Application of incremental unknowns to the Burgers equation

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In this article, we make a few remarks on the role that attractors and inertial manifolds play in fluid mechanics problems. We then describe the role of incremental unknowns for approximating attractors and inertial manifolds when finite difference multigrid discretizations are used. The relation with direct numerical simulation and large eddy simulation is also mentioned.

1. Motivations and objectives

At this time, there are two different methods for mathematically describing a turbulent flow when the permanent regime is established. The first one, more traditional, is related to the concept of ensemble averages and the idea that turbulent flows are statistically well defined and reproducible. The flow is described by a measure in the infinite dimensional function space, the statistics being, of course, that associated to the measure. Probabilities and statistical tools are essential in this approach.

Another more recent approach to turbulence stems from dynamical system theory in which the flow is described by an attractor which can be a complicated (fractal) set (see e.g. Constantin, Foias & Temam 1985). Furthermore, this attractor is expected to have in general a large dimension although some interesting turbulent flows with a low dimensional attractor have been produced in special cases. However, this low dimensional behavior is not likely to appear in general for flows of industrial interest (see Keefe, Moin & Kim 1992). Before making further remarks on the attractor point of view, let us observe that the statistical and attractor approaches are not inconsistent. A measure can be defined on the attractor. Furthermore, it was one of the main results of Constantin, Foias & Temam (1985, 1988) to show that certain segments of the Kolmogorov and Kraichnan theories of turbulence can be rigorously derived from the Navier-Stokes equations using the attractor point of view.

It may not seem practically useful to state that a turbulent flow is represented by a fractal attractor of large dimension. Fortunately, some useful information can be derived from the study of the attractor which sheds some new light on the numerical simulation of turbulence. The first information is that the attractor can be embedded approximately, or perhaps exactly, in smooth manifolds called inertial manifolds (Foias, Sell & Temam 1985, 1988; Foias, Manley & Temam 1987; see also Temam 1991a). These manifolds yield a slaving of the high frequency component of the flow by its low frequency component and reduce the number of modes needed to monitor a flow.

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Incremental unknowns have been introduced in Temam (1990) as a means to approximate attractors and inertial manifolds in the context of finite difference multigrid approximations. Indeed, when spectral discretization is used, the separation between the small and large scale components of the flow occurs naturally in the spectral space. When finite differences are used, all mesh points play the same role however large is the number of mesh points, and there is no obvious way to distinguish between the small scales which carry little energy and the large scales which carry most of the energy. Incremental unknowns of various types have been introduced which produce the separation of scale which is needed (Chen & Temam 1993). The utilization of incremental unknowns for large scale computations of turbulent flows, for direct numerical simulation or large eddy simulation, remains to be done. As a first step towards this important task, we describe hereafter in section 2 the utilization of incremental unknowns for the solution of the Burgers equation with stochastic forces. Indeed, these equations are known to be a good model for turbulent flows (see e.g. Chambers *et al.* 1988; Choi *et al.* 1992).

These computations support two important observations:

—— The small scale component of the flow as defined in the context of incremental unknowns (IU) is indeed small. Its instantaneous variation is very fast while its averaged variation is very slow.

—— The IU method is numerically efficient. It produces an improvement of the CFL stability condition which has not been analytically fully explained, although there are already some partial theoretical justifications (Temam 1990, 1991b). Consequently, the same accuracy can be recovered with less calculations, the gain in computing time in the cases that we consider being a factor of 7.

2. Accomplishments

In this section, we describe the procedure for applying the IU method to the stochastic Burgers equation. This equation contains nonlinear convection and diffusion terms, and its solution exhibits a chaotic nature; these qualities make it a natural model for the more complicated Navier-Stokes equations. Section 2.1 describes the governing equation and boundary conditions. The mathematical and numerical procedures of applying the IU method to the stochastic Burgers equation are described in section 2.2. Numerical results are presented in section 2.3. Application of the IU method to the Navier-Stokes equations is considered in section 2.4.

