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Receptivity in parallel flows: an adjoint approach

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1. Motivation and objectives

Linear receptivity studies in parallel flows are aimed at understanding how external forcing couples to the natural unstable motions which a flow can support. The vibrating ribbon problem (Gaster (1965)) models the original Schubauer and Skramstad (1947) boundary layer experiment and represents the classic boundary layer receptivity problem. The process by which disturbances are initiated in convectively-unstable jets and shear layers has also received attention (Balsa (1988), Huerre and Monkewitz (1985), Tam (1978)).

Gaster (1965) was the first to handle the boundary layer analysis with the recognition that spatial modes, rather than temporal modes, were relevant when studying convectively-unstable flows that are driven by a time-harmonic source. The amplitude of the least stable spatial mode, far downstream of the source, is related to the source strength by a coupling coefficient. The determination of this coefficient is at the heart of this type of linear receptivity study.

Traditionally, the Briggs method is applied to Fourier-inversion integrals to find the asymptotic temporal and spatial behavior after a time-harmonic source has been switched on. Ashpis and Reshotko (1990) give a detailed description of this procedure for the vibrating ribbon problem. Unfortunately, the coupling coefficient, which gives a measure of the amplitude of the asymptotic response relative to the amplitude of the source, does not have a very convenient form, either from the point of view of interpretation or of computation. The expression involves derivatives with respect to wavenumber of either a flow quantity or the dispersion relation (Ashpis and Reshotko (1990), Kozlov and Ryzhov (1990), Balsa (1988), Huerre and Monkewitz (1985), Gaster (1965)).

Earlier work (Hill (1992)) indicates that adjoint eigensolutions for the global temporal modes of a cylinder wake characterize rather simply how receptive the wake is to control forces. Can this approach be adapted for the spatial instabilities that occur in boundary layers, jets, and shear layers? The first objective of the present study was to determine whether the various wavenumber derivative factors, appearing in the coupling coefficients for linear receptivity problems, could be re-expressed in a simpler form involving adjoint eigensolutions. Secondly, it was hoped that the general nature of this simplification could be shown; indeed, a rather elegant characterization of the receptivity properties of spatial instabilities does emerge. The analysis is quite distinct from the usual Fourier-inversion procedures, although a detailed knowledge of the spectrum of the Orr-Sommerfeld equation is still required. Since the cylinder wake analysis proved very useful in addressing control considerations, the final objective was to provide a foundation upon which boundary layer control theory may be developed.

2. Background

2.1 The Lagrange identity

The cornerstone of this work is the *Lagrange identity* (Ince (1944)), from which the adjoint equations are extracted. This remarkably powerful relation is implicit in the work of Salwen and Grosch (1981) (here onwards referred to as SG), and Salwen (1979), although it is never stated explicitly. As has already been intimated, the origins of the adjoint analysis can be traced back to work of Lagrange in the late 18th century (See Lagrange's collected works (1867)). Although Fuchs (1858) wrote in German, Ince (1944) credits the use of the word 'adjoint' to him. Adjoint eigensolutions play a key role in the imposition of solvability conditions (the Fredholm alternative) and as such are indispensable to researchers studying nonlinear phenomena using bifurcation theory (Iooss and Joseph (1980)).

The Lagrange identity is developed with reference to the linearized Navier- Stokes equations. If $\underline{V}(\underline{r})$ is a steady incompressible viscous flow field, then linear velocity disturbances $\underline{v}(\underline{r},t)$ and pressure disturbances $p(\underline{r},t)$ upon this flow satisfy

$$\frac{\partial \underline{v}}{\partial t} + \underline{L}(\underline{V};R)\,\underline{v} + \nabla p = \underline{0}, \tag{1}$$

$$\nabla.\underline{v} = 0, \tag{2}$$

where the ith component of the linear operator $\underline{L}(\underline{V};R)$ is

$$\left(\underline{L}(\underline{V};R)\,\underline{v}\right)_i = V_j\frac{\partial v_i}{\partial x_j} + v_j\frac{\partial V_i}{\partial x_j} - \frac{1}{R}\frac{\partial^2 v_i}{\partial x_j^2}. \tag{3}$$

