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WAVELETS AND SPACETIME SQUEEZE

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Abstract

It is shown that the wavelet is the natural language for the Lorentz covariant description of localized light waves. A model for covariant superposition is constructed for light waves with different frequencies. It is therefore possible to construct a wave function for light waves carrying a covariant probability interpretation. It is shown that the time-energy uncertainty relation $(\Delta t)(\Delta\omega) \simeq 1$ for light waves is a Lorentz-invariant relation. The connection between photons and localized light waves is examined critically.

1 Introduction

The word "squeeze" is relatively new in physics, but the squeeze transformation has been one of the most important transformations in both relativity and quantum mechanics [1]. The geometry of squeeze is very simple. Let us consider the two-dimensional xy coordinate system. We can expand the x coordinate while contracting y in such a way that the product xy is preserved. This transformation is built in many branches of physics, including classical mechanics, special relativity, quantification of uncertainty relations, the Bogoliubov transformation in condensed matter physics, and two-photon coherent states in quantum optics [2]. Indeed, this new word enables us to study the squeeze transformations more effectively and systematically.

The concept of squeeze in quantum optics was developed from the parametric oscillation. Let us start with a simple harmonic oscillator with a given frequency. If we add a small sinusoidal variation to the frequency, the original oscillator will be modulated [3], and the resulting oscillation will be, to a good approximation, a superposition of two oscillations with different frequencies. We can use the mathematics of this oscillator system for the Fock-space description of creation and annihilation of two photons in a coherent or correlated manner, created in a laser cavity where the index of refraction undergoes a sinusoidal variation with respect to time.

Indeed, the mathematics of this two-photon system was worked out by Dirac in 1963 [4]. It is possible to translate the mathematics of this two-photon system into that of the Wigner phase-space distribution function defined over four-dimensional phase space. It is remarkable that the two-photon coherence we observe in laboratories can be described by the squeeze transformations in this four-dimensional phase space [5].

Fourteen years before the appearance of his 1963 paper [4], Dirac observed that the Lorentz boost in a given direction is a squeeze transformation. In his 1949 paper entitled "Forms of Relativistic Dynamics" [6], Dirac observed that the Lorentz boost along a given direction is a squeeze transformation. The application of this idea to relativistic hadronic system was made in 1973 [7].

These days, we have a new mathematical technique called wavelets, which serves a useful purpose in signal analysis [8]. This technique contains many features which are not available in the conventional method of Fourier analysis. It accommodates squeeze transformations. The wavelets can also constitute a representation of the Lorentz group. With these features in mind, we shall examine in this paper whether the wavelet can serve as the proper language for covariant localized light waves.

Photons are important particles in physics. Since they are relativistic particles, the quantum mechanics of photons occupies an important place in relativistic quantum mechanics. The difficulty in formulating the theory of photons is that there is no position operator which is covariant and Hermitian. This is known as the photon localization problem [9]. However, when we discuss photons, we always think of localized light waves in a given Lorentz frame. The question then is whether someone in a different Lorentz frame will think in the same way.

With this point in mind, we considered the covariance of localized light waves [10]. It was noted in our 1987 paper that localized light waves cannot represent photons. It was shown however that, if the momentum distribution is sufficiently narrow, the light wave distribution can numerically be close to that of the photon. For this reason, it is still useful to study the covariance of localized light waves.

The question of relating waves with photons is a well-defined problem in physics [11], even though the problem has not yet been completely solved. In this paper, we shall bring them closer together by using the wavelet formulation of light waves.

2 Localized Light Waves

For light waves, the Fourier relation $(\Delta t)(\Delta\omega) \simeq 1$ was known before the present form of quantum mechanics was formulated [12, 13]. However, the question of whether this is a Lorentz-invariant relation has not yet been fully examined. Let us consider a blinking traffic light. A stationary observer will insist on $(\Delta t)(\Delta\omega) \simeq 1$. An observer in an automobile moving toward the light will see the same blinking light. This observer will also insist on $(\Delta t^*)(\Delta\omega^*) \simeq 1$ on his/her coordinate system. However, these observers may not agree with each other, because neither t nor ω is a Lorentz-invariant variable. The product of two non-invariant quantities does not necessarily lead to an invariant quantity.

