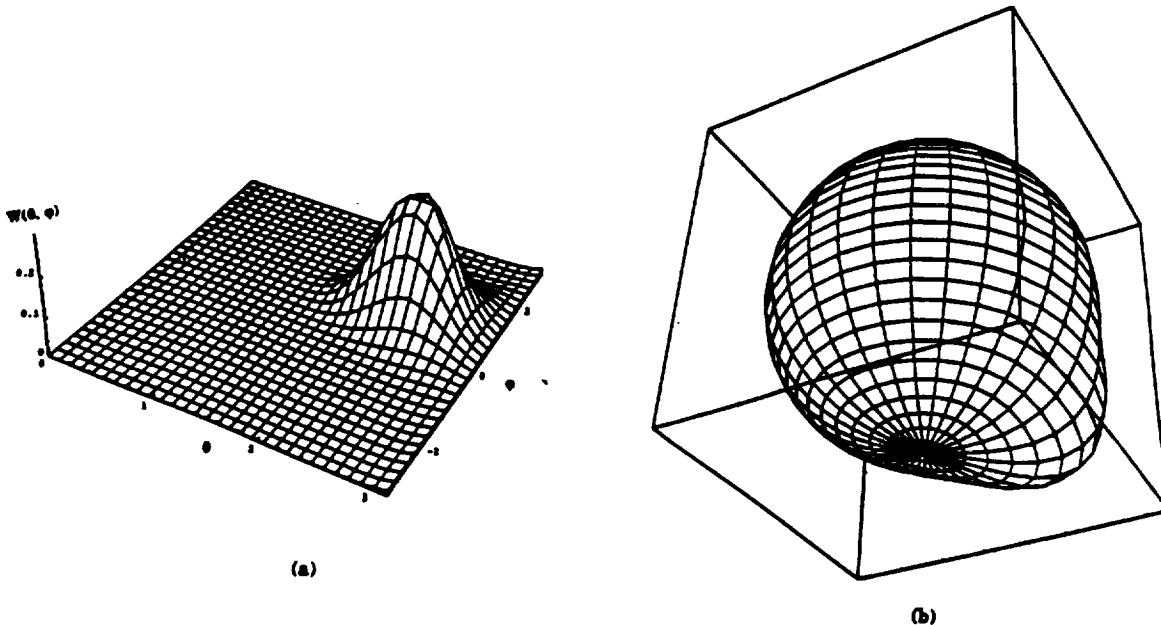

(a)

(b)

FIG. 3. Here we plot the Wigner distribution $W^{coherent}/\sqrt{j(j+1)}$ for the coherent state $|\alpha,\beta\rangle$, Eq. (2). We choose the parameters $\alpha=\beta=\pi/4$ that correspond to a Gaussian distribution localized at $\theta = 3\pi/4$, $\varphi = \pi/4$. This distribution is qualitatively similar to that of the coherent state for photons. Again we present a two-dimensional surface view (a) and a spherical coordinate perspective (b).

4 Atomic Squeezed State $|\zeta,m\rangle$

We finally consider the state [12,13] of the angular momentum system defined by

$$|\zeta,m\rangle = \mathcal{A}_m \exp\left(\Theta J_z\right) \exp\left(-i\pi J_y/2\right)|jm\rangle , \qquad (13)$$

where $\mathcal{A}_m$ is the normalization constant. For $m = -j$, this state — generated by a non-Hermitian operator — describes the behavior of a collection of two-level atoms interacting with a squeezed coherent photon state. In the state, Eq. (13), the x quadrature, i.e., J_x is squeezed as per Eq. (13),

$$\langle(\Delta J_x)^2\rangle - \frac{1}{2}\left|\langle J_z\rangle\right| = -\frac{1}{2}\left|\langle J_z\rangle\left(1 - e^{-|\zeta|}\right)\right| , \qquad (14)$$

where ζ is defined by

$$e^{2\Theta} = \tanh(2|\zeta|) . \qquad (15)$$

This relation implies that $\langle(\Delta J_x)^2\rangle < |\langle J_z\rangle|/2$ that shows a suppression of x noise, ΔJ_x, in uncertainty relation $\langle(\Delta J_x)^2\rangle \langle(\Delta J_y)^2\rangle \geq |\langle J_z\rangle|^2/4$ at the expense of the y fluctuations, ΔJ_y. Thus the states of Eq. (13) can be considered as suitable candidates for squeezed states of the general angular momentum system, besides, Agarwal and Puri [13] have shown that the states, Eq. (13), are the eigenstates of the operator $(J^- \cosh|\zeta| + J^+ \sinh|\zeta|)/\sqrt{2 \sinh 2|\zeta|}$ with the eigenvalue m, and that these states are the analog of the two photon coherent states [2] for photons. Note further that Eq. (13) can be written in terms of the elements of the rotation operator coefficients $d^j_{mm'}(\pi/2)$ via the relationship

$$\langle jm| \zeta p\rangle = A_p e^{\Theta m} d^j_{mp}(\pi/2), \qquad (16)$$

where we define

$$d^j_{mp}(\pi/2) = \frac{\left((j+m)! (j-m)! (j+p)! (j-p)!\right)^{1/2}}{2^j} \sum_{q=-j}^{+j} \frac{(-1)^q}{(j-p-q)! \, q! \, (q+p-m)! \, (j+m-q)!} . \qquad (17)$$

Upon using Eqs. (13), (16), and (17) for the squeezed state, and Eqs. (4) and (5) for the definition of the Wigner function, we find the coefficients of the squeezed density operator G to be

$$G^{squeezed}_{KQ} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} (-1)^{j-m} (2K+1)^{1/2} \begin{pmatrix} j & K & j \\ -m & Q & m' \end{pmatrix} \left[\frac{e^{(m+m')\Theta} d^j_{mp} d^j_{m'p}}{\sum_{m''}\left|d^j_{m''p}\right|^2 e^{2m\Theta}}\right], \qquad (18)$$

where we have also introduced the value of the normalization constant. The Wigner distribution $W^{squeezed}(\theta,\varphi)$ obtained from Eq. (6), using Eqs. (17) and (18), is plotted in

Fig. 4 for $j = 5$ and $p = -5$. We take the squeezing parameter Θ equal to -2.13×10^{-5} which, in the Agarwal and Puri system of two-level atoms interacting with a squeezed photon bath, corresponds to a mean photon occupation number of $\bar{n} = \sinh^2\left(\frac{1}{2}\operatorname{arctanh}\ (e^{2\Theta})\right) = 50$ corresponding to $\zeta = 2.65$. The plot is again normalized so that the elongated Gaussian of the squeezed state appears as a "Wigner banana" draped across the surface of sphere of radius one at the south pole. (To see this, one must take the surface in Fig. 4a and mentally map it onto a sphere of radius one, as in Fig. 4b.) Notice that the localization of the state is squeezed in the x direction at the expense of knowledge about the y location. Agarwal and Puri [13] have shown how the atomic states Eq. (13) can be produced if the collection of two-level atoms interacts with a broad band squeezed bath and if one concentrates only on the steady-state solution for the collective system. The parameter ζ characterizes the squeezed bath with average photon number equal to $\sinh^2\zeta$.
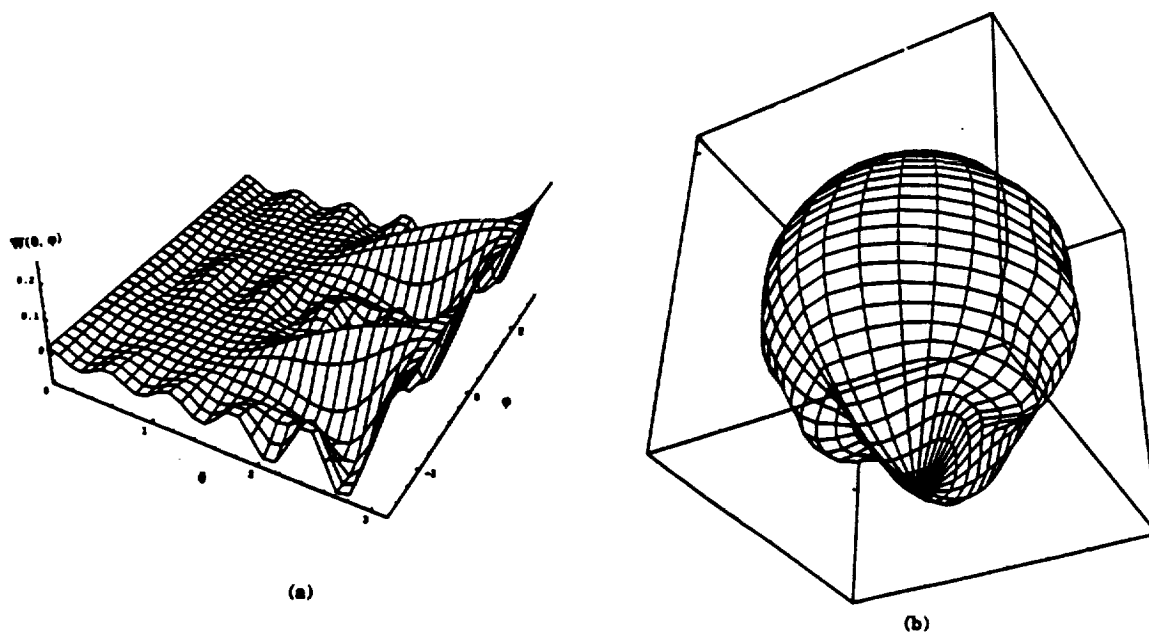

(a)

(b)

FIG. 4. Here we plot the Wigner function for a squeezed angular momentum state $|\zeta, -5\rangle$ defined by Eq. (13). The function $W^{\text{squeezed}}(\theta,\varphi)$ is computed using Eqs. (6), (17), and (18) for a squeezing parameter of $\Theta = -2.13 \times 10^{-5}$ corresponding to a mean occupation number of $\bar{n} = 50$. In (a) we plot the function as a surface $W(\theta,\varphi)$ as before. We have normalized the variation in the surface in spherical coordinates to a sphere of radius $\sqrt{j(j+1)}$ in (b) so that the elongated Gaussian appears here as a "Wigner banana" draped across the surface of the sphere of radius one at the south pole. Notice that the squeezed state is more localized in the x' direction than the coherent state, Fig. 3, at the expense of decreased localization or increased noise in the y' direction.

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5 Summary and Conclusions

In summary the Wigner distribution for a general angular momentum state has been derived and given explicitly for a Fock state, a coherent state, and a squeezed state. Represented as a pseudo-probability distribution on the sphere of radius one, the Wigner function is plotted for these three situations. These plots enable us to understand the nonclassical nature of the states of a collection of identical two-level atoms since the collection is described by the addition of the spin operators for each atom.

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WAVELETS AND SPACETIME SQUEEZE

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Abstract

It is shown that the wavelet is the natural language for the Lorentz covariant description of localized light waves. A model for covariant superposition is constructed for light waves with different frequencies. It is therefore possible to construct a wave function for light waves carrying a covariant probability interpretation. It is shown that the time-energy uncertainty relation $(\Delta t)(\Delta \omega) \simeq 1$ for light waves is a Lorentz-invariant relation. The connection between photons and localized light waves is examined critically.

1 Introduction

The word "squeeze" is relatively new in physics, but the squeeze transformation has been one of the most important transformations in both relativity and quantum mechanics [1]. The geometry of squeeze is very simple. Let us consider the two–dimensional xy coordinate system. We can expand the x coordinate while contracting y in such a way that the product xy is preserved. This transformation is built in many branches of physics, including classical mechanics, special relativity, quantification of uncertainty relations, the Bogoliubov transformation in condensed matter physics, and two–photon coherent states in quantum optics [2]. Indeed, this new word enables us to study the squeeze transformations more effectively and systematically.

The concept of squeeze in quantum optics was developed from the parametric oscillation. Let us start with a simple harmonic oscillator with a given frequency. If we add a small sinusoidal variation to the frequency, the original oscillator will be modulated [3], and the resulting oscillation will be, to a good approximation, a superposition of two oscillations with different frequencies. We can use the mathematics of this oscillator system for the Fock-space description of creation and annihilation of two photons in a coherent or correlated manner, created in a laser cavity where the index of refraction undergoes a sinusoidal variation with respect to time.

Indeed, the mathematics of this two-photon system was worked out by Dirac in 1963 [4]. It is possible to translate the mathematics of this two-photon system into that of the Wigner phase-space distribution function defined over four-dimensional phase space. It is remarkable that the

two–photon coherence we observe in laboratories can be described by the squeeze transformations in this four–dimensional phase space [5].

Fourteen years before the appearance of his 1963 paper [4], Dirac observed that the Lorentz boost in a given direction is a squeeze transformation. In his 1949 paper entitled "Forms of Relativistic Dynamics" [6], Dirac observed that the Lorentz boost along a given direction is a squeeze transformation. The application of this idea to relativistic hadronic system was made in 1973 [7].

These days, we have a new mathematical technique called wavelets, which serves a useful purpose in signal analysis [8]. This technique contains many features which are not available in the conventional method of Fourier analysis. It accommodates squeeze transformations. The wavelets can also constitute a representation of the Lorentz group. With these features in mind, we shall examine in this paper whether the wavelet can serve as the proper language for covariant localized light waves.

Photons are important particles in physics. Since they are relativistic particles, the quantum mechanics of photons occupies an important place in relativistic quantum mechanics. The difficulty in formulating the theory of photons is that there is no position operator which is covariant and Hermitian. This is known as the photon localization problem [9]. However, when we discuss photons, we always think of localized light waves in a given Lorentz frame. The question then is whether someone in a different Lorentz frame will think in the same way.

With this point in mind, we considered the covariance of localized light waves [10]. It was noted in our 1987 paper that localized light waves cannot represent photons. It was shown however that, if the momentum distribution is sufficiently narrow, the light wave distribution can numerically be close to that of the photon. For this reason, it is still useful to study the covariance of localized light waves.

The question of relating waves with photons is a well–defined problem in physics [11], even though the problem has not yet been completely solved. In this paper, we shall bring them closer together by using the wavelet formulation of light waves.

2   Localized Light Waves

For light waves, the Fourier relation $(\Delta t)(\Delta \omega) \simeq 1$ was known before the present form of quantum mechanics was formulated [12, 13]. However, the question of whether this is a Lorentz–invariant relation has not yet been fully examined. Let us consider a blinking traffic light. A stationary observer will insist on $(\Delta t)(\Delta \omega) \simeq 1$. An observer in an automobile moving toward the light will see the same blinking light. This observer will also insist on $(\Delta t^*)(\Delta \omega^*) \simeq 1$ on his/her coordinate system. However, these observers may not agree with each other, because neither t nor ω is a Lorentz–invariant variable. The product of two non–invariant quantities does not necessarily lead to an invariant quantity.

Let us assume that the automobile is moving in the negative z direction with velocity parameter β. Since both t and ω are the time–like components of four-vectors $(\mathbf{x}, t)$ and $(\mathbf{k}, \omega)$ respectively, a Lorentz boost along the z direction will lead to new variables:

$$t^* = (t + \beta z)/(1 - \beta^2)^{1/2}, \qquad \omega^* = (\omega + \beta k)/(1 - \beta^2)^{1/2}, \tag{1}$$

where the light wave is assumed to travel along the z axis with $k = \omega$. In the above transformation, the light wave is boosted along the positive z direction. If the light passes through the point $z = 0$ at $t = 0$, then $t = z$ on the light front, and the transformations of Eq.(1) become

$$t^* = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} t, \qquad \omega^* = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \omega. \qquad (2)$$

These equations will formally lead us to

$$(\Delta t^*)(\Delta \omega^*) = \frac{1+\beta}{1-\beta}(\Delta t)(\Delta \omega), \qquad (3)$$

which indicates that the time-energy uncertainty relation is not a Lorentz–invariant relation, and that Planck's constant depends on the Lorentz frame in which the measurement is taken. This is not correct, and we need a better understanding of the transformation properties of Δt and $\Delta \omega$.

This problem is related to another fundamental problem in physics. We are tempted to say that the above-mentioned Fourier relation is a time-energy uncertainty relation. However, in order that it be an uncertainty relation, the wave function for the light wave should carry a probability interpretation. This problem has a stormy history and is commonly known as the photon localization problem [9]. The traditional way of stating this problem is that there is no self-adjoint position operator for massless particles including photons.

In spite of this theoretical difficulty, it is becoming increasingly clear that single photons can be localized by detectors in laboratories. The question then is whether it is possible to construct the language of the photon localization which we observe through oscilloscopes. Throughout the history of this localization problem, the main issue has been and still is how to construct localized photon wave functions consistent with special relativity.

However, in this paper, we shall approach this problem by constructing covariant localized light waves and comparing them with photon field operators. As we shall see, the task of constructing a covariant light wave is constructing a wavelet representation of a light wave. First, we construct a unitary representation for Lorentz transformations for a monochromatic light wave. It is shown then that a Lorentz-covariant superposition of light waves is possible for different frequencies. After constructing the covariant light wave, we shall observe that there is a gap between the concept of photons and that of localized waves. From the physical point of view, this gap is not significant. However, there is a definite distinction between the mathematics of photons and that of light waves.

3 Affine Symmetry of Wavelets

Like Fourier transformations, wavelets are the superposition of plane waves with different frequencies. In addition, the distribution function has the affine symmetry. Let us briefly examine what the affine transformation is [14].

To a given number, we can add another number, and we can also multiply it by another real number. This combined mathematical operation is called the affine transformation. Since the multiplication does not commute with addition, affine transformations can only be achieved by

matrices. We can write the addition of b to x as

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}. \tag{4}$$

This results in $x' = x + b$. We shall call this operation translation. The inverse of the above translation matrix is

$$\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}. \tag{5}$$

We can represent the multiplication of by e^η as

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} e^\eta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{6}$$

which leads to $x' = e^\eta x$. This multiplication operation is usually called dilation. The inverse of the above dilation matrix is

$$\begin{pmatrix} e^{-\eta} & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

The translation does not commute dilation. If dilation precedes translation, we shall call this the affine transformation of the first kind, and the transformation takes the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^\eta & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^\eta & b \\ 0 & 1 \end{pmatrix} \tag{8}$$

If the translation is made first, we shall call this the affine transformation of the second kind. The transformation takes the form

$$\begin{pmatrix} e^\eta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^\eta & e^\eta b \\ 0 & 1 \end{pmatrix} \tag{9}$$

Indeed, the affine transformations of the first and second kinds lead to

$$x' = e^\eta x + b, \qquad x' = e^\eta(x + b), \tag{10}$$

respectively. Let us next consider inverse transformations. The inverse of the first-kind transformation of Eq.(8) becomes an affine transformation of the second kind:

$$\begin{pmatrix} e^{-\eta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-\eta} & -e^{-\eta}b \\ 0 & 1 \end{pmatrix}, \tag{11}$$

while the inverse of the second kind of Eq.( ) becomes an affine transformation of the first kind:

$$\begin{pmatrix} 0 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\eta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-\eta} & -b \\ 0 & 1 \end{pmatrix}. \tag{12}$$

The distinction between the first and second kinds is not mathematically precise, because the translation subgroup of the affine group is an invariant subgroup. We make this distinction purely for convenience. Whether we choose the first kind or second kind depends on the physical problem

under consideration. For a covariant description of light waves, the affine transformation of the second kind is more appropriate, and this affine transformation takes the form

$$x' = e^{\eta}(x + b), \tag{13}$$

and its inverse is

$$x = e^{-\eta}x' - b = e^{-\eta}(x' - e^{\eta}b). \tag{14}$$

Therefore, the transformation of a function $f(x)$ corresponding to the vector transformation of Eq.(13) is

$$f\left(e^{-\eta}x - b\right) = f\left(e^{-\eta}(x - e^{\eta}b)\right). \tag{15}$$

This translation symmetry leads us to the concept of "window," which will be discussed further in Sec. 4.

Next, let us consider the normalization of the function. The normalization integral

$$\int |f(e^{-\eta}x - b)|^2 dx \tag{16}$$

does not depend on translation parameter b, but it depends on the multiplication parameter η. Indeed,

$$\int |f(e^{-\eta}x - b)|^2 dx = e^{\eta} \int |f(x - b)|^2 dx. \tag{17}$$

In order to preserve the normalization under the affine transformation, we can introduce the form [8]

$$e^{-\eta/2}f(e^{-\eta}x - b). \tag{18}$$

This is the wavelet form of the function $f(e^{-\eta}x - b)$. This is of course the wavelet form of the second kind. The wavelet of the first kind will be

$$e^{-\eta/2}f\left(e^{-\eta}(x - b)\right). \tag{19}$$

Both the first and second kinds of wavelet forms are discussed in the literature [8]

4 Windows

There are in physics many distributions, and their functional forms usually extend from minus infinity to plus infinity. However, the distribution function of physical interest is usually concentrated within a finite interval. It thus is not uncommon in physics that mathematical difficulties in theory come from the region in which the distribution function is almost zero and is physically insignificant. Thus, we are tempted to ignore contributions from outside of the specified region. This is called the "cut-off" procedure.

One of the difficulties of this procedure is that a good cut-off approximation in one Lorentz frame may not remain good in different frames. The translational symmetry of wavelets allows us to define the cut-off procedure which will remain valid in all Lorentz frames.

We can allow the function to be nonzero within the interval

$$a \le x \le a + w, \tag{20}$$

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while demanding that the function vanish everywhere else. The parameter w determines the size of the window. The window can be translated or expanded/contracted according to the operation of the affine group. We can now define the window of the first kind and the window of the second kind. Both widows can be translated according to the transformation given in Eq.(4). The window of the first kind is not affected by the scale transformation. On the other hand, the size and location of the window of the second kind becomes affected by the scale transformation according to Eq.(11). Depending on our need, we can define the window to preserve the information. The idea of introducing the new word "window" is to define the information-preserving boundary conditions.

The window may become a very powerful device in describing the real world, especially in localization problems dealing with distributions concentrated within a finite region. The concept of cut-off in a distribution function is not new. However, the cut-off process causes mathematical difficulties usually introducing undesirable singularities. Also the cut-off process destroys the Lorentz covariance, unless it is done carefully. A good approximation in one Lorentz frame is not necessarily a good approximation in different frames. In this paper, we shall examine possible role of wavelets and windows in discussing localized light waves and their connection to photons.

5 Light Waves and Wave Packets in Quantum Mechanics

We are concerned here with the possibility of constructing wave functions with quantum probability interpretation for relativistic massless particles. The natural starting point for tackling this problem is a free-particle wave packet in nonrelativistic quantum mechanics which we pretend to understand. Let us write down the time-dependent Schrödinger equation for a free particle moving in the z direction:

$$i\frac{\partial}{\partial t}\psi(z,t) = -\frac{1}{2m}\frac{\partial^2}{\partial z^2}\psi(z,t).$$ (21)

The Hamiltonian commutes with the momentum operator. If the momentum is sharply defined, the solution of the above differential equation is

$$\psi(z,t) = \exp[i(pz - p^2t/2m)].$$ (22)

If the momentum is not sharply defined, we have to take the linear superposition:

$$\psi(z,t) = \int \exp\left\{i\left(kz - \frac{k^2}{2m}t\right)\right\} g(k)dk.$$ (23)

The width of the wave function becomes wider as the time variable increases. This is known as the wave packet spread.

Let us study the transformation properties of this wave function. Rotation and translation properties are trivial. In order to study the boost property within the framework of Galilean kinematics, let us imagine an observer moving in the negative z direction. To this observer, the center of the wave function moves along the positive z direction, and the transformed wave function takes the form

$$\psi_v(z,t) = \exp[-im(vz - \frac{1}{2}v^2t)] \int g(k - mv)e^{i(kz - k^2t/2m)}dk,$$ (24)

where v is the boost velocity. This expression is different from the usual expression in textbooks by an exponential factor in front of the integral sign. The origin of this phase factor is well-understood.

In nonrelativistic quantum mechanics, $\psi_v(z,t)$ has a probability interpretation, and there is no difficulty in giving an interpretation for the transformed wave function in spite of the above-mentioned phase factor. The basic unsolved problem is whether the probabilistic interpretation can be extended into the Lorentzian regime. This has been a fundamental unsolved problem for decades, and we do not propose to solve all the problems in this paper. A reasonable starting point for approaching this problem is to see whether a covariant probability interpretation can be given to light waves.

For light waves, we start with the usual expression

$$f(z,t) = \frac{1}{\sqrt{2\pi}} \int g(k)e^{i(kz-\omega t)}dk. \tag{25}$$

Unlike the case of the Schrödinger wave, ω is equal to k, and there is no spread of wave packet. The velocity of propagation is always that of light. We might therefore be led to think that the problem for light waves is simpler than that for nonrelativistic Schrödinger waves. This is not the case for the following reasons.

(1). We like to have a wave function for light waves. However, it is not clear which component of the Maxwell wave should be identified with the quantal wave whose absolute square gives a probability distribution. Should this be the electric or magnetic field, or should it be the four-potential?

