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The 'ideal' Kolmogorov inertial range and constant

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The energy transfer statistics measured in numerically simulated flows are found to be nearly self-similar for wavenumbers in the inertial range. Using the measured self-similar form, we were able to deduce an 'ideal' energy transfer function and the corresponding energy flux rate. From this flux rate, we calculate the Kolmogorov constant to be 1.5, in excellent agreement with experiments.

1. Motivation and objectives

Last year, an entire volume of the *Proceedings of the Royal Society* (434 Compiled and edited by Hunt et al. 1991) was devoted to Kolmogorov's ideas about turbulence. Indeed, Kolmogorov's inertial range theory (Kolmogorov, 1941, Monin and Yaglom, 1975) has formed a foundation for turbulence research for the last fifty years even though the existence of an inertial range requires high Reynolds numbers (Re) normally encountered only in geophysical flows (Monin and Yaglom, 1975, Chapman, 1971).

It is essential to obtain a simulated flow field as close to the Kolmogorov inertial range as possible in order to obtain accurate measurements of the energy transfer process. In this report, the inertial range is represented by statistically stationary flow fields generated using a Fourier spectral code (Rogallo, 1981) in which the $k^{-5/3}$ spectrum is maintained explicitly. The method follows the spirit of Kraichnan's constrained decimation theory (Kraichnan, 1975) and is essentially that of She and Jackson (1992) who reported a simulation at 128^3 resolution. Basically, at each time step, the Fourier modes in each spherical shell are multiplied by the real constant that returns the shell energy to the Kolmogorov $k^{-5/3}$ spectrum. This method can be thought of as a constrained dynamical system. For an N^3 problem, one has placed $N/2$ constraints on the $2N^3$ degrees of freedom. The method is equivalent to the use of forcing at the small wavenumber (via a linear instability) and a spectral eddy viscosity at high wavenumber. To validate the method, we have repeated our analysis using the forced LES dataset of Chasnov (1991)at 128^3 resolution, which was generated using a traditional spectral eddy viscosity (Kraichnan, 1976), and we also performed simulations at 64^3, 128^3, and 256^3 to investigate the effect of mesh size. We analyzed several independent fields at each resolution and found no variation in the statistics. The results reported in this paper were measured in a stationary velocity field on a 256^3 mesh size after 3200 time steps of evolution. The energy spectrum for the inertial range LES is $k^{-5/3}$ over the entire spectral range of the simulation. She and Jackson (1992) found that the measured scaling exponents for flatness factors are in good agreement with experiment (Anselmet et al., 1984).

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As in all numerical simulations, our inertial-range dataset is restricted by the finite computational domain, and separating physics from numerics becomes an important concern. It is necessary to identify and eliminate the numerical artifacts in the measurements. This effort leads to the *construction* of an 'ideal' Kolmogorov inertial range and a determination of the Kolmogorov constant.

2. Self-similarity of the energy transfer in the inertial range

We found that the fractional contributions from interactions between relative scales to the energy flux are essentially independent of k as would be expected in a scale–similar inertial range (Zhou, 1992a,b). This strongly suggests that the transfer process is self-similar, but it is important to confirm this directly.

Kraichnan (1971) pointed out that similarity within a Kolmogorov $k^{-5/3}$ inertial range implies the scaling

$$T(k,p,q) = a^3 T(ak, ap, aq) \tag{1}$$

if all six wave-numbers are in the inertial range. If we take $a = q^{-1}$, (1) reduces to

$$T(k,p,q) = q^{-3} T(k/q, p/q, 1) = q^{-3} F(k/q, p/q), \tag{2}$$

and the number of dependent variables is reduced from three to two. In figure 1, we have plotted $T(k,p,q)$ against k/q for several representative values of p/q. While there is a good collapse of the curves for the various bands, a failure of self-similarity is observed for interactions involving bands near the spectral boundaries of the computation.