Let us assume that the automobile is moving in the negative z direction with velocity parameter β. Since both t and ω are the time-like components of four-vectors $(\mathbf{x}, t)$ and $(\mathbf{k}, \omega)$ respectively, a Lorentz boost along the z direction will lead to new variables:

$$t^* = (t + \beta z)/(1 - \beta^2)^{1/2}, \qquad \omega^* = (\omega + \beta k)/(1 - \beta^2)^{1/2}, \tag{1}$$

where the light wave is assumed to travel along the z axis with $k = \omega$. In the above transformation, the light wave is boosted along the positive z direction. If the light passes through the point $z = 0$ at $t = 0$, then $t = z$ on the light front, and the transformations of Eq.(1) become

$$t^* = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} t, \qquad \omega^* = \left(\frac{1+\beta}{1-\beta}\right)^{1/2} \omega. \tag{2}$$

These equations will formally lead us to

$$(\Delta t^*)(\Delta\omega^*) = \frac{1+\beta}{1-\beta}(\Delta t)(\Delta\omega), \tag{3}$$

which indicates that the time-energy uncertainty relation is not a Lorentz–invariant relation, and that Planck's constant depends on the Lorentz frame in which the measurement is taken. This is not correct, and we need a better understanding of the transformation properties of Δt and $\Delta\omega$.

This problem is related to another fundamental problem in physics. We are tempted to say that the above-mentioned Fourier relation is a time-energy uncertainty relation. However, in order that it be an uncertainty relation, the wave function for the light wave should carry a probability interpretation. This problem has a stormy history and is commonly known as the photon localization problem [9]. The traditional way of stating this problem is that there is no self-adjoint position operator for massless particles including photons.

In spite of this theoretical difficulty, it is becoming increasingly clear that single photons can be localized by detectors in laboratories. The question then is whether it is possible to construct the language of the photon localization which we observe through oscilloscopes. Throughout the history of this localization problem, the main issue has been and still is how to construct localized photon wave functions consistent with special relativity.

However, in this paper, we shall approach this problem by constructing covariant localized light waves and comparing them with photon field operators. As we shall see, the task of constructing a covariant light wave is constructing a wavelet representation of a light wave. First, we construct a unitary representation for Lorentz transformations for a monochromatic light wave. It is shown then that a Lorentz-covariant superposition of light waves is possible for different frequencies. After constructing the covariant light wave, we shall observe that there is a gap between the concept of photons and that of localized waves. From the physical point of view, this gap is not significant. However, there is a definite distinction between the mathematics of photons and that of light waves.

3 Affine Symmetry of Wavelets

Like Fourier transformations, wavelets are the superposition of plane waves with different frequencies. In addition, the distribution function has the affine symmetry. Let us briefly examine what the affine transformation is [14].

To a given number, we can add another number, and we can also multiply it by another real number. This combined mathematical operation is called the affine transformation. Since the multiplication does not commute with addition, affine transformations can only be achieved by matrices. We can write the addition of b to x as

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}. \tag{4}$$

This results in $x' = x + b$. We shall call this operation translation. The inverse of the above translation matrix is

$$\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}. \tag{5}$$

We can represent the multiplication of by e^{η} as

$$\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} e^{\eta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{6}$$

which leads to $x' = e^{\eta}x$. This multiplication operation is usually called dilation. The inverse of the above dilation matrix is

$$\begin{pmatrix} e^{-\eta} & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

The translation does not commute dilation. If dilation precedes translation, we shall call this the affine transformation of the first kind, and the transformation takes the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\eta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\eta} & b \\ 0 & 1 \end{pmatrix} \tag{8}$$

If the translation is made first, we shall call this the affine transformation of the second kind. The transformation takes the form

$$\begin{pmatrix} e^{\eta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\eta} & e^{\eta}b \\ 0 & 1 \end{pmatrix} \tag{9}$$