For *any* pair of fields $(\underline{v},p)$ and $(\tilde{\underline{v}},\tilde{p})$, ($(\underline{v},p)$ does not have to satisfy (1) and (2)), defined over the flow domain, the following Lagrange identity is satisfied:

$$\left[(\frac{\partial \underline{v}}{\partial t} + \underline{L}(\underline{V};R)\,\underline{v} + \nabla p).\tilde{\underline{v}} + \nabla.\underline{v}\,\tilde{p}\,\right] + \left[\underline{v}.(\frac{\partial \tilde{\underline{v}}}{\partial t} + \tilde{\underline{L}}(\underline{V};R)\,\tilde{\underline{v}} + \nabla\tilde{p}) + p\nabla.\tilde{\underline{v}}\right]$$

$$= \frac{\partial}{\partial t}(\underline{v}.\tilde{\underline{v}}) + \nabla.\underline{J}\big((\underline{v},p),(\tilde{\underline{v}},\tilde{p})\big), \tag{4}$$

where $\tilde{\underline{L}}(\underline{V};R)$ is the adjoint linearized Navier-Stokes operator with components

$$\left(\tilde{\underline{L}}(\underline{V};R)\,\tilde{\underline{v}}\right)_i = V_j\frac{\partial \tilde{v}_i}{\partial x_j} - \tilde{v}_j\frac{\partial V_j}{\partial x_i} + \frac{1}{R}\frac{\partial^2 \tilde{v}_i}{\partial x_j^2}. \tag{5}$$

The vector $\underline{J}\big((\underline{v},p),(\tilde{\underline{v}},\tilde{p})\big)$ is the *bilinear concomitant* with components

$$\Big(\underline{J}\big((\underline{v},p),(\tilde{\underline{v}},\tilde{p})\big)\Big)_j = v_i\tilde{\sigma}_{ij} + \sigma_{ij}\tilde{v}_i, \tag{6}$$

where

$$\sigma_{ij} = p\delta_{ij} - \frac{1}{R}\frac{\partial v_i}{\partial x_j} + V_j v_i, \tag{7}$$

$$\tilde{\sigma}_{ij} = \tilde{p}\delta_{ij} + \frac{1}{R}\frac{\partial \tilde{v}_i}{\partial x_j}. \tag{8}$$

Examining the second term in square brackets on the left hand side of the Lagrange identity (4), we define the adjoint equations

$$\frac{\partial \underline{\tilde{v}}}{\partial t} + \underline{\tilde{L}}(\underline{V}; R)\,\underline{\tilde{v}} + \nabla\tilde{p} = \underline{0}, \tag{9}$$

$$\nabla.\underline{\tilde{v}} = 0. \tag{10}$$

2.2 Bi-orthogonality

A brief summary will be given in this section of the procedure by which bi-orthogonal spatial-eigensolution sets can be constructed for the Orr-Sommerfeld equation. This is essentially a review of the work of SG.

Schensted (1960), and Drazin and Reid (1981) show how general disturbances in spanwise-bounded plane parallel flows can be expanded as a sum of temporal eigensolutions of the Orr-Sommerfeld equation. These eigensolutions together with a set of eigensolutions of the corresponding adjoint equation form a bi-orthogonal set, under the action of an appropriate inner product. SG develop this further and demonstrate how expansion of arbitrary disturbances as a sum of temporal or spatial eigensolutions may be carried out in unbounded flows where, in fact, a continuum of eigensolutions forms part of the spectrum.