2.1 The Burgers equation with random forcing

Consider the randomly forced non-dimensionalized Burgers equation with no-slip boundary conditions

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} + \chi(x,t), \qquad 0 < x < 1, \\ &u(x=0,t) = u(x=1,t) = 0, \\ &u(x,t=0) = u_o, \end{aligned} \tag{2.1}$$

where u is the velocity, χ is the random forcing, and Re is the Reynolds number. The initial data u_o is an instantaneous solution of the Burgers equation with random forcing χ rather than an arbitrary function. The forcing function χ is a white noise random process in x with zero mean (see Chambers *et al.* 1988; Choi *et al.* 1992):

$$< \chi >_x = 0, \quad < \chi^2 >_x = 1.$$

Here $< \cdot >_x$ denotes the average value over space. In the absence of forcing ($\chi = 0$), the solutions of equation (2.1) decay to zero from any bounded initial data.

2.2 Description of the IU method for the Burgers solution

The discretized equations for the stochastic Burgers equation with no-slip boundary conditions are obtained using a semi-implicit method in time (Adams-Bashforth for the convection term and Crank-Nicholson for the diffusion term) and a second-order centered difference method in space:

$$\alpha u_i^n - \beta u_{i+1}^n - \beta u_{i-1}^n = g_i, \quad i = 1, \cdots, 2I-1, \tag{2.2}$$

$$u_0 = u_{2I} = 0,$$

where

$$\begin{aligned}
\alpha &= 1 + \frac{\Delta t}{Re \Delta x^2}, \\
\beta &= \frac{\Delta t}{2Re\Delta x^2}, \\
g_i &= u_i^{n-1} - \frac{\Delta t}{2Re\Delta x^2}\left(2u_i^{n-1} - u_{i+1}^{n-1} - u_{i-1}^{n-1}\right) \\
&\quad - \frac{\Delta t}{8\Delta x}\left[3(u_{i+1}^{n-1^2} - u_{i-1}^{n-1^2}) - (u_{i+1}^{n-2^2} - u_{i-1}^{n-2^2})\right] + \Delta t\, \chi_i^{n-\frac{1}{2}},
\end{aligned} \tag{2.3}$$

where Δt is the computational time step and $\Delta x = 1/(2I)$ is the grid spacing.

The incremental unknowns for the present problem consist of the numbers ϕ_{2i} and ψ_{2i+1} (see figure 1):

$$\begin{aligned}
&\phi_{2i} = u_{2i}, \quad i = 0, \cdots, I, \\
&\psi_{2i+1} = u_{2i+1} - \frac{1}{2}(u_{2i} + u_{2i+2}), \quad i = 0, \cdots, I-1,
\end{aligned} \tag{2.4}$$

where ϕ_0 and ϕ_{2I} correspond to the velocities at the boundary points. Thus ψ_{2i+1} is the increment of u to the average of the values at the neighboring points, $2i$ and $2i+2$; hence by Taylor's formula, ψ_{2i+1} is small, of order Δx^2.

At points $2i+1$, equation (2.2) becomes

$$\alpha u_{2i+1}^n - \beta u_{2i+2}^n - \beta u_{2i}^n = g_{2i+1}.$$

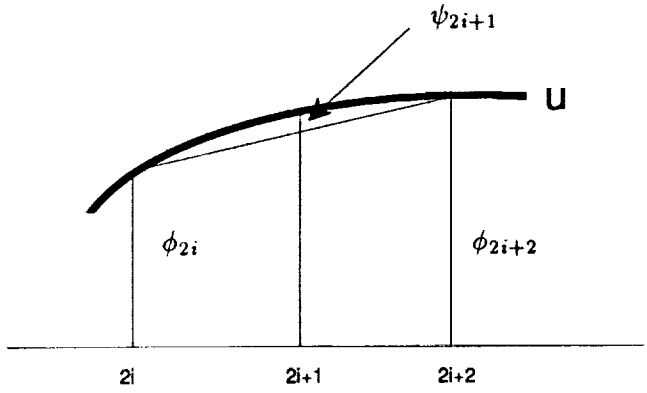

FIGURE 1. Incremental unknowns.