Indeed, the affine transformations of the first and second kinds lead to

$$x' = e^{\eta}x + b, \qquad x' = e^{\eta}(x + b), \tag{10}$$

respectively. Let us next consider inverse transformations. The inverse of the first-kind transformation of Eq.(8) becomes an affine transformation of the second kind:

$$\begin{pmatrix} e^{-\eta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-\eta} & -e^{-\eta}b \\ 0 & 1 \end{pmatrix}, \tag{11}$$

while the inverse of the second kind of Eq.( ) becomes an affine transformation of the first kind:

$$\begin{pmatrix} 0 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\eta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-\eta} & -b \\ 0 & 1 \end{pmatrix}. \tag{12}$$

The distinction between the first and second kinds is not mathematically precise, because the translation subgroup of the affine group is an invariant subgroup. We make this distinction purely for convenience. Whether we choose the first kind or second kind depends on the physical problem under consideration. For a covariant description of light waves, the affine transformation of the second kind is more appropriate, and this affine transformation takes the form

$$x' = e^{\eta}(x + b), \tag{13}$$

and its inverse is

$$x = e^{-\eta}x' - b = e^{-\eta}(x' - e^{\eta}b). \tag{14}$$

Therefore, the transformation of a function $f(x)$ corresponding to the vector transformation of Eq.(13) is

$$f\left(e^{-\eta}x - b\right) = f\left(e^{-\eta}(x - e^{\eta}b)\right). \tag{15}$$

This translation symmetry leads us to the concept of "window," which will be discussed further in Sec. 4.

Next, let us consider the normalization of the function. The normalization integral

$$\int |f(e^{-\eta}x - b)|^2 dx \tag{16}$$

does not depend on translation parameter b, but it depends on the multiplication parameter η. Indeed,

$$\int |f(e^{-\eta}x - b)|^2 dx = e^{\eta} \int |f(x - b)|^2 dx. \tag{17}$$

In order to preserve the normalization under the affine transformation, we can introduce the form [8]

$$e^{-\eta/2} f(e^{-\eta}x - b). \tag{18}$$

This is the wavelet form of the function $f(e^{-\eta}x - b)$. This is of course the wavelet form of the second kind. The wavelet of the first kind will be

$$e^{-\eta/2} f\left(e^{-\eta}(x - b)\right). \tag{19}$$

Both the first and second kinds of wavelet forms are discussed in the literature [8]

4 Windows

There are in physics many distributions, and their functional forms usually extend from minus infinity to plus infinity. However, the distribution function of physical interest is usually concentrated within a finite interval. It thus is not uncommon in physics that mathematical difficulties in theory come from the region in which the distribution function is almost zero and is physically insignificant. Thus, we are tempted to ignore contributions from outside of the specified region. This is called the "cut-off" procedure.

One of the difficulties of this procedure is that a good cut-off approximation in one Lorentz frame may not remain good in different frames. The translational symmetry of wavelets allows us to define the cut-off procedure which will remain valid in all Lorentz frames.

We can allow the function to be nonzero within the interval

$$a \leq x \leq a + w, \tag{20}$$

while demanding that the function vanish everywhere else. The parameter w determines the size of the window. The window can be translated or expanded/contracted according to the operation of the affine group. We can now define the window of the first kind and the window of the second kind. Both widows can be translated according to the transformation given in Eq.(4). The window of the first kind is not affected by the scale transformation. On the other hand, the size and location of the window of the second kind becomes affected by the scale transformation according to Eq.(11). Depending on our need, we can define the window to preserve the information. The idea of introducing the new word "window" is to define the information-preserving boundary conditions.

The window may become a very powerful device in describing the real world, especially in localization problems dealing with distributions concentrated within a finite region. The concept of cut-off in a distribution function is not new. However, the cut-off process causes mathematical difficulties usually introducing undesirable singularities. Also the cut-off process destroys the Lorentz covariance, unless it is done carefully. A good approximation in one Lorentz frame is not necessarily a good approximation in different frames. In this paper, we shall examine possible role of wavelets and windows in discussing localized light waves and their connection to photons.