(2). The expression given in Eq.(25) is valid in a given Lorentz frame. What form does this equation take for an observer in a different frame?

(3). Even if we are able to construct localized light waves, does this solve the photon localization problem?

(4). The photon has spin 1 either parallel or antiparallel to its momentum. The photon also has gauge degrees of freedom. How are these related to the above-mentioned problems?

Indeed, the burden on Eq.(25) is the Lorentz covariance. It is not difficult to carry out a spectral analysis and therefore to give a probability interpretation for the expression of Eq.(25) in a given Lorentz frame. However, this interpretation has to be covariant. This is precisely the problem we are addressing in the present paper.

6 Extended Little Group for Photons

The little group is the maximal subgroup of the Lorentz group which leaves the four-momentum of a given particle invariant. For a massless particle moving along the z direction, the little group is generated by [15]

$$J_3, N_1, N_2, \tag{26}$$

with $N_1 = K_1 - J_2, N_2 = K_2 + J_2$, where J_i and K_i are the generators of rotations and boosts respectively. The above generators satisfy the commutation relations:

$$[N_1, N_2] = 0, \qquad [J_3, N_1] = iN_2, \qquad [J_3, N_2] = -iN_2. \tag{27}$$

These commutation relations are identical to those of the two-dimensional Euclidean group.

In addition, we can consider K_3 which generates boosts along the z direction. This operator satisfies the following commutation relations with the above generators of the little group.

$$[K_3, J_3] = 0, \qquad [K_3, N_1] = -iN_1, \qquad [K_3, N_2] = -iN_2. \tag{28}$$

Since the operators N_1, N_2, J_3, and K_3 form a closed Lie algebra, we shall call the group generated by these four operators the "extended little group."

The boost generated by K_3 has no effect on J_3, while changing the scale of N_1 and N_2. In particular, if we start with a monochromatic light wave whose four–potential is

$$A^\mu(x) = (A, 0, 0, 0)e^{i(kz-\omega t)} \tag{29}$$

in the metric convention: $x^\mu = (x, y, z, t)$, the Lorentz boost generated by K_3 leaves the above expression invariant. Since N_1 and N_2 generate gauge transformations which do not lead to observable consequences, we can stick to the above expression, and ignore the effect of N_1 and N_2. J_3 generates rotations around the z axis. In this case, the rotation leads to a linear combination of the x and y components. This operation is consistent with the fact that the photon has two independent components, which is thoroughly familiar to us. Therefore for all practical purposes, $A^\mu(x)$ has just one component which remains invariant under transformations of the extended little group. We can thus write $A^\mu(x)$ as

$$A^\mu(x) = Ae^{i(kz-\omega t)}. \tag{30}$$

While the group of Lorentz transformations has six generators, the extended little group has only four. This means that the extended little group is a subgroup of the Lorentz group. How can we then generalize the above reasoning to take into account the most general case? The choice of the z axis is purely for convenience, and it was chosen to be the direction of the wave propagation. If this direction is rotated, it is not difficult to deal with the problem. If the boost is made along the direction different from that of propagation, then the operation is equivalent to a gauge transformation followed by a rotation. Therefore, the extended little group, while being simpler than the six-parameter Lorentz group can take care of all possible Lorentz transformations of the monochromatic wave.

The above reasoning remains valid for the case of the superposition of several waves with different frequencies propagating in the same direction:

$$A^\mu(x) = \sum_i A_i e^{i(k_i z - \omega_i it)}, \tag{31}$$

and the norm:

$$N = \sum_i |A_i|^2. \tag{32}$$

remains invariant under transformations of the extended little group.

7 Unitary Representation for Four–potentials

One of the difficulties in dealing with the photon problem has been that the electromagnetic four-potential could not be identified with a unitary irreducible representation of the Poincaré group. The purpose of this section is to resolve this problem. In Ref. [15], we studied unitary transformations associated with Lorentz boosts along the direction perpendicular to the momentum. In this section, we shall deal with the most general case of boosting along an arbitrary direction.

Let us consider a monochromatic light wave travelling along the z axis with four-momentum p. The four-potential takes the form

$$A^\mu(x) = A^\mu e^{i\omega(z-t)}, \tag{33}$$

with

$$A^\mu = (A_1, A_2, A_3, A_0). \tag{34}$$

We use the metric convention: $x^\mu = (x, y, z, t)$. The momentum four-vector in this convention is

$$p^\mu = (0, 0, \omega, \omega). \tag{35}$$

Among many possible forms of the gauge-dependent four-vector A^μ, we are interested in the eigenstates of the helicity operator:

$$S_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

The four-vectors satisfying this condition are

$$A^\mu_\pm = (1, \pm i, 0, 0), \tag{37}$$

where the subscripts + and − specify the positive and negative helicity states respectively. These are commonly called the photon polarization vectors.

In order that the four-vector be a helicity state, it is essential that the time-like and longitudinal components vanish:

$$A_3 = A_0 = 0. \tag{38}$$

This condition is equivalent to the combined effect of the Lorentz condition:

$$\frac{\partial}{\partial x^\mu} A^\mu(x) = 0, \tag{39}$$

and the transversality condition:

$$\nabla \cdot \mathbf{A}(x) = 0. \tag{40}$$

As before, we call this combined condition the helicity gauge.

While the Lorentz condition of Eq.(39) is Lorentz-invariant, the transversality condition of Eq.(40) is not. However, both conditions are invariant under rotations and under boosts along the direction of momentum. We call these helicity preserving transformations. If we make a boost

along an arbitrary direction, this is not a helicity preserving transformation. However, we can express this in terms of helicity preserving transformations preceded by a gauge transformation.

Let us consider in detail the boost along the arbitrary direction. This boost will transform the momentum p to p',

$$p'^{\mu} = B_{\phi}(\eta)p^{\mu}. \tag{41}$$

However, this is not the only way in which p can be transformed to p'. We can boost p along the z direction and rotate it around the y axis. The application of the transformation $[R(\theta)B_z(\xi)]$ on the four-momentum gives the same effect as that of the application of $B_{\phi}(\eta)$. Indeed, the matrix

$$D(\eta) = [B_{\phi}(\eta)]^{-1}R(\theta)B_z(\xi) \tag{42}$$

leaves the four-momentum invariant, and is therefore an element of the $E(2)$-like little group for photons. The effect of the above D matrix on the polarization vectors has been calculated in Appendix A, and the result is

$$D(\eta)A_{\pm}^{\mu} = A_{\pm}^{\mu} + (p^{\mu}/\omega)u(\eta, \theta), \tag{43}$$

where

$$u(\eta, \theta) = \frac{-2\sin(\theta/2)\cosh(\eta/2)}{\cos(\theta/2)\cosh(\eta/2) + \sqrt{(\cos(\theta/2)\cosh(\eta/2))^2 - 1}}. \tag{44}$$

Thus $D(\eta)$ applied to the polarization vector results in the addition of a term which is proportional to the four-momentum. $D(\eta)$ therefore performs a gauge transformation on $A_{\pm}^{\mu}$. With this preparation, let us boost the photon polarization vector:

$$\tilde{A}_{\pm}^{\mu} = B_{\phi}(\eta)A_{\pm}^{\mu}. \tag{45}$$

The four-vector $\tilde{A}_{\pm}^{\mu}$ satisfies the Lorentz condition $p_{\mu}\tilde{A}_{\pm}^{\mu} = 0$, but its fourth component will not vanish. The four-vector $\tilde{A}_{\pm}^{\mu}$ does not satisfy the helicity condition.

On the other hand, if we boost the four-vector $A_{\pm}^{\mu}$ after performing the gauge transformation $D(\eta)$,

$$\begin{aligned} A_{\pm}'^{\mu} &= B_{\phi}(\eta)A_{\pm}^{\mu} \\ &= B_{\phi}(\eta)[B_{\phi}^{-1}(\eta)R(\theta)B_z(\xi)]A_{\pm}^{\mu} \\ &= R(\theta)B_z(\xi)A_{\pm}^{\mu}. \end{aligned} \tag{46}$$

Since $B_z(\xi)$ leaves $A_{\pm}^{\mu}$ invariant, we arrive at the conclusion that

$$A_{\pm}'^{\mu} = R(\theta)A_{\pm}^{\mu}. \tag{47}$$

This means

$$A_{\pm}'^{\mu} = B_x(\eta)D(\eta)A_{\pm}^{\mu} = (\cos\theta, \pm i, -\sin\theta, 0), \tag{48}$$

which satisfies the helicity condition:

$$\mathbf{A}_{\pm}'^0 = 0, \tag{49}$$

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and

$$\mathbf{p}' \cdot \mathbf{A}'_{\pm} = 0. \tag{50}$$

The Lorentz boost $B(\eta)$ on $A^\mu_\pm$ preceded by the gauge transformation $D(\eta)$ leads to the pure rotation $R(\theta)$. This rotation is a finite-dimensional unitary transformation.

The above result indicates, for a monochromatic wave, that all we have to know is how to rotate. If, however, the photon momentum has a distribution, we have to deal with a linear superposition of waves with different momenta. The photon momentum can have both longitudinal and transverse distributions. In this paper, we shall assume that there is only longitudinal distribution. This of course is a limitation of the model we present. However, our apology is limited in view of the fact that laser beams these days can go to the moon and come back after reflection.

With this point in mind, we note first that the above-mentioned unitary transformation preserves the photon polarization. This means that we can drop the polarization index from A^μ assuming that the photon has either positive or negative polarization. $A^\mu(x)$ can now be replaced by $A(x)$.

Next, the transformation matrices discussed in this section depend only on the direction and the magnitude of the boost but not on the photon energy. This is due to the fact that the photon is a massless particle [15]. For the superposition of two different frequency states:

$$A(x) = A_1 e^{i\omega_1(z-t)} + A_2 e^{i\omega_2(z-t)}, \tag{51}$$

a Lorentz boost along an arbitrary direction results in a rotation followed by a boost along the z direction. Since neither the rotation nor the boost along the z axis changes the magnitude of $A_i (i = 1, 2)$, the quantity

$$|A|^2 = |A_1|^2 + |A_2|^2 \tag{52}$$

remains invariant under the Lorentz transformation. This result can be generalized to the super-position of many different frequencies:

$$A(x) = \sum_k A_k e^{i(kz-t)}, \tag{53}$$

with $|A|^2 = \sum_k |A_k|^2$ .The norm $|A|^2$ remains invariant under the Lorentz transformation in the sense that it is invariant under rotations and is invariant under the boost along the z direction.

Can this sum be transformed into an integral form of Eq.(25)? From the physical point of view, the answer should be Yes. Mathematically, the problem is how to construct a Lorentz-invariant integral measure. It is not difficult to see that the norm of Eq.(32) remains invariant under rotations, which perform unitary transformations on the system. The problem is how to construct a measure invariant under the boost along the z direction.

8 Localized Light Wavelets

For light waves, we use the form of Eq.(25). Let us write down the expression again.

$$f(z,t) = \frac{1}{\sqrt{2\pi}} \int g(k) e^{i(kz-\omega t)} dk. \tag{54}$$

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However, the form commonly used in quantum electrodynamics is

$$A(z,t) = \int \sqrt{\frac{1}{\omega}} a(k) e^{i(kz-\omega t)} dk. \tag{55}$$

This is a covariant expression in the sense that the norm

$$\int |a(k)|^2 (1/\omega) dk. \tag{56}$$

is invariant under the Lorentz boost, because the integral measure $(1/\omega)dk$ is Lorentz-invariant. On the other hand, the expression given in Eq.(54) is not covariant if $g(k)$ is a scalar function, because the measure dk is not invariant.

It is possible to give a particle interpretation to Eq.(55) after second quantization. However, $A(z,t)$ cannot be used for the localization of photons. On the other hand, it is possible to give a localized probability interpretation to $f(z,t)$ of Eq.(54), while it does not accept the particle interpretation of quantum field theory.

If $g(k)$ is not a scalar function, what is its transformation property? We shall approach this problem using the light-cone coordinate system. We define the light-cone variables as

$$s = (z+t)/2, \qquad u = (z-t). \tag{57}$$

The Fourier-conjugate momentum variables are

$$k_s = (k-\omega), \qquad k_u = (k+\omega)/2. \tag{58}$$

If we boost the light wave (or move against the wave with velocity parameter β), the new coordinate variables become

$$s' = \alpha_+ s, \qquad u' = \alpha_- u, \tag{59}$$

where $\alpha_\pm = [(1 \pm \beta)/(1 \mp \beta)]^{1/2}$. If we construct a phase space consisting of s and k_s or u and k_u, the effect of the Lorentz boost will simply be the elongation and contraction of the coordinate axes. If the coordinate s is elongated by α_+, then k_s is contracted by α_- with $\alpha_+ \alpha_- = 1$.

In the case of light waves, k_s vanishes, and k_u becomes k or ω. In terms of the light-cone variables, the expression of Eq.(54) becomes

$$f(u) = (1/2\pi)^{1/2} \int g(k) e^{iku} dk. \tag{60}$$

We are interested in a unitary transformation of the above expression into another Lorentz frame. In order that the norm

$$\int |g(k)|^2 dk \tag{61}$$

be Lorentz-invariant, $f(u)$ and $g(k)$ should be transformed like

$$f(u) \to \sqrt{\alpha_+} f(\alpha_+ u), \qquad g(k) \to \sqrt{\alpha_-} g(\alpha_- k). \tag{62}$$

Then Parseval's relation:

$$\int |f(u)|^2 du = \int |g(k)|^2 dk \tag{63}$$

will remain Lorentz-invariant.

It is not difficult to understand why u and k in Eq.(62) are multiplied by α_+ and α_- respectively. However, we still have to give a physical reason for the existence of the multipliers $(\alpha_\pm)^{1/2}$ in front of $f(u)$ and $g(k)$. They are there because the integration measure in Eq.(54) is not Lorentz-invariant.

In Ref. [10], we argued from our experience in the relativistic quark model that the integration measure can become Lorentz invariant if we take into account the remaining light-cone variables in Eq.(57) and Eq.(58). Indeed, the measures $(duds)$ and $(dk_u dk_s)$ are Lorentz invariant. However, this argument is not complete because the s and k_s variables do not exist in the case of light waves. In Ref. [16], Kim and Wigner pointed out that the multipliers in Eq.(62) come from the requirement that the Wigner phase–space distribution function be covariant under Lorentz transformations.

Let us illustrate the wavelet form using a Gaussian form. We can consider the g(k) function of the form

$$g(k) = (1/\pi b)^{1/4} \exp\left\{-(k-p)^2/2b\right\}, \tag{64}$$

where b is a constant and specifies the width of the distribution, and p is the average momentum:

$$p = \int k|g(k)|^2 dk. \tag{65}$$

Under the Lorentz boost according to Eq.(62), $g(k)$ becomes

$$(1/\pi b)^{1/4}\sqrt{\alpha_-} \exp\left\{-\sqrt{\alpha_-}(k-\sqrt{\alpha_+}p)^2/2b\right\}. \tag{66}$$

We note here that the average momentum p is now increased to $\sqrt{\alpha_+}p$. The average momentum therefore is a covariant quantity, and α_- can therefore be written as

$$\alpha_- = \Omega/p, \tag{67}$$

where Ω is the average momentum in the Lorentz frame in which $\alpha_- = 1$.

As a consequence, in order to maintain the covariance, we can replace $f(u)$ and $g(k)$ by $F(u)$ and $G(u)$ respectively, where

$$F(u) = \sqrt{\frac{p}{\Omega}}f(u), \qquad G(k) = \sqrt{\frac{\Omega}{p}}g(k). \tag{68}$$

These functions will satisfy Parseval's equation:

$$\int |F(u)|^2 du = \int |G(k)|^2 dk \tag{69}$$

in every Lorentz frame without the burden of carrying the multipliers $\sqrt{\alpha_+}$ and $\sqrt{\alpha_-}$.

9 The Concept of Photons

It is now possible to construct a localized wave function for a light wave with a Lorentz-invariant normalization. This wave function is now called the wavelet. We shall examine in this section

whether the wavelet can be used for photons. If the answer is NO, we then have to examine how close the wavelet is to the particle description of photons.

Let us see how the mathematics for the light-wave localization is different from that of quantum electrodynamics where photons acquire a particle interpretation through second quantization. In QED, we start with the Klein-Gordon equation with its normalization procedure. As a consequence, we use the expression:

$$g(k) = \frac{1}{\sqrt{k}} a(k), \tag{70}$$

where a(k) is a scalar function. The Lorentz-transformation property of this quantity is the same as that for $G(k)$ of Eq.(68).

However, the basic difference between the above expression and that of Eq.(68) is that the kinematical factor in front of $a(k)$ is $1/\sqrt{k}$ in Eq.(70), while that for $G(k)$ of Eq.(68) is $1/\sqrt{p}$. This is the basic gap between wavelets and photons. The gap becomes narrower when the distribution in k becomes narrower.

Furthermore, we can use the concept of windows to sharpen up the localization. Instead of leaving insignificant non-zero distribution outside the localization region, we can assume that the distribution vanishes outside the region.

I have to do some more writing.

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IV. TIME-DEPENTENT AND DISSIPATION PROBLEMS

NON-STATIONARY AND NON-LINEAR DISPERSIVE MEDIUM AS EXTERNAL FIELD WHICH GENERATES THE SQUEEZED STATES

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Abstract

The theory of quantum effects in non-linear dielectric media influenced by pumping external field based directly on Maxwell equations is developed. The diagonalisation of Hamiltonian of quantized generated field by the canonical Bogoliubov transformations allowed to obtain the general expressions for the number of created photons and for the degree of squeezing. As an example for the case of plane pumping wave the results are calculated in the zero order of secular perturbation theory on small parameter characterizing the medium non-linearity. The Heisenberg equations of motion are obtained for non-stationary case and commonly used effective Hamiltonian derived from the first principles of quantum electrodynamics.

As it is well known for theoretical description of squeezed states the quantum treatment of light is necessary. Consideration of the medium as classical one supposes some effective interaction of the pumping and generated waves. For such description effective Hamiltonians were commonly used. But the problem of correspondence between the Heisenberg equations which follow from the effective Hamiltonians and the Maxwell equations for quantized electromagnetic field in the medium was not investigated up to now.

The main contents of our paper is to treat the theory of quantized electromagnetic field propagating in the medium with time dependent dielectric properties on the base of Maxwell equations. This problem is quite analogical with the theory of quantum effects in non-stationary external fields [1]. But in our case the role of "external field" is played not by the pumping field itself but by the induced non-stationary dielectrical properties of the medium.

The non-linear medium is described by the tensors of dielectric sensibilities of second, third and higher ranks which determinate the medium polarization produced by the electric field. In the frame of semiclassical theory we shall decompose the whole electromagnetic field into the sum of intensive classical pumping field $E_{pk}(x)$ and generated by the medium quantized field $\hat{E}_k(x)$

$$E_k(x) = E_{pk}(x) + \hat{E}_k(x). \tag{1}$$

Supposing the pumping field to be more intensive then the generated one we can omit the terms in the operator of electric induction which are higher than linear in quantized field. From the quantized Maxwell equations in the medium the following integro-differential equation follows for the operator of vector-potential $\hat{A}_k(x)$ (we use gauge with $\hat{A}_0 = 0, \partial_k \hat{A}_k(x) = 0$)

$$\frac{\partial}{\partial t}(K_{ij}\frac{\partial}{\partial t}\hat{A}_j)(x) - \Delta\hat{A}_i(x) = 0, \tag{2}$$

where

$$K_{ij} = 1 + L_{ij} + N_{ij}, \tag{3}$$

$$(L_{ij}\hat{E}_j)(x) = 4\pi \int\limits_{-\infty}^{\infty} \chi_{ij}^{(1)}(t - t'; x)\hat{E}_j(t', x)dt', \tag{4}$$

$$\left(N_{ij}\hat{E}_j\right)(x) = 8\pi \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \chi_{ikj}^{(2)}(t - t', t - t''; x)E_{pk}(t', x)\hat{E}_j(t'', x)dt'dt''. \tag{5}$$

So we are resulting with the problem of quantization in the external field which is included into the kernel of the integral operator K_{ij}.

The ground of secondary quantization is that the quantized field must be decomposed over the complete system of solutions $u_{ip}^{(\delta)\sigma}(x)$ of the classical equation corresponding to the quantized one 2

$$\hat{A}_j(x) = \sum_{\sigma,p}[u_{jp}^{(-)\sigma}(x)a_\sigma(p) + u_{jp}^{(+)\sigma}(x)a_\sigma^+(p)], \tag{6}$$

$$[a_\sigma(p), a_{\sigma'}^+(p')] = \delta_{\sigma\sigma'}\delta_{pp'}. \tag{7}$$

To orthonormalise the set of solutions it is necessary to introduce a scalar product [2]

$$(u, v) = i \int d^3x \int\limits_{-\infty}^{1} d\tau \int\limits_{-\infty}^{\infty} d\tau' \frac{\partial K_{ij}(\tau, \tau - \tau'; x)}{\partial\tau}[u_i^*(\tau, x)\frac{\partial}{\partial\tau'}v_j(\tau', x) - \frac{\partial}{\partial\tau'}u_j^*(\tau', x)v_i(\tau, x)], \tag{8}$$

$$\left(u_p^{(\pm)\sigma}, u_{p'}^{(\pm)\sigma'}\right) = \mp\delta_{\sigma\sigma'}\delta_{pp'}. \tag{9}$$

Operators $a_\sigma(p)$, $a_\sigma^+(p)$ annihilate and create free photons in the medium when time tends to infinity in the state with quantum numbers p, σ. When time increases negative- and positive-frequency solutions will be mixed which has the interpretation as particles-antiparticles creation. As the consequence the role of photons in the medium will be played by quasiparticles which creation-annihilation operators $b_\alpha^+(t), b_\alpha(t)$ (here $\alpha = (\sigma, p)$) diagonalize the Hamiltonian of quantized field in a moment t and which are connected with a_α^+, a_α by the canonical Bogoliubov transformation

$$a_\alpha = \sum_\beta[\Phi_{\alpha\beta}(t)b_\beta(t) + \Psi_{\alpha\beta}(t)b_\beta^+(t)]. \tag{10}$$

The number of the quasiparticles created by the medium in α-state is

$$N_\alpha(t) = <0_{-\infty}|b_\alpha^+(t)b_\alpha(t)|0_{-\infty}> = \sum_\gamma \Psi_{\alpha\gamma}^+(t)\Psi_{\gamma\alpha}(t). \tag{11}$$

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The degree of squeezing is defined by the value of dispersion of quadrature components

$$X_{1\alpha}(t) = \frac{1}{2}\left(b_\alpha^+(t) + b_\alpha(t)\right), X_{2\alpha}(t) = \frac{1}{2i}\left(b_\alpha^+(t) - b_\alpha(t)\right) \tag{12}$$

or by their linear combinations

$$Y_{1,2\beta} = \sum_\alpha Q_{\beta\alpha} X_{1,2\alpha}. \tag{13}$$

The matrix of squeezing is

$$S_{ik,\alpha\beta} = <0_{-\infty}|Y_{i,\alpha}(t)Y_{k,\beta}(t)|0_{-\infty}> = \frac{1}{4}[Q(\Phi \mp \Psi)^+ (\Phi \mp \Psi)Q^T]_{\alpha\beta}, \tag{14}$$

(here minus for $i = k = 1$, plus for $1 = k = 2$).

Let us apply the developed formalism to the quantum process of light generation [3]. We shall suppose the non-linear crystal to be placed in a flat resonator without losses and medium absorption [4]. To obtain solutions of 2 we shall decompose $u_{ip}^{(\delta)\sigma}(x)$ over the space harmonics of resonator. The system of equations for the Fourier coefficients can be solved by the perturbation theory with the small parameter ϵ

$$\epsilon = 8\pi E_{p0} \max_\omega |\frac{\chi^{(2)}(\omega_p, -\omega)}{1 + 4\pi\chi^{(1)}(\omega)}| \ll 1. \tag{15}$$

Because of parametric resonance it is necessary to make use of the secular perturbation theory [5].

From the zero order solution it is easy to obtain the number of created by the medium photons in n-mode

$$N_n(t) = |\theta_n|^2 \sinh^2 \Lambda_n \epsilon t, \tag{16}$$

where θ_n and Λ_n are some constants of the order of 1. From the matrix of squeezing it is seen that the quadrature components dispersions grow exponentially at a large time. However there is a time interval $[0, t_{n\,min}]$ during which the dispersion of one of the quadrature components is squeezed to the value less than the standard quantum limit $\frac{1}{4}$

$$t_{n\,min} = \frac{1}{\Lambda_n\epsilon} Arctanh \ e^{-r_n}, \quad \sinh r_n = |\sigma_n|, \tag{17}$$

$$S_{nn\,min} = \frac{1}{4}\frac{2\sinh^2 r_n}{e^{2r_n} - 1} < \frac{1}{4}, \tag{18}$$

where σ_n is proportional to the difference between the sum of generated waves frequencies and the pumping wave frequency caused by medium dispersion. So the frequency upset caused by the medium dispersion destroys squeezing.