Using equation (2.4) gives

$$\alpha\psi^n_{2i+1} + \left(\frac{\alpha}{2} - \beta\right)\left(\phi^n_{2i} + \phi^n_{2i+2}\right) = g_{2i+1}, \tag{2.5}$$

where $i = 0, \cdots, I-1$. Similarly, at points $2i$, equation (2.2) becomes

$$\alpha u^n_{2i} - \beta u^n_{2i+1} - \beta u^n_{2i-1} = g_{2i}.$$

Substituting equations (2.4) and (2.5) into the above equation gives

$$\frac{1}{\alpha}\left(\alpha^2 - 2\beta^2\right)\phi^n_{2i} - \frac{\beta^2}{\alpha}\phi^n_{2i+2} - \frac{\beta^2}{\alpha}\phi^n_{2i-2} = g_{2i} + \frac{\beta}{\alpha}\left(g_{2i+1} + g_{2i-1}\right), \tag{2.6}$$

where $i = 1, \cdots, I-1$. The system consisting of equation (2.6) is similar to the system consisting of equation (2.2), but involves half as many unknowns.

We can, of course, repeat the procedure. If we start with $\Delta x = 1/(2^l I)$, then after l steps, we reduce the initial system involving $2^l I$ unknowns to a similar one involving I unknowns. Following the same procedure described above, one can get the discretized equations for multi-steps l:

$$\alpha^m\phi^{n,m}_{2i} - \beta^m\phi^{n,m}_{2i+2} - \beta^m\phi^{n,m}_{2i-2} = g^m_{2i}, \quad i = 1, \cdots, 2^{l-m}I - 1, \tag{2.7}$$

$$\psi^{n,m}_{2i+1} = \frac{1}{\alpha^m}\left[g^m_{2i+1} - \left(\frac{\alpha^m}{2} - \beta^m\right)\left(\phi^{n,m}_{2i} + \phi^{n,m}_{2i+2}\right)\right], \quad i = 0, \cdots, 2^{l-m}I - 1, \tag{2.8}$$

$$\begin{aligned}
\alpha^m &= \frac{1}{\alpha^{m-1}}\left(\alpha^{m-1^2} - 2\beta^{m-1^2}\right), \\
\beta^m &= \frac{\beta^{m-1^2}}{\alpha^{m-1}}, \\
g^m_i &= g^{m-1}_{2i} + \frac{\beta^{m-1}}{\alpha^{m-1}}\left(g^{m-1}_{2i+1} + g^{m-1}_{2i-1}\right), \quad i = 1, \cdots, 2^{l-m+1}I - 1,
\end{aligned} \tag{2.9}$$

where $m = 1, \cdots, l$ denotes the level of grids and $m = 1$ and l correspond to the finest and coarsest grids, respectively;

$\alpha^0 = \alpha$, $\beta^0 = \beta$ and $g_i^0 = g_i$, $i = 1, \cdots, 2^l I - 1$ (equation (2.3)).

Numerical procedure

The numerical procedure of applying the IU method to the stochastic Burgers equation can be written as follows:

Step 1: Start from an initial velocity field u_o, $u_i^{n-1} = u_{oi}$.
Step 2: Get α^m, β^m, and g_{2i}^m from equation (2.9) for $m = 1, \cdots, l$.
Step 3: Solve equation (2.7) for the coarsest grid ($m = l$) to obtain $\phi_{2i}^{n,l}$.
Step 4: Obtain $\psi_{2i+1}^{n,m}$ from equation (2.8) with $\phi_{2i}^{n,m}$.
Step 5: Obtain $u^{n,m}$; $u_{2i}^{n,m} = \phi_{2i}^{n,m}$ and $u_{2i+1}^{n,m} = \psi_{2i+1}^{n,m} + \frac{1}{2}\left(\phi_{2i}^{n,m} + \phi_{2i+2}^{n,m}\right)$.
Step 6: Obtain $\phi_{2i}^{n,m-1} = u_i^{n,m}$.
Step 7: Repeat Steps 4 and 6 until $u^{n,1}$ is obtained.

