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NONLINEAR DAMPING MODEL FOR FLEXIBLE STRUCTURES

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Nonlinear Damping Model

for Flexible Structures

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By

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ABSTRACT OF THE DISSERTATION

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This dissertation is on the study of nonlinear damping problem of flexible structures. Both passive and active damping, both finite dimensional and infinite dimensional models are studied.

In the first part of this dissertation, the spectral density and the correlation function of the following single DOF nonlinear damping model is investigated

$$\ddot{x} + 2\xi\omega_0\dot{x} + \gamma D(x,\dot{x}) + \omega_0^2 x = \sigma n(t)$$

where $\gamma > 0$ is a small parameter. A formula for the spectral density is established with $O(\gamma^2)$ accuracy based upon Fokker-Planck technique and perturbation. The spectral density depends upon certain first order statistics which could be obtained

if the stationary density is known. A method is proposed to find the approximate stationary density explicitly.

In the second part of this dissertation, the spectral density of the following multi-DOF nonlinear damping model is investigated

$$M\ddot{x} + D_0\dot{x} + \gamma D(x,\dot{x}) + Kx = \sigma n(t)$$

where $\gamma > 0$ is a small parameter.

In multi-DOF case, x and $\dot{x}$ are generally not uncorrelated in stationary state, even in linear case, which is one of the features of the multi-DOF model. A necessary and sufficient condition for uncorrelatedness is given for the linear model.

In the third part of this dissertation, energy type nonliner damping model in an infinite dimensional setting is studied. According to its geometry of the structures considered, the nonlinear damping models are divided into two types. The existence and uniqueness result of the nonlinear damping model is based on the work of A. Lunardi.

Then a Krylov-Bogoliubov type approximation is established for the nonlinear damping model in the case the linear damping part is neglected. In general, the generalization of Krylov-Bogoliubov approximation method, which applies only to single DOF model, to multi-DOF model has been a formidable task. The result presented here is based upon the specific form of nonlinearity - energy type damping. From its Krylov-Bogoliubov approximation, we can see that there is no exchange of energy between modes, i.e., internal resonance does not exist.

The notions of *Characteristic equation* and its *Root locus* are extended to actively damped distributed parameter systems. The root locus provides an insight of the nature of active damping. Sufficient conditions of strong stabilizability are provided, which are the weakest sufficient conditions obtained so far.

Chapter 1

Introduction

1.1 Background and Survey

A new generation of spacecrafts often contain very large flexible components such as truss structures, solar panels, dish antennas, radar arrays, space telescopes and space stations. In general, these structures are characterized by weak damping, and interconnection of rigid and flexible parts. The tasks of controlling the rotation, pointing with high accuracy in minimum time or with minimum energy consumption, and at the same time stabilizing the vibrations, pose very difficult control problems. This need is evident for various ongoing government programs such as space shuttle and space station. In 1984, NASA Langley Research Center and Dr. A. V. Balakrishnan initiated a design challenge for the SCOLE (Spacecraft Control Laboratory Experiment) problem[4], the objective of which includes the task of directing the line-of-sight of the shuttle/antenna configuration toward a fixed target, under condition of noisy data, limited control authority and random disturbances [4] - [13].

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In most applications, damping is important to the structural dynamics, and in many applications it is in fact critical. Two examples may suffice to illustrate the point: for spinning (or partly spinning) spacecraft, the level of energy dissipation in the structure determine whether an initial wobbling motion grows or decays (dynamical stability); and an automatically controlled flexible spacecraft may act either unmanageable or docile (control system stability) depending on the level of damping in the higher order vibration modes.

In addition, the advance of modern material science and technology has provided us with useful structural material which are generally light weight. Their applications in spacrcrafts, high-performance helicopters have sharply increased in the past decade. However, such viscoelastic materials have highly nonlinear characteristics that cause significant nonlinear response in the system. The questions of analysis, design and control appear more difficult.

Literature Survey

The following literature survey is made along the lines of nonlinear damping and linear damping. In the nonlinear damping case, literature is devided into two groups: (1) Finite dimensional model; (2) Infinite dimensional model. While in the linear damping case, literature on distributed parameter system is divided into three groups: (1) Strictly proportional and asymptotic proportional damping operator; (2)Boundary damping model; (3)Interior point damping model. In the end, the work on finite dimensional linear damping modelling is briefly reported .

Nonlinear Damping Model

- Finite Dimensional Model

    There has been large amount of literature on finite dimensional nonlinear damping model, among the authors, T. K. Caughey [25] - [27], S. H. Crandall

[34] - [36], P -T. Spanos [64] [65], V. V. Bolotin [21], T. S. Atalik and S. Utku [1], I. I. Orabi and G. Ahemadi [59]. Hysteretic type nonlinear damping has been studied by Y. K. Wen [70] [71]. An excellent survey is given by S. H. Crandall [33]. These works have primarily focused on the study of the response of nonlinear damping model to random excitations. Generally, three methods are used in the analysis of nonlinear damping systems under random excitations:

1. The Fokker-Planck approach;

2. The perturbation approach;

3. The equivalent (stochastic) linearization approach.

The main advantage of Fokker-Planck method over all the others is that, theoretically, exact solutions may be obtained when the excitations are Gaussian white noise. Unfortunately, its use is limited because of the severe restrictions that must be placed on the form of nonlinearities and on the spectral density matrix of the excitations. For a more detailed study, refer to [25].

If the dynamical system has weak nonlinearities, then the approximate random response may be obtained using the classical perturbation theory. First developed by Crandall [34], the approach has been later generalized to multi-degree of freedom systems by Tung [68].

Among the methods mentioned above, the equivalent linearization technique has the widest range of applicability. Basically, the method is the statistical extension of Krylov Bogoliubov approximation method [49]. Although the equivalent linearization method is widely used, it is incapable of displaying the nature of nonlinearity.

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- Infinite Dimensional Model

  To the best of my knowledge, research work on nonlinear damping in a distributed parameter system is very limited.