The transfer function

$$T(k,p) = \sum_q T(k,p,q) \tag{3}$$

gives the transfer of energy into k resulting from all interactions involving band p. Analogous to (1), the self-similar scaling law for $T(k,p)$ in the inertial range is

$$T(k,p) = a^2 T(ak, ap). \tag{4}$$

We can further reduce (4) to

$$T(k,p) = p^{-2} T(k/p, 1) = p^{-2} H(k/p). \tag{5}$$

This self-similar law is also well satisfied except for p near the computational boundaries, as shown in figure 2.

In both figures 1 and 2, self-similar profiles can be found by averaging over the collapsed curves, and such averaged $T(k,p)$ values have been marked in figure 2.

Note that the question of the locality of dominant interactions can be answered in terms of figure 2. When $s = \max(k,p,q)/\min(k,p,q)$ is large, the three wavenumbers in a triad effectively reduce to two scales. $T(k,p)$ provides a direct measurement of the locality since self-similarity further reduces the variables to one, implying an
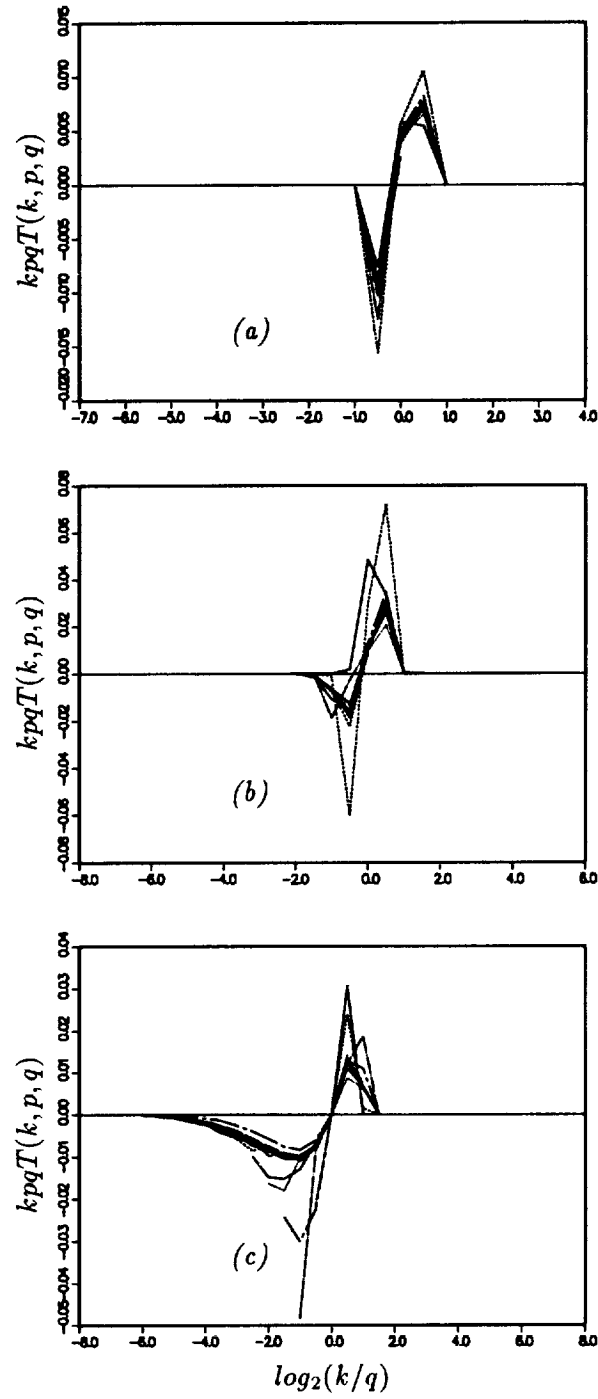
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FIGURE 1. Direct verification of self-similarity (1) of the transfer $T(k, p, q)$ in the inertial range. (a) $p/q = 1/8$; (b) $p/q = 1/4$; (c) $p/q = 1$.
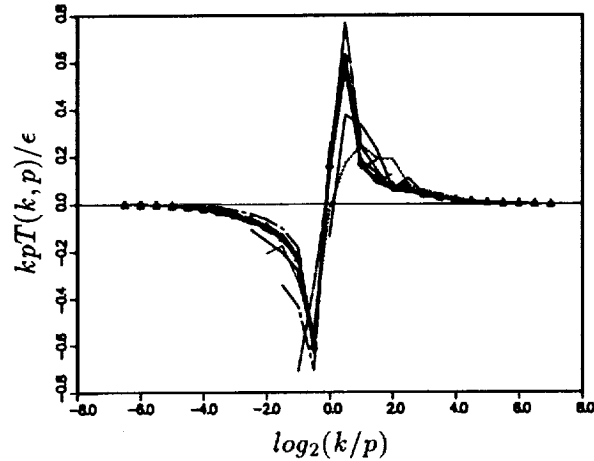
FIGURE 2. Self similarity of the transfer function $T(k,p)$ in an inertial range. The curves are for the various p bands of the inertial range LES. The points △ are the average values of $H(k/p)$ used to represent the "ideal" inertial range.
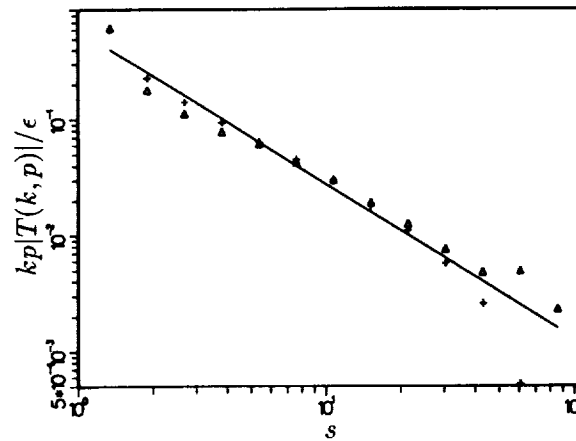

