Similarity states of homogeneous stably-stratified turbulence at infinite Froude number

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1. Motivation and objectives

Turbulent flow in stably-stratified fluids is commonly encountered in geophysical settings, and an improved understanding of these flows may result in better ocean and environmental turbulence models. Much of the fundamental physics of stably-stratified turbulence can be studied under the assumption of statistical homogeneity, leading to a considerable simplification of the problem. A study of homogeneous stably-stratified turbulence may also be useful as a vehicle for the more general study of turbulence in the presence of additional sources and sinks of energy.

Our main purpose here is to report on recent progress in an ongoing study of asymptotically long-time similarity states of stably-stratified homogeneous turbulence which may develop at high Reynolds numbers. A similarity state is characterized by the predictability of future flow statistics from current values by a simple rescaling of the statistics. The rescaling is typically based on a dimensional invariant of the flow. Knowledge of the existence of an asymptotic similarity state allows a prediction of the ultimate statistical evolution of a turbulent flow without detailed knowledge of the very complicated and not well-understood non-linear transfer processes.

We present in this report evidence of similarity states which may develop in homogeneous stably-stratified flows if a dimensionless group in addition to the Reynolds number, the so-called Froude number, is sufficiently large. Here, we define the Froude number as the ratio of the internal wave time-scale to the turbulence time-scale; its precise definition will be given below. In this report, we will examine three different similarity states which may develop depending on the initial conditions of the velocity and density fields. Theoretical arguments and results of large-eddy simulations will be presented. We will conclude this report with some speculative thoughts on similarity states which may develop in stably-stratified turbulence at arbitrary Froude number as well as our future research plans in this area.

2. The governing equations

Choosing our co-ordinate system such that the z-axis is pointed vertically upwards, we assume a stable density distribution

$$\rho = \rho_0 - \beta z + \rho',$$

where ρ_0 is a constant, uniform reference density, $\beta > 0$ is a constant, uniform density gradient along z, and ρ' is the density deviation from the horizontal average. The kinematic viscosity ν and molecular diffusivity D of the fluid are assumed constant and uniform. After application of the Boussinesq approximation, the governing equations for the fluid velocity $\mathbf{u}$ and the density fluctuation ρ' are

$$\nabla \cdot \mathbf{u} = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{\rho' \mathbf{g}}{\rho_0} - \frac{\nabla(p + \rho_0 g z)}{\rho_0} + \nu \nabla^2 \mathbf{u}, \tag{2.2}$$

$$\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho' = \beta u_3 + D \nabla^2 \rho', \tag{2.3}$$

where $\mathbf{g} = -\mathbf{j} g$ with $g > 0$, $\mathbf{j}$ is the vertical (upwards) unit vector, and p is the fluid pressure.

We will consider three limiting flows which may occur in a stably-stratified fluid. Firstly, we will consider decaying isotropic turbulence with an isotropic passive scalar, whose governing equations are obtained from (2.1) - (2.3) when $g, \beta = 0$. Secondly, we will consider decaying isotropic turbulence in a mean passive scalar gradient, obtained when $g = 0$ only, and; thirdly, we will consider buoyancy-generated turbulence (Batchelor, Canuto & Chasnov, 1992), obtained when $\beta = 0$ only. The conditions under which these limiting flows may develop in a stably-stratified fluid where both g and β are nonzero are most easily determined after a transformation of the equations to dimensionless variables. First, to make the equations more symmetric in the velocity and density fields, we define following Cambon (private communication) a normalized density fluctuation θ such that it has units of velocity,

$$\theta = \sqrt{\frac{g}{\rho_0 \beta}} \rho'. \tag{2.4}$$

Use of θ instead of ρ' in (2.2) - (2.3) modifies the terms proportional to g and β into terms proportional to N, where

$$N = \sqrt{\frac{g\beta}{\rho_0}} \tag{2.5}$$

is the Brunt-Vaisala frequency associated with the internal waves of the stably stratified flow. Furthermore, $\frac{1}{2}\langle \mathbf{u}^2 \rangle$ is the kinetic energy and $\frac{1}{2}\langle \theta^2 \rangle$ is the potential energy of the fluid per unit mass, and the equations of motion conserve the total energy (kinetic + potential) in the absence of viscous and diffusive dissipation.