  A. V. Balakrishnan [13], for the first time, established energy nonlinear damping model for distributed parameter systems. The damping consists of asymptotically proportional linear damping term and energy nonlinear damping term.

  A. Lunardi [55] considered the transverse deflection of an extensible beam with hinged ends and the nonlinear damping term considered is nonlinear viscous damping. By reformulating the nonlinear damping model as a semilinear abstract parabolic initial value problem, the author studied the stability and the instability of all the stationary solutions and of small periodic orbits near stationary solutions for the various ranges of the associated parameter in the model.

  H. K Wang and G. Chen [69] considered a vibrating string with one end fixed and the other end is installed on a nonlinear damping device whose velocity-frictional force relationship as determined by material testing is given by

  $$T\frac{\partial y(0,t)}{\partial x} = f[\frac{\partial y(0,t)}{\partial t}]$$

  where $f(x)$ is a a multivalued function. The authors used the method of characteristics and nonlinear semigroup theory to study the effect of nonlinear boundary damping and analyzed the asymptotic behavior of the solutions of such systems. The ω-limit set of the dynamical system and the asymptotic rates of various solutions to the ω-limit set are determined.

Linear Damping (Distributed Parameter Systems)

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- Strictly proportional and asymptotic proportional damping

To formulate the internal passive damping, strictly proportional and asymptotic proportional damping operators have been reported in A. V. Balakrishnan [10] [11], as well as in G. Chen and D. Russell [28], S. Chen and R. Triggiani [32]. F. Huang [42] [43] studied the spectral property of the systems in the form

$$\ddot{x}(t) + B\dot{x}(t) + Ax(t) = 0$$

where B is a closed linear operator related in various ways to A^α with $1/2 \leq \alpha \leq 1$. The author obtained some fundamental results for the holomorphic property and the exponential stability of the semigroups associated with these systems.

Strictly proportional damping operator is essentially the square root of the stiffness operator A. In this case, the eigenvalues have the proportionality property

$$\frac{\Re(\lambda_n)}{\Im(\lambda_n)} = -\frac{\xi}{\sqrt{1 - \xi^2}}$$

The drawback of strictly proportional damping is that the damping operator contains nonlocal feature, which is unnatural if we consider that internal passive damping is due to the structure material itself. However, if strict proportionality is relaxed to asymptotic proportionality, i.e.

$$\lim_{n \to \infty} \frac{\Re(\lambda_n)}{\Im(\lambda_n)} = constant$$

then, the nonlocal feature can be avoided.

A. V. Balakrishnan, based upon his theory on the fractional power of closed linear operators [14], explicitly calculated the strictly proportional and asymptotic proportional damping operators for the beam bending model [11], in

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which one end of the beam is clamped and the other end has an end-body attached to it. In [10], the strictly proportional damping operator is given explicitly for beam torsion model.

- Boundary Damping

There has been considerable amount of literature on the distributed parameter systems with boundary damping (including boundary active control). G. Chen [29] [30]. J. Lagnese [50] [51], and R. Triggiani [52] [67], to name a few, have studied the energy decay of solutions of wave equation on bounded domain with damping only on the boundary. In general, in this kind of study, the concrete question to be answered is:

Under what conditions is it true that there is an *exponential decay rate* for $E(t)$, i.e.

$$E(t) \leq Ce^{-\omega t}E(0), \quad t \geq 0$$

for some positive ω? In the above, $E(t)$ stands for the total energy of the vibrating systems, which needs to be properly defined.

- Interior Point Damping

G. Chen, M. Coleman and H. H. West [31] and K. Liu [54] studied the energy decay rate of a coupled vibrating string with a point damper installed at the coupling point. In these studies, it is assumed that a damper applies damping at an interior point where two strings couple. Possible mechanical designs are also proposed in [31].

As far as linear damping problem in finite dimensional space, certain work has been done. Modern computer-based techniques (such as finite element method)

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enable the structural engineers to make highly precise calculations of the mass and stiffness matrices of elastic structures. These calculations, in turn, lead to quite accurate estimates of natural frequencies and mode shapes, particularly for the lower modes. Structural damping properties, on the other hand, tend to be much less accurately calculated; indeed, one usually simply guesses at modal damping factors. Limited work has been done in this area [2] [3] [40]. D. F. Golla and P. C. Hughes [40] developed a method of constructing linear damping matrix for viscoelastic structures in the framework of finite element method. They assume that certain material constants are available (ultimately by measurements) for each constituent material of the structure, just as known densities determine mass properties, and known elastic constants, such as Young's modulus, determine stiffness properties. These measured viscoelastic material constants permit a set of equations to be formulated for the dissipative properties over all parts of the structure. The method merges naturally with finite element method and is a natural extension of it.

1.2 Objectives and Contributions of the Dissertation

Experimental data has clearly indicated the nonlinear nature of the internal friction damping of large flexible space structures. Then, what is the frequency response of a nonlinearly damped structure? Until now , Monte Carlo simulation has been the only method of computing the frequency response, due to the lack of the parallel theory as in linear systems, in which the frequency response can simply be obtained from transfer functions. This dissertation provides an analytical method of computing the frequency response of single DOF oscillator with nonlinear damping. In

spite of its fundamental importance, the nature of internal damping has been little known. In the modeling aspect, this dissertation proposes an energy type nonlinear damping model and the corresponding stationary probability density with white noise input can be obtained explicitly. *Theorem 3* gives an interesting result, in terms of Krylov-Bogoliubov approximation, concerning the modeling and identification of nonlinear internal damping in flexible structures. This work also serves a contribution to the random vibration theory by providing a method of computing the first and the second order statistics (steady state probability density, correlation function and spectral density) of nonlinearly damped oscillators with white noise input.

A Krylov-Bogoliubov type approximation is established for systems having infinite number of DOF's and its error estimate is obtained. Comparisons are made between nonlinear damping (linear stiffness) models and nonlinear stiffness (linear damping) models, and between nonlinear damping (linear stiffness) models and linear models.