FIGURE 3. The self-similar transfer function $T(k,p)$: △ , $s = k/p$; +, $s = p/k$. The line indicates a $s^{-4/3}$ behavior. The scatter at large s is due to numerical error.

equivalence between s and the ratios k/p or p/k when they are large. While an interaction range of $s = 50$ as seen in figure 2 may seem rather "non-local", the basic question really is whether the interaction range is large enough to contain both the energetic and dissipation scales at large Reynolds number. The rapid $s^{-4/3}$ decay shown in figure 3 would seem to rule that out.

From the detailed balance, one expects that $T(k,p)$ is antisymmetric at large s, that is $H(s) = -H(1/s)$. Figure 3 shows that this is indeed the case. The deviation at very large s is due to numerical error.
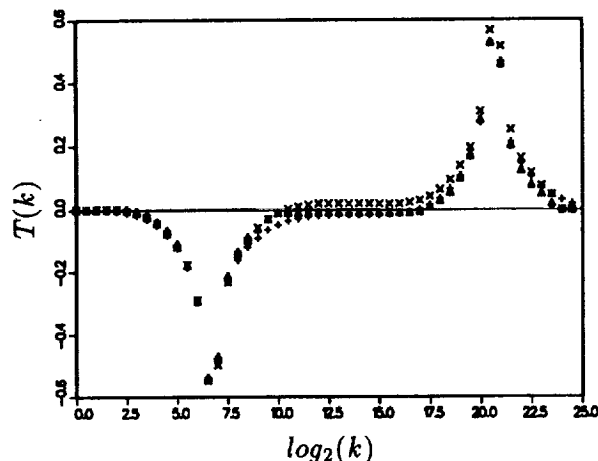
FIGURE 4. The 'ideal' self-similar transfer $T(k)$: $\triangle$, 64^3 mesh; +, 128^3 mesh; ×, 256^3 mesh.