Now, defining dimensionless variables as

$$T = t\frac{u_0}{l_0}, \quad X = \frac{x}{l_0}, \quad \mathbf{U} = \frac{\mathbf{u}}{u_0}, \quad P = \frac{(p + \rho_0 g z)}{\rho_0 {u_0}^2}, \quad \Theta = \frac{\theta}{\theta_0}, \tag{2.5}$$

where l_0, u_0, and θ_0, are as yet unspecified length, velocity, and normalized density scales, the equations of motion become

$$\nabla \cdot \mathbf{U} = 0, \tag{2.6}$$

$$\frac{\partial \mathbf{U}}{\partial T} + \mathbf{U} \cdot \nabla \mathbf{U} = -\mathbf{j}\frac{1}{F_0}\frac{\theta_0}{u_0}\Theta - \nabla P + \frac{1}{R_0}\nabla^2 \mathbf{U}, \tag{2.7}$$

$$\frac{\partial \Theta}{\partial T} + \mathbf{U} \cdot \nabla \Theta = \frac{1}{F_0}\frac{u_0}{\theta_0}U_3 + \frac{1}{\sigma R_0}\nabla^2 \Theta, \tag{2.8}$$

where

$$F_0 = \frac{u_0}{Nl_0}, \quad R_0 = \frac{u_0 l_0}{\nu}, \quad \sigma = \frac{\nu}{D}. \tag{2.9}$$

F_0 and R_0 can be regarded as an initial Froude number and Reynolds number of the flow, respectively, although their precise definition is yet dependent on our specification of l_0, u_0, and θ_0; σ is the Schmidt (or Prandtl) number of the fluid.

2.1 Isotropic turbulence with an isotropic passive scalar

This limiting flow may be obtained by initializing the flow with an isotropic velocity and density field with given kinetic and potential energy spectrum of comparable integral scales. The unspecified dimensional parameter l_0 may be taken equal to the initial integral scale of the flow, and u_0 and θ_0 may be taken equal to the initial root-mean-square values of the velocity and normalized density fluctuations. The non-dimensional variables of (2.5) ensure that the maximum values of U and Θ and the non-dimensional integral length scale of the flow is of order unity at the initial instant, and, provided that u_0 is of order θ_0, implying comparable amounts of kinetic and potential energy in the initial flow field, and $F_0 \gg 1$, both of the terms multiplied by $1/F_0$ in (2.7) and (2.8) are small initially. Over times in which these terms remain small, the resulting equations govern the evolution of a decaying isotropic turbulence convecting a decaying isotropic passive scalar field.

2.2 Isotropic turbulence in a passive scalar gradient

Here, the flow is initialized with an isotropic velocity field with given kinetic energy spectrum and no initial density fluctuations. Again we take the dimensional parameter l_0 to be the initial integral scale of the flow and u_0 equal to the initial root-mean-square value of the velocity field. The maximum value of U and the non-dimensional integral length scale of the flow are then of order unity. However, the initial conditions introduce no intrinsic density scale, and such a scale needs to be constructed from other dimensional parameters in the problem. If at some time in the flow-evolution not too far from the initial instant the maximum of the dimensionless density fluctuation Θ is also to be of order unity, then the dimensionless group multiplying U_3 in equation (2.8) must necessarily be of order unity. Setting this group exactly equal to unity yields an equation for θ_0 with solution $\theta_0 = Nl_0$. Thus defining θ_0, we find that the dimensionless group multiplying Θ in equation (2.7) is equal to $1/F_0^2$ so that, in the limit of $F_0 \gg 1$, this term is small at the initial instant and may be neglected for some as yet to be determined period of time. The resulting equations then govern the evolution of decaying isotropic turbulence in the presence of a mean passive scalar gradient over this period of time.

2.3 Buoyancy-generated turbulence

Here, the flow is initialized with an isotropic density field with given potential energy spectrum and no initial velocity fluctuations. Similarly as above, we take the dimensional parameter l_0 to be the initial integral scale of the density field and θ_0 to be equal to the initial root-mean-square value of the θ-field. The maximum value of Θ and the dimensionless integral scale of the flow is then of order unity. However, here the initial conditions introduce no intrinsic velocity scale. If at a time in the flow-evolution not too far from the initial instant we wish the maximum of the dimensionless velocity fluctuation U to also be of order unity, then the dimensionless group multiplying Θ in equation (2.7) must necessarily be of order unity. Setting this group exactly equal to unity yields a simple quadratic equation for u_0, with solution $u_0 = \sqrt{N l_0 \theta_0}$, or equivalently, $u_0 = \sqrt{g l_0 \rho_0' / \rho_0}$, where ρ_0' is the value of $\langle \rho'^2 \rangle^{1/2}$ at the initial instant. We note that this is the same velocity scale chosen previously by Batchelor *et al.* (1992) in their study of buoyancy-generated turbulence. Upon use of the identity $\theta_0 = u_0^2 / N l_0$, we find that the dimensionless group multiplying U_3 in (2.8) is exactly equal to $1/F_0^2$ so that, in the limit of $F_0 \gg 1$, this term is small at the initial instant. Using the definition of u_0, the initial Froude number here is seen to be equal to $F_0 = \rho_0' / \beta l_0$. For times over which the term multiplied by F_0^{-2} may be neglected, the resulting equations then govern the evolution of buoyancy-generated turbulence.