As it is commonly known, the diagonalization of Hamiltonian is equivalent to the solution of Heisenberg equations. Now we shall introduce the Heisenberg operators of creation and annihilation and deduce the equations for these operators for the case of non-stationary external field.

Simple differentiation of $b_\alpha(t)$ with the help of Bogoliubov transformation 10 provides that the quasiparticles operators satisfies the following equation

$$\dot{b}_\alpha(t) = \sum_\beta \{[\dot{\Phi}^+(t)\Phi(t) - \dot{\Psi}^T(t)\Psi^*(t)]_{\alpha\beta}b_\beta(t) + [\dot{\Phi}^+(t)\Psi(t) - \dot{\Psi}^T(t)\Phi^*]_{\alpha\beta}b_\beta^+(t)\}. \tag{19}$$

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The operators of quasiparticles differ from the Heisenberg operators $c_\alpha(t), c_\alpha^+(t)$ extended to the non-stationary case only by a phase [1]

$$c_\alpha(t) = e^{-i\theta_\alpha(t)} b_\alpha(t), \qquad \theta_\alpha(t) = 2 \int_{-\infty}^{t} \omega_\alpha(\tau) d\tau, \tag{20}$$

where $\omega_\alpha(\tau)$ is the instant energy of quasiparticle. Remembering that in terms of Heisenberg operators the Hamiltonian is also diagonal and with the help of 19, 20 we obtain the generalized Heisenberg equations

$$\dot{c}_\alpha(t) = -i[c_\alpha(t), H(t)] +$$

$$\sum_\beta [e^{-i\theta_\alpha(t)} \left(\dot{\Phi}^+ \Phi - \dot{\Psi}^T \Psi^* \right)_{\alpha\beta} e^{+i\theta_\beta(t)} c_\beta(t) + e^{-i\theta_\alpha(t)} \left(\dot{\Phi}^+ \Psi - \dot{\Psi}^T \Phi^* \right)_{\alpha\beta} e^{-i\theta_\beta(t)} c_\beta^+(t)]. \tag{21}$$

In the limits when time tends to infinity Bogoliubov coefficients tend to constants and we are resulting with the ordinary Heisenberg equations.

Inserting the expressions for the Bogoliubov transformation coefficients for zero order perturbation theory into generalized Heisenberg equation 21 we obtain the following equation describing the process of parametric generation of photons in n and $l - n$ modes

$$\dot{c}_n(t) = -i\Omega c_n(t) + \epsilon \Lambda_n \theta_n^* e^{-i2\Omega t} c_{l-n}^+(t), \tag{22}$$

where Ω is the energy in mode n or $l - n$. It is clearly seen that this equation may be provided as the usual Heisenberg equation $\dot{c}_n = -i[c_n, H]$ by the standard effective Hamiltonian [3]

$$H_{eff}(t) = \Omega c_n^+(t) c_n(t) + \Omega c_{l-n}^+(t) c_{l-n}(t) + \epsilon \Lambda_n [\theta_n^* e^{-i2\Omega t} c_n^+(t) c_{l-n}^+(t) + \theta_n e^{i2\Omega t} c_n(t) c_{l-n}(t)]. \tag{23}$$

So the standard quadratic effective Hamiltonian is obtained as the zero order of secular perturbation theory applied to the exact integro-differential equation which describes the propagation of quantized electromagnetic field in non-stationary medium. The corrections to it also can be obtained in the frame of exposed here self-consistent description of the process of squeezed states generation based on the first principles of quantum electrodynamics.

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RELATION OF SQUEEZED STATES BETWEEN DAMPED HARMONIC
AND SIMPLE HARMONIC OSCILLATORS

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Abstract

The minimum uncertainty and other relations are evaluated in the framework of the coherent states of the damped harmonic oscillator. It is shown that the coherent states of the damped harmonic oscillator are the squeezed coherent states of the simple harmonic oscillator. The unitary operator is also constructed, that connects coherent states between damped harmonic and simple harmonic oscillators.

1 Introduction

Recently there has been a surge of interest in the minimum uncertainty state which is one of the fundamental features of quantum mechanics[1]. Introducing the canonical conjugate variables for the harmonic oscillator, position x and momentum p in the appropriate dimensionless units, the coherent states can be described by a symmetric uncertainty in x and p with $\Delta p \cdot \Delta x = 1$ and $\Delta x = \Delta p = 1$. From the restriction of the uncertainty principle, $\Delta x \cdot \Delta p$, we may consider a more precise position $\Delta x < 1$ and a more uncertain momentum $\Delta p > 1$. These states, i.e., one variable is squeezed at the expense of its conjugate, are called squeezed states or minimum uncertainty states, which can not be obtained from the optical sources generating the coherent states[2], but from two-photon coherent states[3] including ordinary coherent states as a special case. This kind

of change in the variable corresponds to the measurement of either x or p in a rotating frame in phase space. This new space is the quadrature phase, that is directly related with a homodyne or heterodyne detection. Recently, two-photon devices have produced the squeezed states of light[4] with high precision interferometers[5].

The two-photon coherent states or minimum uncertainty can be distinguished from a coherent state in many ways, i.e., different photon processes, quantum statistical properties and coherence properties. The coherent state can be generated from one-photon stimulated processes, while the two-photon coherent states are generated from two-photon processes for two photons of the same mode. For the photon annihilation operator with frequency ω, we may define the coherent states $\mid \alpha >$ ($a \mid \alpha >= \alpha \mid \alpha >$), and for the case of a two-photon process, a self-adjoint operator $a \equiv a_1 + ia_2$ yields $< \Delta a_1^2 >=< \Delta a_2^2 >= 1/4$ for the coherent state $\mid \alpha >$, as derived in Sec. 3 below. However, the states with a more precise quantity $< \Delta a_1^2 > \ll 1/4$ and a more uncertain $< \Delta a_1^2 > \gg 1/4$ are permitted by the uncertainty $< \Delta a_1^2 >< \Delta a_2^2 > \geq 1/16$ with minimum uncertainty $< \Delta a_1^2 >< \Delta a_2^2 >= 1/16$. This indicates that the ordinary coherent states are different from the minimum uncertainty.

The purpose of this paper is to show that our previous results[6] of the coherent states for the damped harmonic oscillator (DHO) are the squeezed states of simple harmonic oscillator (SHO). Introducing the Caldirola-Kanai Hamiltonian[7], we review the propagator, wave function, uncertainty relation and coherent states[8] of the Caldirola-Kanai Hamiltonian in Sec. 2. In Sec. 3 we define the self-adjoint operator and construct the coherent states for DHO. We determine the properties and structure of the unitary transformation of the coherent states of DHO and SHO in Sec. 4. The results and discussion will be given in Sec. 5 together with graphs.

2 Propagator and Wave Function of DHO

We introduce the Caldirola-Kanai Hamiltonian for DHO as

$$\mathcal{H} = e^{-\gamma t} \frac{p^2}{2m} + e^{\gamma t} \frac{1}{2} m \omega_0^2 x^2 , \tag{1}$$

where γ is the positive constant. As we have developed the quantum theory or damped driven harmonic oscillator by the path integral method[8], the propagator and wave function of DHO are given as

$$K(x, t; x_0, 0) = \left[\frac{m \omega e^{\frac{\gamma t}{2}}}{2\pi i \hbar \sin \omega t} \right]^{1/2} \exp[\frac{im}{4\hbar} \{\gamma(x_0^2 - e^{\gamma t} x^2)$$
$$+ \frac{2\omega}{\sin \omega t}((x^2 e^{\gamma t} + x_0^2) \cos \omega t - 2e^{\frac{\gamma t}{2}} x x_0)\}] , \tag{2}$$

$$\Psi_n(x, t) = \frac{N}{(2^n n!)^{1/2}} H_n(Dx) \exp\left[-i(n + \frac{1}{2}) \cot^{-1}(\frac{\gamma}{2\omega} + \cot \omega t) - Ax^2 \right] , \tag{3}$$

where

$$\omega = \left(\omega_0^2 - \frac{\gamma^2}{4} \right)^{1/2} ,$$

$$N(t) = \left[\frac{m\omega}{\pi\hbar}\right]\frac{e^{-\gamma t}}{\xi(t)^2} \; ,$$

$$\xi(t)^2 = \frac{\gamma^2}{4\omega^2}\sin^2\omega t + \frac{\gamma}{2\omega}\sin 2\omega t + 1 \; , \tag{4}$$

$$A(t) = \frac{m\omega e^{\gamma t}}{2\hbar\xi(t)^2}\left\{1 + i\left[\left(\frac{\gamma^3}{8\omega^3} + \frac{\gamma}{\omega}\right)\sin^2\omega t + \frac{\gamma^2}{8\omega^2}\sin 2\omega t\right]\right\} \; ,$$

$$D(t) = \left(\frac{m\omega}{\hbar}\right)^{1/2}\frac{e^{\frac{\gamma t}{2}}}{\xi(t)} \; .$$

To construct the coherent states ($\mid \alpha >$) for DHO, we define the annihilation operator a and creation operator $a^\dagger$ as

$$a = \frac{1}{i\hbar}(\eta x - \mu p) \; , \tag{5}$$

$$a^\dagger = \frac{1}{i\hbar}(\mu^* p - \eta^* x) \; , \tag{6}$$

where $\mu(t)$ and $\eta(t)$ are

$$\mu(t) = \frac{1}{2}(ReA)^{-1/2}\exp\left\{i\cot^{-1}\left(\frac{\gamma}{2\omega} + \cot\omega t\right)\right\} \; , \tag{7}$$

$$\eta(t) = \sqrt{2}i\hbar\frac{A}{D}\exp\left\{i\cot^{-1}\left(\frac{\gamma}{2\omega} + \cot\omega t\right)\right\} \; . \tag{8}$$

Equations. (5)-(6) satisfy the commutation relation $[a, a^\dagger] = 1$, which corresponds to $[x, p] = i\hbar$. The coherent states in the coordinate representation $\mid x >$ can be expressed by

$$< x \mid \alpha >= (2\pi\mu\mu^*)^{-1/4}\exp\left[-\frac{1}{2i\hbar}\frac{\eta}{\mu}x^2 + \frac{\alpha}{\mu}x - \frac{1}{2}\mid \alpha \mid^2 -\frac{1}{2}\frac{\mu^*}{\mu}\alpha^2\right] \; . \tag{9}$$

With the use of Eqs. (5)-(8) the uncertainty relation can be easily obtained as

$$(\Delta x \Delta p) = \mid \mu \mid\mid \eta \mid= \frac{\hbar}{2}\beta(t)$$

$$= \frac{\hbar}{2}\left\{1 + \left[\left(\frac{\gamma^3}{8\omega^3} + \frac{\gamma}{\omega}\right)\sin^2\omega t + \frac{\gamma^2}{8\omega^2}\sin 2\omega t\right]^2\right\}^{1/2} \tag{10}$$

Here, Eq. (10) is the minimum uncertainty corresponding to the (0,0) states. All of the formulas derived above reduce to those of simple harmonic oscillator (SHO) when $\gamma = 0$. The propagator [Eq. (2)] has a very similar form to those of Cheng[9] and others[10], but Eq. (3) is of a new form.

3 Two-Dimensional Self-Adjoint Operators

We introduce the dimensionless two self-adjoint operators

$$a \equiv a_1 + ia_2, \quad a_1 = a_1^\dagger, \quad a_2 = a_2^\dagger \tag{11}$$

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and the corresponding eigenstates

$$| \alpha > = | \alpha_1 >_1 + i | \alpha_2 >_2 , \tag{12}$$

where α_1 and α_2 are real. We refer to (a_1, a_2) or (α_1, α_2) as the quadrature components, and the relation between Eqs. (11) and (12) are given by

$$a_1 | \alpha_1 >_1 \quad = \quad \alpha_1 | \alpha_1 >_1 ,$$

$$\tag{13}$$

$$a_2 | \alpha_2 >_2 \quad = \quad \alpha_2 | \alpha_2 >_2 .$$

Using Eqs. (5)-(6) we may express Eq. (11) as

$$a_1 \quad = \quad \frac{1}{2i\hbar}[(\eta - \eta^*)x + (\mu^* - \mu)p] , \tag{14}$$

$$a_2 \quad = \quad \frac{1}{2\hbar}[-(\eta + \eta^*)x + (\mu + \mu^*)p] . \tag{15}$$

Rewriting Eqs. (14) and (15) as the representation of x and p, we get

$$x \quad = \quad (\mu + \mu^*)a_1 - i(\mu - \mu^*)a_2 , \tag{16}$$

$$p \quad = \quad (\eta + \eta^*)a_1 - i(\eta - \eta^*)a_2 . \tag{17}$$

With the use of the wave function expressed as Eq. (3) and through the following definition

$$< \Delta a_1^2 >_{m,n} \equiv < (a_1 - < a_1 >_{m,n})(a_1 - < a_1 >_{m,n})^* >_{m,n} , \tag{18}$$

we obtain the uncertainty relations at various states as

$$< \Delta a_1^2 >_{n+2,n} < \Delta a_2^2 >_{n+2,n} \quad = \quad \frac{1}{16}(n + 2)(n + 1) \xrightarrow{min.} \frac{1}{8} , \tag{19}$$

$$< \Delta a_1^2 >_{n+1,n} < \Delta a_2^2 >_{n+1,n} \quad = \quad \frac{1}{16}(n + 1)^2 \xrightarrow{min.} \frac{1}{16} . \tag{20}$$

$$< \Delta a_1^2 >_{n,n} < \Delta a_2^2 >_{n,n} \quad = \quad \frac{1}{16}(2n + 1)^2 \xrightarrow{min.} \frac{1}{16} , \tag{21}$$

$$< \Delta a_1^2 >_{n-1,n} < \Delta a_2^2 >_{n-1,n} \quad = \quad \frac{1}{16}n^2 \xrightarrow{min.} \frac{1}{16} , \tag{22}$$

$$< \Delta a_1^2 >_{n-2,n} < \Delta a_2^2 >_{n-2,n} \quad = \quad \frac{1}{16}n(n - 1) \xrightarrow{min.} \frac{1}{8} . \tag{23}$$

Averages in the coherent states can be defined as

$$< \alpha | a | \alpha > = < a > = \alpha , \tag{24}$$

and thus we have

$$< a_1 > = \frac{1}{2}(\alpha + \alpha^*) = \alpha_1 , \tag{25}$$

$$< a_2 > = \frac{i}{2}(\alpha^* - \alpha) = \alpha_2 , \tag{26}$$

$$< a_1^* a_1 > \;=\; \alpha_1^2 + \frac{1}{4} \;, \tag{27}$$

$$< a_2^* a_2 > \;=\; \alpha_2^2 + \frac{1}{4} \;, \tag{28}$$

$$< \Delta a_1^2 > \;=\; < \Delta a_2^2 > = \frac{1}{4} \;, \tag{29}$$

and the following α_1 representation

$$< \alpha_1 \mid \alpha > = \left(\frac{2}{\pi}\right)^{1/4} \exp\left[-(\alpha_1' - \alpha)^2 + \frac{i}{4}\alpha Im\alpha \right] \;, \tag{30}$$

where $a_1 \mid \alpha_1 > = \alpha_1' \mid \alpha >$.

4 Unitary Transformation

Now we will construct the unitary operator that transforms the coherent states for SHO to that of the two-photon coherent state of DHO and *vice versa*. From Eqs. (5)-(6), we can easily show the relation

$$a \;=\; \nu a_0 - \lambda a_0^\dagger \;, \tag{31}$$

$$a^\dagger \;=\; -\lambda^* a_0 + \nu^* a_0^\dagger \;, \tag{32}$$

where the expressions of a_0 and $a_0^\dagger$ by a and $a^\dagger$ are

$$a_0 \;=\; \nu^* a + \lambda a^\dagger \;, \tag{33}$$

$$a_0^\dagger \;=\; \lambda^* a + \nu a^\dagger \;, \tag{34}$$

for a pair of numbers λ and ν satisfying

$$\mid \nu \mid^2 - \mid \lambda \mid^2 = 1 \;. \tag{35}$$

We take the values of ν and λ as the following:

$$
\begin{aligned}
\nu \;&=\; \sqrt{\frac{m\omega_0}{2\hbar}}\mu - i\frac{1}{\sqrt{2m\omega_0\hbar}}\eta \\
&=\; \frac{1}{2\xi}\sqrt{\frac{\omega}{\omega_0}}e^{\frac{\gamma t}{2}}\left[\frac{\omega}{\omega_0}e^{-\gamma t} + (1 - i\sqrt{1-\beta^2})\right]\exp\left[i\cot^{-1}(\frac{\gamma}{2\omega} + \cos\omega t)\right] \\
&=\; \frac{1}{2}\left[\sqrt{\frac{\omega_0}{\omega}}e^{-\frac{\gamma t}{2}}(\frac{\gamma}{2\omega}\sin\omega t + \cos\omega t) - \sqrt{\frac{\omega}{\omega_0}}e^{\frac{\gamma t}{2}}(\cos\omega t - \frac{\gamma}{2\omega}\sin\omega t)\right] \\
&\quad + \frac{i}{2}\left[\sqrt{\frac{\omega_0}{\omega}}e^{-\frac{\gamma t}{2}}\sin\omega t - \sqrt{\frac{\omega}{\omega_0}}e^{\frac{\gamma t}{2}}(\frac{\gamma^2}{4\omega^2}\sin\omega t + \sin\omega t)\right] \\
&=\; \frac{1}{2\xi}\left[\frac{\omega_0}{\omega}e^{-\gamma t} + 2 + \frac{\omega}{\omega_0}\beta^2\right]^{1/2}\exp\left[i\cot^{-1}(\frac{\gamma}{2\omega} + \cot\omega t) + \tan^{-1}\left(\frac{\sqrt{1-\beta^2}}{\frac{\omega_0}{\omega}e^{-\gamma t} + 1}\right)\right] \;,
\end{aligned}
\tag{36}
$$

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$$\lambda = \sqrt{\frac{m\omega_0}{2\hbar}}\mu + i\frac{1}{\sqrt{2m\omega_0\hbar}}\eta$$

$$= \frac{1}{2\xi}\sqrt{\frac{\omega}{\omega_0}}e^{\frac{\gamma t}{2}}\left[\frac{\omega}{\omega_0}e^{-\gamma t} - (1 + i\sqrt{1-\beta^2})\right]\exp\left[i\cot^{-1}(\frac{\gamma}{2\omega} + \cot\omega t)\right]$$

$$= \frac{1}{2}\left[\sqrt{\frac{\omega^0}{\omega}}e^{-\frac{\gamma t}{2}}(\frac{\gamma}{2\omega}\sin\omega t + \cos\omega t) - \sqrt{\frac{\omega}{\omega_0}}e^{\frac{\gamma t}{2}}(\cos\omega t - \frac{\gamma}{2\omega}\sin\omega t)\right] \tag{37}$$

$$+\frac{i}{2}\left[\sqrt{\frac{\omega^0}{\omega}}e^{-\frac{\gamma t}{2}}\sin\omega t - \sqrt{\frac{\omega}{\omega_0}}e^{\frac{\gamma t}{2}}(\frac{\gamma^2}{4\omega^2}\sin\omega t + \cos\omega t)\right]$$

$$= \frac{1}{2\xi}\left[\frac{\omega_0}{\omega}e^{-\gamma t} - 2 + \frac{\omega}{\omega_0}\beta^2\right]^{1/2}\exp\left[i\cot^{-1}(\frac{\gamma}{2\omega} + \cot\omega t) + \tan^{-1}\left(\frac{\sqrt{1-\beta^2}}{\frac{\omega_0}{\omega}e^{-\gamma t} - 1}\right)\right] .$$

Since a canonical transformation is defined as any transformation which keeps the commutator invariant, we can confirm that the transformation of variables from $(a_0, a_0^\dagger)$ to $(a, a^\dagger)$ given in Eqs. (31)-(32) and (35) is a canonical linear transformation. According to a theorem of Von Neumann[11], there exists a unitary operator U_α which yields all the linear canonical transformations, i.e.,

$$b(a_0, a_0^\dagger) = U_\alpha a_0 U_\alpha^\dagger = \nu a_0 - \lambda a_0^\dagger . \tag{38}$$

The commutation relation $\left[[a, a^\dagger] = 1\right]$ and unitary transformation [Eq. (38)] provide a with properties exactly similar to those of a_0. Therefore, we may obtain the usual properties of a as

$$N = a^\dagger a , \tag{39}$$

$$N\,|\,n> = n\,|\,n>, \qquad N\,|\,0>= 0 , \tag{40}$$

$$|\,n> = U_\alpha\,|\,n>_0 , \tag{41}$$

and a coherent state for DHO is given by

$$|\,\alpha> = U_\alpha\,|\,\alpha>_0 , \tag{42}$$

where $|\,\alpha>_0$ is a coherent states for SHO. The representation of coherent states for DHO in the SHO space is given by

$$< \alpha\,|\,\alpha>_0 = \frac{1}{\sqrt{\nu^*}}\exp\left[-\frac{1}{2}\,|\,\alpha\,|^2 - \frac{1}{2}\,|\,\alpha_0\,|^2 + \frac{\lambda^*}{2\nu^*}\alpha_0^2 - \frac{\lambda}{2\nu^*} + \frac{1}{\nu^*}\alpha^*\alpha_0\right] , \tag{43}$$

where the coefficients are

$$\frac{1}{\sqrt{\nu^*}} = \frac{-\sqrt{2\xi}}{\left(2 + \frac{\omega_0}{\omega}e^{-\frac{\gamma t}{2}} + \frac{\omega}{\omega_0}\beta^2\right)^{1/4}} ,$$

$$\theta_\nu = \tan^{-1}\left[\frac{(\frac{\gamma}{2\omega}\sin\omega t + \cos\omega t)\sqrt{\beta^2 - 1} + (\frac{\omega_0}{\omega}e^{-\gamma t} + 1)\sin\omega t}{(\beta\xi + \frac{\gamma}{2\omega}\sin\omega t + \cos\omega t)(\frac{\omega_0}{\omega}e^{-\gamma t} + 1) - \sin\omega t\sqrt{\beta^2 - 1}}\right] ,$$

$$\frac{\lambda^*}{2\nu^*} = -\frac{1}{2}\left(\frac{\omega_0^2 e^{-\gamma t} - 2\omega\omega_0 + \omega^2\beta^2}{\omega_0^2 e^{-\gamma t} + 2\omega\omega_0 + \omega^2\beta^2}\right)^{1/2}\exp\left\{i\tan^{-1}\left[\frac{2\sqrt{\beta^2 - 1}}{2 - \beta^2 - (\frac{\omega_0}{\omega})^2 e^{-2\gamma t}}\right]\right\} , \tag{44}$$

$$\frac{\lambda}{2\nu^*} = -\frac{1}{2}\left(\frac{\omega_0^2 e^{-\gamma t} - 2\omega\omega_0 + \omega^2\beta^2}{\omega_0^2 e^{-\gamma t} + 2\omega\omega_0 + \omega^2\beta^2}\right)^{1/2} \exp\left\{i\tan^{-1}\left[\frac{2\frac{\omega_0}{\omega}e^{-\gamma t}\sqrt{\beta^2-1}}{\beta^2-(\frac{\omega_0}{\omega})^2 e^{-2\gamma t}}\right]\right\} ,$$

$$\frac{1}{\nu^*} = \frac{2\xi}{\left(2 + \frac{\omega_0}{\omega}e^{\frac{-\gamma t}{2}} + \frac{\omega}{\omega_0}\beta^2\right)^{1/2}} e^{i2\theta_\nu} .$$