2.3 Numerical results using the IU method

In this section, we apply the numerical procedure described in the previous section to the Burgers equation. A uniform computational mesh of 2049 points is used in x ($\Delta x = 1/2048$) and $l = 4$ (number of the grid levels); therefore, the coarsest grid has 257 points.

2.3.1 Properties of the incremental unknowns

An initial velocity field u_o is obtained for $Re = 1500$ and $\Delta t_r = 0.01$, where Δt_r is the time scale of the random forcing (for more details see Choi *et al.* 1992). Figure 2 (a) shows the root-mean-square values of ϕ and ψ^m as a function of t, where $m = 1, 2, 3$; $m = 1, 2$, and 3 correspond to the grid points of 2049, 1025, and 513, respectively. It can be seen that the ψ^m's are several orders of magnitude smaller than ϕ. Also, note that the incremental unknowns ψ at the coarser grid level have larger magnitude as compared to those at the denser grid level.

The root-mean-square values of $\partial\phi/\partial t$ and $\partial\psi^m/\partial t$ as a function of t are shown in figure 2 (b). Contrary to the results of ϕ and ψ^m (figure 2 (a)), the magnitude of $\partial\psi^m/\partial t$ is comparable to that of $\partial\phi/\partial t$, indicating that $\partial\psi^m/\partial t$ cannot be neglected pointwise as compared to $\partial\phi/\partial t$ when there is a stochastic motion in flow. In case there is no small-scale motion in flow, however, the magnitude of $\partial\psi^m/\partial t$ is several orders of magnitude smaller than that of $\partial\phi/\partial t$: we have tested an initial velocity field, $\sin(2\pi x)$, without random forcing, and the result showed this is indeed the case. When a small-scale motion exists in the flow, we expect the derivatives $\partial\psi^m/\partial t$ to be small in average as evidenced in figure 3. This point will be discussed elsewhere.

Figure 4 shows $\partial\psi^1/\partial t$ at $x = 0.5$ as a function of t and $\partial\psi^1/\partial t$ at $t = 2$ as a function of x; $\partial\psi^1/\partial t(t)$ shows an intermittent behavior while a chaotic behavior is shown in $\partial\psi^1/\partial t(x)$. The power spectra of these $\partial\psi^1/\partial t$'s show that the incremental unknowns have nearly same power at all wavenumbers (a white noise) while they have higher power at the highest frequency.