A group of sufficient conditions for strong stabilizability is provided for general distributed parameter oscillation system, taking the actuator saturation into consideration. These are the weakest sufficient conditions obtained so far and it is found that the nature of internal damping is not crucial in guaranteeing the strong stabilizability.

By extending the notions of characteristic equation and its root locus to our distributed parameter oscillation system, we studied the nature of active damping.

1.3   Organizations of the Dissertation

This dissertation is organized in the order of dimensions of the model:

1. Chapters 2 and 3 are on the single degree of freedom nonlinear damping models;

2. Chapters 4 and 5 are on the multi- but finite- degree of freedom nonlinear damping models;

3. Chapters 6 is on infinite dimensional energy type nonlinear passive damping models;

4. Chapter 7 is on the active damping of a uniform Euler-Bernoulli beam with one end clamped and the other end free, with a tip mass. The notions of *Characteristic equation* and *Root locus* are extended to distributed parameter systems.

5. Chapter 8 is a summary of conclusions and a list of some related open problems.

Chapter 2

The Spectral Density of Nonlinear Damping Model: Single DOF Case

2.1 Introduction

The problem of characterizing the damping mechanism in flexible structures has received renewed attention in recent years in connection with the need to stablize flexible flight structures such as antennas deployed in space. Experimental evidence suggests the need for nonlinear damping model and the need to consider the effect of random disturbances due to the uncertainties in system parameters and the environment. One of the most important subjects in nonlinear random vibration is to obtain the second order statistics, i.e., correlation function and spectral density of the stationary response, because they provide average amplitude and frequency

information about the sample histories. Unfortunately, up to now, the only practical method available is Monte Carlo simulation and there is no analytical technique for the second order statistics of nonlinear systems [33]. This paper presents an analytical technique for computing correlation function and spectral density of the stationary response of nonlinear damping model subject to white noise excitation.

The basic nonlinear damping model we consider is

$$\ddot{x} + 2\xi\omega_0\dot{x} + \gamma D(x,\dot{x}) + \omega_0^2 x = \sigma n(t) \tag{1}$$

where $\gamma > 0$ is a small constant because the damping in flexible space structures, whatever its nature, is small.

The corresponding Fokker-Planck equation is given by

$$
\begin{aligned}
\frac{\partial p}{\partial t} &= -y\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}[(\omega_0^2 x + 2\xi\omega_0 y + \gamma D(x,y))p] + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial y^2} \\
&= \mathbf{L}_0 p + \gamma\frac{\partial}{\partial y}[D(x,y)p] \\
&\quad \lim_{t \to 0} p(t,x,y|x_0,y_0) = \delta(x - x_0)\delta(y - y_0)
\end{aligned}
\tag{2}
$$

<u>Notations</u>

$z(t) \overset{\text{def}}{=} D(x(t),y(t))$;

$p_s(x,y)$: stationary density of $(x(t),y(t))$, i.e., the invariant measure;

$m_{i,j} \overset{\text{def}}{=} \iint_{R^2} x^i y^j p_s(x,y)dxdy, \; i,j = 0,1,2,\cdots$;

$p(t,x,y|x_0,y_0)$: the fundamental solution of (2);

$p_0(t,x,y|x_0,y_0)$: the fundamental solution of (2) with $\gamma = 0$;

$\mathbf{T}(t) \overset{\text{def}}{=} exp(\mathbf{L}_0 t)$;

$q(t,s,x,y|x_0,y_0) \overset{\text{def}}{=} \iint_{R^2} D(u,v)p_0(t - s,x,y|u,v)p(s,u,v|x_0,y_0)dudv$.

It is well-known that $p_0(t, x, y | x_0, y_0)$ is a two-dimensional Gaussian density function. Its mean vector and covariance matrix can be found by straightforward calculation, as

$$
\begin{pmatrix} m_x(t) \\ m_y(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}
$$

$$
= \frac{e^{-\xi\omega_0 t}}{\omega_n} \begin{pmatrix} \omega_0 \cos(\omega_n t - \theta) & \sin \omega_n t \\ -\omega_0^2 \sin \omega_n t & \omega_0 \cos(\omega_n t + \theta) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{3}
$$

$$
\Sigma(t) \overset{\text{def}}{=} \begin{pmatrix} \sigma_x^2(t) & \sigma_{xy}(t) \\ \sigma_{xy}(t) & \sigma_y^2(t) \end{pmatrix}
$$

$$
= \frac{\sigma^2}{4\xi\omega_0} \begin{pmatrix} \frac{1}{\omega_0^2} & 0 \\ 0 & 1 \end{pmatrix}
$$

$$
+ \frac{\sigma^2}{4\omega_n^2} e^{-2\xi\omega_0 t} \begin{pmatrix} -\frac{1}{\xi\omega_0}[\xi \sin(2\omega_n t - \theta) + 1] & 1 - \cos 2\omega_n t \\ 1 - \cos 2\omega_n t & \frac{\omega_0}{\xi}[\xi \sin(2\omega_n t + \theta) - 1] \end{pmatrix} \tag{4}
$$

where

$$
\omega_n = \omega_0 \sqrt{1 - \xi^2}
$$

$$
\theta = \tan^{-1} \frac{\xi}{\sqrt{1 - \xi^2}}
$$

Later on, we will need the notation

$$
\Sigma_0 \overset{\text{def}}{=} \lim_{t \to \infty} \Sigma(t) = \frac{\sigma^2}{4\xi\omega_0} \begin{pmatrix} \frac{1}{\omega_0^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{5}
$$

and without ambiguity we will often denote $\Sigma(t)$ by Σ.

Assumptions on $D(x, y)$

(A1) $D(x, y)$ is differentiable with respect to y;

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(A2) $\exists K > 0, k > 0$ such that $|D(x,y)| \leq K[1 + (x^2 + y^2)^k]$ for $(x,y) \in \mathbb{R}^2$;

(A3) $m_{i,j}$ are finite for all nonnegative integers i, j.