The wave function $< n \mid \alpha >$ for a coherent state of DHO in the state of SHO can be obtained from Eq. (43). Using the following formula with the nth Hermite polynomial,

$$e^{2zt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n , \qquad \mid t \mid < \infty , \tag{45}$$

and through the similar derivation of Eq. (9), we can easily obtain

$$_0< n \mid \alpha > = \frac{1}{\sqrt{2^n n!}} \frac{\sqrt{2\xi}}{(\frac{\omega_0}{\omega}e^{-\gamma t} + 2 + \frac{\omega}{\omega_0}\beta^2)} \left[\frac{\omega_0^2 e^{-\gamma t} - 2\omega\omega_0 + \omega^2\beta^2}{\omega_0^2 e^{-\gamma t} + 2\omega\omega_0 + \omega^2\beta^2}\right]^{\frac{n}{4}}$$

$$\times \quad H_n[(-2\nu\lambda)^{-1/2}\alpha]\exp(-\frac{1}{2}\mid\alpha\mid^2 - \frac{\lambda^*}{2\nu}\alpha^2) , \tag{46}$$

where the coefficients in Eq. (46) are given as

$$(-2\nu\lambda)^{-1/2} = \frac{1}{2\xi}\left[(\frac{\omega_0}{\omega}e^{-\gamma t} + 2 + \frac{\omega}{\omega_0}\beta^2)(\frac{\omega_0}{\omega}e^{-\gamma t} - 2 + \frac{\omega}{\omega_0}\beta^2)\right]^{1/4} e^{i\theta_{\nu\lambda}},$$

$$\theta_{\nu\lambda} = \tan^{-1}\frac{\sqrt{\xi^2-1} + \sin\omega t}{\sqrt{\xi^2-1} - A_{\nu\lambda}\sin\omega t} ,$$

$$A_{\nu\lambda} = \frac{\sqrt{(\frac{\omega_0}{\omega}e^{-\gamma t})^2 + 1 - \beta^2} - 2\frac{\omega_0}{\omega}e^{-\gamma t} + \sqrt{(\frac{\omega_0}{\omega}e^{-\gamma t} + 1)^2 + 1 - \beta^2}}{B_{\nu\lambda}} , \tag{47}$$

$$B_{\nu\lambda} = (\frac{\omega_0}{\omega}e^{-\gamma t} + 1)(\frac{\omega_0}{\omega}e^{-\gamma t} - 1) - (\frac{\omega_0}{\omega}e^{-\gamma t} + 1)^2\sqrt{(\frac{\omega_0}{\omega}e^{-\gamma t} - 1)^2 + 1 - \beta^2}$$

$$-(\frac{\omega_0}{\omega}e^{-\gamma t} + 1)\ sqrt(\frac{\omega_0}{\omega}e^{-\gamma t} - 1)^2 + 1 - \beta^2$$

$$+\sqrt{[(\frac{\omega_0}{\omega}e^{-\gamma t} - 1)^2 + 1 - \beta^2][(\frac{\omega_0}{\omega}e^{-\gamma t} + 1)^2 + 1 - \beta^2]} .$$

If we represent the annihilation operator a_0 in the state of DHO, we get

$$< a_0 > \equiv < \alpha \mid \alpha_0 \mid \alpha > = \nu^*\alpha + \lambda\alpha^* \equiv \alpha_{d1} + i\alpha_{d\bar{2}} , \tag{48}$$

from the definition of Eq. (18) we obtain the quantities

$$< \Delta a_{01}^2 > \equiv < \alpha \mid (a_{01} - \alpha_{d1})^2 \mid \alpha >$$

$$= \frac{1}{4}\mid \nu + \lambda \mid^2$$

$$= \frac{m\omega_0}{2\hbar}\mid\mu\mid^2$$

$$= \frac{1}{4}\frac{\omega_0}{\omega}e^{-\gamma t}\xi^2 , \tag{49}$$

$$< \Delta a_{02}^2 > \equiv \frac{1}{4} \mid \nu - \lambda \mid^2$$

$$= \frac{1}{2m\omega_0 \hbar} \mid \eta \mid^2$$

$$= \frac{1}{4} \frac{\omega}{\omega_0} e^{\gamma t} \frac{\beta^2}{\xi^2} , \tag{50}$$

$$< \Delta a_{01}^2 >< \Delta a_{02}^2 >= \frac{1}{16} \beta^2 , \tag{51}$$

$$< \Delta a_0^2 > = \mid \lambda \mid^2$$

$$= \frac{1}{2\xi} \left[\frac{\omega_0}{\omega} e^{-\gamma t} - 2 + \frac{\omega}{\omega_0} \beta^2 \right]^{1/2} . \tag{52}$$

The repetition of representation for the annihilation operator a in the state of SHO yields

$$< a >_0 = < \alpha_0 \mid a \mid \alpha_0 >= \nu \alpha_0 - \lambda \alpha_0^* \equiv \alpha_h = \alpha_{h1} + i \alpha_{h2} , \tag{53}$$

$$< \Delta a_1^2 >_0 \equiv < \alpha_0 \mid (a_1 - \alpha_{h1})^2 \mid \alpha_0 > ,$$

$$= \frac{1}{4} \mid \nu^* - \lambda \mid$$

$$= \frac{1}{4} \left[\frac{\omega_0}{\omega} e^{-\gamma t} \sin^2 \omega t + \frac{\omega}{\omega_0} e^{\gamma t} (\cos \omega t - \frac{\gamma}{2\omega} \sin \omega t)^2 \right] , \tag{54}$$

$$< \Delta a_2^2 >_0 = \frac{1}{4} \mid \nu^* + \lambda \mid$$

$$= \frac{1}{4} \left[\frac{\omega_0}{\omega} e^{-\gamma t} (\frac{\gamma}{2\omega} \sin \omega t + \cos \omega t)^2 + \frac{\omega}{\omega_0} e^{\gamma t} \sin^2 \omega t (\frac{\gamma^2}{4\omega^2} + 1) \right] , \tag{55}$$

$$< \Delta a_1^2 >_0< \Delta a_2^2 >_0 = \frac{1}{16} \{ [\frac{\omega_0}{\omega} e^{-\gamma t} (\frac{\gamma}{2\omega} \sin \omega t + \cos \omega t)^2$$

$$+ \frac{\omega}{\omega_0} e^{\gamma t} (\cos \omega t - \frac{\gamma}{2\omega} \sin \omega t)(\frac{\gamma^2}{2\omega^2} + 1) \sin \omega t]^2 \tag{56}$$

$$+ [\cos 2\omega t - \frac{\gamma^2}{2\omega^2} \sin^2 \omega t]^2 \} ,$$

$$< \Delta a^2 >_0 = \mid \lambda \mid^2$$

$$= \frac{1}{2\xi} (\frac{\omega_0}{\omega} e^{-\gamma t} - 2 + \frac{\omega}{\omega_0} \beta^2)^2 . \tag{57}$$

In Eq. (42) we have defined the unitary operator that is a linear canonical transformation. From this equation we have

$$< \alpha_0 \mid \beta > = < \alpha_0 \mid U_L \mid \beta_0 >$$

$$= U_L^n(\beta_0, \alpha_0^*) < \alpha_0 \mid \beta > . \tag{58}$$

A direct application of the following formulas[12]

$$e^{c_1 A} e^{c_2 B} e^{c_3 C} = \exp(c_1 A + c_2 B + c_3 C + \frac{1}{2}\{c_1 c_2 [A,B] + c_1 c_3 [A,C] + c_2 c_3 [B,C]\}$$

$$+ \sum_{i=1} \sum_{j=1} \overbrace{[A,[A,\cdots,[A,}^{i} \overbrace{[B,[B,\cdots,[B,}^{j} C]\cdots]\frac{c_1^i c_2^j c_3}{(i+1)!(j+1)!} \tag{59}$$

$$e^{c_1 a^{\dagger 2}} e^{c_2 a^{\dagger} a} e^{c_3 a^2} = \exp\{c_1 a^{\dagger 2} + c_2 a^{\dagger} a + c_3 a^2 - (a^{\dagger 2} + a^{\dagger} a + a a^{\dagger} + a^2)$$

$$+ \sum_{j=1}^{\infty} \frac{(-2j)^j}{(j+1)!}[-2c_1 c_3 (a^{\dagger} a + a a^{\dagger}) + 8c_1^2 c_3 a^{\dagger 2}]\}$$

$$= \exp[(c_1 - 8c_1^2 c_3 \frac{e^{2c_1}-1}{2c_2} - c_1 c_2)a^{\dagger 2}(c_2 - c_1 c_3 + 2c_1 c_3 \frac{e^{-2c_2}-1}{2c_2})a^{\dagger} a \tag{60}$$

$$+(-c_1 c_3 + 2c_1 c_3 \frac{e^{-2c_2}-1}{2c_2})a a^{\dagger} + (c_1 - c_2 c_3)a^2]$$

gives the unitary operator in the $|\alpha>_0$ representation,

$$U_L^{(n)}(\alpha_0, \alpha_0^*) = \frac{1}{\sqrt{\nu}}\exp\left[\frac{\lambda}{2\nu}\alpha_0^{*2} + (\frac{1}{\nu}-1)\alpha_0^*\alpha_0 - \frac{\lambda^*}{2\nu}\alpha_0^2\right] \tag{61}$$

$$= \frac{1}{\sqrt{\nu\pi}}\exp\left[\frac{\lambda}{2\nu}a_0^{\dagger 2} - \ln\nu a_0^{\dagger} a_0 - \frac{\lambda^*}{2\nu}a_0^2\right] \tag{62}$$

$$= \frac{1}{\sqrt{\nu\pi}}\exp\left[A_u a_0^{\dagger 2} + B_u a_0^{\dagger} a_0 + C_u a_0 a_0^{\dagger} + D_u a_0^2\right] , \tag{63}$$

where the coefficients are

$$A_u = \frac{\lambda}{2\nu} + \frac{\lambda}{2\nu}\ln\nu + \frac{2}{3}\left(\frac{\lambda}{2\nu}\right)^2\left(\frac{\lambda^*}{2\nu}\right)\frac{1-\nu^2}{\ln\nu} ,$$

$$B_u = -\ln\nu - \frac{\frac{\lambda}{2\nu}\frac{\lambda^*}{2\nu}}{2\ln\nu}(1-\nu^2) , \tag{64}$$

$$C_u = -\frac{\frac{\lambda}{2\nu}\frac{\lambda^*}{2\nu}}{2\ln\nu}(1-\nu^2) ,$$

$$D_u = \frac{\frac{\lambda}{2\nu}}{2\ln\nu}(1-\nu^2) .$$

5 Results and Discussion

Starting from the coherent states of DHO, we have shown that these states are the squeezed states of SHO and *vice versa*. We have also evaluated the averages of the operators $a_0, a, \Delta a_1^2$ and Δa_2^2 in both spaces of DHO and SHO. We have constructed the unitary operator which transforms the
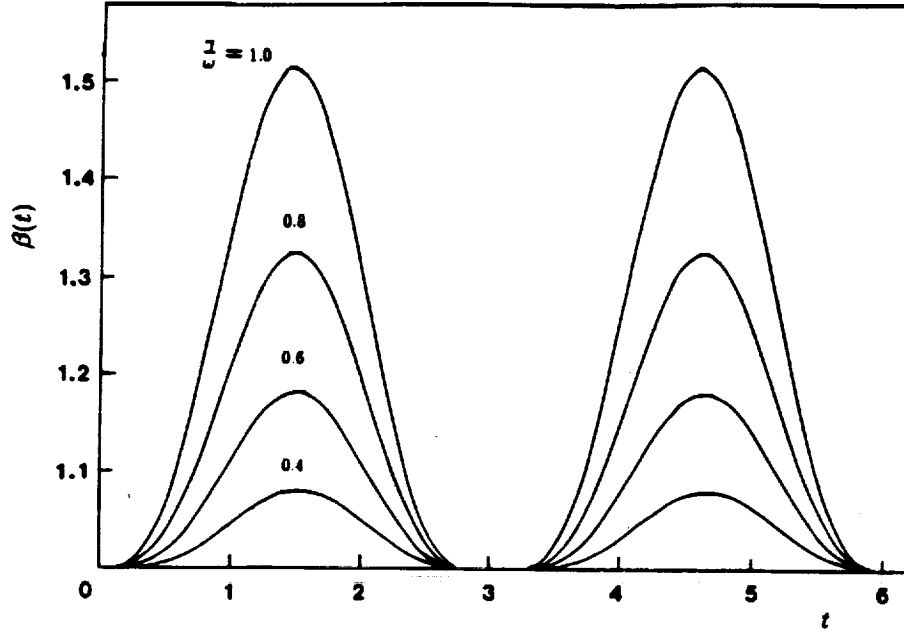
FIG. 1. $\beta(t)$ as a funtion of ωt at various values of $x = \gamma/\omega$.

coherent states ($\mid \alpha >_0$) to the coherent states ($\mid \alpha >$), i.e., $\mid \alpha >= U_\alpha \mid \alpha >_0$.

Figure 1 illustrates the behavior of $\beta(t)$ [Eq. (10)] as a function of t and $x = \gamma/\omega$. As x increases, the amplitude of the oscillation becomes large. For the condition $\gamma \ll \omega_0$, $\omega \simeq \omega_0$ and $\gamma \to 0$, $\beta(t)$ approaches to unity, with DHO reducing to SHO. Therefore, the uncertainty relation for the (n,n) state [Eq. (10)] oscillates with the period π.

From the definition of the self adjoint operator and Eq. (18), we have evaluated the minimum uncertainty for various states in Eqs. (19)-(23). The minimum uncertainties for the diagonal and first off-diagonal states have the value of 1/16, and the minimum values for the second off-diagonal states are 1/8. For $< \Delta a_1^2 > \ll 1/4$, the corresponding canonical part results in more uncertainty.

The creation and annihilation operators ($a^\dagger$ and a) in Sec. 4 can be shown under the condition $\mid \nu \mid^2 - \mid \lambda \mid^2 = 1$. The operators ($a_0^\dagger, a_0$) are transformed to the operators ($a^\dagger, a$) through unitary operator U_α. The behaviors of $\mid \nu \mid$ and $\mid \lambda \mid$ are depicted in Figures 2 and 3, respectively. We can confirm that $\mid \nu \mid$ oscillates periodically in general, but $\mid \lambda \mid$ behaves in a more complicated fashion, and as $x = \gamma/\omega$ increases to larger than unity, the oscillation decays rapidly .

The average of Δa_{01}^2 and Δa_{02}^2 in the states of DHO are given in Eqs. (49)-(50). $< \Delta a_{01}^2 >$ oscillates with exponential decrease, while $< \Delta a_{02}^2 >$ does so with exponential increase. The minimum value of $< \Delta a_{01}^2 > < \Delta a_{02}^2 >$ is 1/16 at $\beta(t) = 1$. The averages of Δa_1^2 and Δa_2^2 in the space of SHO are evaluated in Eqs. (54)-(55). The uncertainty relation [Eq. (56)] has a minimum value of 1/16 at $t = \sin^{-1} n\pi$ or $t = \cos^{-1}(\gamma/2\omega - 4\omega/\gamma)$, and maximum value at $\beta^2 = (\omega_0/\omega)^2 e^{-2\gamma t}$ (Figure 4).

Equations (61)-(63) represent the unitary operator which transforms $\mid \alpha >_0$ to $\mid \alpha >$ and vice versa. Therefore, we can obtain the scaled state through $< x \mid \alpha >=< x \mid U \mid \alpha >_0$.

In conclusion, we have shown the uncertainties and their relations in the states of SHO and
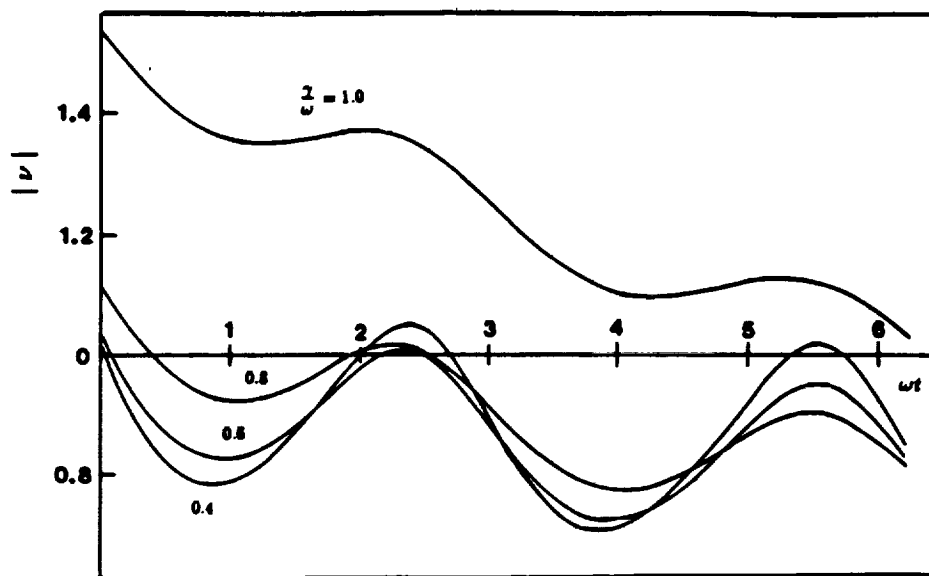
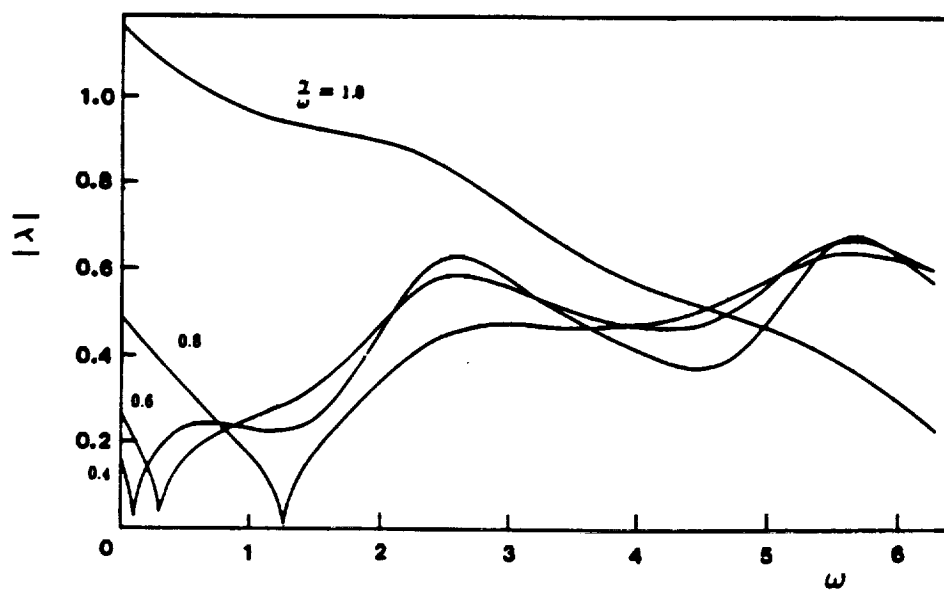FIG. 2. $|\nu|$ versus ωt at various values of γ/ω.

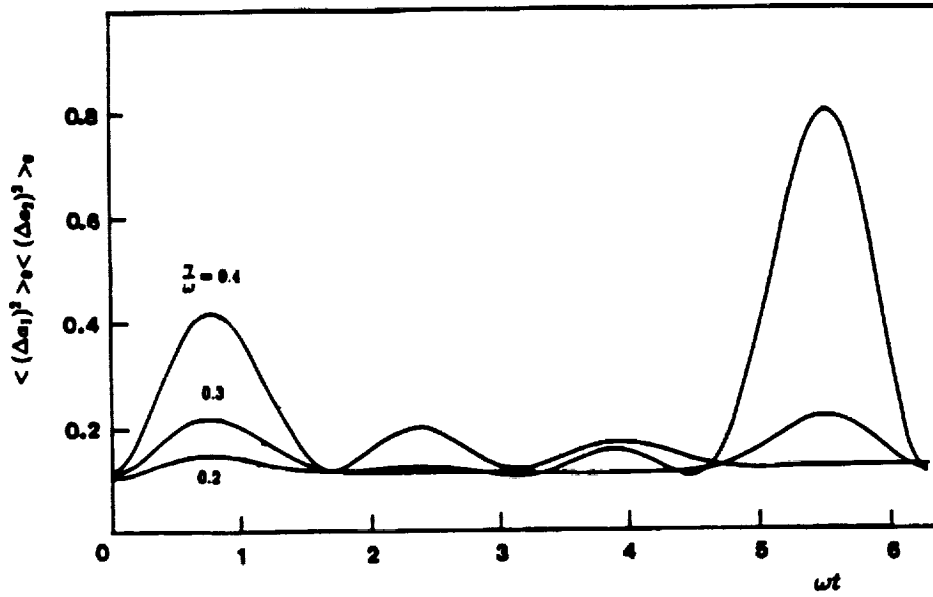FIG. 3. $|\lambda|$ versus ωt at various values of γ/ω.

FIG. 4. Uncertainty relations versus ωt.

DHO. We have also shown that there exists a unitary operator to connect the coherent states of SHO with those of DHO.

6 Acknowledgments

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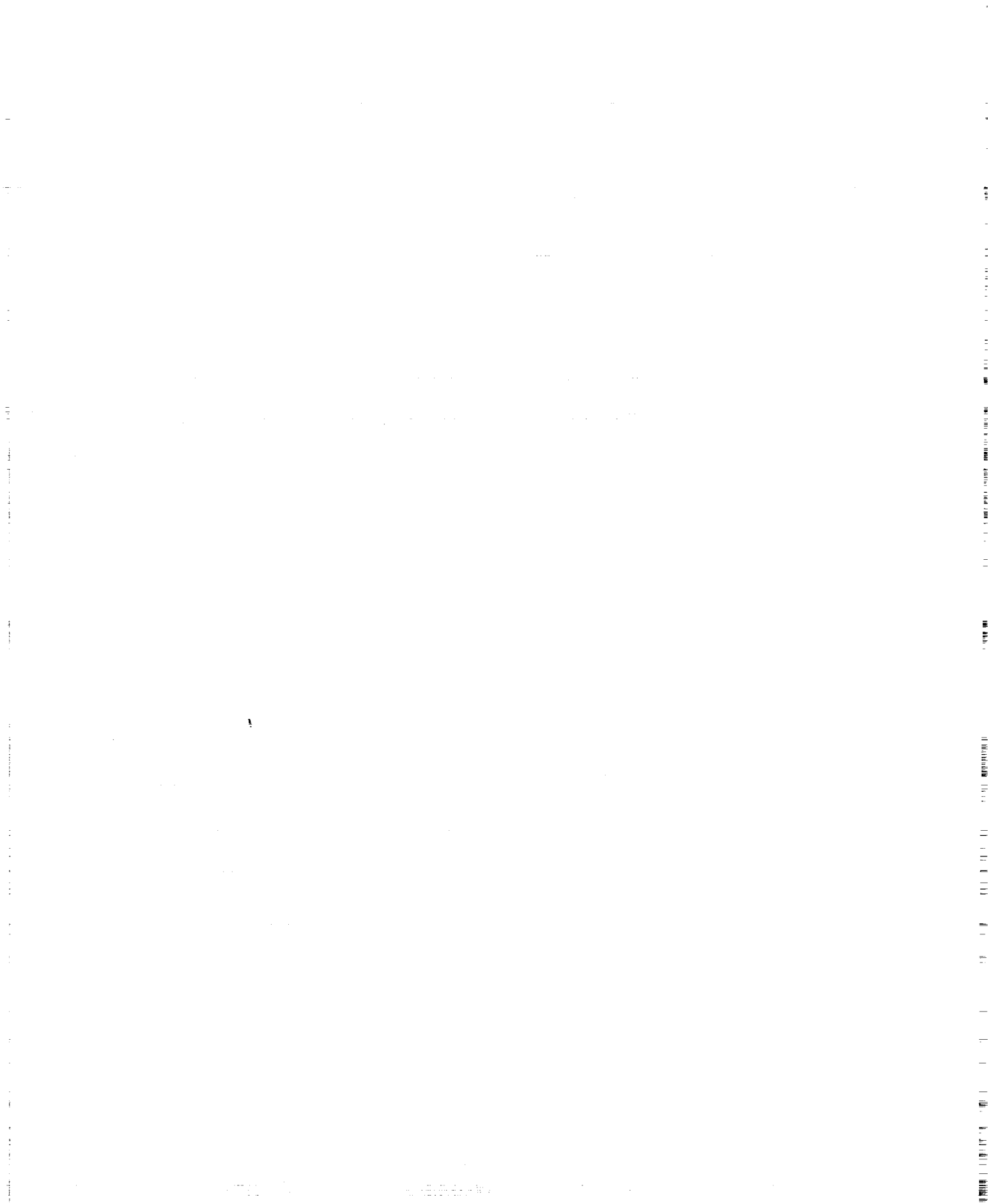
WAVE AND PSEUDO-DIFFUSION EQUATIONS
FROM SQUEEZED STATES

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Abstract

We show that the probability distributions $P_n(q,p;y) := |\langle n|p,q;y\rangle|^2$, which are obtained from squeezed states, obey an interesting partial differential equation, to which we give two intuitive interpretations: as a wave equation in one space dimension and also as a pseudo-diffusion equation. We also study the corresponding Wehrl entropies $S_n(y)$, and show that they have minima at zero squeezing, $y = 0$.

1 Introduction

This talk is based mainly on a work which was done in collaboration with Salomon Mizrahi from Brazil.