Following SG, let $\underline{V}(\underline{r}) = U(y)\hat{x}$ define a two-dimensional parallel flow in the xy-plane, for $y_1 < y < y_2$. For a boundary layer we may have $y_1 \equiv 0$ and $y_2 \equiv \infty$, whereas for a shear layer $y_1 \equiv -\infty$, and $y_2 \equiv \infty$. Writing $\underline{v} = \nabla \times (\psi(x,y,t)\hat{z})$ for some stream function $\psi(x,y,t)$, $\hat{z}$ being the unit vector normal to the plane of the flow, the governing equation is given by the z-component of the curl of (1)

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\psi - \frac{d^2U}{dy^2}\frac{\partial \psi}{\partial x} - \frac{1}{R}\nabla^4\psi = 0. \tag{11}$$

Similarly, writing $\underline{\tilde{v}} = \nabla \times (\tilde{\psi}(x,y,t)\hat{z})$, the adjoint equation is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\tilde{\psi} + 2\frac{dU}{dy}\frac{\partial^2 \tilde{\psi}}{\partial x \partial y} + \frac{1}{R}\nabla^4\tilde{\psi} = 0. \tag{12}$$

No-slip boundary conditions

$$\psi = \frac{\partial \psi}{\partial y} = 0, \text{ and } \tilde{\psi} = \frac{\partial \tilde{\psi}}{\partial y} = 0 \tag{13}$$

are imposed on any walls. As $|y| \to \infty$, disturbances are required to decay to zero. As a direct consequence, the y-component of $\underline{J}$ approaches zero as y approaches the limits of the flow domain.

With $\alpha, \tilde{\alpha}$ being wavenumbers and $\omega, \tilde{\omega}$ being frequencies, let

$$\psi(x,y,t) = \phi_{\alpha\omega}(y)e^{i(\omega t+\alpha x)}, \quad \tilde{\psi}(x,y,t) = \tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}(y)e^{-i(\tilde{\omega}t+\tilde{\alpha}x)} \tag{14}$$

be solutions of equation (11) and the adjoint (12), respectively (i.e. $\phi_{\alpha\omega}(y)$ satisfies the Orr-Sommerfeld equation, and $\tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}(y)$ an adjoint Orr- Sommerfeld equation). An auxiliary problem can then be solved to obtain the pressure eigensolution and the adjoint

$$p(x,y,t) = p_{\alpha\omega}(y)e^{i(\omega t+\alpha x)}, \quad \tilde{p}(x,y,t) = \tilde{p}_{\tilde{\alpha}\tilde{\omega}}(y)e^{-i(\tilde{\omega}t+\tilde{\alpha}x)}. \tag{15}$$

Substituting these velocity and pressure fields into the Lagrange identity (with $\underline{V}(\underline{r}) = U(y)\hat{x}$), the entire left hand side is zero, from which it follows (SG, Salwen (1979)) that

$$(\omega - \tilde{\omega})\langle\phi_{\alpha\omega}, \tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}\rangle + (\alpha - \tilde{\alpha})[\phi_{\alpha\omega}, \tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}] = 0, \tag{16}$$

with

$$\langle\phi_{\alpha\omega}, \tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}\rangle = \int_{y_1}^{y_2} \left\{ \frac{\partial\phi_{\alpha\omega}}{\partial y}\frac{\partial\tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}}{\partial y} + \alpha\tilde{\alpha}\phi_{\alpha\omega}\tilde{\phi}_{\tilde{\alpha}\tilde{\omega}} \right\} dy, \tag{17}$$

$$\begin{aligned}
[\phi_{\alpha\omega}, \tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}] = \int_{y_1}^{y_2} \Big\{ &(\omega\alpha + \tilde{\omega}\tilde{\alpha})\phi_{\alpha\omega}\tilde{\phi}_{\tilde{\alpha}\tilde{\omega}} \\
&+ U\left((\alpha^2 + \alpha\tilde{\alpha} + \tilde{\alpha}^2)\phi_{\alpha\omega}\tilde{\phi}_{\tilde{\alpha}\tilde{\omega}} + \phi_{\alpha\omega}\frac{\partial^2\tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}}{\partial y^2} + 2\frac{\partial\phi_{\alpha\omega}}{\partial y}\frac{\partial\tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}}{\partial y}\right) \\
&- \frac{i(\alpha+\tilde{\alpha})}{R}\left((\alpha^2 + \tilde{\alpha}^2)\phi_{\alpha\omega}\tilde{\phi}_{\tilde{\alpha}\tilde{\omega}} + 2\frac{\partial\phi_{\alpha\omega}}{\partial y}\frac{\partial\tilde{\phi}_{\tilde{\alpha}\tilde{\omega}}}{\partial y}\right)\Big\} dy.
\end{aligned} \tag{18}$$