Of course, to satisfy the energy nonincrease requirement, we also need

$$2\xi\omega_0 y^2 + D(x,y)y \geq 0 \quad (x,y) \in \mathbb{R}^2$$

2.2 Results

2.2.1 An equation of spectral density

Lemma 1 *Under assumption (A2) and (A3), it holds*

$$\lim_{x^2+y^2 \to \infty} q(t,s,x,y|x_0,y_0) = 0 \quad \forall \, 0 \leq s \leq t, \; (x_0,y_0) \in \mathbb{R}^2 \tag{6}$$

provided

$$\lim_{x^2+y^2 \to \infty} p(t,x,y|x_0,y_0) = 0 \quad \forall t > 0, \; (x_0,y_0) \in \mathbb{R}^2 \tag{7}$$

Proof: First, by Schwarz inequality, we have

$$
\begin{aligned}
|q(t,s,x,y|x_0,y_0)| &\leq \iint_{\mathbb{R}^2} |D(u,v)| p_0(t-s,x,y|u,v) p(s,u,v|x_0,y_0) du dv \\
&\leq [\iint_{\mathbb{R}^2} |D(u,v)|^2 p_0(t-s,x,y|u,v) p(s,u,v|x_0,y_0) du dv]^{1/2} \\
&\quad \times [\iint_{\mathbb{R}^2} p_0(t-s,x,y|u,v) p(s,u,v|x_0,y_0) du dv]^{1/2}
\end{aligned}
$$

The second sqare root term goes to 0 as $x^2 + y^2 \longrightarrow \infty$ by the assumption (7). Therefore it is sufficient to show that the first sqare root term is bounded for all $(x,y) \in \mathbb{R}^2$.

In fact, we have

$$
\begin{aligned}
&\iint_{\mathbb{R}^2} |D(u,v)|^2 p_0(t-s,x,y|u,v) p(s,u,v|x_0,y_0) du dv \\
&\leq \frac{1}{2\pi\sqrt{det\Sigma(t-s)}} \iint_{\mathbb{R}^2} |D(u,v)|^2 p(s,u,v|x_0,y_0) du dv
\end{aligned}
\tag{8}
$$

In addition, by assumptions (A2) and (A3), we have

$$\iint_{I\!R^2} \iint_{I\!R^2} |D(x,y)|^2 p(t,x,y|x_0,y_0) p_s(x_0,y_0) dx_0 dy_0 dx dy$$

$$= \iint_{I\!R^2} |D(x,y)|^2 p_s(x,y) dx dy$$

$$\leq \iint_{I\!R^2} K^2 [1 + (x^2 + y^2)^k]^2 p_s(x,y) dx dy < \infty$$

By *Tonelli*'s lemma, we know that

$$|D(x,y)|^2 p(t,x,y|x_0,y_0) p_s(x_0,y_0) \in L^1(I\!R^2 \otimes I\!R^2)$$

Then by *Fubini*'s theorem, we know that

$$\iint_{I\!R^2} |D(x,y)|^2 p(t,x,y|\cdot,\cdot) dx dy p_s(\cdot,\cdot) \in L^1(I\!R^2)$$

which implies $|D(\cdot,\cdot)|^2 p(t,\cdot,\cdot|x_0,y_0) \in L^1(I\!R^2)$. Therefore, by (8), the first sqare root term is bounded for all $(x,y) \in I\!R^2$. □

Theorem 1 *Under the assumptions (A1)-(A3), the following equations hold:*

$$\Phi_{xx}(\omega) = \frac{4\xi \omega_0^3 m_{2,0}}{(\omega^2 - \omega_0^2)^2 + 4\xi^2 \omega_0^2 \omega^2} + 2\gamma \Re[\frac{\Psi_{xz}(\omega)}{\omega^2 - \omega_0^2 + i2\xi \omega_0 \omega}] \qquad (9)$$

$$\Phi_{yy}(\omega) = \frac{4\xi \omega_0 m_{0,2} \omega^2}{(\omega^2 - \omega_0^2)^2 + 4\xi^2 \omega_0^2 \omega^2} + 2\gamma \omega \Im[\frac{\Psi_{yz}(\omega)}{\omega^2 - \omega_0^2 + i2\xi \omega_0 \omega}] \qquad (10)$$

where

$$\Psi_{xz}(\omega) = \int_0^\infty R_{xz}(\tau) e^{i\omega\tau} d\tau$$

$$\Psi_{yz}(\omega) = \int_0^\infty R_{yz}(\tau) e^{i\omega\tau} d\tau$$

and $\Re$, $\Im$ denote real, imaginary parts.

Proof:(2) is equivalent to the following integral equation

$$p(t,x,y|x_0,y_0) = p_0(t,x,y|x_0,y_0)$$

$$+\gamma \int_0^t T(t-s) \frac{\partial}{\partial y} [D(x,y) p(s,x,y|x_0,y_0)] ds$$

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or more explicitly,

$$p(t,x,y|x_0,y_0) = p_0(t,x,y|x_0,y_0)$$

$$+ \gamma \int_0^t \iint_{I\!\!R^2} p_0(t-s,x,y|u,v) \frac{\partial}{\partial v}[D(u,v)p(s,u,v|x_0,y_0)] du\,dv\,ds \quad (11)$$

Performing integration by part in the second term and noticing the relation

$$\frac{\partial p_0}{\partial v}(t,x,y|u,v) = -\frac{e^{-\xi\omega_0 t}}{\omega_n} \sin\omega_n t \frac{\partial p_0}{\partial x}(t,x,y|u,v)$$

$$-\frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}} \cos(\omega_n t + \theta)\frac{\partial p_0}{\partial y}(t,x,y|u,v)$$

we obtain the following

$$\gamma \int_0^t \iint_{I\!\!R^2} p_0(t-s,x,y|u,v)\frac{\partial}{\partial v}[D(u,v)p(s,u,v|x_0,y_0)] du\,dv\,ds$$

$$= -\gamma \int_0^t \iint_{I\!\!R^2} \frac{\partial p_0}{\partial v}(t-s,x,y|u,v)D(u,v)p(s,u,v|x_0,y_0) du\,dv\,ds$$

$$= \gamma \frac{\partial}{\partial x} \int_0^t \frac{e^{-\xi\omega_0(t-s)}}{\omega_n} \sin\omega_n(t-s)q(t,s,x,y|x_0,y_0) ds$$

$$+ \gamma \frac{\partial}{\partial y} \int_0^t \frac{e^{-\xi\omega_0(t-s)}}{\sqrt{1-\xi^2}} \cos[\omega_n(t-s)+\theta]q(t,s,x,y|x_0,y_0) ds$$