5 Light Waves and Wave Packets in Quantum Mechanics

We are concerned here with the possibility of constructing wave functions with quantum probability interpretation for relativistic massless particles. The natural starting point for tackling this problem is a free-particle wave packet in nonrelativistic quantum mechanics which we pretend to understand. Let us write down the time-dependent Schrödinger equation for a free particle moving in the z direction:

$$i\frac{\partial}{\partial t}\psi(z,t) = -\frac{1}{2m}\frac{\partial^2}{\partial z^2}\psi(z,t). \tag{21}$$

The Hamiltonian commutes with the momentum operator. If the momentum is sharply defined, the solution of the above differential equation is

$$\psi(z,t) = \exp[i(pz - p^2 t/2m)]. \tag{22}$$

If the momentum is not sharply defined, we have to take the linear superposition:

$$\psi(z,t) = \int \exp\left\{i\left(kz - \frac{k^2}{2m}t\right)\right\} g(k)dk. \tag{23}$$

The width of the wave function becomes wider as the time variable increases. This is known as the wave packet spread.

Let us study the transformation properties of this wave function. Rotation and translation properties are trivial. In order to study the boost property within the framework of Galilean kinematics, let us imagine an observer moving in the negative z direction. To this observer, the center of the wave function moves along the positive z direction, and the transformed wave function takes the form

$$\psi_v(z,t) = \exp[-im(vz - \frac{1}{2}v^2 t)] \int g(k - mv) e^{i(kz - k^2 t/2m)} dk, \tag{24}$$

where v is the boost velocity. This expression is different from the usual expression in textbooks by an exponential factor in front of the integral sign. The origin of this phase factor is well-understood.

In nonrelativistic quantum mechanics, $\psi_v(z,t)$ has a probability interpretation, and there is no difficulty in giving an interpretation for the transformed wave function in spite of the above-mentioned phase factor. The basic unsolved problem is whether the probabilistic interpretation can be extended into the Lorentzian regime. This has been a fundamental unsolved problem for decades, and we do not propose to solve all the problems in this paper. A reasonable starting point for approaching this problem is to see whether a covariant probability interpretation can be given to light waves.

For light waves, we start with the usual expression

$$f(z,t) = \frac{1}{\sqrt{2\pi}} \int g(k) e^{i(kz-\omega t)} dk. \tag{25}$$

Unlike the case of the Schrödinger wave, ω is equal to k, and there is no spread of wave packet. The velocity of propagation is always that of light. We might therefore be led to think that the problem for light waves is simpler than that for nonrelativistic Schrödinger waves. This is not the case for the following reasons.

(1). We like to have a wave function for light waves. However, it is not clear which component of the Maxwell wave should be identified with the quantal wave whose absolute square gives a probability distribution. Should this be the electric or magnetic field, or should it be the four–potential?

(2). The expression given in Eq.(25) is valid in a given Lorentz frame. What form does this equation take for an observer in a different frame?

(3). Even if we are able to construct localized light waves, does this solve the photon localization problem?

(4). The photon has spin 1 either parallel or antiparallel to its momentum. The photon also has gauge degrees of freedom. How are these related to the above-mentioned problems?

Indeed, the burden on Eq.(25) is the Lorentz covariance. It is not difficult to carry out a spectral analysis and therefore to give a probability interpretation for the expression of Eq.(25) in a given Lorentz frame. However, this interpretation has to be covariant. This is precisely the problem we are addressing in the present paper.

6 Extended Little Group for Photons

The little group is the maximal subgroup of the Lorentz group which leaves the four-momentum of a given particle invariant. For a massless particle moving along the z direction, the little group is generated by [15]

$$J_3, N_1, N_2, \tag{26}$$

with $N_1 = K_1 - J_2, N_2 = K_2 + J_2$, where J_i and K_i are the generators of rotations and boosts respectively. The above generators satisfy the commutation relations:

$$[N_1, N_2] = 0, \qquad [J_3, N_1] = iN_2, \qquad [J_3, N_2] = -iN_2. \tag{27}$$

These commutation relations are identical to those of the two-dimensional Euclidean group.