When considering the spatial stability problem, ω is chosen to be purely real, and from the Orr-Sommerfeld equation, a set of characteristic values for α can then be found. This set can consist of discrete and continuum modes. For the adjoint problem, likewise selecting $\tilde{\omega} = \omega$, a set of possible values for $\tilde{\alpha}$ may be obtained. From (16), we have

$$(\alpha - \tilde{\alpha})[\phi_{\alpha\omega}, \tilde{\phi}_{\tilde{\alpha}\omega}] = 0, \tag{19}$$

from which it follows that

$$\text{if } [\phi_{\alpha\omega}, \tilde{\phi}_{\tilde{\alpha}\omega}] \neq 0 \text{ , then } \alpha = \tilde{\alpha}. \tag{20}$$

With appropriate care taken in handling the continuum modes, SG proceed to construct a bi-orthogonal eigensolution set in which for every solution $\phi_{\alpha\omega}(y)e^{i(\omega t+\alpha x)}$ to (11), there corresponds a normalized adjoint solution $\tilde{\phi}_{\alpha\omega}(y)e^{-i(\omega t+\alpha x)}$ to (12) such that, for example, for discrete modes with wavenumbers α_i,

$$[\phi_{\alpha_i\omega}, \tilde{\phi}_{\alpha_j\omega}] = \delta_{ij}. \tag{21}$$

A similar result is quoted by SG for the case of temporal modes, based on the inner product $\langle\cdot,\cdot\rangle$.

For an arbitrary solution $\underline{s} = (\underline{v}(x,y), p(x,y))e^{i\omega t}$ of the linearized equations, the adjoint solutions can be used to decompose the disturbance field. For a chosen eigenmode, if $\underline{\tilde{s}} = (\underline{\tilde{v}}_{\alpha\omega}(y), \tilde{p}_{\alpha\omega}(y))e^{-i(\omega t+\alpha x)}$ is the adjoint eigenmode, the filtering operation

$$\int_{y_1}^{y_2} \hat{x}.\underline{J}(\underline{s},\underline{\tilde{s}})dy \tag{22}$$

returns the modal amplitude.

3. Accomplishments

In this section I will give examples of how the adjoint problem provides a reformulation of various aspects of the receptivity problem.

3.1 Failure of bi-orthogonality

It can be demonstrated that

$$\frac{d\omega}{d\alpha} = -\frac{[\phi_{\alpha\omega}, \tilde{\phi}_{\alpha\omega}]}{\langle\phi_{\alpha\omega}, \tilde{\phi}_{\alpha\omega}\rangle}, \tag{23}$$

which I believe to be a previously-unknown form for the group velocity of Orr-Sommerfeld eigenmodes. This form has implications for the formation of orthogonal solution sets. In the event that the chosen eigensolution is at a branch point singularity, for which $d\omega/d\alpha = 0$, then clearly

$$[\phi_{\alpha\omega}, \tilde{\phi}_{\alpha\omega}] = 0, \tag{24}$$

(the denominator in (23) is always bounded). The normalization condition (21) would require that the bilinear operation (24) take the value 1. The failure of (21) is connected to degeneracy of the wavenumber eigenvalues as the branch point is approached in the complex-ω plane.

3.2 The vibrating ribbon problem

The classical vibrating ribbon problem was discussed in some detail recently by Ashpis and Reshotko (1990). A vibrating ribbon is placed at the wall beneath a parallel boundary layer flow, and, after $t = 0$, it oscillates with velocity

$$\underline{v}_b(x,t) = \delta(x)e^{i\omega t}\hat{y}. \tag{25}$$

The classical result indicates that a long time after the initiation of the excitation and far downstream of the ribbon, the y-velocity of disturbances is

$$v(x,y,t) = i\frac{v(y;\alpha,\omega)}{\frac{\partial v}{\partial\alpha}(0;\alpha,\omega)}e^{i(\omega t+\alpha x)}, \tag{26}$$

where $\alpha(\omega)$ is wavenumber of the fastest growing discrete spatial mode of the Orr-Sommerfeld equation at frequency ω. The field $v(y;\alpha,\omega)$ is the y- velocity field of the mode. The factor $\partial v/\partial\alpha$ arises from contour integration around a pole in the complex wavenumber plane.