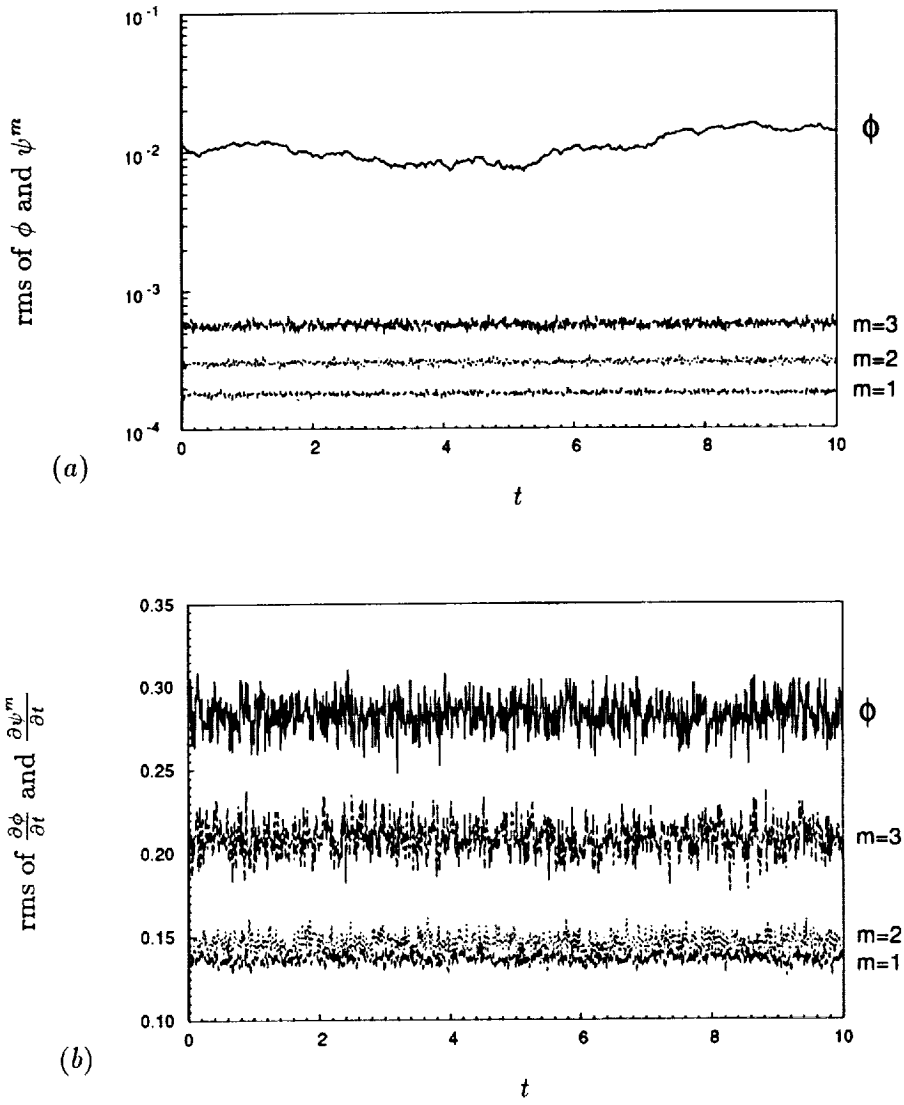

FIGURE 2. Root-mean-square values of the incremental unknowns as a function of t: (a) ——— , ϕ; - - - - , ψ^1; ····· , ψ^2; — - — , ψ^3; (b) ——— , $\frac{\partial\phi}{\partial t}$; - - - - , $\frac{\partial\psi^1}{\partial t}$; ····· , $\frac{\partial\psi^2}{\partial t}$; — - — , $\frac{\partial\psi^3}{\partial t}$. Note that $m = 1$ corresponds to the finest grid.