Therefore, we obtain the following integro-differential equation for $p(t,x,y|x_0,y_0)$:

$$p(t,x,y|x_0,y_0) = p_0(t,x,y|x_0,y_0)$$

$$+\gamma\frac{\partial}{\partial x}\int_0^t \frac{e^{-\xi\omega_0(t-s)}}{\omega_n}\sin\omega_n(t-s)q(t,s,x,y|x_0,y_0)ds$$

$$+\gamma\frac{\partial}{\partial y}\int_0^t \frac{e^{-\xi\omega_0(t-s)}}{\sqrt{1-\xi^2}}\cos[\omega_n(t-s)+\theta]q(t,s,x,y|x_0,y_0)ds \quad (12)$$

Based upon (A2) and hence *Lemma 1*, we find that the marginal transition probability density $p(t,x|x_0,y_0)$ satisfies the following equation, by integrating (12) with respect to y over $I\!\!R^1$,

$$p(t,x|x_0,y_0) = p_0(t,x|x_0,y_0) + \gamma\frac{\partial}{\partial x}\int_0^t \frac{e^{-\xi\omega_0(t-s)}}{\omega_n}\sin\omega_n(t-s)$$

$$\times \iint_{I\!\!R^2} p_0(t-s,x|u,v)D(u,v)p(s,u,v|x_0,y_0)du\,dv\,ds \quad (13)$$

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After integration, the third term vanishes by *Lemma 1*. And in (13), $p_0(t, x | x_0, y_0)$ is a one-dimensional Gaussian density with mean

$$\frac{e^{-\xi \omega_0 t}}{\omega_n} [\omega_0 x_0 \cos(\omega_n t - \theta) + y_0 \sin \omega_n t]$$

and variance given by

$$\frac{\sigma^2}{4\xi \omega_0^3} [1 - \frac{e^{-2\xi \omega_0 t}}{1 - \xi^2} (\xi \sin(2\omega_n t - \theta) + 1)]$$

Multiplying (13) by $x_0 x p_s(x_0, y_0)$ and integrating with respect to (x, x_0, y_0) over $I\!R^3$, we obtain, for $t > 0$,

$$
\begin{aligned}
R_{xx}(t) &= \iint_{I\!R^2} x_0 \int_{-\infty}^{\infty} x p_0(t, x | x_0, y_0) dx p_s(x_0, y_0) dx_0 dy_0 \\
&\quad - \gamma \int_0^t \frac{e^{-\xi \omega_0 (t-s)}}{\omega_n} \sin \omega_n(t - s) \iint_{I\!R^2} \iint_{I\!R^2} \int_{-\infty}^{\infty} p_0(t - s, x | u, v) dx \\
&\quad \times D(u, v) p(s, u, v | x_0, y_0) x_0 p_s(x_0, y_0) du dv dx_0 dy_0 ds \\
&= \iint_{I\!R^2} x_0 \frac{e^{-\xi \omega_0 t}}{\omega_n} [\omega_0 x_0 \cos(\omega_n t - \theta) + y_0 \sin \omega_n t] p_s(x_0, y_0) dx_0 dy_0 \\
&\quad - \gamma \int_0^t \frac{e^{-\xi \omega_0 (t-s)}}{\omega_n} \sin \omega_n(t - s) \\
&\quad \times \iint_{I\!R^2} \iint_{I\!R^2} x_0 D(x, y) p(s, x, y | x_0, y_0) p_s(x_0, y_0) dx_0 dy_0 dx dy ds \\
&= m_{2,0} \frac{e^{-\xi \omega_0 t}}{\sqrt{1 - \xi^2}} \cos(\omega_n t - \theta) \\
&\quad - \gamma \int_0^t \frac{e^{-\xi \omega_0 (t-s)}}{\omega_n} \sin \omega_n(t - s) R_{xx}(s) ds
\end{aligned}
\tag{14}
$$

In the above, we used the fact that

$$m_{1,1} = \iint_{I\!R^2} x_0 y_0 p_s(x_0, y_0) dx_0 dy_0 = 0$$

which is generally true in nonlinear mechanics due to symmetry. If $m_{1,1} \neq 0$, we can get similar result with similar proof, and the only difference is an extra term with $m_{1,1}$ as coefficient on the right of (9) and (10).

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Since $R_{xx}(\tau)$ is even, hence

$$\int_{-\infty}^{\infty} R_{xx}(\tau)e^{i\omega\tau}d\tau = 2\int_0^{\infty} R_{xx}(\tau)\cos\omega\tau d\tau \qquad (15)$$

And in general, we have the following relation

$$\int_0^{\infty}\int_0^t f(t-s)g(s)ds\cos\omega t dt$$
$$= \int_0^{\infty} f(t)\cos\omega t dt \int_0^{\infty} g(t)\cos\omega t dt - \int_0^{\infty} f(t)\sin\omega t dt \int_0^{\infty} g(t)\sin\omega t dt \quad (16)$$

if each of the integrals in (16) exists. Also, from the identity

$$\int_0^{\infty} \frac{e^{-\xi\omega_0 t}}{\omega_n}\sin\omega_n t e^{i\omega t}dt = \frac{1}{\omega_0^2 - \omega^2 - i2\xi\omega\omega_0} \qquad (17)$$

one can easily obtain two identities corresponding to the real and imaginary parts of both sides.