Beginning with (26), by differentiating the linearized equations of motion with respect to α and making use of the Lagrange identity, I have shown that

$$v(x,y,t) = \tilde{p}_{\alpha\omega}(0)\ v(y;\alpha,\omega)e^{i(\omega t+\alpha x)}. \tag{27}$$

Details of the analysis are not given here. The adjoint pressure at the wall, $\tilde{p}_{\alpha\omega}(0)$, gives the amplitude of the response to the ribbon oscillation.

3.3 Excitation of a free shear layer

Huerre and Monkewitz (1985) consider the response of an inviscid incompressible free shear layer ($U(y)$ defined for $-\infty < y < \infty$) to excitation by a point vorticity source positioned at $x = 0, y = y_0$,

$$\Omega(x,y,t) = \delta(x)\delta(y-y_0)e^{i\omega t}. \tag{28}$$

Quoting their result, far downstream the stream function behaves as

$$\psi(x,y,t) = \frac{\phi_{\alpha\omega}(y_0)\phi_{\alpha\omega}(y)}{(\alpha U(y_0)+\omega)\frac{\partial D}{\partial\alpha}(y_0;\alpha,\omega)}e^{i(\omega t+\alpha x)}, \tag{29}$$

where $\phi_{\alpha\omega}(y)$ is the least stable spatial eigensolution of the Rayleigh equation, and

$$D(y_0;\alpha,\omega) = \left(\Phi^+\frac{\partial\Phi^-}{\partial y} - \Phi^-\frac{\partial\Phi^+}{\partial y}\right)_{y=y_0}. \tag{30}$$

Here $\Phi^+(y;\alpha,\omega)$ and $\Phi^-(y;\alpha,\omega)$ are the upper ($y \geq y_0$) and lower ($y \leq y_0$) eigensolutions of the Rayleigh equation, which decay as $|y| \to \infty$.

I have taken (29) and simplified it to obtain

$$\psi(x,y,t) = \tilde{\phi}_{\alpha\omega}(y_0)\ \phi(y)e^{i(\omega t+\alpha x)}. \tag{31}$$

In this case, the modal amplitude is simply the value of the adjoint stream function, $\tilde{\phi}_{\alpha\omega}(y_0)$, at the source location.

3.4 Response to a general harmonic source distribution

Consider a source distribution oscillating harmonically at real frequency ω, including sources of momenta of strength $\underline{q}(\underline{r};\omega)$, and mass sources $\varphi(\underline{r};\omega)$. To keep the analysis as general as possible, we consider a boundary layer flow in the region $y > 0$ so that boundary velocities $\underline{v}_b(x;\omega)$ can be specified at $y = 0$. The governing equations are then

$$i\omega\underline{v} + \underline{L}(\underline{V};R)\,\underline{v} + \nabla p = \underline{q}(\underline{r};\omega), \tag{32}$$

$$\nabla.\underline{v} = \varphi(\underline{r};\omega), \tag{33}$$

with

$$\underline{v} = \underline{v}_b(x;\omega) \text{ on } y = 0. \tag{34}$$

The sources are assumed to be localized in so far as they disappear for $|x|$ larger than some value, say $X > 0$.

Let the discrete spatial mode of interest have stream function $\phi_{\alpha\omega}(y)e^{i(\omega t+\alpha x)}$. Typically the mode with largest value of $-Im(\alpha(\omega))$ would be considered. We wish to determine the amplitude of this mode far downstream as a result of the excitation by the various sources.