3. The 'ideal' Kolmogorov energy transfer and inertial range

The failure of self-similarity near the computational boundaries is a numerical artifact of the forcing and eddy viscosity used in the LES. This suggests that the numerical artifacts can be eliminated, or at least reduced, by using the self-similar scaling to filter the raw data. Essentially, the data redundancy implied by the scaling law's reduction of three variables to two allows us to reduce the error associated with end-effects of the computational domain. To obtain the corrected data, we have simply removed the curves associated with bands near the boundaries that did not collapse and averaged the remaining ones. Such an operation reduces the data to a single curve that can be viewed as the 'ideal' one, that is, the one that would be obtained in an infinitely long inertial range. As a result, we are able to *construct* the 'ideal' energy transfer function $T(k)$ in an infinite inertial range by integrating the self-similar $T(k, p)$ over a finite range of p.

A suitable analogy for such an infinitely long inertial range is an infinitely long 'pipe' without leaks. To illustrate the interaction of scales, we 'cut' a finite section from this 'pipe' and view its inflow and outflow. The finite section of pipe corresponds to the finite range of the integral over p mentioned above. The "ideal" $T(k)$ constructed from simulations of size 64^3, 128^3, and 256^3 are shown in figure 4. The negative and positive peaks correspond to inflow and outflow. Since the flow is statistically steady and $\int dk T(k) = 0$, we have shifted the peaks so that the three mesh sizes overlap. Because the 'ideal' pipe does not leak, its length is not important. This is a direct visualization of the Kolmogorov energy transfer process in a finite section of the 'ideal' inertial range, and the 'ideal' inflow and outflow profiles are quite different from actual measured transfer spectra (figure 5). Indeed, the 'pipe' concept is suggested by the long range of scales in figure 5 in which the net transfer is very small.
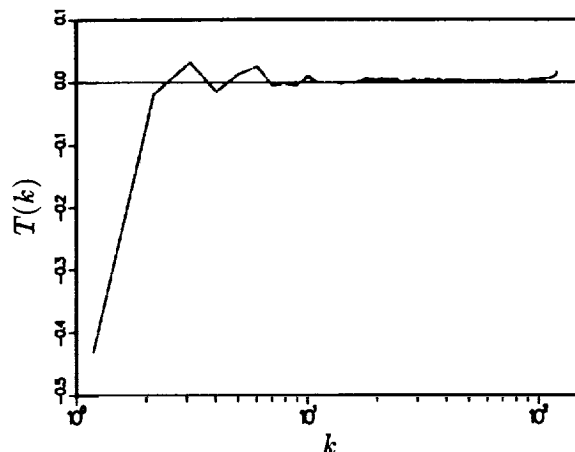
FIGURE 5.   Transfer spectrum of the simulations: inertial-range LES at one instant.

4. A determination of the Kolmogorov constant

Experiments at high Reynolds number give values of the Kolmogorov constant in the range of $C_K \sim 1.5$ (Monin and Yaglom, 1975), but values determined directly from spectra in numerical simulations are usually around 2. (Vincent and Meneguzzi,1991; Sanada, 1992; Chasnov, 1991).

For the inertial range LES data, the dissipation rate estimated from the maximum resolved energy flux is .45, giving a value of the Kolmogorov constant of 1.7 (Zhou, 1992b). (Recall that the energy spectrum was held constant at $E(k) = k^{-5/3}$ so that $C_K \epsilon^{2/3} = 1$).

We can also measure the energy flux as the integral of the inflow or outflow of the 'ideal' pipe (figure 4). This gives a flux value of about .64 and a corresponding Kolmogorov constant $C_K \sim 1.5$. This 'ideal' energy dissipation rate, evaluated using the self-similar law, has hopefully eliminated most of the computational artifacts resulting from the limited computational domain. She and Jackson (1992) estimated the Kolmogorov constant using the calculated eddy viscosity, which they showed to be self-similar, and also found $C_K \sim 1.5$.

5. Summary

The measured energy transfer is reasonably self-similar for wavenumbers in the inertial range. Artifacts of the finite computational domain, the LES models, can be identified and to some extent eliminated by constructing an 'ideal' energy transfer function. The energy flux, corrected for the loss due to the finite computation domain, was used to calculate the Kolmogorov constant 1.5, in excellent agreement with experiments (Monin and Yaglom, 1975).

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