Squeezed oscillator states are defined in terms of the bosonic creation and annihilation operators, $a^\dagger := \frac{1}{\sqrt{2}}(x - \frac{\partial}{\partial x})$, and $a := \frac{1}{\sqrt{2}}(x + \frac{\partial}{\partial x})$, as follows:

$$|z;\xi\rangle = |p,q;\xi\rangle := \mathcal{D}(q,p)\mathcal{S}(\xi)|0\rangle, \quad \text{where} \quad z := (q + ip)/\sqrt{2}, \tag{1}$$

and $|0\rangle$ is the ground state of the harmonic oscillator. Both $\mathcal{D}$ and $\mathcal{S}$ are unitary operators. $\mathcal{D}$ creates the coherent state, and is defined by

$$\mathcal{D}(q,p) := \exp[za^\dagger - z^*a] = \exp[ipx - q\frac{\partial}{\partial x}], \tag{2}$$

and $\mathcal{S}(\xi)$ is the **squeezing operator**:

$$\mathcal{S}(\xi) := \exp[\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)], \tag{3}$$

where ξ is a complex variable. For $\xi = 0$, we recover the ordinary (unsqueezed) coherent states. The squeezed states satisfy the completness relation, $\int |p,q;\xi\rangle\langle p,q;\xi| \frac{dpdq}{2\pi} = 1$, for every ξ. Therefore,

$$\int P_n(q,p;\xi) \frac{dpdq}{2\pi} = 1, \quad \text{where} \quad P_n(q,p;\xi) := |\langle p,q;\xi|n\rangle|^2, \tag{4}$$

where $|n\rangle$ is the number state. If we interpret the real parameters q and p as the position and momentum variables, then (4) allows us to interpret the non-negative functions P_n as probability distributions in the (q,p)-phase plane, for every n and ξ.

In this talk, I shall consider these P_n for real values of the squeezing parameter ξ, which will be denoted by y. In particular, I shall show that the $P_n(q,p;y)$ satisfy the interesting partial differential equations (9) and (12), to which two intuitive interpretations can be given. Finally, I shall show that the Wehrl entropy $S_n(y)$ (14) of the P_n must have their minima at zero squeezing, $y = 0$.

2 Explicit Form of the Distributions P_n

The distribution $P_n(q,p;\xi) := |\langle n|p,q;\xi\rangle|^2$ gives the probability of finiding n bosons (photons) in the squeezed states $|q,y;\xi\rangle$. It is a physically important quantity, and it has been calculated by different methods. The dependence of $P_n(q,p;\xi)$ on n was studied by Schleich and Wheeler [2]. For $\xi = y$, the P_n is given by the following complicated expression [1,3,7]:

$$P_n(q,p;y) := |\langle p,q;y|n\rangle|^2 = \frac{2\sqrt{\gamma}}{2^n\, n!(\gamma+1)}|\tilde{H}_n(2,\eta;w)|^2 \exp\left[-\frac{q^2+\gamma p^2}{1+\gamma}\right], \quad n \geq 0, \quad (5)$$

where

$$\gamma := e^{2y}, \quad \eta := \frac{1-\gamma}{1+\gamma}, \quad \text{and} \quad w := \frac{q+i\gamma p}{\gamma+1}, \quad (6)$$

and where $\tilde{H}_n(2,\eta;w)$ are the generalized Hermite polynomials ($\mathcal{GHP}$), which are defined in terms of the raising operatores $R(\alpha,\beta;x) = \alpha x - \beta\frac{\partial}{\partial x}$, as follows [1]:

$$\tilde{H}_n(\alpha,\beta;x) = R^n(\alpha,\beta;x)\cdot 1 = \sum_{s=0}^{[n/2]} \frac{n!}{(n-2s)!s!}\left(-\frac{\alpha\beta}{2}\right)^s (\alpha x)^{n-2s}. \quad (7)$$

These polynomials are equal to the standard Hermite polynomials for $\alpha = 2$ and $\beta = 1$. In the limit, $\beta \to 0$, these $\tilde{H}_n(x)$ becomes simple powers of x: $\tilde{H}_n(\alpha,0,x) = \alpha^n x^n$. Therefore, in the limit of zero squeezing, $\gamma \to 1$, we have $\eta \to 0$, so that the above $\mathcal{GHP}$'s become simple powers of w. Thus, for $y \to 0$, equation (5) gives the following well-known Poisson distribution of the unsqueezed coherent states:

$$P_n(q,p;0) = \frac{\rho^{2n}}{2^n n!} \exp\left[-\frac{\rho^2}{2}\right], \quad n \geq 0, \quad \text{where} \quad \rho^2 := q^2 + p^2, \quad (8)$$

When discussing probability distributions, it is useful to think of the regions that are surrounded by the **equipotential curves**, $P_n(q,p;y) = const.$; I shall call these regions **potential regions**. Thus, the potential regions of the above Poisson distribution $P_n(q,p;0)$ are concentric circles in the (q,p)-plane. But for $y \neq 0$, these regions will have approximately elliptical shapes, whose the major axes lie along the p-axis for $y < 0$ and along the q-axis for $y > 0$. These regions become more elongated in one direction and narrower in the other, as $|y|$ increases.

3 The Partial Differential Equation for the P_n

Since the integral (4) of the distributions $P_n(q,p;y)$ over the whole (q,p)-space remains constant under squeezing, it is useful to think of the change of $P_n(q,p;y)$ as functions of y as a

redistribution of probability densities in phase space, which maintains the positivity condition $P_n(q, p; y) \geq 0$ for all y. This redistribution of the $P_n(q, p; y)$ is governed by the following interesting and amazingly simple partial differential equation:

$$\frac{\partial}{\partial \gamma} P_n(q, p; y(\gamma)) = \frac{1}{4} \left(\frac{\partial^2}{\partial q^2} - \frac{1}{\gamma^2} \frac{\partial^2}{\partial p^2} \right) P_n(q, p; y(\gamma)) , \quad \text{where} \quad \gamma := e^{2\nu} . \tag{9}$$

This equation was originally obtained [1] by straightforward but lengthy differentiation of the expression (5) , and by using the following property of the $\mathcal{GHP}$ [1]:

$$\frac{\partial}{\partial \eta} \tilde{H}_n(\alpha, \eta, w) = -\frac{1}{4} \frac{\partial^2}{\partial w^2} \tilde{H}_n(\alpha, \eta, w) . \tag{10}$$

However, we can now derive it by two other more general methods [5], as reported in the summary section.

4 Interpretation as Wave and Pseudo-Diffusion Equations

I shall now present two possible intuitive interpretations of the above differential equation:

(I) **D'Alembert or Wave Equation:** The following is a new interpretation, which was not discussed in [1]: For a *fixed squeezing parameter* y, equation (9) looks like the wave equation for one space dimension q, if we think of the p variable in (9) as the time variable t:

$$\left(\frac{\partial^2}{\partial q^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \Phi(q, t; y) = -4\pi \rho(q, t; y) , \quad \text{where} \quad \rho(q, t; y) = -\frac{1}{\pi} \frac{\partial}{\partial \gamma} P_n(q, t; y(\gamma)) , \tag{11}$$

In this interpretation, the parameter γ would then play the role of the speed of light $c(n)$ in matter, which depends on the parameter y, similar to the dependence of $c(n)$ on the index of refraction index n. If the P_n are thought of as electromagnetic potentials $\Phi(q, t; y)$, then $4\frac{\partial}{\partial \gamma} P_n(q, p; y(\gamma))$ will play the role of a time-dependent charge distributions $-4\pi\rho(q, t; y)$.

(II) **Pseudo-Diffusion Equation:** By substituting $\frac{\partial}{\partial y} = 2e^{2\nu} \frac{\partial}{\partial \gamma}$ into (9) , we obtain a more symmetric differential equation for the P_n:

$$\frac{\partial}{\partial y} P_n(q, p; y) = \frac{1}{2} \left(e^{2\nu} \frac{\partial^2}{\partial q^2} - e^{-2\nu} \frac{\partial^2}{\partial p^2} \right) P_n(q, p; y) . \tag{12}$$

This equation is also new and permits a more pertinent intuitive understanding of the redistribution process of the P_n, by comparing (12) with the diffusion equation in two dimensions [6]:

$$\frac{\partial}{\partial t} T(q, p; t) = \sigma \left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2} \right) T(q, p; t) , \tag{13}$$

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where σ is the diffusion coefficient. Equations (12) and (13) are similar, if we interpret the squeezing parameter y as the time variable. However, the two equations differ in two interesting aspects:

(1) The sign in front of $\frac{\partial^2}{\partial p^2}$ in (12) is negative rather than positive. Such a *"negative diffusion coefficient"* leads to *"infusion" rather than diffusion in the p-direction*. Consequently, as y increases, the equi-probability curves, $P_n(q,p;y) = const.$, move <u>towards</u> the origin along the p-axis, but <u>away from</u> the origin along the q-axis. Therefore, we expect the probability regions to be concentric elongated "quasi ellipses" which are extended along the p-axis for $y \to -\infty$. They become more and more circular as y approaches zero, and then stretch outwards along the q-axis, as $y \to \infty$. For the above reasons, we shall call equations (9) and (12) **"pseudo diffusion equation"**.

(2) The "diffusion coefficients" $\exp[2y]/2$ and $-\exp[-2y]/2$ and in front of $\frac{\partial^2}{\partial q^2}$ and $\frac{\partial^2}{\partial p^2}$ in (12) depend on y. For $y \to +\infty$, the term $\frac{1}{2}e^{2y}\frac{\partial^2}{\partial q^2}P_n$ dominates the r.h.s. of (12), whereas for $y \to -\infty$, the second term dominates. This dependence on y can be given an interesting intuitive explanation: Let us consider the redistribution process when y is very large: In this case the probability densities $P_n(q,p;y)$ are extended in the q-direction and tightly squeezed or compressed in the p-axis, which makes it difficult to compress them further along the p-axis. For this reason the "infusion coefficient" becomes so small, namely $\propto \exp[-2y]$. In contrast, the diffusion along the q-axis must become faster and faster, in order to diffuse all the incoming density flux from the other orthogonal p-direction, which is entering the cigar-shaped potential regions through their lengthy boundaries.

5   The Wehrl Entropy for the P_n

A useful measure for the information content of the probability distributions $P_n(q,p;y)$ is the Gibbs or Wehrl entropy [7], which is defined by

$$S_n(y) := -\int P_n(q,p;y)\ln P_n(q,p;y)\,\frac{dpdq}{2\pi} \; . \tag{14}$$

Because of the symmetry $P_n(q,p;-y) = P_n(p,q;y)$, the entropy (14) is even in y: $S_n(-y) = S_n(y)$. Therefore, at $y = 0$ each $S_n(y)$ must have either a maximum or a minimum. We shall now argue that $S_n(0)$ should correspond to a minumum: We assume that $S_n(y)$ does not oscillate as a function of y. Therefore, it is enough to argue that $S_n(y)$ grows with $|y|$ for large values of $|y|$. For large positive y, equation (12) behaves essentially like a one-dimensional diffusion equation in the q-variable. But it is well-known that the solutions of diffusion equations lead to entropies which increase with time [6]. Therefore, the $S_n(y)$ must increase as $y \to \infty$. But since the $S_n(y)$ are even in y, they must also grow as $y \to -\infty$. Hence, the $S_n(0)$ must lie at the bottom of the curves $S(y)$ vs. y.

Finally, we note that the von Neumann entropy $S_{vN}(\rho) := -\text{Tr}(\rho ln\rho)$ for the pure states $\rho := |n\rangle\langle n|$ must vanisch. In contrast, explicit calculations of the Wehrl entropies of the Poisson

distributions (8) shows that $S_n(0) \geq 1$ for all n, in accordance with a conjecture by Wehrl [7], which was proved by Lieb [8].

To summarize this section: in contrast to diffusion equations, where the entropies of their solutions always increase with time, the entropies $S_n(y)$ for the solutions of the above pseudo-diffusion equation first decrease monotonically as y grows from $-\infty$ to zero, but then increase monotonically as y grows from zero to $+\infty$.

6 Summary and Outlook

Two equivalent partial differential equations (9) and (12) were presented and then interpreted, as wave and as pseudo-diffusion equations. The probability densities $P_n(q, p; y)$ (5) provide infinite number of their solutions.

By the time of writing the present lecture notes, we succeeded in proving, by two general methods, that the expectation values $\langle q, p; \xi | O | q, p; \xi \rangle$ of an arbitrary operator O, satisfy a generalized version of the above partial differential equations, which also include rotations, i.e for the general squeezing $\xi = re^{i\phi}$. Interesting examples of O are the number operators N and N^2; their expectation values provide the simplest solutions of (9) and (12) . Also the projection operator $|q, p; \xi\rangle\langle q, p; \xi|$, and consequently its Wigner function, satisfy these equations.

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Infinite–Mode Squeezed Coherent States and Non–equilibrium Statistical Mechanics (Phase–Space–Picture Approach) *

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Abstract

The phase–space–picture approach to quantum non–equilibrium statistical mechanics via the characteristic function of infinite–mode squeezed coherent states is introduced. We use quantum Brownian motion as an example to show how this approach provides an interesting geometrical interpretation of quantum non–equilibrium phenomena.

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1 Introduction

The standard approach of quantum statistical mechanics uses the density operator $\hat{\rho}$ to describe the (mixed) state of the physical system of interest. Since $\hat{\rho}$ is an operator in the Hilbert space, we usually need some representation to perform the practical calculations. There are many equivalent representations in the literature, e.g., the coordinate representation, P–representation, Q–representation, Fock space representation, Wigner function and characteristic function (Chi function hereafter), etc. In this paper we will use the last two representations since they provide a "phase–space picture" for the quantum–mechanical problems [1].

In quantum equilibrium statistical mechanics, a system (A) immersed in a heat bath (B) with temperature T is described by the canonical ensemble. According to ensemble theory, the density operator is:

$$\hat{\rho} = \frac{\exp(-\beta\hat{H})}{Tr[\exp(-\beta\hat{H})]}, \tag{1}$$

where $\beta = \frac{1}{kT}$ is the inverse temperature and $\hat{H}$ is the Hamiltonian of (A). The structure of (B) and the interaction between (A) and (B) are irrelevant to the density operator. If $\hat{H}$ is (inhomogeneously) quadratic and with a finite number of degrees of freedom, the density operator will correspond to a finite–mode thermal Squeezed Coherent State (SqCS) [2].

In quantum non–equilibrium statistical mechanics, ensemble theory is no longer valid and we have to build a model for the heat bath (B) and consider (A)+(B) as a total system. The total Hamiltonian then contains three parts—the Hamiltonian of (A) and (B) and the interaction Hamiltonian.

It is well known that the number of degrees of freedom of a heat bath must be infinite (the thermodynamic limit), otherwise, due to the Poincaré recurrence theorem there will be no phenomena such as approach to equilibrium, damping or dissipation. The simplest model of a heat bath is an assembly of infinitely many harmonic oscillators with linear couplings to (A). In this kind of model, the total Hamiltonian is quadratic if the Hamiltonian of (A) is quadratic. Since for quadratic Hamiltonians we have a phase–space picture of quantum mechanics with the help of Wigner and/or Chi function, we can construct an infinite–mode

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(pure) SqCS for the total system in an infinite–dimensional phase space using these functions. After reduction, i.e., ignoring the heat bath but keeping its "influence", we will get a finite–mode SqCS for (A). In the limit as time approaches infinity, it can be shown that (A) will approach equilibrium, and the finite–mode SqCS will become a thermal SqCS consistent with the fluctuation–dissipation theorem [3, 4, 5, 6, 7, 8].

In this paper we introduce a geometric interpretation of these non–equilibrium phenomena via the Chi–function representation of infinite–mode SqCS. In Sec. 2 notations, conventions and a lemma on matrix are introduced for the mathematics used in this paper. In Sec. 3 we study finite–mode SqCS's by Wigner and Chi functions and then extend them to infinite mode. In Sec. 4 we use the quantum Brownian motion as an example to illustrate geometrical reduction in phase space.

2 Mathematical Preliminaries

Throughout this paper, $\hbar$ is set equal to 1; "T" denotes the transpose of a matrix and "$-$T"denotes the inverse of the transpose of a matrix. The physical system under consideration is of $N = n + 1$ degrees of freedom, where n is either finite or equal to infinity. The symbols $\vec{x} = (x^0, x^1, x^2, \ldots, x^n)$ and $\vec{k} = (k^0, k^1, k^2, \ldots, k^n)$ denote the N–dimensional canonical coordinate and momentum respectively, thus $\vec{z} \equiv (\vec{x}; \vec{k})$ is a vector in $2N$–dimensional phase space. $\vec{q}$ and $\vec{p}$ denote N–dimensional position and momentum operators corresponding to the canonical variables $\vec{x}$ and $\vec{k}$.

The metaplectic group $\mathbf{Mp(2N,R)}$ is an $N(2N + 1)$–dimensional Lie group. It is the quantum analogue of symplectic group $\mathbf{Sp(2N,R)}$. The elements of the Lie algebra of $\mathbf{Mp(2N,R)}$ can be organized as anti–hermitian operators in the following form:

$$
\begin{aligned}
\hat{\Phi}(m) &= \frac{i}{2} \sum_{i,j=0}^{n} [\alpha_{ij}\hat{q}_i\hat{q}_j + \beta_{ij}\hat{p}_i\hat{p}_j + \gamma_{ij}(\hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i)] \\
&= \frac{i}{2}(\vec{q}; \vec{p}) \begin{pmatrix} \alpha & \gamma \\ \gamma^{\mathsf{T}} & \beta \end{pmatrix} (\vec{q}; \vec{p})^{\mathsf{T}} \\
&= \frac{i}{2}(\vec{q}; \vec{p}) Jm (\vec{q}; \vec{p})^{\mathsf{T}},
\end{aligned}
\tag{2}
$$

where $\alpha_{ij} = \alpha_{ji}$, $\beta_{ij} = \beta_{ji}$ and

$$m = \begin{pmatrix} -\gamma^{\mathsf{T}} & -\beta \\ \alpha & \gamma \end{pmatrix} \in \mathrm{sp}(2N, r) \tag{3}$$

is a $2N \times 2N$ real matrix [9], while

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad 1 = N \times N \text{ unit matrix.} \tag{4}$$

The Lie algebra of $\mathbf{Mp(2N,R)}$ is isomorphic to that of $\mathbf{Sp(2N,R)}$. The action of $\exp[\hat{\Phi}(m)] \in \mathbf{Mp(2N,R)}$ on $(\vec{q}; \vec{p})$ is:

$$\exp[\hat{\Phi}(m)](\vec{q}; \vec{p})^{\mathsf{T}} \exp[-\hat{\Phi}(m)] = \exp(-m)(\vec{q}; \vec{p})^{\mathsf{T}}, \tag{5}$$

where $\exp(-m) \in \mathbf{Sp(2N,R)}$.

Lemma [10]

If M is a symmetric and positive definite $2N \times 2N$ matrix, then there exist two matrices $S_1, S_2 \in \mathbf{Sp(2N,R)}$ (but not unique), such that

$$M = S_1^{\mathsf{T}} \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} S_1 = S_2^{\mathsf{T}} \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix} S_2, \tag{6}$$

where $\omega = \mathrm{diag}(\omega_0, \omega_1, \omega_2, \dots, \omega_n)$, $\omega_j > 0$ for all j, and

$$S_2 = \begin{pmatrix} \omega^{-\frac{1}{2}} & 0 \\ 0 & \omega^{\frac{1}{2}} \end{pmatrix} S_1. \tag{7}$$

Remarks:

(1) $S \in \mathbf{Sp(2N,R)}$ if and only if $S^{\mathsf{T}} J S = J$ by definition.

(2) ω_j's are not eigenvalues of M in general. We will call them the "symplectic eigenvalues" of matrix M.

(3) The eigenvalues of JM are $\pm i\omega_j$'s, hence we can calculate ω_j's from JM as an ordinary eigenvalue problem.

(4) If the matrix C_j corresponds to a 2–dimensional rotation on the (x_j, k_j) plane, then

$$C_j^{\mathsf{T}} \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} C_j = C_j^{\mathsf{T}} C_j \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}. \tag{8}$$

Therefore S_1 in (6) can be replaced by $C_j S_1$ and hence is not unique.

3 Squeezed Coherent States in Phase Space

3.1 Wigner Function

The Wigner function of an N–mode density operator $\hat{\rho}$ is defined as [11, 12]:

$$W(\vec{x}; \vec{k}) = \pi^{-N} \int_{-\infty}^{\infty} d\vec{y} \exp(2i\vec{k} \cdot \vec{y}) \rho(\vec{x} - \vec{y}, \vec{x} + \vec{y}), \tag{9}$$

where $\rho(\vec{x}, \vec{x'})$ is the coordinate representation of the density operator $\hat{\rho}$.

The Wigner function is real and normalized by definition:

$$\int_{-\infty}^{\infty} d\vec{x} d\vec{k} W(\vec{x}; \vec{k}) = 1. \tag{10}$$

However, it is not always positive–definite and is thus called the quasi–probability distribution function over the "phase space" $\vec{z} = (\vec{x}; \vec{k})$.

If the density operator is an exponential of a quadratic form of position and momentum operators, then the Wigner function is a Gaussian distribution in $\vec{z}$:

$$W(\vec{z}) = C_N \exp[-(\vec{z} - \vec{z}_c) M (\vec{z} - \vec{z}_c)^{\top}]. \tag{11}$$

where $C_N = \pi^{-N} \sqrt{\det(M)}$ is the normalization constant, $\vec{z}_c$ is a constant vector in the 2N–dimensional phase space, and M is a symmetric, positive definite matrix with all its symplectic eigenvalues smaller or equal to 1. (Otherwise (11) will not correspond to a physical state.) The Gaussian Wigner function (11) corresponds to the multimode thermal SqCS in general [2]. It contains the ordinary coherent states (when M is a unit matrix) and the ordinary SqCS (when all the symplectic eigenvalues of M equal to 1) as special cases.

The "Wigner ellipsoid" of (11) is defined as:

$$(\vec{z} - \vec{z}_c) M (\vec{z} - \vec{z}_c)^{\top} = 1, \tag{12}$$

which is an ellipsoid in the 2N–dimensional phase space with its center at $\vec{z}_c$ and its shape determined by M. We can take the Wigner ellipsoid as a geometric representation of the Gaussian Wigner function.

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3.2 Characteristic Function (Chi Function)

The Chi function of a density operator $\hat{\rho}$ is defined as:

$$\chi(\vec{x}; \vec{k}) = Tr[\hat{\rho}\hat{D}(-\vec{x}; -\vec{k})],$$

(13)

It can be shown that Chi function is the symplectic Fourier transformation of the Wigner function:

$$\chi(\vec{x}; \vec{k}) = \int_{-\infty}^{\infty} d\vec{x}' d\vec{k}' W(\vec{x}'; \vec{k}') \exp[-i(\vec{x}' \cdot \vec{k} - \vec{k}' \cdot \vec{x})].$$

(14)

The normalization condition of the Wigner function corresponds to $\chi(\vec{0}; \vec{0}) = 1$ in the Chi function. Since the operator $\hat{D}(-\vec{x}; -\vec{k})$ is unitary, $\chi(\vec{x}; \vec{k})$ is complex in general.

The Chi function which corresponds to the Gaussian Wigner function (11) is also Gaussian:

$$\chi(\vec{z}) = \exp[-\frac{1}{4}\vec{z}JM^{-1}J^{\mathsf{T}}\vec{z}^{\mathsf{T}} + i\vec{z}J\vec{z}_c^{\mathsf{T}}].$$

(15)

Analogue to the Wigner ellipsoid, we can also define the "Chi ellipsoid" for a Gaussian Chi function as:

$$(\vec{z} - \vec{z}_c)JM^{-1}J^{\mathsf{T}}(\vec{z} - \vec{z}_c)^{\mathsf{T}} = 1.$$

(16)

The center of Chi ellipsoid is the same as that of the Wigner ellipsoid, while the shape of this ellipsoid is determined by the matrix $JM^{-1}J^{\mathsf{T}}$.