For a convectively-unstable mode which is traveling downstream, the stream function behaves as

$$a\ \phi_{\alpha\omega}(y)e^{i(\omega t+\alpha x)} \tag{35}$$

in the region downstream of all sources. The amplitude a is given explicitly by

$$a = \int_{x_1}^{x_2}\int_0^\infty \underline{q}(\underline{r};\omega).\underline{\tilde{v}}_{\alpha\omega}(y)e^{-i\alpha x}dydx + \int_{x_1}^{x_2}\int_0^\infty \varphi(\underline{r};\omega)\tilde{p}_{\alpha\omega}(y)e^{-i\alpha x}dydx + \int_{x_1}^{x_2}\underline{v}_b(x;\omega).\underline{\tilde{S}}_{\alpha\omega}e^{-i\alpha x}dx, \tag{36}$$

where $x_1 < -X, x_2 > X$. Clearly, if there are no sources, then $a = 0$. The field

$$\underline{\tilde{v}}_{\alpha\omega}(y) = \left(\frac{\partial\tilde{\phi}_{\alpha\omega}}{\partial y}, i\alpha\tilde{\phi}_{\alpha\omega}\right) \tag{37}$$

is the adjoint velocity field, $\tilde{p}_{\alpha\omega}(y)$ is the corresponding adjoint pressure, and the complex vector

$$\underline{\tilde{S}}_{\alpha\omega} = \left(\tilde{p}_{\alpha\omega}\hat{y} + \frac{1}{R}\frac{\partial\tilde{u}_{\alpha\omega}}{\partial y}\hat{x}\right)_{y=0} \tag{38}$$

is the *adjoint stress.*

The effect of some other types of sources can also be deduced. For sources of vorticity, the term involving $\underline{q}(\underline{r};\omega)$ is rewritten as

$$\int_{x_1}^{x_2}\int_0^\infty \underline{q}(\underline{r};\omega).\nabla\times(\tilde{\psi}\hat{z})dydx = \int_{x_1}^{x_2}\int_0^\infty \Omega(\underline{r};\omega)\tilde{\phi}_{\alpha\omega}(y)e^{-i\alpha x}dydx \tag{39}$$

where $\Omega(\underline{r};\omega) = \hat{z}.(\nabla\times\underline{q}(\underline{r};\omega))$ is the vorticity source distribution which would appear, for example, on the right-hand side of the Orr-Sommerfeld equation. Vorticity sources in the flow are weighted by the adjoint stream function.

Rather than specify a velocity at the wall (i.e. at $y = 0$), suppose that the wall is in motion, oscillating about its mean position with a small velocity $\underline{v}_d(x;\omega)$. Linearizing this boundary condition, it follows that the boundary integral in (36) can be re-expressed as

$$\int_{x_1}^{x_2}\underline{v}_d(x;\omega).\underline{\tilde{S}}'_{\alpha\omega}e^{-i\alpha x}dx \tag{40}$$

where

$$\underline{\tilde{S}}'_{\alpha\omega} = \left(\left(\tilde{p}_{\alpha\omega} + \frac{i}{\omega R}\frac{dU}{dy}\frac{\partial \tilde{u}_{\alpha\omega}}{\partial y}\right)\hat{y} + \frac{1}{R}\frac{\partial \tilde{u}_{\alpha\omega}}{\partial y}\hat{x}\right)_{y=0} \tag{41}$$

we will call the ***modified adjoint stress.***

In each instance, the streamwise integration is weighted by $e^{-i\alpha x}$. Since, typically, $e^{i\alpha x}$ grows downstream, $e^{-i\alpha x}$ will grow *upstream.* There is no surprise here since sources further upstream will have a greater contribution to the far field disturbance amplitude; the response to such sources has convected further and hence has grown more.

The deductions made in sections 3.2 and 3.3 can now be reconfirmed. For the vibrating ribbon problem, the response to boundary motion (25) ($x_1 < 0, x_2 > 0$) predicted by (36) is

$$\int_{x_1}^{x_2} \delta(x)\hat{y}.\underline{\tilde{S}}_{\alpha\omega} e^{-i\alpha x} dx = \tilde{p}_{\alpha\omega}(0). \tag{42}$$

For excitation of a free shear layer by a vorticity source (28) with the y integration in (39) now extending from $-\infty$ to ∞, the amplitude of the response is

$$\int_{x_1}^{x_2}\int_{-\infty}^{\infty} \delta(x)\delta(y-y_0)\tilde{\phi}_{\alpha\omega}(y)e^{-i\alpha x} dy dx = \tilde{\phi}_{\alpha\omega}(y_0). \tag{43}$$

The amplitude of a particular spatial eigensolution generated by an upstream time-harmonic source distribution can be expressed as a weighted integral of the sources whether they are within the flow or upon a flow boundary. The weighting functions are simply different field quantities of the adjoint eigensolution corresponding to the mode being considered. The field quantity which is appropriate depends upon the nature of the source. The following table summarizes the cases considered in this section.