Next, performing Fourier transform on both sides of (14) using (15), (16) and (17), one obtains

$$\begin{aligned}\Phi_{xx}(\omega) &= \frac{4\xi\omega_0^3 m_{2,0}}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2}\\ &\quad + 2\gamma\frac{(\omega^2 - \omega_0^2)\int_0^{\infty} R_{xz}(t)\cos\omega t dt + 2\xi\omega_0\omega\int_0^{\infty} R_{xz}(t)\sin\omega t dt}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2}\\ &= \frac{4\xi\omega_0^3 m_{2,0}}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2} + 2\gamma\frac{\Re[(\omega^2 - \omega_0^2 - i2\xi\omega_0\omega)\Psi_{xz}(\omega)]}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2}\\ &= \frac{4\xi\omega_0^3 m_{2,0}}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2} + 2\gamma\Re\left[\frac{\Psi_{xz}(\omega)}{\omega^2 - \omega_0^2 + i2\xi\omega_0\omega}\right]\end{aligned}$$

For the second identity (10), one can similarly first show

$$\begin{aligned}R_{yy}(t) &= m_{0,2}\frac{e^{-\xi\omega_0 t}}{\sqrt{1-\xi^2}}\cos(\omega_n t + \theta)\\ &\quad -\gamma\int_0^t \frac{e^{-\xi\omega_0(t-s)}}{\sqrt{1-\xi^2}}\cos[\omega_n(t-s) + \theta]R_{yz}(s)ds \qquad (18)\end{aligned}$$

from which (10) follows. $\qquad\qquad\square$

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In linear case, i.e. $\gamma = 0$ in (1), it is well known that the spectral densities of x and $\dot{x}$ have the following relation

$$\omega^2 \Phi_{xx}(\omega) = \Phi_{yy}(\omega)$$

The natural question to ask is whether we have similar relation for nonlinear damping model (1). The answer to this question is provided in the following

Corollary 1 *Under the assumptions (A1)-(A3), it holds for (1)*

$$\omega^2 \Phi_{xx}(\omega) = \Phi_{yy}(\omega) \tag{19}$$

$$\Im\left[\frac{\Psi_{yz}(\omega)}{\omega^2 - \omega_0^2 + i2\xi\omega_0\omega}\right] = \omega\Re\left[\frac{\Psi_{xz}(\omega)}{\omega^2 - \omega_0^2 + i2\xi\omega_0\omega}\right] \tag{20}$$

$$\omega_0^2 m_{2,0} = m_{0,2} \tag{21}$$

Proof: (19) follows from

$$\begin{aligned}
R_{yy}(\tau) &= \frac{d}{d\tau} E y(t) x(t + \tau) \\
&= -\frac{d^2}{d\tau^2} E x(t - \tau) x(t) \\
&= -\frac{d^2}{d\tau^2} R_{xx}(\tau)
\end{aligned}$$

For (20), first one has

$$\begin{aligned}
\frac{d}{d\tau} R_{xz}(\tau) &= -E y(t - \tau) z(t) \\
&= -R_{yz}(\tau)
\end{aligned}$$

Then integration by part gives

$$\begin{aligned}
\Psi_{xz}(\omega) &= -\frac{1}{i\omega} \int_0^\infty \frac{dR_{xz}(\tau)}{d\tau} e^{i\omega\tau} d\tau \\
&= \frac{1}{i\omega} \int_0^\infty R_{yz}(\tau) e^{i\omega\tau} d\tau \\
&= \frac{1}{i\omega} \Psi_{yz}(\omega)
\end{aligned} \tag{22}$$

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in which we have assumed that

$$R_{xz}(0) = \iint_{I\!\!R^2} x D(x,y) p_s(x,y) dx dy = 0$$

$$\lim_{\tau \to \infty} R_{xz}(\tau) = 0$$

Therefore, from (22), we have

$$\Im[\frac{\Psi_{yz}(\omega)}{\omega^2 - \omega_0^2 + i2\xi\omega_0\omega}] = \Im[\frac{i\omega\Psi_{xz}(\omega)}{\omega^2 - \omega_0^2 + i2\xi\omega_0\omega}]$$

$$= \omega\Re[\frac{\Psi_{xz}(\omega)}{\omega^2 - \omega_0^2 + i2\xi\omega_0\omega}]$$

from which (10) becomes

$$\Phi_{yy}(\omega) = \frac{4\xi\omega_0 m_{0,2}\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2} + 2\gamma\omega^2\Re[\frac{\Psi_{xz}(\omega)}{\omega^2 - \omega_0^2 + i2\xi\omega_0\omega}]$$

$$= \frac{4\xi\omega_0 m_{0,2}\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2} + \omega^2[\Phi_{xx}(\omega) - \frac{4\xi\omega_0^3 m_{2,0}}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2}]$$

$$= \omega^2\Phi_{xx}(\omega) - \frac{4\xi\omega_0(\omega_0^2 m_{2,0} - m_{0,2})\omega^2}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2} \tag{23}$$

Comparing (23) with (19), one can immediately obtains (21). □

2.2.2   The spectral density

In this subsection, we establish the perturbation formula of the spectral density of
(1) with $O(\gamma^2)$ accuracy. First, we need two lemmas.

Lemma 2 *The following matrix relations hold*

$$\Sigma(t) = \Sigma_0 - e^{At}\Sigma_0 e^{A^T t} \tag{24}$$

$$\Sigma^{-1} - \Sigma^{-1}e^{At}(\Sigma_0^{-1} + e^{A^T t}\Sigma^{-1}e^{At})^{-1}e^{A^T t}\Sigma^{-1} = \Sigma_0^{-1} \tag{25}$$

$$(\Sigma_0^{-1} + e^{A^T t}\Sigma^{-1}e^{At})^{-1}e^{A^T t} = \Sigma_0 e^{A^T t}\Sigma_0^{-1}\Sigma \tag{26}$$

Proof: Define

$$\tilde{\Sigma}(t) \overset{\text{def}}{=} \Sigma_0 - e^{At}\Sigma_0 e^{A^T t}$$

To show (24) is equivalent to showing $\Sigma(t) \equiv \tilde{\Sigma}(t)$.