3.3 Mean Vector and Covariance Matrix

For an N–mode (mixed) state with the density operator $\hat{\rho}$, the mean vector is defined as $(\langle\vec{\hat{q}}\rangle; \langle\vec{\hat{p}}\rangle)$ in the $2N$–dimensional phase space, where $\langle\hat{q}_i\rangle \equiv Tr(\hat{\rho}\hat{q}_i)$, etc. The covariance matrix is defined as a $2N \times 2N$ matrix of the form:

$$\begin{pmatrix} U & Q \\ Q^{\mathsf{T}} & V \end{pmatrix},$$

(17)

$$U_{ij} \equiv \langle(\hat{q}_i - \langle\hat{q}_i\rangle)(\hat{q}_j - \langle\hat{q}_j\rangle)\rangle = \langle\hat{q}_i\hat{q}_j\rangle - \langle\hat{q}_i\rangle\langle\hat{q}_j\rangle,$$

(18)

$$V_{ij} \equiv \langle(\hat{p}_i - \langle\hat{p}_i\rangle)(\hat{p}_j - \langle\hat{p}_j\rangle)\rangle = \langle\hat{p}_i\hat{p}_j\rangle - \langle\hat{p}_i\rangle\langle\hat{p}_j\rangle,$$

(19)

$$Q_{ij} \equiv \frac{1}{2}\langle(\hat{q}_i - \langle\hat{q}_i\rangle)(\hat{p}_j - \langle\hat{p}_j\rangle) + (\hat{p}_j - \langle\hat{p}_j\rangle)(\hat{q}_i - \langle\hat{q}_i\rangle)\rangle$$

$$= \langle\frac{1}{2}(\hat{q}_i\hat{p}_j + \hat{p}_j\hat{q}_i)\rangle - \langle\hat{q}_i\rangle\langle\hat{p}_j\rangle. \tag{20}$$

For the Gaussian Wigner function (11) or Gaussian Chi function (15), the mean vector is $\vec{z}_c$, and the covariance matrix takes the form:

$$\begin{pmatrix} U & Q \\ Q^\mathsf{T} & V \end{pmatrix} = \frac{1}{2}M^{-1}. \tag{21}$$

Therefore (15) can be re-written as:

$$\chi(\vec{z}) = \exp[-\frac{1}{2}\vec{z}\begin{pmatrix} V & -Q^\mathsf{T} \\ -Q & U \end{pmatrix}\vec{z}^\mathsf{T} + i\vec{z}J\vec{z}_c^\mathsf{T}]. \tag{22}$$

3.4 Time Evolution of Wigner and Chi Functions

Consider an N-mode Hamiltonian:

$$\hat{H} = \frac{1}{2}(\vec{q};\vec{p})K(\vec{q};\vec{p})^\mathsf{T}, \tag{23}$$

where K is a $2N \times 2N$ positive definite symmetric matrix . According to the lemma introduced in Sec. 1, this kind of Hamiltonian can be transformed into the following form:

$$\hat{H} = \frac{1}{2}(\vec{q};\vec{p})S^\mathsf{T}\begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix}S(\vec{q};\vec{p}), \tag{24}$$

where $\omega = \operatorname{diag}(\omega_0,\omega_1,\ldots,\omega_n)$ and the ω_i's are symplectic eigenvalues of the matrix K. The time–evolution operator $\exp(-it\hat{H})$ is an element in $\mathbf{Mp(2N,}$ $\mathbf{R})$ and the time evolution of $(\vec{q};\vec{p})$ is a special case of (5):

$$\exp(it\hat{H})(\vec{q};\vec{p})^\mathsf{T}\exp(-it\hat{H}) = R(t)(\vec{q};\vec{p})^\mathsf{T}, \tag{25}$$

where

$$R(t) = \exp(tJK) = S^{-1}\begin{pmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix}S \tag{26}$$

is an element in $\mathbf{Sp(2N,R)}$. $\{R(t)|\, t \in \mathbf{R}\}$ forms a one–parameter subgroup of $\mathbf{Sp(2N,R)}$ describing the phase flow in the $2N$–dimensional classical phase space:

$$(\vec{x}(t);\vec{k}(t))^\mathsf{T} = R(t)(\vec{x}(0);\vec{k}(0))^\mathsf{T}. \tag{27}$$

It is well known that for the quadratic Hamiltonian (23), the time evolution of Wigner function and Chi function follow exactly this phase flow:

$$W(\vec{z};t) = C_N \exp[-(\vec{z} - \vec{z}_c(t))R(t)^{-\mathsf{T}} M R(t)^{-1}(\vec{z} - \vec{z}_c(t))^{\mathsf{T}}], \qquad (28)$$

$$\chi(\vec{z};t) = \exp[-\frac{1}{2}\vec{z} R(t)^{-\mathsf{T}} \begin{pmatrix} V & -Q^{\mathsf{T}} \\ -Q & U \end{pmatrix} R(t)^{-1}\vec{z}^{\mathsf{T}} + i\vec{z}J\vec{z}_c^{\mathsf{T}}(t)]$$

$$\equiv \exp[-\frac{1}{2}\vec{z}\begin{pmatrix} V(t) & -Q(t)^{\mathsf{T}} \\ -Q(t) & U(t) \end{pmatrix}\vec{z}^{\mathsf{T}} + i\vec{z}J\vec{z}_c^{\mathsf{T}}(t)], \qquad (29)$$

where $\vec{z}_c^{\mathsf{T}}(t) = R(t)\vec{z}_c^{\mathsf{T}}$.

3.5 Reduction of Multimode Squeezed Coherent States

Consider the quantum system (A)+(B) discussed in Sec. 1 whose density operator is $\hat{\rho}_{AB}$. If we reduce this system by ignoring (B), the expectation value of an operator $\hat{O}_A$ which corresponds to a measurement on (A) will become:

$$\langle \hat{O}_A \rangle = Tr[\hat{\rho}_A \hat{O}_A], \qquad (30)$$

where $\hat{\rho}_A = Tr_{(B)}(\hat{\rho}_{AB})$ is a well–defined reduced density operator for (A) which contains the "influence" of (B) on (A), $Tr_{(B)}$ represents the "partial trace" which only takes trace with respect to the degrees of freedom of (B).

If the Wigner function $W(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B)$ corresponds to the original density operator $\hat{\rho}_{AB}$, then the reduced Wigner function corresponding to $\hat{\rho}_A$ will be [12]:

$$W_A(\vec{x}_A; \vec{k}_A) = \int_{-\infty}^{\infty} d\vec{x}_B d\vec{k}_B W(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B). \qquad (31)$$

As for the Chi function, if $X(\vec{x}_A, \vec{x}_B; \vec{k}_A, \vec{k}_B)$ corresponds to $\hat{\rho}_{AB}$, the reduced Chi function corresponding to $\hat{\rho}_A$ will take the form:

$$X_A(\vec{x}_A; \vec{k}_A) = X(\vec{x}_A, \vec{0}; \vec{k}_A, \vec{0}), \qquad (32)$$

which is a restriction of the original $X(\vec{x}_A; \vec{x}_B, \vec{k}_A; \vec{k}_B)$ to a subspace in the $2N$–dimensional phase space. From the mathematical point of view, it is easier to use the Chi function to perform the reduction.

Now let us use N-mode to one-mode reduction as an example. For a given N-mode Gaussian Chi function (22), we want to make a reduction by ignoring all the degrees of freedom which correspond to modes $1, 2, \ldots, n$ and leave only the 0-th mode. Without any calculation, we can write down the reduced Chi function directly :

$$\chi(x^0, k^0) = \exp[-\frac{1}{2}(x^0, k^0) \begin{pmatrix} V_{00} & -Q_{00} \\ -Q_{00} & U_{00} \end{pmatrix} (x^0, k^0)^\top + i(x^0 k_c^0 - k_0 x_c^0)], \quad (33)$$

which is a one-mode Gaussian Chi function.

The geometrical interpretation of this reduction process is cutting the original Chi ellipsoid in the $2N$-dimensional phase space by a "shifted (x^0, k^0) plane"—the plane which is parallel to (x^0, k^0) plane and passes through the center of the Chi ellipsoid. The section on the Chi ellipsoid gives the "Chi ellipse" on the shifted (x^0, k^0) plane which represents the reduced one-mode Gaussian Chi function. A schematic graph of this geometrical reduction is shown in Fig. 1.

3.6 Infinite–Mode Squeezed Coherent States

The infinite–mode SqCS is a naive generalization of finite–mode SqCS. Comparing the three equivalent representations of finite–mode SqCS's: (11) , (15) and (22), we see that (22) can be directly generalized to infinite mode without any ambiguity or convergence problem. So we will take (22) in the infinite–dimensional phase space as the definition of infinite–mode SqCS, all the above formulas which involve (22) can be applied to infinite–mode case.

4 Quantum Brownian Motion

In this section we shall study quantum Brownian motion of a harmonic oscillator. The physical picture is a quantum harmonic oscillator immersed in a dissipative heat bath. In classical statistical mechanics, this problem can be studied via the Langevin equation:

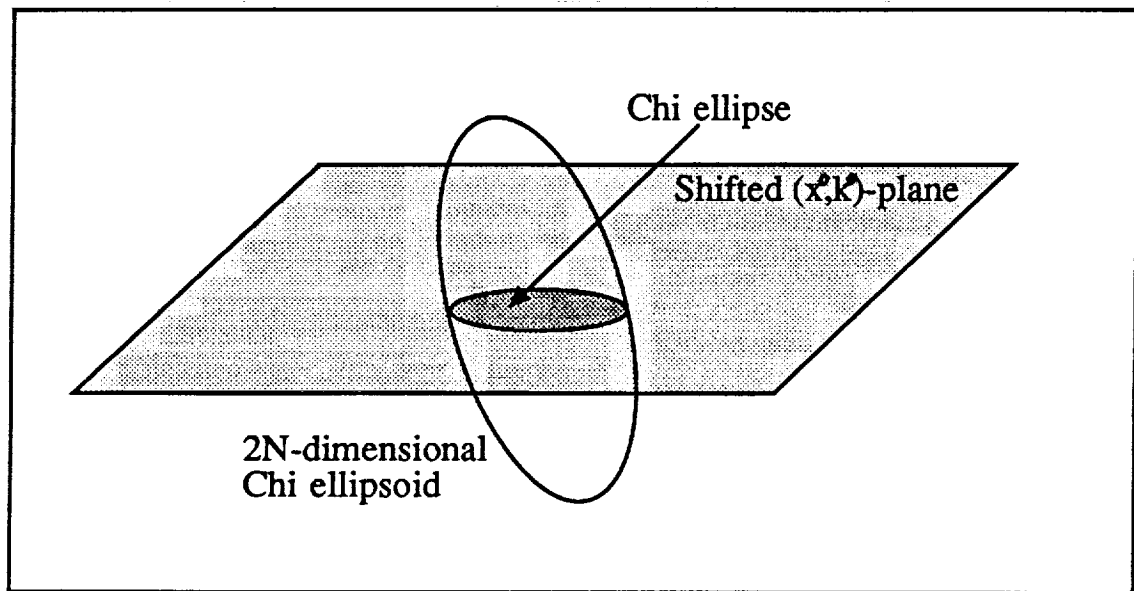$$M \ddot{X} + M\gamma \dot{X} + M\Omega^2 X = 0, \quad (34)$$

Fig. 1. Reduction as a geometrical operation in phase space.

where X, M and Ω are the coordinate, mass and frequency of the oscillator, and $M\gamma$ is the friction constant.

The quantum analogue of the Langevin equation can be achieved by several quantum–mechanical heat–bath models, e.g., the FKM model [3], linear coupling model [4, 5], independent–oscillator model [13], etc. Actually it can be proved that all these models are equivalent [14]. In this paper we will use the independent–oscillator model since it is the simplest and most intuitive.

4.1 Independent–Oscillator Heat–Bath Model

Consider the Brownian particle to be a harmonic oscillator immersed in a dissipative heat bath with inverse temperature β. Using the independent–oscillator heat–bath model, the total Hamiltonian of the system is [13]:

$$H = \frac{\hat{P}^2}{2M} + \frac{1}{2}M\Omega^2\hat{Q}^2 + \sum_{i=1}^{\infty}[\frac{\hat{p}_i^2}{2m_i} + \frac{1}{2}m_i\omega_i^2(\hat{q}_i - \hat{Q})^2], \qquad (35)$$

where $\hat{Q}$ and $\hat{P}$ are the position and momentum operators of the Brownian particle; $\hat{q}_i$, $\hat{p}_i$, m_i and ω_i are the position operator, momentum operator, mass and frequency of the i-th heat–bath oscillator, $i = 1, 2, 3, \ldots$. This Hamiltonian is a special case of (23).

It can be proved that in order to make the Brownian particle satisfy the quantum Langevin equation:

$$M\ddot{\hat{Q}} + M\gamma\dot{\hat{Q}} + M\Omega^2\hat{Q} = 0, \qquad (36)$$

the spectral distribution of heat–bath oscillators must obey: ·

$$\sum_{i=1}^{n} m_i\omega_i^2\delta(\omega - \omega_i) = \frac{2}{\pi}M\gamma. \qquad (37)$$

4.2 Quantum Brown Motion in Phase Space

In the following we will study time evolution of the Brownian particle by the reduced Chi function. The initial condition is chosen to be $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$, where $\hat{\rho}_A$ is the initial density operator of the Brownian particle which corresponds to an

arbitrary Gaussian Wigner/Chi function, and $\hat{\rho}_B$ is the initial density operator of heat bath which is in thermal equilibrium at the inverse temperature β. Since the detailed calculations can be obtained by combining the calculations in [6] and [13], here we will only discuss the result and the geometrical interpretation.

Let the degree of freedom of the Brownian particle correspond to the 0-th mode, and those of the heat bath correspond to other modes. The infinite-mode Chi function for the initial condition state is (22) with the following parameters: U_{00}, V_{00}, Q_{00} and $\vec{z}_c^0$, which correspond to the initial conditions of the Brownian particle, are arbitrary; $\vec{z}_c$ has only two non-zero components corresponding to $\vec{z}_c^0$ since the mean vectors for all heat-bath oscillators equal to zero; and other elements in the covariance matrix are:

$$U_{ij} = \frac{1}{2m_i\omega_i}\coth(\frac{1}{2}\beta\omega_i)\delta_{ij}, \tag{38}$$

$$V_{ij} = \frac{1}{2}m_i\omega_i\coth(\frac{1}{2}\beta\omega_i)\delta_{ij}, \tag{39}$$

$$U_{i0} = U_{0j} = V_{i0} = V_{0j} = Q_{i0} = Q_{0j} = Q_{ij} = 0, \tag{40}$$

for all $i,j = 1, 2, 3, \ldots$.

Combining (29) and (33), we get the time-dependent reduced Chi function of the Brownian particle (the index 0 for the canonical coordinates is suppressed):

$$\chi(x, k; t) = \exp[-\frac{1}{2}(x, k)\begin{pmatrix} V_{00}(t) & -Q_{00}(t) \\ -Q_{00}(t) & U_{00}(t) \end{pmatrix}(x, k)^\mathsf{T} + i(xk_c(t) - kx_c(t))]. \tag{41}$$

It is easy to write down the corresponding Wigner function by comparing (11) and (15):

$$W(x, k; t) = C_1 \exp[-(x - x_c(t), k - k_c(t))M(t)(x - x_c(t), k - k_c(t))^\mathsf{T}], \tag{42}$$

where

$$M(t) = \frac{1}{2[U_{00}(t)V_{00}(t) - Q_{00}^2(t)]}\begin{pmatrix} V_{00}(t) & -Q_{00}(t) \\ -Q_{00}(t) & U_{00}(t) \end{pmatrix}. \tag{43}$$

Unlike ordinary reduction methods [5, 6, 7], we obtained this reduced Wigner function without using integration over the heat-bath degrees of freedom.

Comparing (41) with (42) and (43), we see that at any moment the Wigner ellipse and the Chi ellipse are similar and their areas inversely proportional to

each other. (Although both areas are time–dependent in general.) When time approaches infinity, the Brownian particle will approach the equilibrium state which is independent of its initial condition and consistent with the fluctuation-dissipation theorem. In Fig. 2, we plot the time evolution of Wigner ellipse and Chi ellipse of the Brownian particle in phase space.

5   Conclusion and Outlook

The method and result discussed in this paper are valid as long as: (1) The initial state of (A) is a finite–mode (not necessary one–mode) Gaussian Wigner/Chi function. (2) The Hamiltonian of (A) is quadratic and with finite number of degrees of freedom.

If (1) no longer holds, then we will not be able to use an ellipsoid in phase space to represent the state. However, the phase–space picture continues to be valid since time evolution of the Wigner/Chi function will still follow the phase flow in classical phase space. On the contrary if (2) is not true, e.g., as in quantum tunneling problems, then time evolution of the Wigner/Chi function will not follow the phase flow exactly and the phase–space picture will fail. In order to relieve this limitation, some authors introduced the idea of "effective potential"[15, 16] so that time evolution of the Wigner/Chi function can be still expressed in terms of the phase flow. Integration of this modified phase–space picture with the dissipation mechanism is an interesting question and worth pursuing further.
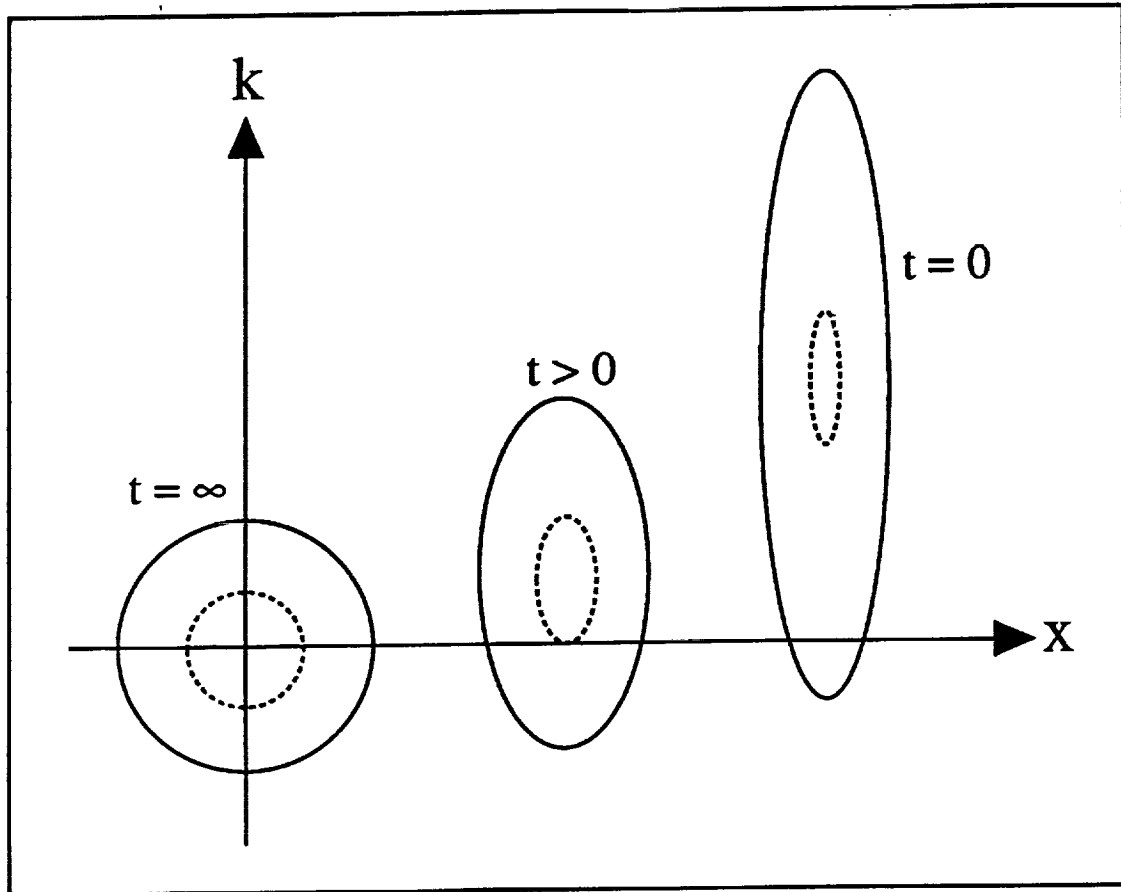
Acknowledgements

Fig. 2. Time evolution of Wigner ellipse (solid) and Chi ellipse (dotted) of the Brownian particle. Note that the area of the former is always larger than that of the latter.

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Disclaimer

CORRELATED STATES OF A QUANTUM OSCILLATOR ACTED BY
SHORT PULSES

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ABSTRACT

Correlated squeezed states for a quantum oscillator
acted by very short in time pulses modeled by special
dependence on time of frequency of oscillator in the form
of sequence of three delta-kickings of frequency are
constructed based on the method of quantum integrals of
motion. Also the correlation coefficient and quantum
variances of operators of coordinate and momenta are
written in explicit form.

The aim of the paper is to discuss the squeezing phenomenon
and correlations in the system of quantum parametric oscillator
with special dependence on time of frequency of oscilllator. We
consider the case when oscillator is acted by very short in time
pulses. This dependence on time we will model by δ-kickings of
frequency. In this paper we will consider the case of sequence of
three δ-kickings of frequency. The cases of one and two δ-kickings
of frequency were considered in [1]. Short pulses in the form of
δ-kicking were discussed briefly in the case of two-mode squeezing
[2] and for the chain of quantum oscillators [3].
   Let us consider the quantum parametric oscillator which is
acted by very short in time pulses. We are modeling this action by
the special dependence on time of frequency of oscillator. We will
use the model of δ-kickings of frequency.
   Let the first kick be at initial moment of time $t_1=0$, the
second one in the moment $t_2=\tau$, and the third one at $t_3=2\tau$.
   The Hamiltonian of the system is of the form
$$\hat{H} = \hat{p}^2/2m + m\omega^2(t)\hat{q}^2/2 \qquad (1)$$

where $\hat{q}$ is coordinate operator $\hat{p}$ is momentum operator, m is the mass and $\omega(t)$ is time-dependent frequency. We choose the following dependence on time of oscillator frequency

$$\omega^2(t) = \omega_0^2 - 2\sum_{n=1}^{3} \kappa\delta(t-t_n). \tag{2}$$

The equations of motion corresponding to Hamiltonian (1) are of the form

$$\ddot{q} + (\omega_0^2 - 2\kappa\delta(t) - 2\kappa\delta(t-\tau) - 2\kappa\delta(t-2\tau))q = 0 \tag{3}$$

Following the usual scheme [4] one can construct integral of motion for the Hamiltonian (1)

$$\hat{A}(t) = \frac{i}{2}\left\{\frac{\hat{p}\varepsilon}{(\hbar m\omega_0)^{1/2}} - \frac{\hat{q}\dot{\varepsilon}}{(\hbar\omega_0/m)^{1/2}}\right\} \tag{4}$$

where function ε is the solution of equation of motion (3). If function ε satisfy additional condition

$$\dot{\varepsilon}\varepsilon^* - \dot{\varepsilon}^*\varepsilon = 2i\omega_0.$$

the integral of motion (4) and its hermitian conjugate satisfy boson commutation relations. The ground state of the system can be found from the condition

$$\hat{A}(t)\psi_0(q,t) = 0$$

and has the form

$$\psi_0(q,t) = \pi^{-1/4}\left[\frac{\hbar\varepsilon}{m\omega_0}\right]^{-1/2}\exp\left\{\frac{i\dot{\varepsilon}}{2\varepsilon}\frac{q^2}{(\hbar/m\omega)}\right\}. \tag{5}$$

The coherent states of the system can be found as eigenfunctions of the integral of motion $\hat{A}(t)$

$$\hat{A}(t)\,\psi_\alpha(q,t) = \alpha\psi_\alpha(q,t),$$

where α is complex number and has the form

$$\psi_\alpha(q,t) = \psi_0(q,t)\exp\left\{-\frac{|\alpha|^2}{2} + \frac{\sqrt{2}\,\alpha q}{\varepsilon\sqrt{\hbar/m\omega_0}} - \frac{\alpha^2\varepsilon^*}{2\varepsilon}\right\}. \tag{6}$$

One can see that ground and coherent states are Gaussian wavepackets with time-dependent coefficients in quadratic form under exponential function.

In order to write integral of motion in explicit form one has to solve equations (3) for the function ε in the case of sequence

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of three δ-kicks of frequencies (2). For the function $\varepsilon(t)$ one can write following solutions

$$\varepsilon_o(t) = A_o e^{i\omega_o t} + B_o e^{-i\omega_o t} , \quad t<0.$$
$$\varepsilon_1(t) = A_1 e^{i\omega_o t} + B_1 e^{-i\omega_o t} , \quad 0<t<t_2,$$
$$\varepsilon_2(t) = A_2 e^{i\omega_o t} + B_2 e^{-i\omega_o t} , \quad t_2<t<t_3, \quad (7)$$
$$\varepsilon_3(t) = A_3 e^{i\omega_o t} + B_3 e^{-i\omega_o t} , \quad t>t_3.$$

So, one has four regions of changing the function $\varepsilon(t)$. At three points of time t_1, t_2, t_3 we have following conditions for functions ε_i

$$\varepsilon_i(t_i) = \varepsilon_{i-1}(t_i),$$
$$\dot{\varepsilon}_i(t_i) - \dot{\varepsilon}_{i-1}(t_i) = 2\kappa \, \varepsilon_{i-1}(t_i).$$

From this conditions one can find the conditions for coefficients A_i and B_i. Taking in the initial moment of time the wave with $A_o=1$ and $B_o=0$ one has the solutions for ε-function after δ-kickings

$$\varepsilon_o = e^{i\omega_o t} , \quad t<0. \quad (8)$$

$$\varepsilon_1(t) = (1-i\kappa/\omega_o) \, e^{i\omega_o t} + i\kappa/\omega_o e^{-i\omega_o t} , \quad 0<t<\tau , \quad (9)$$

$$\varepsilon_2(t) = \left[(1-i\kappa/\omega_o)^2 + \frac{\kappa^2}{\omega_o^2} e^{-2i\omega_o t} \right] e^{i\omega_o t} + \left[(i\kappa/\omega_o)(1+i\kappa/\omega_o) \right.$$

$$\left. + (i\kappa/\omega_o)(1-i\kappa/\omega_o)e^{2i\omega_o\tau} \right] e^{-i\omega_o t} , \quad \tau<t<2\tau. \quad (10)$$

$$\varepsilon_3(t) = \left[(1-i\kappa/\omega_o)(\chi^2 - 1)e^{-i\omega_o\tau} - \chi e^{-2i\omega_o\tau} \right] e^{i\omega_o t} +$$

$$+ \frac{i\kappa}{\omega_o} (1 + \chi^2)e^{3i\omega_o\tau} e^{-i\omega_o t} , \quad t>2\tau . \quad (11)$$

where $\chi=2\cos\omega_o\tau + \dfrac{2\kappa}{\omega_o} \sin\omega_o\tau$.