In addition, we can consider K_3 which generates boosts along the z direction. This operator satisfies the following commutation relations with the above generators of the little group.

$$[K_3, J_3] = 0, \qquad [K_3, N_1] = -\mathrm{i}\mathrm{N}_1, \qquad [K_3, N_2] = -\mathrm{i}\mathrm{N}_2. \tag{28}$$

Since the operators N_1, N_2, J_3, and K_3 form a closed Lie algebra, we shall call the group generated by these four operators the "extended little group."

The boost generated by K_3 has no effect on J_3, while changing the scale of N_1 and N_2. In particular, if we start with a monochromatic light wave whose four-potential is

$$A^\mu(x) = (A, 0, 0, 0)e^{i(kz-\omega t)} \tag{29}$$

in the metric convention: $x^\mu = (x, y, z, t)$, the Lorentz boost generated by K_3 leaves the above expression invariant. Since N_1 and N_2 generate gauge transformations which do not lead to observable consequences, we can stick to the above expression, and ignore the effect of N_1 and N_2.J_3 generates rotations around the z axis. In this case, the rotation leads to a linear combination of the x and y components. This operation is consistent with the fact that the photon has two independent components, which is thoroughly familiar to us. Therefore for all practical purposes, $A^\mu(x)$ has just one component which remains invariant under transformations of the extended little group. We can thus write $A^\mu(x)$ as

$$A^\mu(x) = Ae^{i(kz-\omega t)}. \tag{30}$$

While the group of Lorentz transformations has six generators, the extended little group has only four. This means that the extended little group is a subgroup of the Lorentz group. How can we then generalize the above reasoning to take into account the most general case? The choice of the z axis is purely for convenience, and it was chosen to be the direction of the wave propagation. If this direction is rotated, it is not difficult to deal with the problem. If the boost is made along the direction different from that of propagation, then the operation is equivalent to a gauge transformation followed by a rotation. Therefore, the extended little group, while being simpler than the six-parameter Lorentz group can take care of all possible Lorentz transformations of the monochromatic wave.

The above reasoning remains valid for the case of the superposition of several waves with different frequencies propagating in the same direction:

$$A^\mu(x) = \sum_i A_i e^{i(k_i z - \omega_i t)}, \tag{31}$$

and the norm:

$$N = \sum_i |A_i|^2. \tag{32}$$

remains invariant under transformations of the extended little group.

7 Unitary Representation for Four–potentials

One of the difficulties in dealing with the photon problem has been that the electromagnetic four-potential could not be identified with a unitary irreducible representation of the Poincaré group. The purpose of this section is to resolve this problem. In Ref. [15], we studied unitary transformations associated with Lorentz boosts along the direction perpendicular to the momentum. In this section, we shall deal with the most general case of boosting along an arbitrary direction.

Let us consider a monochromatic light wave travelling along the z axis with four-momentum p. The four-potential takes the form

$$A^{\mu}(x) = A^{\mu} e^{i\omega(z-t)}, \tag{33}$$

with

$$A^{\mu} = (A_1, A_2, A_3, A_0). \tag{34}$$

We use the metric convention: $x^{\mu} = (x, y, z, t)$. The momentum four-vector in this convention is

$$p^{\mu} = (0, 0, \omega, \omega). \tag{35}$$

Among many possible forms of the gauge-dependent four-vector A^{μ}, we are interested in the eigenstates of the helicity operator:

$$S_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

The tour-vectors satisfying this condition are

$$A^{\mu}_{\pm} = (1, \pm i, 0, 0), \tag{37}$$

where the subscripts + and − specify the positive and negative helicity states respectively. These are commonly called the photon polarization vectors.

In order that the four-vector be a helicity state, it is essential that the time-like and longitudinal components vanish:

$$A_3 = A_0 = 0. \tag{38}$$

This condition is equivalent to the combined effect of the Lorentz condition:

$$\frac{\partial}{\partial x^{\mu}} A^{\mu}(x) = 0, \tag{39}$$

and the transversality condition:

$$\nabla \cdot \mathbf{A}(x) = 0. \tag{40}$$

As before, we call this combined condition the helicity gauge.