3. Asymptotic similarity states

3.1. Final period of decay

Exact analytical treatment of (2.1) - (2.3) is rendered difficult because of the quadratic terms. Under conditions of a final period of decay (Batchelor, 1948), these terms may be neglected and an exact analytical solution of (2.1)- (2.3) may be determined. Although most of the results concerning the final period are well-known or easily found, we recall them here since the ideas which arise in a consideration of the final period are relevant to our high Reynolds number analysis.

During the final period, viscous and diffusive effects dissipate the high wavenumber components of the energy and scalar-variance spectra, and, at late times, the only relevant part of the spectra are their forms at small wavenumbers at an earlier time. Defining the kinetic energy spectrum $E(k,t)$ and the density-variance spectrum $G(k,t)$ to be the spherically-integrated three dimensional Fourier transform of the co-variances $\frac{1}{2}\langle u_i(\mathbf{x},t) u_i(\mathbf{x}+\mathbf{r},t)\rangle$ and $\langle \rho'(\mathbf{x},t) \rho'(\mathbf{x}+\mathbf{r},t)\rangle$, an expansion of the spectra near $k=0$ can be written as

$$E(k,t) = 2\pi k^2 (B_0 + B_2 k^2 + \ldots) \tag{3.1}$$

$$G(k,t) = 4\pi k^2 (C_0 + C_2 k^2 + \ldots), \tag{3.2}$$

where $B_0, B_2, \ldots$, and $C_0, C_2, \ldots$ are the Taylor series coefficients of the expansion. In a consideration of isotropic turbulence, Batchelor and Proudman (1956) assumed the spectral tensor of the velocity correlation $\langle u_i(\mathbf{x}) u_j(\mathbf{x}+\mathbf{r})\rangle$ to be analytic at

$k = 0$ and determined that $B_0 = 0$ and that non-linear interactions (which are important during the initial period) necessarily result in a time-dependent non-zero value of B_2. Saffman (1967a) later showed that it is physically possible for turbulence to be initially created with a non-zero value of B_0 and that, for decaying isotropic turbulence, B_0 is invariant in time throughout the evolution of the flow. By analogous arguments, it can be shown that the spectrum of the density correlation is itself analytic at $k = 0$ when $C_0 \neq 0$, and, for an isotropic decaying density (scalar) field, C_0 is invariant in time (Corrsin, 1951).

Here, rather than present an exact derivation of the final period results, we will demonstrate how a simple dimensional analysis can recover the correct decay laws. We consider separately the three different limiting flows envisioned above.

Isotropic turbulence with an isotropic passive scalar

The evolution of the mean-square velocity may be found by dimensional analysis assuming the only relevant dimensional quantities are the low wavenumber invariant of the energy spectrum B_0, if non-zero initially, viscosity ν, and time t. The equations of motion are assumed to be linear in the velocity field during the final period so that $\langle \mathbf{u}^2 \rangle$ must linearly depend on B_0, and we find

$$\langle \mathbf{u}^2 \rangle \propto B_0 \nu^{-\frac{3}{2}} t^{-\frac{3}{2}}, \tag{3.3}$$

as determined by Saffman (1967a). If B_0 is initially zero, then B_2 is necessarily non-zero and is also invariant during the final period when nonlinear interactions are negligible. A corresponding dimensional analysis based on B_2 instead of B_0 yields

$$\langle \mathbf{u}^2 \rangle \propto B_2 \nu^{-\frac{5}{2}} t^{-\frac{5}{2}}, \tag{3.4}$$

as originally determined by Batchelor (1948). Analogous arguments applied to the isotropic passive density (scalar) field, which is seen to be uncoupled from the velocity field during the final period, implies a dependence on C_0, necessarily linear, the diffusivity D, and time t, yielding

$$\langle \rho'^2 \rangle \propto C_0 D^{-\frac{3}{2}} t^{-\frac{3}{2}}, \tag{3.5}$$

as originally determined by Corrsin (1951), and, if C_0 is initially zero,

$$\langle \rho'^2 \rangle \propto C_2 D^{-\frac{5}{2}} t^{-\frac{5}{2}}. \tag{3.6}$$

Isotropic turbulence with passive scalar gradient

The passive density (scalar) field for this flow is driven by velocity fluctuations, and the low wavenumber coefficient of the density-variance spectrum is no longer invariant in time. In fact, an exact relation, valid even when nonlinear terms are non-negligible, holds between C_0 and B_0 and is

$$C_0(t) = \frac{1}{3}\beta^2 B_0 t^2, \tag{3.7}$$

indicating that there is now only one invariant, namely B_0, which is relevant to our dimensional analysis. Making use of this invariant, we find in the final period the decay law

$$\langle \rho'^2 \rangle \propto \beta^2 B_0 \nu^{-\frac{3}{2}} t^{\frac{1}{2}}, \tag{3.8}$$

which is most simply found by substitution of (3.7) directly into (3.5). If B_0 is zero, then B_2 is invariant during the final period, and C_2 is related to B_2 (exact only when nonlinear terms are negligible) by

$$C_2(t) = \frac{1}{3}\beta^2 B_2 t^2. \tag{3.9}$$

The decay law in the final period is then

$$\langle \rho'^2 \rangle \propto \beta^2 B_2 \nu^{-\frac{5}{2}} t^{-\frac{1}{2}}. \tag{3.10}$$

We have thus found the interesting result that the density-variance may either increase or decrease during the final period depending on the form of the low-wavenumber energy spectrum. This result may be of use to researchers interested in determining the form of the low wavenumber energy spectrum in homogeneous turbulence under experimental conditions.