In fact, since

$$\Sigma(t) = \int_0^t e^{A(t-s)} \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} e^{A^T(t-s)} ds$$

we know that $\Sigma(t)$ satisfies the following linear differential equation

$$\frac{d}{dt}\Sigma(t) = A\Sigma(t) + \Sigma(t)A^T + \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} \tag{27}$$

And, based upon the simple fact

$$-A\Sigma_0 - \Sigma_0 A^T = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

we also have

$$\begin{aligned}
\frac{d}{dt}\tilde{\Sigma}(t) &= A(\tilde{\Sigma}(t) - \Sigma_0) + (\tilde{\Sigma}(t) - \Sigma_0)A^T \\
&= A\tilde{\Sigma}(t) + \tilde{\Sigma}(t)A^T + \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix}
\end{aligned} \tag{28}$$

Obviously, $\Sigma(t)$ and $\tilde{\Sigma}(t)$ have the same initial condition

$$\Sigma(0) = \tilde{\Sigma}(0) = 0$$

Therefore, the uniqueness of the solutions of linear differential equations implies $\Sigma(t) \equiv \tilde{\Sigma}(t)$.

By the matrix inversion formula

$$(M_1 + M_2 M_3 M_4)^{-1} = M_1^{-1} - M_1^{-1} M_2 (M_3^{-1} + M_4 M_1^{-1} M_2)^{-1} M_4 M_1^{-1} \tag{29}$$

we have, noticing (24),

$$\Sigma^{-1} - \Sigma^{-1}e^{At}(\Sigma_0^{-1} + e^{A^Tt}\Sigma^{-1}e^{At})^{-1}e^{A^Tt}\Sigma^{-1}$$

$$= (\Sigma + e^{At}\Sigma_0 e^{A^Tt})^{-1}$$

$$= \Sigma_0^{-1}$$

For the proof of (26), we again use (29) and (24) to obtain

$$(\Sigma_0^{-1} + e^{A^Tt}\Sigma^{-1}e^{At})^{-1}e^{A^Tt}$$

$$= [\Sigma_0 - \Sigma_0 e^{A^Tt}(\Sigma + e^{At}\Sigma_0 e^{A^Tt})^{-1}e^{At}\Sigma_0]e^{A^Tt}$$

$$= \Sigma_0 e^{A^Tt}(I - \Sigma_0^{-1}e^{At}\Sigma_0 e^{A^Tt})$$

$$= \Sigma_0 e^{A^Tt}[I - \Sigma_0^{-1}(\Sigma_0 - \Sigma)]$$

$$= \Sigma_0 e^{A^Tt}\Sigma_0^{-1}\Sigma$$

$\square$

Lemma 3 *For linear damping model*

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = \sigma n(t)$$

we have

$$E\left(\begin{array}{c} x(t) \\ y(t) \end{array}\right) D(x(t+\tau), y(t+\tau)) = \Sigma_0 e^{A^T\tau}\Sigma_0^{-1}\left(\begin{array}{c} \psi_x \\ \psi_y \end{array}\right) \qquad (30)$$

where

$$\left(\begin{array}{c} \psi_x \\ \psi_y \end{array}\right) = \iint_{\mathbf{R}^2} D(x,y)\left(\begin{array}{c} x \\ y \end{array}\right)\frac{2\xi\omega_0^2}{\pi\sigma^2}\exp[-\frac{2\xi\omega_0}{\sigma^2}(\omega_0^2 x^2 + y^2)]dxdy \qquad (31)$$

Proof: Let us use the following notations:

$$X \stackrel{\text{def}}{=} \left(\begin{array}{c} x \\ y \end{array}\right) \qquad X_0 \stackrel{\text{def}}{=} \left(\begin{array}{c} x_0 \\ y_0 \end{array}\right)$$

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By (25) and (26), we have

$$(X - e^{At}X_0)^T\Sigma^{-1}(X - e^{At}X_0) + X_0^T\Sigma_0^{-1}X_0$$

$$= X^T[\Sigma^{-1} - \Sigma^{-1}e^{At}(\Sigma_0^{-1} + e^{A^Tt}\Sigma^{-1}e^{At})^{-1}e^{A^Tt}\Sigma^{-1}]X$$

$$+[X_0 - (e^{A^Tt}\Sigma^{-1}e^{At} + \Sigma_0^{-1})^{-1}e^{A^Tt}\Sigma^{-1}X]^T(e^{A^Tt}\Sigma^{-1}e^{At} + \Sigma_0^{-1})$$

$$\times[X_0 - (e^{A^Tt}\Sigma^{-1}e^{At} + \Sigma_0^{-1})^{-1}e^{A^Tt}\Sigma^{-1}X]$$

$$= X^T\Sigma_0^{-1}X$$

$$+(X_0 - \Sigma_0 e^{A^Tt}\Sigma_0^{-1}X)^T(e^{A^Tt}\Sigma^{-1}e^{At} + \Sigma_0^{-1})(X_0 - \Sigma_0 e^{A^Tt}\Sigma_0^{-1}X)$$