While the Lorentz condition of Eq.(39) is Lorentz-invariant, the transversality condition of Eq.(40) is not. However, both conditions are invariant under rotations and under boosts along the direction of momentum. We call these helicity preserving transformations. If we make a boost along an arbitrary direction, this is not a helicity preserving transformation. However, we can express this in terms of helicity preserving transformations preceded by a gauge transformation.

Let us consider in detail the boost along the arbitrary direction. This boost will transform the momentum p to p',

$$p'^{\mu} = B_{\phi}(\eta)p^{\mu}. \tag{41}$$

However, this is not the only way in which p can be transformed to p'. We can boost p along the z direction and rotate it around the y axis. The application of the transformation $[R(\theta)B_z(\xi)]$ on the four-momentum gives the same effect as that of the application of $B_{\phi}(\eta)$. Indeed, the matrix

$$D(\eta) = [B_{\phi}(\eta)]^{-1}R(\theta)B_z(\xi) \tag{42}$$

leaves the four-momentum invariant, and is therefore an element of the $E(2)$-like little group for photons. The effect of the above D matrix on the polarization vectors has been calculated in Appendix A, and the result is

$$D(\eta)A^{\mu}_{\pm} = A^{\mu}_{\pm} + (p^{\mu}/\omega)u(\eta,\theta), \tag{43}$$

where

$$u(\eta,\theta) = \frac{-2\sin(\theta/2)\cosh(\eta/2)}{\cos(\theta/2)\cosh(\eta/2) + \sqrt{(\cos(\theta/2)\cosh(\eta/2))^2 - 1}}. \tag{44}$$

Thus $D(\eta)$ applied to the polarization vector results in the addition of a term which is proportional to the four-momentum. $D(\eta)$ therefore performs a gauge transformation on $A^{\mu}_{\pm}$. With this preparation, let us boost the photon polarization vector:

$$\tilde{A}^{\mu}_{\pm} = B_{\phi}(\eta)A^{\mu}_{\pm}. \tag{45}$$

The four-vector $\tilde{A}^{\mu}_{\pm}$ satisfies the Lorentz condition $p_{\mu}\tilde{A}^{\mu}_{\pm} = 0$, but its fourth component will not vanish. The four-vector $\tilde{A}^{\mu}_{\pm}$ does not satisfy the helicity condition.

On the other hand, if we boost the four-vector $A^{\mu}_{\pm}$ after performing the gauge transformation $D(\eta)$,

$$\begin{aligned} A'^{\mu}_{\pm} &= B_{\phi}(\eta)A^{\mu}_{\pm} \\ &= B_{\phi}(\eta)[B_{\phi}^{-1}(\eta)R(\theta)B_z(\xi)]A^{\mu}_{\pm} \\ &= R(\theta)B_z(\xi)A^{\mu}_{\pm}. \end{aligned} \tag{46}$$

Since $B_z(\xi)$ leaves $A^{\mu}_{\pm}$ invariant, we arrive at the conclusion that

$$A'^{\mu}_{\pm} = R(\theta)A^{\mu}_{\pm}. \tag{47}$$

This means

$$A'^{\mu}_{\pm} = B_x(\eta)D(\eta)A^{\mu}_{\pm} = (\cos\theta, \pm i, -\sin\theta, 0), \tag{48}$$

which satisfies the helicity condition:

$$\mathbf{A}'^{0}_{\pm} = 0, \tag{49}$$

and

$$\mathrm{p}' \cdot \mathrm{A}'_{\pm} = 0. \tag{50}$$

The Lorentz boost $B(\eta)$ on $A^{\mu}_{\pm}$ preceded by the gauge transformation $D(\eta)$ leads to the pure rotation $R(\theta)$. This rotation is a finite-dimensional unitary transformation.

The above result indicates, for a monochromatic wave, that all we have to know is how to rotate. If, however, the photon momentum has a distribution, we have to deal with a linear superposition of waves with different momenta. The photon momentum can have both longitudinal and transverse distributions. In this paper, we shall assume that there is only longitudinal distribution. This of course is a limitation of the model we present. However, our apology is limited in view of the fact that laser beams these days can go to the moon and come back after reflection.