Buoyancy-driven flow

Here, it is the velocity field which is driven by density fluctuations, and the low wavenumber coefficient B_0 of the kinetic energy spectrum is no longer an invariant. The relevant invariant here is the low wavenumber coefficient C_0 of the density-variance spectrum. As before, an exact relation holds between B_0 and C_0, valid even when nonlinear terms are non-negligible, and is

$$B_0(t) = \frac{2}{3}\frac{g^2 C_0}{\rho_0^2} t^2. \tag{3.11}$$

Making use of the invariant C_0, we find another form for the mean-square velocity fluctuation under an assumption of a final period in which non-linear terms are negligible

$$\langle \mathbf{u}^2 \rangle \propto \frac{g^2 C_0}{\rho_0^2} \nu^{-\frac{3}{2}} t^{\frac{1}{2}}. \tag{3.12}$$

The mean-square density fluctuations decay as for the isotropic passive scalar flow. Clearly, an increase in $\langle \mathbf{u}^2 \rangle$ during a "final period" contradicts the very existence of a final period since the Reynolds number of the flow is increasing in time. If C_0 is initially zero, then C_2 is necessarily non-zero and is invariant when nonlinear terms are negligible. B_2 is now related to C_2 by

$$B_2(t) = \frac{2}{3}\frac{g^2 C_2}{\rho_0^2} t^2, \tag{3.13}$$

and the mean-square velocity follows

$$\langle \mathbf{u}^2 \rangle \propto \frac{g^2 C_2}{\rho_0^2} \nu^{-\frac{5}{2}} t^{-\frac{1}{2}}. \tag{3.14}$$

Although the mean-square velocity decays in this case, the integral scale grows like $t^{1/2}$ so that the Reynolds number increases in time, again contradicting the existence of a final period.

3.2. Exact high Reynolds number similarity states

At high Reynolds numbers, direct effects of viscosity and diffusivity occur at much larger wavenumber magnitudes than those scales which contain most of the energy and density-variance so that the asymptotic forms of $\langle \mathbf{u}^2 \rangle$ and $\langle \rho'^2 \rangle$ can be expected to be independent of ν and D. Viscous and diffusive smoothing of the energy and density-variance containing components of the spectra are now replaced by nonlinear transfer processes so that one can still reasonably expect the asymptotic scaling of $\langle \mathbf{u}^2 \rangle$ and $\langle \rho'^2 \rangle$ to be on the form of the spectra at low wavenumbers. The low wavenumber coefficient B_0 is an invariant, even at high Reynolds numbers, for decaying isotropic turbulence and so is C_0 for a decaying isotropic passive scalar. If there is a mean passive scalar gradient, then C_0 is asymptotically related to B_0 by (3.7). For buoyancy-driven flows, C_0 is an invariant and B_0 is asymptotically related to C_0 by (3.11).

Based on dimensional analysis, we can now determine the high Reynolds number, long-time evolution of the energy and density-variance when B_0 and C_0 are non-zero for our three limiting flows.

Isotropic turbulence with an isotropic passive scalar

The low wavenumber coefficients B_0 and C_0 are separately invariant and the high-Reynolds number asymptotic results are the Saffman (1967b) decay law

$$\langle \mathbf{u}^2 \rangle \propto B_0^{\frac{2}{5}} t^{-\frac{6}{5}}, \tag{3.15}$$

and its analogous law for the passive density-variance

$$\langle \rho'^2 \rangle \propto C_0 B_0^{-\frac{3}{5}} t^{-\frac{6}{5}}. \tag{3.16}$$

The nonlinearity of the governing equations is reflected by the nonlinear dependence of $\langle \mathbf{u}^2 \rangle$ and $\langle \rho'^2 \rangle$ on B_0, in contrast with the results of the final period. Note, however, that the linearity of the density equation in ρ' results in a linear dependence of $\langle \rho'^2 \rangle$ on C_0.

Dimensional arguments can also determine the asymptotic behavior of the velocity and density integral scales, and one finds

$$L_u, L_\theta \propto B_0^{\frac{1}{5}} t^{\frac{2}{5}}. \tag{3.17}$$

Isotropic turbulence with passive scalar gradient

Here C_0 is no longer invariant, but depends on B_0 asymptotically as (3.7) so that the density-variance now evolves as

$$\langle \rho'^2 \rangle \propto \beta^2 B_0^{\frac{2}{5}} t^{\frac{4}{5}}, \tag{3.18}$$

when B_0 is non-zero.