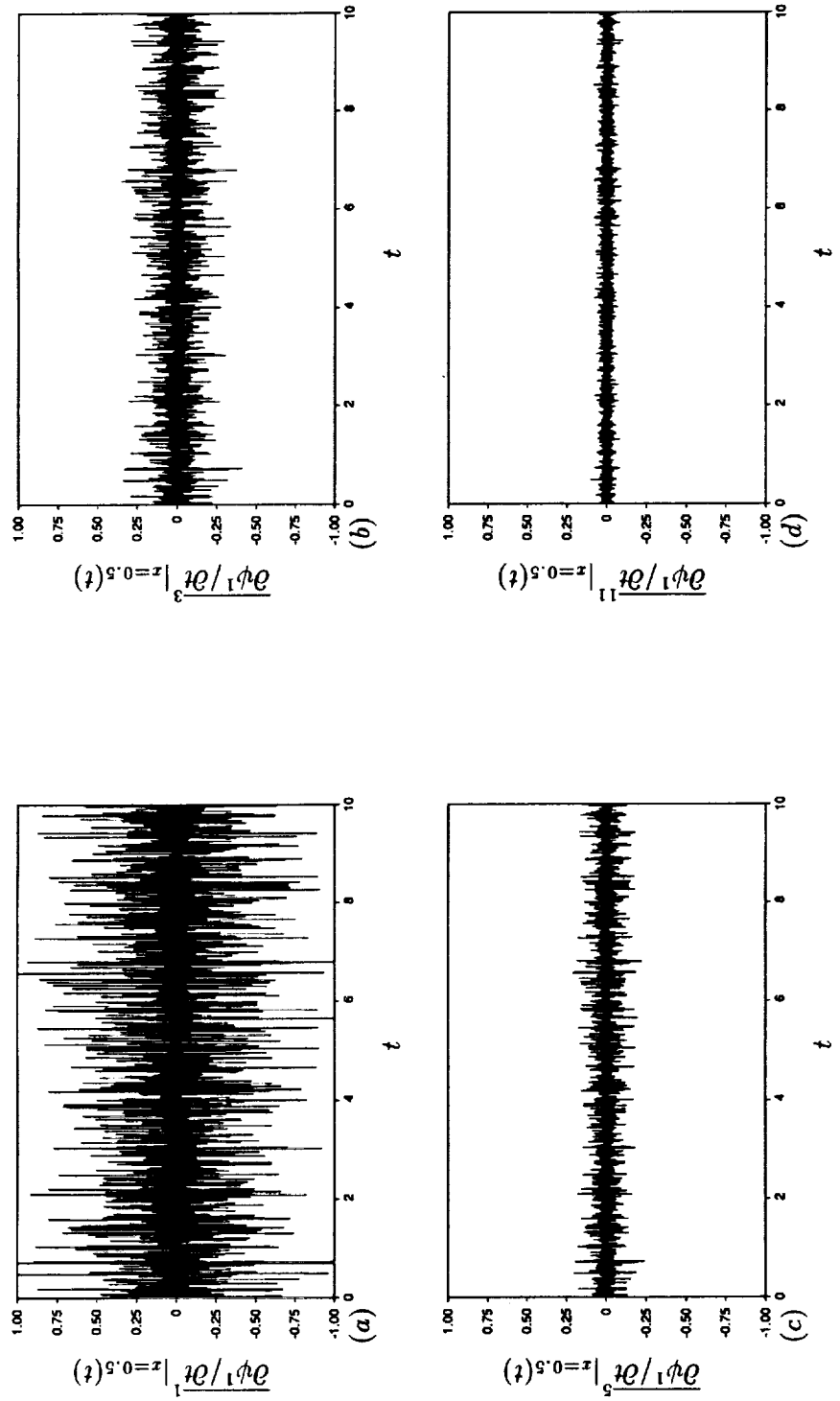

FIGURE 3. Time-averaged values of the incremental unknowns, $\overline{\partial\psi^1/\partial t}^k|_{x=0.5}(t)$: (*a*) $k = 1$; (*b*) $k = 3$; (*c*) $k = 5$; (*d*) $k = 11$, where $\overline{\frac{\partial\psi^1}{\partial t}}^k|_{x=0.5}(t) = \frac{1}{k\Delta t}\int_{t-\frac{k}{2}\Delta t}^{t+\frac{k}{2}\Delta t} \frac{\partial\psi^1}{\partial t}(\tau)\, d\tau$.