If before δ-kickings the system was in coherent states then after the sequence of δ-kickings of frequency the oscillator will be in correlated squeezed state determined by formulae (6) with function ε given by formulae (11). In order to have explicit expression for these states in another periods of time one has to put in formulae (6) the explicit expression for ε function in this period of time given by formulae (7).

The disperssion of coordinate after sequence of δ-kickings will be equal to

$$\sigma_q = \langle\psi_\alpha|\hat{q}^2|\psi_\alpha\rangle - \langle\psi_\alpha|\hat{q}|\psi_\alpha\rangle^2 = \frac{\hbar}{2m\omega_0}\left\{1 + \frac{4\kappa^2}{\omega_0^2}(\chi^2-1)^2\sin^2\omega_0(t-2\tau) + \frac{2\kappa}{\omega_0}\left[\chi^2-1\right]^2\sin2\omega_0(t-2\tau) + \frac{2\kappa}{\omega_0}\chi\left[\chi^2-1\right]\sin(2\omega_0 t-5\omega_0\tau)\right.$$

The correlation between coordinate and momenta in this state is not equal to zero and is of the form

$$\sigma_{qp} = \frac{1}{2}\langle\psi_\alpha|\hat{q}\hat{p}+\hat{p}\hat{q}|\psi_\alpha\rangle - \langle\psi_\alpha|\hat{q}|\psi_\alpha\rangle\langle\psi_\alpha|\hat{p}|\psi_\alpha\rangle = = \frac{\hbar}{2}\left\{\left[1 + \frac{4\kappa^2}{\omega_0^2}(\chi^2-1)^2\sin^2\omega_0(t-2\tau) + \frac{2\kappa}{\omega_0}\left[\chi^2-1\right]^2\sin2\omega_0(t-2\tau) + \frac{2\kappa}{\omega_0}\chi\left[\chi^2-1\right]\sin(2\omega_0 t-5\omega_0\tau)\right]\odot\left[1 + \frac{4\kappa^2}{\omega_0^2}(\chi^2-1)^2\cos^2\omega_0(t-2\tau) - \frac{2\kappa}{\omega_0}\left[\chi^2-1\right]^2\sin2\omega_0(t-2\tau) + \frac{2\kappa}{\omega_0}\chi\left[\chi^2-1\right]\sin(2\omega_0 t-5\omega_0\tau)\right] - 1\right\}^{1/2}.$$

So one has statistical dependence of operators of coordinate and momentum after series of δ-kickings and in some periods of time the disperssion of coordinate is less then before δ-kickings. So we have two phenomena due to seria of short in time pulses acted on oscillator: squeezing phenomenon and phenomenon of statistical dependence of operators of coordinates and momenta.

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FLOQUET OPERATOR AS INTEGRAL OF MOTION

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Nonstationary quantum systems have no energy levels. However, for time-dependent periodical quantum systems, the notion of quasi–energy levels has been introduced in Ref. [1, 2]. The main point of the quasi–energy concept is to relate quasienergies to eigenvalues of the Floquet operator or monodromy operators which is equal to the evolution operator of a quantum system taken at the moment coinciding with the period of the system. The purpose of this article is to relate the Floquet operator to integrals of the motion and to introduce new operator which is the integral of motion and has the same quasienergy spectrum that the Floquet operator has. Implicitly, the result of the article was contained in Ref. [3], but we wish to have an explicit formula for this new integral of motion.

If one has the system with Hermitian Hamiltonian $H(t)$ such that $H(t+T) = H(t)$, the unitary evolution operator $U(t)$ is defined as

$$|\Psi, t\rangle = U(t)|\Psi, 0\rangle, \tag{1}$$

where $|\Psi, 0\rangle$ is the state vector of the system at the initial time. Then, by definition, the operator $U(T)$ is the Floquet operator and its eigenvalues have the form

$$f = \exp(-i\epsilon T), \tag{2}$$

with $\hbar = 1$, where ϵ is called the quasienergy state vector. The spectrum of quasienergy may be discrete or continuous for different quantum systems [3]. We wish to answer the following question. Is the quasienergy a conserved observable or not? This question is related to another question. Is the Floquet operator F(T) an integral of motion or not? The answer to the second question is negative. The operator U(T) does not satisfy the relation

$$\frac{\partial I(t)}{\partial t} + i[H(t), I(t)] = 0, \tag{3}$$

which defines the integral of motion $I(t)$. Thus, the Floquet operator $U(T)$ is not the integral of motion for the periodical nonstationary quantum systems. But as it was found in Ref. [3], the operator of the form

$$I(t) = U(t)I(0)U^{-1}(t) \tag{4}$$

satisfies equation (3) and this operator is the integral of motion of the quantum system. Thus, for periodical quantum systems, let us introduce the unitary operator M(t) which has the form

$$M(t) = U(t)U(T)U^{-1}(t). \tag{5}$$

This operator is the integral of motion due to the construction given by the formula (4) for any integral of motion. The spectrum of the new invariant operator $M(t)$ coincides with the spectrum

of the Floquet operator $U(T)$. We have therefore answered the question about quasienergies. Since these numbers are defined as eigenvalues of the integral of motion $M(t)$, they are conserved quantities. Thus we generalize the concept of quasienergies connecting these quantum observables with the integral of motion of periodical quantum systems.

The construction given above allows us to introduce new invariant labels for nonperiodical systems, for example, with the time-dependence of the Hamiltonian corresponding to quasicrystal structure in time. For such systems, the analogue of the invariant Floquet operator (5) will be the operator

$$M_1(t) = U(t)U(t_1)U(t_2)U^{-1}(t).\tag{6}$$

This integral of motion is connected with the two characteristic times of the quasicrystal structure t_1 and t_2. For poly-dimensional structure, we can introduce the integral of motion

$$M_2(t) = U(t)\left[\prod_{i=1}^{i=n} U(t_i)\right] U^{-1}(t),\tag{7}$$

where $t_1, t_2, ..t_n$ are the characteristic times of the system. The eigenvalues of the operators $M_1(t)$ and $M_2(t)$ are conserved quantities, and they characterize the nonperiodical quantum systems with quasicrystal structure in time in the same manner as quasienergy describes the states of periodical quantum systems.

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COHERENT STATES AND UNCERTAINTY RELATIONS FOR THE DAMPED HARMONIC OSCILLATOR WITH TIME-DEPENDENT FREQUENCY

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Abstract

Starting with evaluations of propagator and wave function for the damped harmonic oscillator with time-dependent frequency, exact coherent states are constructed. These coherent states satisfy the properties which coherent states should generally have.

Since Schrödinger[1] constructed the coherent states for the harmonic oscillator, they have been widely used to describe many fields of physics[2,3,4]. Glauber[5] has used coherent states to discuss photon statistics of the radiation field, and Nieto and Simmons[6] have constructed coherent states for particles in various potentials. Hartley and Ray[7] have obtained exact coherent states for a time-dependent harmonic oscillator on the basis of Lewis and Riesenfeld[8]. Recently Yeon, Um and George obtained exact coherent states for a damped harmonic oscillator with constant frequency[9] and also the propagator, wave function, energy expectation values, uncertainty relations and coherent states for a quantum forced time-dependent harmonic oscillator[10].

In this paper we evaluate the wave function and uncertainty relations and construct exact coherent states for the damped harmonic oscillator with time-dependent frequency described by the modified Caldirola-Kanai Hamitonian through the path integral method,

$$H = f(t)[e^{-\gamma t}\frac{p^2}{2m} + e^{\gamma t}\frac{m}{2}(\omega^2 + \frac{\gamma^2}{4f(t)} - \frac{f(t)'\gamma}{2f(t)^3})x^2] , \qquad (1)$$

where $f(t)$ is dimensionless time-dependent function and has the value $f(t) \mid_{t=0} = 1$.

Very recently, we have obtained the propagators and wave functions for the damped driven harmonic oscillator with an external driving force $F(t)$[11], driven coupled harmonic oscillator[12], quantum oscillator chains[13] and a mode of the electromagnetic field in a resonator with time-dependent characteristics of the internal medium[14] by the path integral method. Through similar calculations in the above papers we may evaluate the propagator for the Hamiltonian of Eq.(1) :

$$
\begin{aligned}
K(x,t;x',t') &= [\frac{m\omega e^{\frac{\gamma}{2}(t+t')}}{2\pi i\hbar \sin(\omega \int_{t'}^{t} f(t)\, dt)}]^{1/2} \exp\{\frac{im\omega}{2\hbar}[\cot(\omega \int_{t'}^{t} f(t)\, dt) - \frac{\gamma}{2\omega f(t)}]e^{\gamma t}x^2 \\
&\quad - \frac{2e^{\frac{\gamma}{2}(t+t')}xx'}{\sin(\omega \int_{t'}^{t} f(t)\, dt)} + [\cot(\omega \int_{t'}^{t} f(t)\, dt) + \frac{\gamma}{2\omega f(t)'}]e^{\gamma t'}x'^2\} \ .
\end{aligned}
\tag{2}
$$

The solution of the Schrödinger equation is given as the path-dependent integral equation with propagator K, ·

$$
\psi(x,t) = \int K(x,t;x',0)\psi(x',0)\, dx' \ ,
\tag{3}
$$

which gives the wave function $\psi(x,t)$ at time t in terms of the wave function $\psi(x',0)$ at time $t = 0$. At $t = 0$ the Hamiltonian [Eq.(1)] reduces to the Hamiltonian of a simple harmonic oscillator, and the corresponding wave function becomes

$$
\psi(x',0) = (\frac{\sqrt{m\omega_0/\hbar}}{2^n n!\sqrt{\pi}})^{1/2} H_n(\sqrt{\frac{m\omega_0}{\hbar}}x')e^{-\frac{m\omega_0}{2\hbar}x'^2} \ .
\tag{4}
$$

Substitution of Eqs.(2) and (4) into Eq.(3) yields the wave function

$$
\begin{aligned}
\psi(x,t) &= (\frac{\sqrt{m\omega_0/\hbar}}{2^n n!\sqrt{\pi}})^{1/2}\frac{e^{\frac{\gamma}{2}t}}{\xi} \exp\{-i(n+1/2)\cot[\omega/\omega_0 \cot(\omega \int_{t'}^{t} f(t)\, dt) + \frac{\gamma}{2\omega_0}]\} \\
&\quad \times e^{Ax^2} H_n(Dx) \ ,
\end{aligned}
\tag{5}
$$

where

$$
\xi^2 = \frac{\gamma^4}{16\omega^2\omega_0^2}\sin^2(\omega \int_0^t f(t)\, dt) + \frac{\gamma}{2\omega}\sin(2\omega \int_0^t f(t)\, dt) + 1 \ ,
\tag{6}
$$

$$
\begin{aligned}
A &= -\frac{m\omega_0}{2\hbar}\frac{e^{\gamma t}}{\xi^2} + i\frac{m\omega}{2\hbar}\frac{e^{\gamma t}}{\xi^2}\{\xi^2[\cot(\omega \int_0^t f(t)\, dt) - \frac{\gamma}{2\omega f(t)}] \\
&\quad - [\cot(\omega \int_0^t f(t)\, dt) + \frac{\gamma}{2\omega}]\} \ ,
\end{aligned}
\tag{7}
$$

$$
D = \sqrt{\frac{m\omega_0}{\hbar}}\frac{e^{\frac{\gamma}{2}t}}{\xi} \ , \quad Re\, A = -\frac{D^2}{2} \ , \quad \omega^2 = \omega_0^2 - \frac{\gamma}{4} \ .
\tag{8}
$$

To evaluate the uncertainty values, we calculate the quantities

$$
\begin{aligned}
<x>_{mn} &= \int_{-\infty}^{\infty} \psi_m^*(x,t)x\psi_n(x,t)\, dx \\
&= \frac{\sqrt{n+1}}{\sqrt{2D}}e^{i\theta(t)}\delta_{m,n+1} + \frac{\sqrt{n}}{\sqrt{2D}}e^{-i\theta(t)}\delta_{m,n-1} \\
&= \mu\delta_{m,n+1} + \mu^*\delta_{m,n-1} \ ,
\end{aligned}
\tag{9}
$$

$$< m \mid p \mid n > = \int_{-\infty}^{\infty} \psi_m^*(x,t) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi_n(x,t)\, dx$$

$$= \sqrt{n+1}(-i\frac{\sqrt{2}A\hbar}{D})e^{i\theta(t)}\delta_{m,n+1} + \sqrt{n}(-i\frac{\sqrt{2}A\hbar}{D})^* e^{-i\theta(t)}\delta_{m,n-1}$$

$$= \eta\, \delta_{m,n+1} + \eta^*\delta_{m,n-1}\,, \tag{10}$$

$$< m \mid x^2 \mid n > = \sqrt{(n+2)(n+1)}\, \mu^2\, \delta_{m,n+2} + (2n+1)\mu\mu^*\delta_{m,n} + \sqrt{n(n-1)}\, \mu^{*2}\delta_{m,n-2}\,, \tag{11}$$

$$< m \mid p^2 \mid n > = \sqrt{(n+2)(n+1)}\, \eta^2\delta_{m,n+2} + (2n+1)\eta\eta^*\delta_{m,n} + \sqrt{n(n-1)}\, \eta^{*2}\delta_{m,n-2}\,, \tag{12}$$

$$< m \mid \frac{1}{2}(xp+px) \mid n > = \sqrt{(n+2)(n+1)}\, \mu\eta\, \delta_{m,n+2} + \hbar\frac{ImA}{D^2}(2n+1)\delta_{m,n}$$

$$+ \sqrt{n(n-1)}\, \mu^*\eta^*\delta_{m,n-2}\,, \tag{13}$$

where

$$\theta(t) = \cot^{-1}[\frac{\omega}{\omega_0}\cot(\omega\int_0^t f(t)\,dt) + \frac{\gamma}{2\omega_0}]\,, \tag{14}$$

$$\mu(t) = \frac{e^{i\theta(t)}}{\sqrt{2}D} = \sqrt{\frac{\hbar}{2m\omega_0}}\, \xi e^{-\frac{1}{2}\gamma t}e^{i\theta(t)}\,, \tag{15}$$

$$\eta(t) = -i\hbar\frac{\sqrt{2}A}{D}e^{i\theta(t)}$$

$$= i\sqrt{\frac{m\omega_0\hbar}{2}}\frac{1}{\xi}\exp[\frac{1}{2}\gamma t\{1 - i\frac{\omega}{\omega_0}[\xi^2(\cot(\omega\int_0^t f(t)\,dt) - \frac{\gamma}{2\omega f(t)})$$

$$- (\cot(\omega\int_0^t f(t)\,dt) + \frac{\gamma}{2\omega})]\}]$$

$$= \sqrt{\frac{m\omega_0\hbar}{2}}\frac{1}{\xi}e^{\frac{1}{2}\gamma t}\beta(t)e^{i[\cot^{-1}\sigma(t)+\theta(t)]}\,, \tag{16}$$

$$\sigma(t) = \frac{\omega}{\omega_0}\{\xi^2[\cot(\omega\int_0^t f(t)\,dt) - \frac{\gamma}{2\omega f(t)}] - [\cot(\omega\int_0^t f(t)\,dt) + \frac{\gamma}{2\omega}]\}\,, \tag{17}$$

$$\beta(t) = \sqrt{1 + \sigma^2(t)}\,. \tag{18}$$

With the help of Eqs.(9)-(12), the uncertainty relations in the various states can be obtained as

$$[(\Delta x)^2(\Delta p)^2]_{n+2,n}^{1/2} = [(<x^2> - <x>^2)(<p^2> - <p>^2)]_{n+2,n}^{1/2}$$

$$= \sqrt{(n+2)(n+1)}\mid \mu \mid\mid \eta \mid$$

$$= \frac{\hbar}{2}\sqrt{(n+2)(n+1)}\, \beta(t)\,, \tag{19}$$

$$[(\Delta x)^2(\Delta p)^2]_{n+1,n}^{1/2} = \frac{\hbar}{2}(n+1)\beta(t)\,, \tag{20}$$

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$$[(\Delta x)^2(\Delta p)^2]^{1/2}_{n,n} = \frac{\hbar}{2}(2n+1)\beta(t) \,. \tag{21}$$

Changing $(n+1)$ to n and $(n+2)$ to n in Eqs. (20) and (21), respectively, we can easily obtain the uncertainty in the $(n-1,n)$ state and $(n-2,n)$ state.

Now we return to the coherent states. Before we construct the annihilation operator a and creation operator $a^\dagger$, we will briefly discuss the properties of the coherent states. These states can be defined by the eigenstates of the nonhermitian operator a,

$$a \mid \alpha >= \alpha \mid \alpha > \,. \tag{22}$$

Using the completeness relation for the number representations, we expand $\mid \alpha >$ as

$$\begin{aligned}
\mid \alpha > &= e^{-(1/2)|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mid n > \\
&= e^{-(1/2)|\alpha|^2} e^{\alpha a^\dagger} \mid 0 > \,,
\end{aligned} \tag{23}$$

where $\mid 0 >$ is the vacuum or ground state and is independent of n. The calculation of $< \beta \mid \alpha >$ in Eq. (23) gives

$$< \beta \mid \alpha >= e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\alpha\beta^*} \tag{24}$$

Here, Eq. (24) has nonzero values for $\alpha \neq \beta$, and thus the states are not orthogonal, but when $\mid \alpha - \beta \mid^2 \longrightarrow \beta$ the states become orthogonal.

Since the eigenvalues α of the coherent states are complex numbers $u + iv$, the completeness relation of the coherent states is written as

$$\int \mid \alpha >< \alpha \mid \frac{d^2\alpha}{\pi} = 1 \,, \tag{25}$$

where 1 is the identity operator and $d^2\alpha$ is given by $d(Re\ u)d(Im\ v)$.

From Eqs. (9),(10),(15) and (16), we have the relation

$$\eta\mu^* - \eta^*\mu = i\hbar \,. \tag{26}$$

We can define the annihilation operator a and creation operator $a^\dagger$ for the damped harmonic oscillator with time dependent frequency as

$$\begin{aligned}
a &= \frac{1}{i\hbar}(\eta x - \mu p) \,, \\
a^\dagger &= \frac{1}{i\hbar}(\mu^* p - \eta^* x) \,,
\end{aligned} \tag{27}$$

where the expressions of x and p by a and $a^\dagger$ are

$$\begin{aligned}
x &= \mu^* a + \mu a^\dagger \,, \\
p &= \eta^* a + \eta a \,.
\end{aligned} \tag{28}$$

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Since η is not equal to μ in Eqs. (15) and (16), we can easily confirm that a and $a\dagger$ are not Hermitian operators, but the following relations are preserved :

$$[x, p] = i\hbar ,$$

$$[a, a^\dagger] = 1 . \tag{29}$$

Here, the operators a and $a^\dagger$ are different from $a_0^\dagger$ and a_0, i.e., creation and annihilation operators of the harmonic oscillator, and can be expressed as

$$
\begin{aligned}
a &= \lambda a_0 + \nu a_0^\dagger , \\
a^\dagger &= \mu^* a_0 + \lambda^* a_0^\dagger .
\end{aligned} \tag{30}
$$

Therefore, the coherent states of the damped harmonic oscillator with time-dependent frequency are the squeezed states of the simple harmonic oscillator.

We can evaluate the transformation function $< x \mid \alpha >$ from the coherent states to the coordinate representation $\mid x >$. From Eqs. (22) and (27) we have

$$[\eta x - \mu \frac{\hbar}{i}\frac{\partial}{\partial x}] < x \mid \alpha >= i\hbar\alpha < x \mid \alpha > . \tag{31}$$

Solving this equation, we obtain the coordinate representation

$$< x \mid \alpha >= N \exp[\frac{1}{\mu}\alpha x - (2i\hbar\mu)^{-1}\eta x^2] . \tag{32}$$

Here, N is the integal constant constant. Choosing N to satisfy Eq. (25), we find the eigenvectors of the operator a given in the coordinate representation $\mid x >$ as

$$< x \mid \alpha >= \frac{1}{(2\pi\mu\mu^*)^{1/4}} \exp[\frac{1}{2i\hbar}\frac{\eta}{\mu}x^2 + \frac{\alpha}{\mu}x - \frac{1}{2}\mid \alpha \mid^2 - \frac{1}{2}\frac{\mu^*}{\mu}\alpha^2] , \tag{33}$$

where

$$
\begin{aligned}
(2\pi\mu\mu^*)^{-1/4} &= (\frac{m\omega_0}{\pi\hbar})^{1/4}\zeta^{1/2}e^{-\frac{\gamma}{2}t} , \\
\frac{i\eta}{2\hbar\mu} &= -\frac{m\omega_0}{2\hbar}\frac{1}{\zeta^2}e^{\gamma t}[1 - i\sigma(t)] , \\
\mu^*\mu &= e^{-2i\theta(t)} .
\end{aligned} \tag{34}
$$

Next, we prove that a coherent state represents a minimum uncertainty state. With the help of the relation between $a, a^\dagger, x$ and p, we evaluate the expectation values of x, p, x^2 and p^2 in state $\mid \alpha >$ as follows :

$$
\begin{aligned}
< x > &= < \alpha \mid \mu^* a + \mu a^\dagger \mid \alpha >= \mu^*\alpha + \mu\alpha^* , \\
< p > &= < \alpha \mid \eta^* a + \eta a^\dagger \mid \alpha >= \eta^*\alpha + \eta\alpha^* , \\
< x^2 > &= \mu^{*2}\alpha^2 + \mu\mu*(1 + 2\alpha\alpha^*) + \mu^2\alpha^{*2} , \\
< p^2 > &= \eta^{*2}\alpha^2 + \eta\eta^*(1 + 2\alpha\alpha^*) + \eta^2\alpha^{*2} .
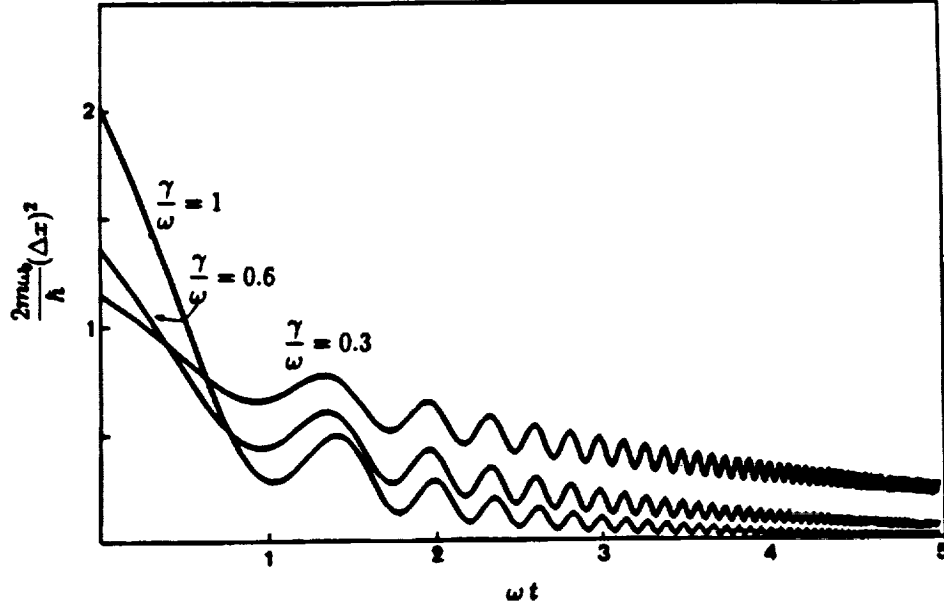\end{aligned} \tag{35}
$$

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FIG. 1. $(\Delta x)^2$ for the (0,0) state as a function of ωt at various values of γ/ω with $\omega/\zeta = 1$.