Buoyancy-driven flow

Here C_0 is invariant and B_0 is not, and B_0 depends on C_0 asymptotically as (3.11). By dimensional arguments (Batchelor *et al.*, 1992), the mean-square velocity and density-variance evolve as

$$\langle \mathbf{u}^2 \rangle \propto (g^2 C_0/\rho_0^2)^{\frac{2}{5}} t^{-\frac{2}{5}}, \tag{3.19}$$

$$g^2 \langle \rho'^2/\rho_0^2 \rangle \propto (g^2 C_0/\rho_0^2)^{\frac{2}{5}} t^{-\frac{12}{5}}, \tag{3.20}$$

and the integral scales evolve as

$$L_u, L_{\rho'} \propto (g^2 C_0/\rho_0^2)^{\frac{1}{5}} t^{\frac{4}{5}}. \tag{3.21}$$

A computation of a Reynolds number based on the root-mean square velocity fluctuation and the integral scale shows that $Re \propto t^{3/5}$ increases asymptotically, precluding the development of a final period in this flow.

3.3. Approximate high Reynolds number similarity states

When either B_0 or C_0 are zero, there are no longer strictly invariant quantities on which to base asymptotic similarity states. The coefficients B_2 and C_2 are affected by nonlinear transfer processes, and exact results as found above become unobtainable. Nevertheless, if we make an additional assumption, which needs to be verified by numerical or experimental data, that the time-variation of B_2 or C_2 due to nonlinear processes are small compared to the rate of change of the energy or density-variance, then approximate asymptotic similarity states may still be based on the nearly-invariant low wavenumber coefficients. The analysis proceeds in exact analogy to that above, and, for use in comparison to the numerical simulation data, we state the results below.

Isotropic turbulence with an isotropic passive scalar

For $B_0 = 0$, we have the Kolmogorov (1941) decay law

$$\langle \mathbf{u}^2 \rangle \propto B_2^{\frac{2}{7}} t^{-\frac{10}{7}}. \tag{3.22}$$

Three additional approximate similarity states exist for the decaying isotropic passive scalar depending on which of B_0 or C_0 are zero (Lesieur, 1990):

$$\langle \rho'^2 \rangle \propto C_2 B_0^{-1} t^{-2} \tag{3.23}$$

$$\langle \rho'^2 \rangle \propto C_0 B_2^{-\frac{3}{7}} t^{-\frac{6}{7}} \tag{3.24}$$

$$\langle \rho'^2 \rangle \propto C_2 B_2^{-\frac{5}{7}} t^{-\frac{10}{7}}, \tag{3.25}$$

Isotropic turbulence with passive scalar gradient

For $B_0 = 0$, we assume that B_2 is approximately invariant. The low wavenumber scalar-variance spectrum coefficient C_2 is approximately related to B_2 by (3.9). The density-variance is found to evolve as

$$\langle \rho'^2 \rangle \propto \beta^2 B_0^{\frac{2}{7}} t^{\frac{4}{7}}. \tag{3.26}$$

Buoyancy-driven flow

For $C_0 = 0$, we assume that C_2 is approximately invariant. The low wavenumber energy spectrum coefficient B_2 is approximately related to C_2 by (3.13). The mean-square velocity and density-variance evolve approximately as

$$\langle \mathbf{u}^2 \rangle \propto (g^2 C_0/\rho_0^2)^{\frac{6}{7}} t^{-\frac{6}{7}}, \tag{3.27}$$

$$g^2 \langle \rho'^2/\rho_0^2 \rangle \propto (g^2 C_0/\rho_0^2)^{\frac{6}{7}} t^{-\frac{20}{7}}, \tag{3.28}$$

and the integral scales evolve as

$$L_u, L_{\rho'} \propto (g^2 C_0/\rho_0^2)^{\frac{1}{7}} t^{\frac{4}{7}}. \tag{3.29}$$

A computation of a Reynolds number based on the root-mean square velocity fluctuation and the integral scale shows that Re again increases asymptotically, but now as $t^{1/7}$.