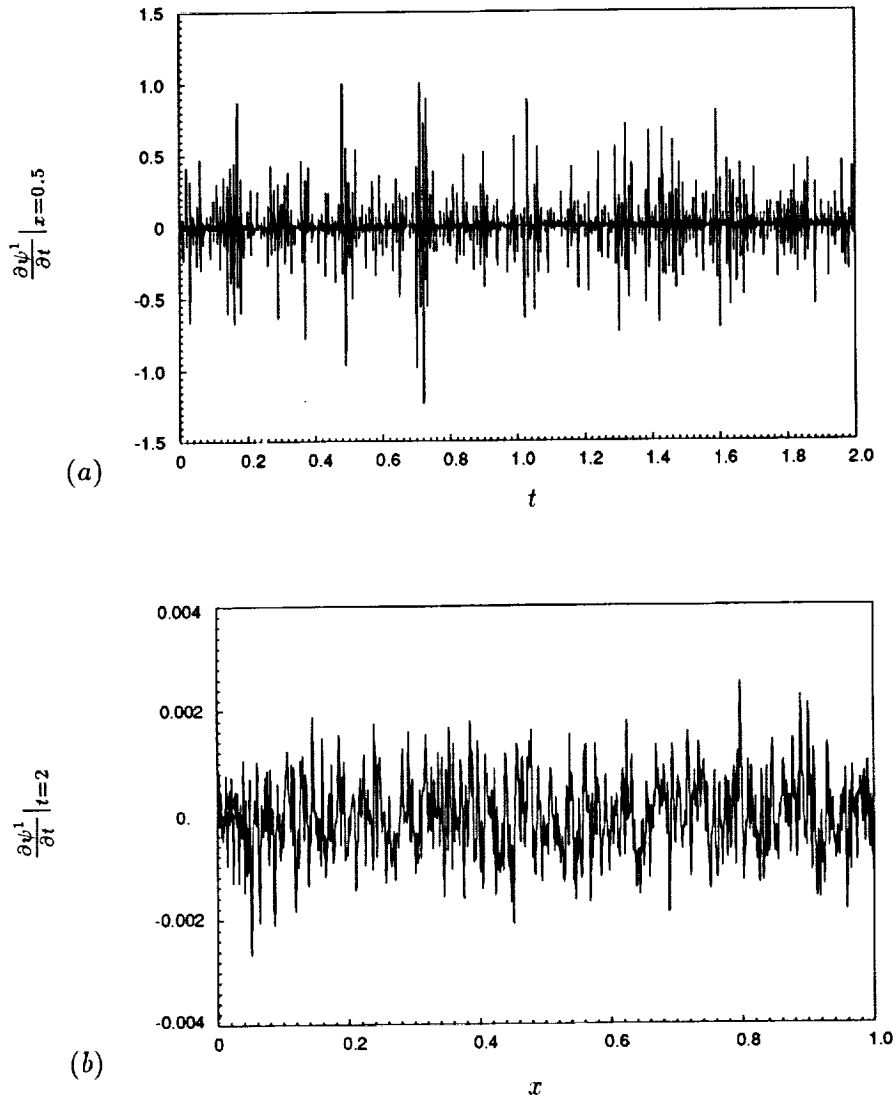

FIGURE 4. Instantaneous $\frac{\partial \psi^1}{\partial t}$ as a function of t or x.

2.3.2 Efficiency of the IU method

In this section, we investigate the effect of the IU method on the numerical stability. A classical semi-implicit method has a limit of the computational time step due to the explicit treatment of the nonlinear term. The maximum CFL (Courant-Friedrichs-Lewy) number is restricted to one with the present semi-implicit method (equation (2.2)). Hence, the maximum time step without the IU method is restricted

Case	$l = 1$	$l = 2$	$l = 3$	$l = 4$
Δt_{max}	0.0051	0.017	0.054	0.054
Total CPU (sec)	0.232	0.088	0.032	0.034

Table 1. Maximum computational time step and total CPU time using the IU method.

as

$$\Delta t \leq \frac{\Delta x}{u}. \tag{2.10}$$

Maximum computational time steps Δt_{max} using the IU method have been obtained for the initial value problem (2.1): $u_o = \sin(2\pi x)$, $Re = 100$, and $\chi = 0$ (no random forcing). Table 1 shows the maximum computational time step and total CPU time using the IU method, where l denotes the level of the coarsest grid. Note that the IU method is not used when $l = 1$. A maximum computational time step is clearly increased when $l > 1$, indicating that the numerical stability is enhanced using the IU method. However, there seems to exist a limit of level l of increasing Δt_{max}; for the present problem, $l = 3$ (see table 1).

The CPU time to advance one time step with the IU method is larger than that without the IU method due to the calculation procedure of α^m, β^m, and g^m, although the matrix size to be inverted is reduced from $2^l I$ to I. Total CPU times to reach $t = 1$ are shown in table 1. Computational cost is significantly reduced by a factor of 7 using the IU method. The mathematical analysis of the stability enhancement using the IU method for the Burgers equation is very complex and will be investigated in the future. Theoretical indications that the IU method produces stability enhancement appear in Temam (1990, 1991).

2.4 Application of the IU method to the unsteady Navier-Stokes equations

The IU method can be applied to the unsteady three-dimensional Navier-Stokes equations as follows: Firstly, the first step of the fractional-step method (Kim & Moin 1985) provides a second-order-accurate approximation of the three-dimensional Navier-Stokes equations with pressure term excluded. Secondly, an approximate factorization technique (Beam & Warming 1978) splits three-dimensional equations to three one-dimensional equations. Finally, the IU method as presented above is applied to each one-dimensional equation. This case will be addressed in the future.

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