With this point in mind, we note first that the above-mentioned unitary transformation preserves the photon polarization. This means that we can drop the polarization index from A^{μ} assuming that the photon has either positive or negative polarization. $A^{\mu}(x)$ can now be replaced by $A(x)$.

Next, the transformation matrices discussed in this section depend only on the direction and the magnitude of the boost but not on the photon energy. This is due to the fact that the photon is a massless particle [15]. For the superposition of two different frequency states:

$$A(x) = A_1 e^{i\omega_1(z-t)} + A_2 e^{i\omega_2(z-t)}, \tag{51}$$

a Lorentz boost along an arbitrary direction results in a rotation followed by a boost along the z direction. Since neither the rotation nor the boost along the z axis changes the magnitude of $A_i (i = 1, 2)$, the quantity

$$|A|^2 = |A_1|^2 + |A_2|^2 \tag{52}$$

remains invariant under the Lorentz transformation. This result can be generalized to the superposition of many different frequencies:

$$A(x) = \sum_k A_k e^{i(kz-t)}, \tag{53}$$

with $|A|^2 = \sum_k |A_k|^2$ .The norm $|A|^2$ remains invariant under the Lorentz transformation in the sense that it is invariant under rotations and is invariant under the boost along the z direction.

Can this sum be transformed into an integral form of Eq.(25)? From the physical point of view, the answer should be Yes. Mathematically, the problem is how to construct a Lorentz-invariant integral measure. It is not difficult to see that the norm of Eq.(32) remains invariant under rotations, which perform unitary transformations on the system. The problem is how to construct a measure invariant under the boost along the z direction.

8 Localized Light Wavelets

For light waves, we use the form of Eq.(25). Let us write down the expression again.

$$f(z,t) = \frac{1}{\sqrt{2\pi}} \int g(k) e^{i(kz-\omega t)} dk. \tag{54}$$

However, the form commonly used in quantum electrodynamics is

$$A(z,t) = \int \sqrt{\frac{1}{\omega}} a(k) e^{i(kz-\omega t)} dk. \tag{55}$$

This is a covariant expression in the sense that the norm

$$\int |a(k)|^2 (1/\omega) dk. \tag{56}$$

is invariant under the Lorentz boost, because the integral measure $(1/\omega)dk$ is Lorentz-invariant. On the other hand, the expression given in Eq.(54) is not covariant if $g(k)$ is a scalar function, because the measure dk is not invariant.

It is possible to give a particle interpretation to Eq.(55) after second quantization. However, $A(z,t)$ cannot be used for the localization of photons. On the other hand, it is possible to give a localized probability interpretation to $f(z,t)$ of Eq.(54), while it does not accept the particle interpretation of quantum field theory.

If $g(k)$ is not a scalar function, what is its transformation property? We shall approach this problem using the light-cone coordinate system. We define the light-cone variables as

$$s = (z+t)/2, \qquad u = (z-t). \tag{57}$$

The Fourier-conjugate momentum variables are

$$k_s = (k-\omega), \qquad k_u = (k+\omega)/2. \tag{58}$$

If we boost the light wave (or move against the wave with velocity parameter β), the new coordinate variables become

$$s' = \alpha_+ s, \qquad u' = \alpha_- u, \tag{59}$$

where $\alpha_\pm = [(1 \pm \beta)/(1 \mp \beta)]^{1/2}$. If we construct a phase space consisting of s and k_s or u and k_u, the effect of the Lorentz boost will simply be the elongation and contraction of the coordinate axes. If the coordinate s is elongated by α_+, then k_s is contracted by α_- with $\alpha_+\alpha_- = 1$.

In the case of light waves, k_s vanishes, and k_u becomes k or ω. In terms of the light-cone variables, the expression of Eq.(54) becomes

$$f(u) = (1/2\pi)^{1/2} \int g(k) e^{iku} dk. \tag{60}$$

We are interested in a unitary transformation of the above expression into another Lorentz frame. In order that the norm

$$\int |g(k)|^2 dk \tag{61}$$

be Lorentz-invariant, $f(u)$ and $g(k)$ should be transformed like

$$f(u) \to \sqrt{\alpha_+} f(\alpha_+ u), \qquad g(k) \to \sqrt{\alpha_-} g(\alpha_- k). \tag{62}$$

Then Parseval's relation:

$$\int |f(u)|^2 du = \int |g(k)|^2 dk \tag{63}$$

will remain Lorentz-invariant.