4. Large-eddy simulations

The high Reynolds numbers required to test the asymptotic scaling determined above may be obtained by a large-eddy simulation (LES) of Eqs. (2.1) - (2.3) using a pseudo-spectral code for homogeneous turbulence (Rogallo, 1981). For the subgrid scale model, we employ a spectral eddy-viscosity and eddy-diffusivity (Kraichnan 1976; Chollet and Lesieur 1981) parametrized by

$$\nu_e(k|k_m, t) = \left[0.145 + 5.01 \exp\left(\frac{-3.03 k_m}{k}\right)\right] \left[\frac{E(k_m, t)}{k_m}\right]^{1/2}, \tag{4.1}$$

and

$$D_e(k|k_m, t) = \frac{\nu_e(k|k_m, t)}{\sigma_e}, \tag{4.2}$$

where k_m is the maximum wavenumber magnitude of the simulation and σ_e is an eddy Schmidt number, assumed here to be constant and equal to 0.6. We take the initial energy spectrum to be

$$E(k, 0) = A_n k_p^{-1} (k/k_p)^n \exp\left(-(n/2)(k/k_p)^2\right), \tag{4.3}$$

where n is equal to 2 or 4, A_n is chosen so that $\langle \mathbf{u}^2 \rangle = 1$, and k_p is the wavenumber at which the initial energy spectrum is maximum. The case $n = 2$ corresponds to $B_0 \neq 0$, and the case $n = 4$ corresponds to $B_0 = 0$. In the 256^3 numerical simulations presented here, the minimum computational wavenumber is 1, the maximum wavenumber is about 120, and we take $k_p = 100$. The initial energy spectrum is set to zero for wavenumbers greater than 118 to allow the subgrid scale eddy-viscosity and eddy-diffusivity to build up from zero values. The relatively large value of k_p chosen here allows an attainment of an asymptotic similarity state before the integral scales of the flow become comparable to the periodicity length. A velocity field with initial energy spectrum given by (4.3) is realized in the simulation by requiring the spectral energy content at each wavenumber to satisfy (4.3) but randomly generating the phase and velocity component distributions (Rogallo 1981).

In the simulations of decaying isotropic turbulence with a decaying isotropic passive scalar, the passive scalar-variance spectrum is also initialized with the spectrum given by (4.3) with A_n chosen so that $\langle \rho'^2 \rangle = 1$. We present the results of two simulations for this flow: the first with an initial energy spectrum with $n = 2$ convecting two passive scalar fields with initial spectra with $n = 2$ and $n = 4$, and the second with an energy spectrum with $n = 4$ convecting two passive scalar fields with $n = 2$ and $n = 4$. Computations of these two velocity fields and four scalar fields are sufficient to test the theoretical scaling discussed in §3.

In the simulations of decaying isotropic turbulence in a passive scalar gradient, the initial fluctuating passive density field is taken identically equal to zero, and two simulations are presented with an initial energy spectrum with $n = 2$ and $n = 4$. The exact value of β is inconsequential provided it is non-zero, and we choose $\beta = 1$.

In the simulations of buoyancy-generated turbulence, the initial fluctuating velocity field is taken identically equal to zero, and two simulations are presented with an initial density spectrum with $n = 2$ and $n = 4$. Here, the exact value of g is inconsequential provided it is non-zero, and we choose units such that $g/\rho_0 = 1$.

4.1. Results

In the interest of brevity, we present here only results from the large-eddy simulations pertaining to the power-law predictions of §3. More detailed results concerning decaying isotropic turbulence with and without a passive scalar gradient will be published in Chasnov (1993) and Chasnov & Lesieur (1993) - slightly lower-resolution simulations (128^3) of buoyancy-generated turbulence have already been published in Batchelor, Canuto & Chasnov (1992). In figures 1 and 2, we plot the instantaneous power-law exponents (logarithmic derivatives) versus time normalized by the initial large-eddy turnover time $\tau(0)$, of the mean-square velocity decay and the passive density-variance decay when $\beta, g = 0$ in (2.2) and (2.3), appropriate for the study of decaying isotropic turbulence with a decaying isotropic passive scalar. In figure 3, we plot the time-evolution of the power-law exponent of the passive density-variance when $\beta = 1$ and $g = 0$, appropriate for the study of decaying isotropic turbulence in the presence of a passive scalar gradient, and, in figures 4 and 5 we plot the time-evolution of the power-law exponent of the mean-square velocity and density-variance when $\beta = 0$ and $g/\rho_0 = 1$, appropriate for the study

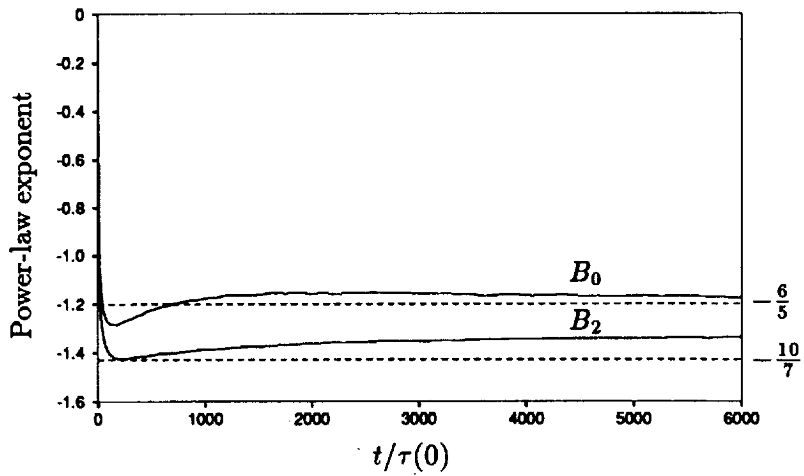

FIGURE 1. Time-evolution of the power-law exponent of $\langle \mathbf{u}^2 \rangle$ for decaying isotropic turbulence. The solid lines are the results of the large-eddy simulations and the dashed lines are the exact and approximate analytical results discussed in §3.