Source type		*Adjoint weighting factor*		
description	*symbol*	*symbol*	*description*	*integrand*
momentum	$\underline{q}(\underline{r};\omega)$	$\underline{\tilde{v}}_{\alpha\omega}(y)e^{-i\alpha x}$	velocity	$\underline{q}(\underline{r};\omega).\underline{\tilde{v}}_{\alpha\omega}(y)e^{-i\alpha x}$
mass	$\varphi(\underline{r};\omega)$	$\tilde{p}_{\alpha\omega}(y)e^{-i\alpha x}$	pressure	$\varphi(\underline{r};\omega)\tilde{p}_{\alpha\omega}(y)e^{-i\alpha x}$
vorticity	$\Omega(\underline{r};\omega)$	$\tilde{\phi}_{\alpha\omega}(y)e^{-i\alpha x}$	stream function	$\Omega(\underline{r};\omega)\tilde{\phi}_{\alpha\omega}(y)e^{-i\alpha x}$
velocity at boundary	$\underline{v}_b(x;\omega)$	$\underline{\tilde{S}}_{\alpha\omega}e^{-i\alpha x}$	adjoint stress	$\underline{v}_b(x;\omega).\underline{\tilde{S}}_{\alpha\omega}e^{-i\alpha x}$
velocity of boundary	$\underline{v}_d(x;\omega)$	$\underline{\tilde{S}}'_{\alpha\omega}e^{-i\alpha x}$	modified adjoint stress	$\underline{v}_d(x;\omega).\underline{\tilde{S}}'_{\alpha\omega}e^{-i\alpha x}$

For momentum, mass, and vorticity sources, the integration is made over the entire flow domain in which sources are present. The resulting value gives the amplitude of the mode far downstream. For boundary sources, the integration is made over the boundary.

4. Future plans

The next step in this work will be to obtain simple numerical solutions of the adjoint fields in flows such as the Blasius boundary layer. A map of the receptivity characteristics for these flows can then be found. This will supplement analytical studies of boundary layer receptivity (Goldstein (1983), Goldstein *et al.* (1983)).

The coupling of free stream disturbances to boundary layer motions as a consequence of surface roughness is an important receptivity path (Goldstein (1985)). This has not yet been considered in the present work, and efforts will be made to extend the analysis to handle this scenario.

In the area of control, a means of analyzing boundary layer control strategies will be pursued. Suppression of a global (temporal) instability, such as occurs in a cylinder wake, can be achieved by a small permanent alteration of the flow field. The corresponding spatial problem is more complex since the control forces in practice are localized in space (for example, a region of suction in a boundary layer (Saric and Nayfeh (1977)), with their effect being felt both upstream and downstream.

Although reduction of the spatial growth rate of instabilities may be important, consideration will also be given to the alteration in the receptivity characteristics as a consequence of the presence of the control system. This can be quantified by examining changes in the adjoint field as a result of the control.

The global instability problem for strongly non-parallel flow has already been handled successfully (Hill (1992)). After the control of spatial instabilities in parallel flows has been fully investigated here, it will remain to consider the spatial problem in non-parallel flows.

It would seem inevitable that a connection will be established with the work of Herbert and Bertolotti (1987) on the Parabolised Stability Equations. Their studies on the evolution of a disturbance amplitude in a slowly evolving non-parallel flow would appear to be intimately connected to the present work, though they do not consider the receptivity problem explicitly.

In the longer term, a study will be carried out of the crossflow instability on an infinite swept airfoil leading edge. This is a phenomenon of major technological importance, and it is hoped that a systematic means of analyzing control possibilities may be found. Providing the ability to analyze how secondary instabilities and turbulent flows respond to control forces also remains a long term goal.

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