From above expressions, we get

$$(\Delta x)^2 = \mu\mu^* = \frac{\hbar}{2m\omega_0}\xi^2 e^{-\gamma t} , \tag{36}$$

$$(\Delta p)^2 = \eta\eta^* = \frac{m\omega_0\hbar}{2}\xi^{-2}e^{\gamma t}\beta^2(t) , \tag{37}$$

and thus we finally obtain the uncertainty relation

$$(\Delta x)(\Delta p) = |\mu||\eta| = \frac{\hbar}{2}\beta(t) . \tag{38}$$

Equation (37) is the minimum uncertainty corresponding to Eq. (13) in the (0,0) state.

Taking $\gamma = 0$ and $f(t) = 1$, all the formulas we have derived are reduced to those of the simple harmonic oscillator. The propagator [Eq.(2)] and the wave function [Eq. (5)] do not have similar forms to those of Cheng[15] and others[16], but are of new form. We should point out that the same classical equation of motion can be obtained from many different action, and thus one may have many different propagators correspoding to the actions.

Figures 1, 2 and 3 illustrate the behaviors of $(\Delta x)^2, (\Delta p)^2$ and $\Delta p \cdot \Delta x$ as a function of ωt at various values of γ/ω and ω/ζ for $F(t) = e^{\zeta t}$ at $\gamma \neq 0$. When oscillation starts, $(\Delta x)^2$ and $(\Delta p)^2$ have the period Π, but their periods decrease rapidly with increasing time, and the amplitude of $(\Delta x)^2$ decreases exponentially, while that of $(\Delta p)^2$ increases exponentially. The uncertainty for the (0.0) state with period Π is reduced to that of the harmonic oscillator of $0°$ and $180°$.

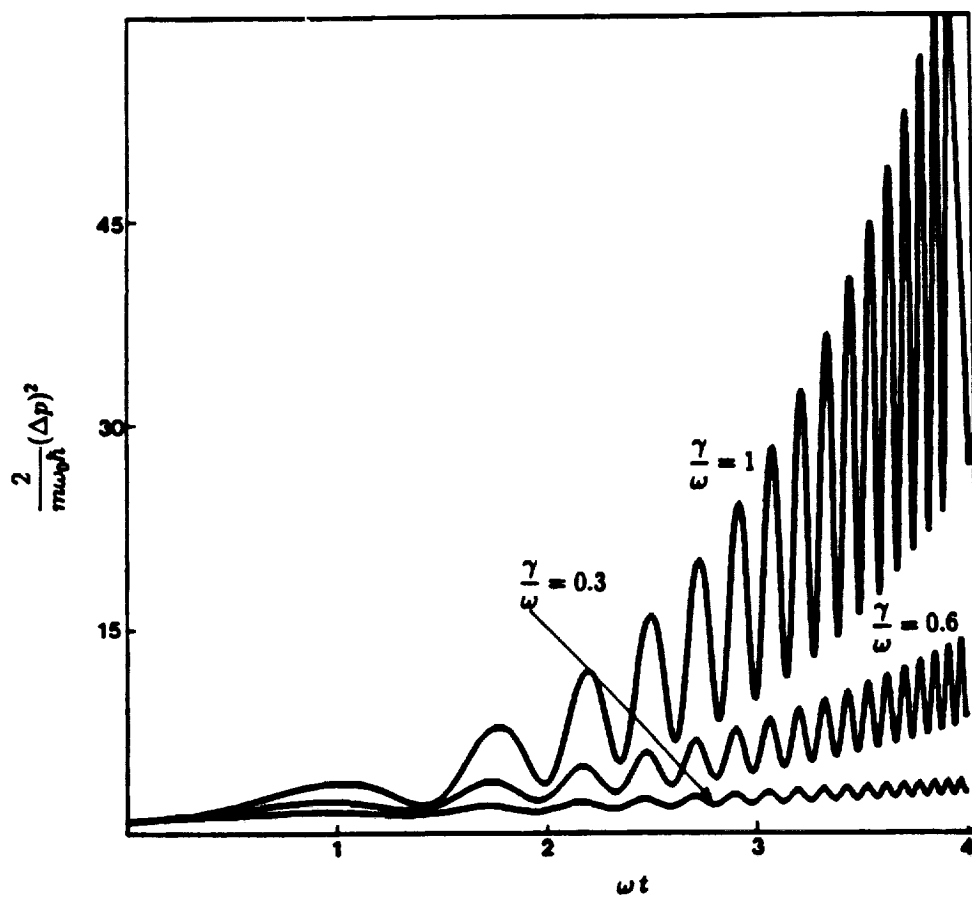From all of the above results, we conclude that the coherent states for the damped harmonic

FIG. 2. $(\Delta p)^2$ for the (0,0) state as a function of ωt at various values of γ/ω with $\omega/\zeta = 1$.
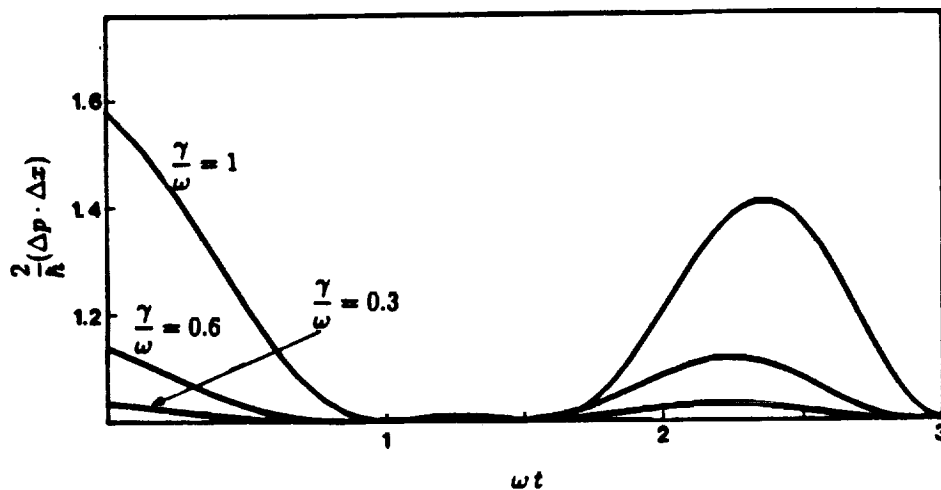

FIG. 3. $\Delta p \cdot \Delta x$ for the (0,0) state versus ωt at various values of γ/ω with $\omega/\zeta = 5$.

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oscillator with the time-dependent frequency described by the modified Caldirola-Kanai Hamiltonian which we have constructed satisfy the renowned properties of coherent states.

Acknowledgments

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QUANTUM PROCESSES IN RESONATORS WITH MOVING WALLS

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Abstract.

The behavior of the electromagnetic field in an ideal cavity with oscillating boundary is considered in the resonance long-time limit. The rates of photons creation from vacuum and thermal states are evaluated. The squeezing coefficients for the field modes are found, as well as the backward reaction of the field on the vibrating wall.

1. Field Quantization in a Cavity of Variable Length

Here we give the results of our recent investigations relating to the behavior of the quantized modes of the electromagnetic field inside a resonator with oscillationg walls. We consider the electro-magnetic field in an empty resonator formed by two ideal conducting plain boundaries $x = 0$ and $x = L(t)$, and restrict ourselves to the linearly polarized modes with the electric vector parallel to the boundaries. Then the field can be described by means of the single scalar equation for the corresponding component of the vector poten-tial with the nonstationary boundary conditions [1] (we assume $c = 1$)

$$\varphi_{tt} - \varphi_{xx} = 0, \quad 0 < x < L(t); \qquad \varphi(0,t) = \varphi(L(t),t) = 0 \qquad (1)$$

The quantization procedure in this case was proposed by Moore [1]. (Another approach including the case of a massive boson scalar field was investigated in ref.[2].) The starting point of Moore's method is the following choice of the fundamental solutions of eq.(1),

$$\psi_n(x,t) = (4\pi n)^{-1/2}\left\{\exp\left[-i\pi n R(t-x)\right] - \exp\left[-i\pi n R(t+x)\right]\right\}, \qquad (2)$$

function $R(\xi)$ being a solution of the functional equation

$$R\left[t+L(t)\right] - R\left[t-L(t)\right] = 2 \qquad (3)$$

In the stationary case $L(t) = L_0$ the solution of eq.(3) is trivial: $R^{(0)}(\xi) = \xi/L_0$. Thus mode functions are usual standing waves

$$\psi_n^{(0)}(x,t) = i(\pi n)^{-1/2}\sin(\pi n x/L_0)\exp(-i\pi n t/L_0). \qquad (4)$$

An approximate solution of (3) for a slowly moving wall was found in [1]. But in the most interesting case of the parametric resonance

$$L(t) = L_0\left[1 + \varepsilon\cdot\sin(\omega_q t)\right], \quad \omega_q = \pi q/L_0, \quad q = 1,2\cdots, \quad |\varepsilon| \ll 1 \qquad (5)$$

that solution appears valid only for not very large values of time satisfying the restriction $\varepsilon c t/L_0 \ll 1$. The correct asymptotic expression for the function $\rho(t) = R(t) - t$ in the long-time limit $\varepsilon\omega t \gg 1$ was obtained in refs. [3-5] ($L_0 = c = 1$, $\zeta = \exp\left[(-1)^{q+1}\cdot\pi q\varepsilon t\right]$):

$$\rho(t) = -(2/\pi q)\cdot\mathrm{Im}\left\{\ln\left[1 + \zeta + \exp(i\pi q t)(1-\zeta)\right]\right\}, \qquad (6)$$

For the motionless walls the field operator $\hat{\varphi}$ in the Heisenberg picture can be developed over the set of functions $\psi_n^{(0)}(x,t)$:

$$\hat{\varphi}(x,t) = \sum_n\left\{\hat{b}_n\psi_n^{(0)}(x,t) + \hat{b}_n^+[\psi_n^{(0)}(x,t)]^*\right\}, \qquad [\hat{b}_n,\hat{b}_m^+] = \delta_{nm}. \qquad (7)$$

If the right wall oscillates only during the time interval $0 \le t \le T$, then for $t > T$ the field operator can be written in two forms,

$$\hat{\varphi} = \sum_n\left\{\hat{b}_n\psi_n + \hat{b}_n^+\psi_n^*\right\} = \sum_m\left\{\hat{a}_m\psi_m^{(0)} + \hat{a}_m^+[\psi_m^{(0)}]^*\right\}. \qquad (8)$$

where $\psi_n(x,t)$ is the solution of the nonstationary problem (1) coinciding with $\psi_n^{(0)}$ at $t < 0$. It seems reasonable to assume that measuring devices react to *steady-state standing waves* (4) which are wave functions of *physical* quantum states possessing definite energy values. Then just the set of operators $(\hat{a},\hat{a}^+)$ has the physical sense at $t > T$. Since all quantum properties of the field were defined with respect to the state determined by the set of operators $(\hat{b},\hat{b}^+)$ (which were "physical" operators for $t < 0$), we have to expand the "new" operators $(\hat{a},\hat{a}^+)$ over the "old" ones $(\hat{b},\hat{b}^+)$,

$$\hat{a}_m = \sum_n\left\{\hat{b}_n\alpha_{nm} + \hat{b}_n^+\beta_{nm}^*\right\}. \qquad (9)$$

To calculate the Bogoliubov coefficients α_{nm} and β_{nm} one should take into account that both systems of mode functions (2) and (4) consti-

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ate complete orthonormal sets with respect to the scalar product [1]

$$(\psi, \chi) = -\iota \int_0^{L(t)} dx \left\{ \psi \chi_t^* - \psi_t \chi^* \right\}, \qquad (\chi_t \equiv \partial \chi / \partial t) \tag{10}$$

Then the following relations can be obtained [3-5],

$$\left. \begin{array}{c} \alpha_{nm} \\ \\ \beta_{nm} \end{array} \right\} = \frac{1}{2} (m/n)^{1/2} \int_{t/L_0 - 1}^{t/L_0 + 1} dx \cdot \exp\left\{ -\iota \pi \left[n\rho(L_0 x) + (n \mp mx) \right] \right\}, \tag{11}$$

The detailed calculations of these integrals were performed in [3-5]. The final result for $q = 2r$ is as follows ($\delta = \sqrt{\varepsilon}/\pi r$),

$$\left. \begin{array}{c} \beta_{nm} \\ \\ \alpha_{nm} \end{array} \right\} = \sqrt{\frac{m}{n}} \frac{\sin\left[\pi (2rn\delta \pm m)/2r \right]}{\pi (2rn\delta \pm m)} \cdot \frac{\sin\left[\pi (n \pm m) \right]}{\sin\left[\pi (n \pm m)/2r \right]} \exp\left[\iota \pi (n \pm m)(1 - 1/2r) \right]. \tag{12}$$

In the main resonance case of $r=1$ the following expression for the moduli squared of the Bogoliubov coefficients can be obtained,

$$\left. \begin{array}{c} |\beta_{nm}|^2 \\ \\ |\alpha_{nm}|^2 \end{array} \right\} = \frac{m}{n\pi^2} \cdot \frac{\left[1 - (-1)^m \cdot \cos(2n\pi\delta) \right]}{(2n\delta \pm m)^2} \cdot \left[1 + (-1)^{m+n} \right]. \tag{13}$$

2. Rates of Photons Generation

The total number of photons created in the m-th mode from the vacuum state to the time instant t equals

$$P_m = \langle 0 | \hat{a}_m^+ \hat{a}_m | 0 \rangle = \sum_n |\beta_{nm}|^2. \tag{14}$$

Omitting the detailes of calculations given in [3-5] we present the final result ($r = 1$)

$$P_m \approx (m\pi^2)^{-1} \left[\ln(m/2\delta) - (-1)^m \cdot \ln(1/2\pi\delta) \right]. \tag{15}$$

Since in the case under study $\delta(t) = \exp(-\pi \varepsilon t)/\pi$, we get the following rate of photons generation in the m-th mode when the wall vibrates at the twice frequency of the first resonator eigenmode for $\varepsilon t \gg 1$:

$$dP_m/dt = (\varepsilon/\pi m) \left[1 - (-1)^m \right]. \tag{16}$$

This result is valid in fact only for not very large numbers m. Since in real situations we should limit time t by the resonator relaxation time τ (due to the dissipation inside walls), the maximum number of photons generated in the m-th mode equals approximately

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$$F_m^{max} \approx (\pi^2 m)^{-1}\left\{2\varepsilon Q(m)/m + O\left(\ln(m+1)\right)\right\}, \tag{17}$$

where $Q(m)$ is the quality factor of the resonator's m-th mode.

Formulas (15)-(17) essentially differ from the results of refs.[6,7], where the problem of photons creation in a resonator with oscillating ideal mirror was also considered. However the authors of that papers did not take into account the deep reconstruction of the field modes inside the resonator in the long-time limit. Therefore the rate of photons generation obtained in [6] and [7] was proportional in essence to $(\varepsilon\omega_q)^2$, whereas our formulas show that this rate is proportional to the first power of the product $\varepsilon\omega_q$. The quadratic law $F_q \propto (\varepsilon\omega_q t)^2$ is valid only in the short-time approximation under the condition $F_q \ll 1$, as was shown in [8].

If the initial density matrix of the field corresponds to the Planck distribution with finite temperature, then the average number of additional "thermal" quanta created in the m-th mode equals [4]

$$\Delta P_m = P_m - P_m^{vac} = \sum_k \left\{|\alpha_{km}|^2 + |\beta_{km}|^2\right\} \cdot \left\{\exp(k/\Theta) - 1\right\}^{-1}$$

$$= 2(\pi^2 m)^{-1}\left[1 - (-1)^m\right] \sum_{j=1}^{\infty} \ln\left\{\frac{\exp(j/\Theta) + 1}{\exp(j/\Theta) - 1}\right\} + O(2\delta/m), \tag{18}$$

where $\Theta = \kappa T L_0/\pi\hbar c$, κ is Boltzmann's constant, T — temperature.

The final number of "thermal" photons does not depend on time. Moreover, in the even modes it is almost zero up to the terms of the order of $2\delta/m$. In the low temperature limit $\Theta \ll 1$ and for $\varepsilon t \gg 1$ [4]

$$\Delta P_m = 4(\pi^2 m)^{-1}\left[1 - (-1)^m\right] \cdot e^{-1/\Theta} \ll P_m^{vac}. \tag{19}$$

In the high temperature limit one gets [4]

$$\Delta P_m = (\Theta/2m)\left[1 - (-1)^m\right] + O(\ln(2\Theta)). \tag{20}$$

If the resonator has a finite quality factor $Q(m)$ in the m-th mode, then the temperature corrections can be neglected provided

$$\Theta \ll 4\varepsilon Q(m)/(\pi^2 m); \qquad \varepsilon Q \gg 1. \tag{21}$$

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3. Squeezing Coefficients

Now let us consider variances of canonical coordinates and moments operators (quadrature components)

$$\hat{x}_m = \left(\hat{a}_m + \hat{a}_m^+\right)/\sqrt{2}, \qquad \hat{p}_m = \iota\left(\hat{a}_m^+ - \hat{a}_m\right)/\sqrt{2}. \tag{22}$$

If the initial quantum state of the field was vacuum or coherent one, then the following general formulas are valid [8],

$$\sigma_{x_m x_m} = \frac{1}{2} + \sum_n \left\{ |\beta_{nm}|^2 + \text{Re}\left(\alpha_{nm} \cdot \beta_{nm}^*\right) \right\}, \tag{23}$$

$$\sigma_{p_m p_m} = \frac{1}{2} + \sum_n \left\{ |\beta_{nm}|^2 - \text{Re}\left(\alpha_{nm} \cdot \beta_{nm}^*\right) \right\}, \tag{24}$$

$$\sigma_{x_m p_m} = \sum_n \text{Im}\left(\alpha_{nm} \cdot \beta_{nm}^*\right). \tag{25}$$

In the case of $q=2$ we have [4,5]

$$\sigma_{x_m x_m}(t \to \infty) = \frac{1}{2} - (\pi^2 m)^{-1}\left\{ 1 - (-1)^m - (\pi m)\cdot \text{si}(\pi m) \right\}, \tag{26}$$

where $\text{si}(x)$ means the integral sine function:

$$\text{si}(x) = \int_\infty^x dt \cdot \sin(t)/t. \tag{27}$$

We see that the variance is always *less* than its value in the vacuum state $\sigma^{vac} = 1/2$. This means that the field occurs in the *squeezed state*. The relative squeezing coefficient $K_m = 1 - 2\sigma_{x_m x_m}$ assumes the maximum value $K_1 = 0.22$ for $m = 1$. For large $m \gg 1$ this coefficient slowly decreases according to the asymptotic formula

$$K_m \approx 2/(\pi^2 m). \tag{28}$$

The canonical momentum variance increases in time according to the same law as the number of created quanta (15). The general dependences of variances on time are rather intricate. As was shown in [8], in the short time limit $\varepsilon t \ll 1$ there is a small squeezing in the canonical *momentum variance*: $\sigma_{pp} \approx \frac{1}{2}(1 - \pi \varepsilon t)$ (for $m = 1$). Meanwhile in the long time limit the situation is quite opposite: there is some squeezing of the canonical coordinate, and unlimitedly growing in time variance of the canonical momentum. As to the covariance of the coordinate and momentum (25), it turns out to be equal to zero up to the terms of the order of $(\varepsilon t)^{-1}$. This means that the field occurs in a squeezed but uncorrelated state. Nonetheless, this state is not a

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minimum uncertainty state, since $\sigma_{pp}\sigma_{xx} \gg 1$ when $\epsilon t \gtrsim 1$. This is explained by a strong intermode interaction.

4. Back Reaction on the Oscillating Wall from the Field

It is well known that vacuum fluctuations of electromagnetic field result in an attractive Casimir's force between uncharged conducting plates [9-11]. The general expression for the force pressing the moving wall (more precisely the T_{11}-component of the energy-momentum tensor of the field) was calculated in [10,12]:

$$F = - [g(t-L(t)) + g(t+L(t))],\qquad(29)$$

where function $g(y)$ is expressed through R-function introduced according to eqs. (2) and (3) as follows (in dimensionless units; remind that we consider the case of "one-dimensional" electrodynamics),

$$g(y) = \frac{1}{24\pi}\left\{ \frac{R'''(y)}{R'(y)} - \frac{3}{2}\left[\frac{R''(y)}{R'(y)}\right]^2 + \frac{\pi^2}{2}[R'(y)]^2 \right\}.\qquad(30)$$

In the case of motionless wall (29) and (30) lead to the known expression for Casimir's force in one dimension

$$F^{(0)} = -\pi\hbar c/24L_0^2\qquad(31)$$

The corrections to (31) in the limit of small velocities of the wall (with respect to the velocity of light) were calculated in [13]. The additional force appears attractive and proportional to the square of wall's velocity. Here we calculate the same force using the long-time asymptotics of R-function (6). Since $|d\xi/dt| \approx |\epsilon\xi| \ll \xi$, we can differentiate R-function with respect to time believing parameter ξ to be constant. Then the first three derivatives are as follows,

$$R'(t) = 2\xi\Psi(t),\qquad(32)$$

$$R''(t) = 2\xi(1-\xi^2)\,\pi q\,\sin(\pi q t)\,\Psi^2(t),\qquad(33)$$

$$R'''(t) = 2\xi(1-\xi^2)(\pi q)^2\left[(1+\xi^2)\cos(\pi q t)+(1-\xi^2)\left[1+\sin^2(\pi q t)\right]\right]\Psi^3(t).\qquad(34)$$

$$\Psi(t) = \left[1 + \xi^2 + (1-\xi^2)\,\cos(\pi q t)\right]^{-1}\qquad(35)$$

Since the force exhibits rapid oscillations, it seems reasonable to average all time dependent functions contained in (35) over the period of oscillations $T = 2/q$. All integrals can be calculated exactly with the aid of formula ([14], eq.2.5.16(22))

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$$\int_{0}^{\pi} \frac{\cos(nx)}{a + b\cos(x)} \, dx = \frac{\pi}{(a^2 - b^2)^{1/2}} \left[\frac{(a^2 - b^2)^{1/2} - a}{b} \right]^{n}$$ (36)

so that we have

$$\langle (R')^2 \rangle = \frac{1}{2} \langle \xi + \xi^{-1} \rangle,$$ (37)

$$\langle (R'' \, R')^2 \rangle = \langle (R''' \, R') \rangle = \frac{1}{2} (nq)^2 \langle \xi + \xi^{-1} - 2 \rangle,$$ (38)

Inserting these expressions into (29) and (30) we get finally in dimensionless units

$$\langle F \rangle = -\frac{\pi}{24} \left[q^2 + \frac{1}{2} (1 - q^2)(\xi + \xi^{-1}) \right]$$ (39)

For $q = 1$ this formula coincides with (31). Note that this is not the resonance case (the minimal resonance value is $q = 2$), so that photons are not created inside the resonator, and the force conserves its vacuum value. For $q \geq 2$ we have not attraction, but an exponentially increasing pressure on the oscillating wall due to the creation of real photons in the cavity. By the way, formula (39) shows distinctly that for $t \to \infty$ the physical results do not depend on the sign of the parameter ε characterizing the dimensionless amplitude of wall's vibrations, since $\langle F \rangle$ is proportional to $\exp(|\varepsilon| nqt)$.

5. Discussion.

Let us summarize the main results. We have presented a new solution for mode functions of the electromagnetic field inside an ideal cavity with oscillating wall in the long-time resonance limit. It appears that the field modes structure is significantly changed in this limit in comparison with the case of motionless boundaries. It is seen distinctly if one compares, e.g., the time derivatives of functions $R^{(0)}(\xi)$ and $R(\xi)$ given by (6): in the motionless case one gets unity (in dimensionless units), whereas in the long-time resonance limit the corresponding value appears much less than unity for almost all instants of time excepting those when $\cos(nqt)$ is very close to ± 1 (see eq. (32)). Physically this change of the field modes structure manifests itself in the transition from the quadratic law of photons generation in the short-time approximation to the linear law in the long-time asymptotics. We have established also the possi-

bility of obtaining some squeezing (although rather moderate) in the resonance modes.

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13. ABSTRACT *(Maximum 200 words)*

This conference publication contains the proceedings of the Second International Workshop on Squeezed States and Uncertainty Relations held in Moscow, Russia, on May 25–29, 1992. This workshop was organized jointly by the P. N. Lebedev Physics Institute of the Russian Academy of Sciences and the University of Maryland at College Park, Maryland.

The purpose of this workshop was to study possible applications of squeezed states of light. The Workshop brought together many active researchers in squeezed states of light and those who may find the concept of squeezed states useful in their research, particularly in understanding the uncertainty relations.

It was found at this workshop that the squeezed state has a much broader implication than the two–photon coherent states in quantum optics, since the squeeze transformation is one of the most fundamental transformations in physics. The Workshop was attended by many researchers in the squeezed states of light as well as those who worked on related fields even before the squeezed state of light became one of the important subjects in physics.

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