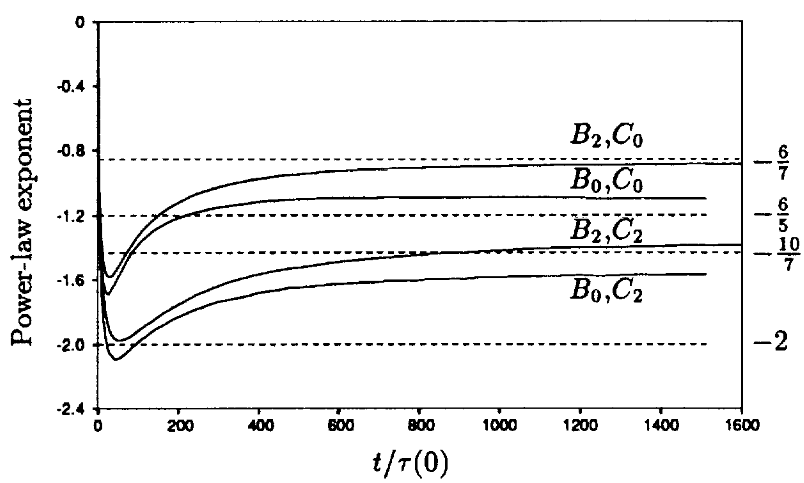

FIGURE 2. Time-evolution of the power-law exponent of $\langle \rho'^2 \rangle$ for a decaying isotropic passive density field.

of buoyancy-generated turbulence. The dashed lines in all of the figures correspond to the exact and approximate asymptotic similarity state results discussed in §3. The solid lines are the results of the large-eddy simulations and are labeled according to the low-wavenumber spectral coefficient which is non-zero and invariant, or postulated to be nearly-invariant.

Overall good agreement is observed between the analytical predictions and the numerical results, lending support to our simple analytical arguments.

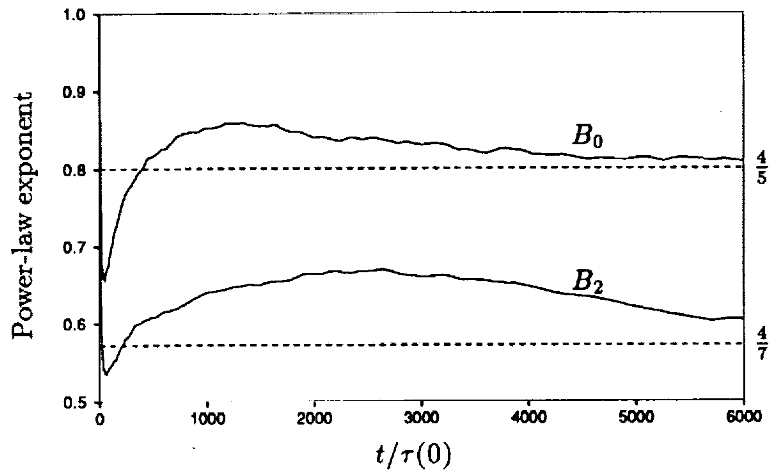

FIGURE 3. Time-evolution of the power-law exponent of $\langle \rho'^2 \rangle$ for decaying isotropic turbulence with a passive mean density gradient.

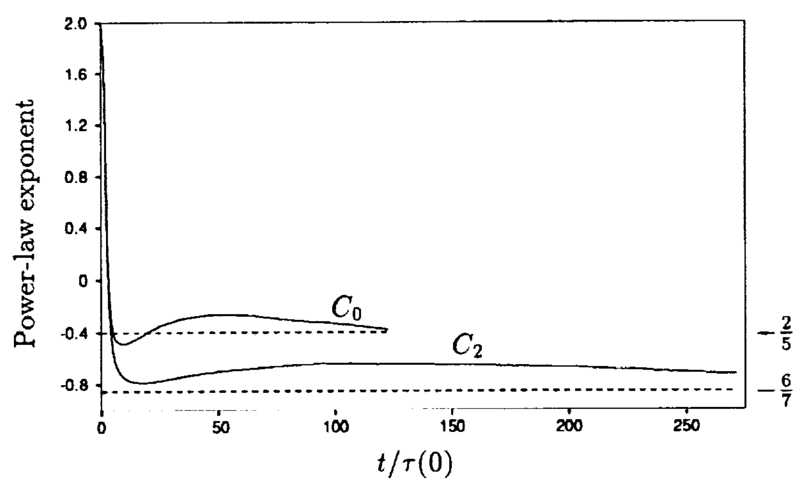

FIGURE 4. Time-evolution of the power-law exponent of $\langle \mathbf{u}^2 \rangle$ for buoyancy-generated turbulence.