It is not difficult to understand why u and k in Eq.(62) are multiplied by α_+ and α_- respectively. However, we still have to give a physical reason for the existence of the multipliers $(\alpha_\pm)^{1/2}$ in front of $f(u)$ and $g(k)$. They are there because the integration measure in Eq.(54) is not Lorentz-invariant.

In Ref. [10], we argued from our experience in the relativistic quark model that the integration measure can become Lorentz invariant if we take into account the remaining light-cone variables in Eq.(57) and Eq.(58). Indeed, the measures $(duds)$ and $(dk_u dk_s)$ are Lorentz invariant. However, this argument is not complete because the s and k_s variables do not exist in the case of light waves. In Ref. [16], Kim and Wigner pointed out that the multipliers in Eq.(62) come from the requirement that the Wigner phase-space distribution function be covariant under Lorentz transformations.

Let us illustrate the wavelet form using a Gaussian form. We can consider the g(k) function of the form

$$g(k) = (1/\pi b)^{1/4} \exp\left\{-(k-p)^2/2b\right\}, \tag{64}$$

where b is a constant and specifies the width of the distribution, and p is the average momentum:

$$p = \int k|g(k)|^2 dk. \tag{65}$$

Under the Lorentz boost according to Eq.(62), $g(k)$ becomes

$$(1/\pi b)^{1/4}\sqrt{\alpha_-} \exp\left\{-\sqrt{\alpha_-}(k - \sqrt{\alpha_+}p)^2/2b\right\}. \tag{66}$$

We note here that the average momentum p is now increased to $\sqrt{\alpha_+}p$. The average momentum therefore is a covariant quantity, and α_- can therefore be written as

$$\alpha_- = \Omega/p, \tag{67}$$

where Ω is the average momentum in the Lorentz frame in which $\alpha_- = 1$.

As a consequence, in order to maintain the covariance, we can replace $f(u)$ and $g(k)$ by $F(u)$ and $G(u)$ respectively, where

$$F(u) = \sqrt{\frac{p}{\Omega}} f(u), \qquad G(k) = \sqrt{\frac{\Omega}{p}} g(k). \tag{68}$$

These functions will satisfy Parseval's equation:

$$\int |F(u)|^2 du = \int |G(k)|^2 dk \tag{69}$$

in every Lorentz frame without the burden of carrying the multipliers $\sqrt{\alpha_+}$ and $\sqrt{\alpha_-}$.

9 The Concept of Photons

It is now possible to construct a localized wave function for a light wave with a Lorentz-invariant normalization. This wave function is now called the wavelet. We shall examine in this section whether the wavelet can be used for photons. If the answer is NO, we then have to examine how close the wavelet is to the particle description of photons.

Let us see how the mathematics for the light-wave localization is different from that of quantum electrodynamics where photons acquire a particle interpretation through second quantization. In QED, we start with the Klein-Gordon equation with its normalization procedure. As a consequence, we use the expression:

$$g(k) = \frac{1}{\sqrt{k}} a(k), \tag{70}$$

where a(k) is a scalar function. The Lorentz-transformation property of this quantity is the same as that for $G(k)$ of Eq.(68).

However, the basic difference between the above expression and that of Eq.(68) is that the kinematical factor in front of $a(k)$ is $1/\sqrt{k}$ in Eq.(70), while that for $G(k)$ of Eq.(68) is $1/\sqrt{p}$. This is the basic gap between wavelets and photons. The gap becomes narrower when the distribution in k becomes narrower.

Furthermore, we can use the concept of windows to sharpen up the localization. Instead of leaving insignificant non-zero distribution outside the localization region, we can assume that the distribution vanishes outside the region.

I have to do some more writing.

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IV. TIME-DEPENTENT AND DISSIPATION PROBLEMS