Although the total evolution time of the simulations appears to be quite long, we note that the number of large-eddy turnover times undergone by the flow at time t is proportional to $log(t)$ so that the approach to asymptotic behavior may be quite slow. The small deviations from the analytical results which are thought to be exact observed in the simulations could very well be due to an insufficient time-evolution. Longer evolution times than presented here must await larger simulations.

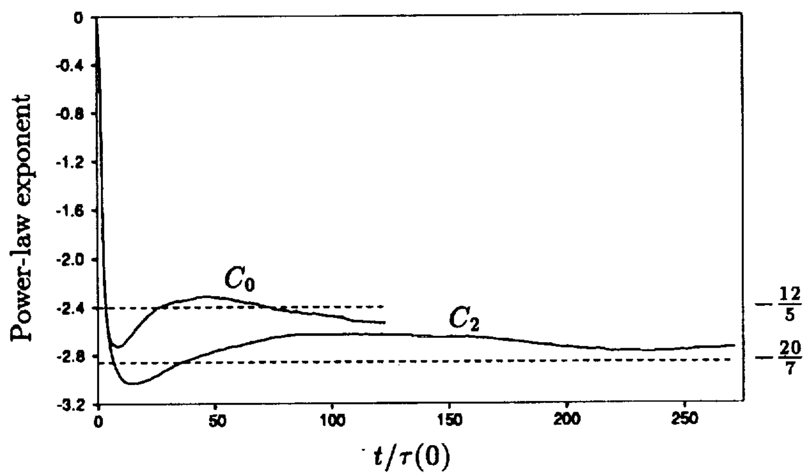

FIGURE 5. Time-evolution of the power-law exponent of $\langle\rho'^2\rangle$ for buoyancy-generated turbulence.

5. Future plans

In this research brief, we have discussed results pertaining to three limiting flows possible in a stably-stratified fluid if the initial Reynolds number and Froude number is sufficiently large. The Froude number, measured at time t, can be shown to be proportional to $1/Nt$ in all of the above similarity states so that at times of order $1/N$ we expect the neglected terms in the equations of motion to become important. In particular, an internal wave field associated with the stable stratification will be generated. The main question which we are currently trying to answer is whether some new and different similarity state is developed asymptotically as the Froude number decreases. Here too, there is an invariant in the flow associated with the low wavenumber spectral coefficients, namely

$$B_0 + \frac{g}{\beta\rho_0}C_0,$$

which is the low wavenumber coefficient of the total (kinetic + potential) energy spectrum. We have tried to base a similarity state of the total energy on this invariant according to

$$\langle \mathbf{u}^2\rangle + \frac{g}{\beta\rho_0}\langle\rho'^2\rangle = (B_0 + \frac{g}{\beta\rho_0}C_0)^{\frac{2}{5}}t^{-\frac{6}{5}} \tag{5.1}$$

and to verify this power-law decay by large-eddy simulation. The results of the large-eddy simulations do indicate a possible similarity state but with a decay power-law exponent about a factor of two smaller than expected by (5.1). Another unusual feature of the simulation results was that the vertical integral scale associated with the total energy in this flow approaches a near-constant value while the horizontal integral scale continues to grow indefinitely. In a similarity state such as those presented in this report, one would naively expect that all length scales would asymptotically behave in the same fashion. The difficulty in theoretically determining the correct similarity state (if one exists) which develops in the low Froude number flow lies with the additional non-dimensional parameter relevant to this flow, namely Nt, and a lack of intuition as to how this parameter should enter into the correct scaling laws. Work on this problem proceeds.

There is also some current interest in stably-stratified flows with regards to the formation of well-mixed layers separated by large density gradients. Such a flow is not statistically homogeneous, and there is some theoretical speculations, as well as experimental evidence, that under certain conditions an initially statistically homogeneous flow may in fact become unstable and form these layers. We wish to determine if such an effect may be observed and studied by numerical simulation.

While the physics behind the formation of these well-mixed layers requires further study, it is also of interest to see how the mixing process may proceed in each individual layer. To this end, it has been proposed (Batchelor, private communication) to study stably-stratified plane Couette flow in order to observe the competition between the generation of turbulence by the externally imposed shear and the stabilization of the flow by the stratification. Stably-stratified Couette flow can be simulated using a modified channel flow code (Lee & Kim, 1991). For high Reynolds numbers, use of a subgrid scale model would be required, and utilization of a modified version of the dynamic subgrid scale model (Cabot, this volume) might be appropriate for this purpose.

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