Then

$$E\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} D(x(t+\tau), y(t+\tau))$$

$$= \iint_{\mathbf{R}^2}\iint_{\mathbf{R}^2} X_0 D(x,y)\frac{1}{2\pi\sqrt{|\Sigma(\tau)|}}\exp[-1/2(X - e^{A\tau}X_0)^T\Sigma^{-1}(\tau)(X - e^{A\tau}X_0)]$$

$$\times\frac{1}{2\pi\sqrt{|\Sigma_0|}}\exp(-1/2X_0^T\Sigma_0^{-1}X_0)d|X|d|X_0|$$

$$= \iint_{\mathbf{R}^2} D(x,y)\frac{1}{2\pi\sqrt{|\Sigma_0|}}\exp(-1/2X^T\Sigma_0^{-1}X)\iint_{\mathbf{R}^2} X_0$$

$$\times\frac{1}{2\pi\sqrt{|\Sigma(\tau)|}}\exp[-1/2(X_0 - \Sigma_0 e^{A^T\tau}\Sigma_0^{-1}X)^T(\Sigma_0^{-1} + e^{A^T\tau}\Sigma^{-1}e^{A\tau})$$

$$\times(X_0 - \Sigma_0 e^{A^T\tau}\Sigma_0^{-1}X)]d|X_0|d|X|$$

$$= \iint_{\mathbf{R}^2} D(x,y)\frac{1}{2\pi\sqrt{|\Sigma_0|}}\exp(-1/2X^T\Sigma_0^{-1}X)$$

$$\times\frac{\Sigma_0 e^{A^T\tau}\Sigma_0^{-1}X}{\sqrt{|\Sigma(\tau)||\Sigma_0^{-1} + e^{A^T\tau}\Sigma^{-1}(\tau)e^{A\tau}|}}dxdy$$

$$= \Sigma_0 e^{A^T\tau}\Sigma_0^{-1}\iint_{\mathbf{R}^2} X D(x,y)\frac{1}{2\pi\sqrt{|\Sigma_0|}}\exp(-1/2X^T\Sigma_0^{-1}X)dxdy$$

$$= \Sigma_0 e^{A^T \tau} \Sigma_0^{-1} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

In the above, we used the fact that

$$|\Sigma(t)||\Sigma_0^{-1} + e^{A^T t}\Sigma^{-1}(t)e^{At}| = 1$$

which could be easily verified by using (26). □

Theorem 2 *Under assumptions (A1)-(A3), the spectral density of (1) is given by*

$$\Phi_{xx}(\omega) = \frac{4\xi\omega_0^3 m_{2,0}}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2}$$

$$+ 2\gamma \frac{2\xi\omega_0\psi_x(\omega^4 - \omega_0^4 + 4\xi^2\omega_0^2\omega^2) + \psi_y[(\omega^2 - \omega_0^2)^2 - 4\xi^2\omega_0^2\omega^2]}{[(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2]^2} + O(\gamma^2) \quad (32)$$

Proof: If we write

$$\begin{pmatrix} R_{xz}(\tau) \\ R_{yz}(\tau) \end{pmatrix} = \begin{pmatrix} R_{xz}^{(0)}(\tau) \\ R_{yz}^{(0)}(\tau) \end{pmatrix} + \gamma \begin{pmatrix} R_{xz}^{(1)}(\tau) \\ R_{yz}^{(1)}(\tau) \end{pmatrix} + \cdots \quad (33)$$

then $(R_{xz}^{(0)}(\tau), R_{yz}^{(0)}(\tau))$ is given by (30), i.e.,

$$\begin{pmatrix} R_{xz}^{(0)}(\tau) \\ R_{yz}^{(0)}(\tau) \end{pmatrix} = \Sigma_0 e^{A^T \tau} \Sigma_0^{-1} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

$$= \frac{e^{-\xi\omega_0 t}}{\omega_n} \begin{pmatrix} \omega_0 \cos(\omega_n t - \theta) & -\sin \omega_n t \\ \omega_0^2 \sin \omega_n t & \omega_0 \cos(\omega_n t + \theta) \end{pmatrix} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

Through one-sided Fourier transform, we obtain

$$\Psi_{xz}^{(0)}(\omega) = \frac{\psi_x(i\omega - 2\xi\omega_0) + \psi_y}{\omega^2 - \omega_0^2 + i2\xi\omega_0\omega} \quad (34)$$

which, together with (9), gives (32). □

EXAMPLE 1: Consider the nonlinear damping model with

$$D(x,y) = x^{2m}|x|^{\alpha}y^{2n+1}|y|^{\beta}$$

where m, n are nonnegative integers and $0 \le \alpha,\ \beta < 1$.

Since $D(x,y)$ is an even function of x, $\psi_x = 0$. And computation gives

$$\psi_y = \frac{1}{\pi\omega_0^{2m+\alpha}}\left(\frac{\sigma^2}{2\xi\omega_0}\right)^{m+n+1+\frac{\alpha+\beta}{2}}\Gamma(m+\frac{1+\alpha}{2})\Gamma(n+1+\frac{1+\beta}{2})$$

EXAMPLE 2: Consider the following saturation type active damping model,

$$\ddot{x} + 2\xi\omega_0\dot{x} + \gamma\tan^{-1}b\dot{x} + \omega_0^2 x = \sigma n(t) \tag{35}$$

Similarly, we have $\psi_x = 0$. And

$$
\begin{aligned}
\psi_y &= \int_{-\infty}^{\infty} y\tan^{-1}by\frac{\sqrt{2\xi\omega_0}}{\sqrt{\pi}\sigma}\exp(-\frac{2\xi\omega_0}{\sigma^2}y^2)dy \\
&= \frac{\sigma b}{2\sqrt{2\pi\xi\omega_0}}\int_{-\infty}^{\infty}\frac{1}{1+b^2y^2}\exp(-\frac{2\xi\omega_0}{\sigma^2}y^2)dy \\
&= \frac{\sigma\sqrt{\pi}}{2\sqrt{2\xi\omega_0}}[1 - \Phi(\frac{\sqrt{2\xi\omega_0}}{b\sigma})]e^{\frac{2\xi\omega_0}{b^2\sigma^2}}
\end{aligned}
$$

where the last equality is based upon the identity [40, p931]

$$\Phi(xy) = 1 - \frac{2x}{\pi}e^{-x^2y^2}\int_0^{\infty}\frac{e^{-t^2y^2}}{t^2+x^2}dt \quad x,\ y \ge 0 \tag{36}$$

in which

$$\Phi(x) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

It is not surprising to see that the second order statistics depends upon the first order statistics $m_{i,j}$. Therefore it is important to compute those numbers $m_{i,j}$, which is our next topic.

Chapter 3

On the Stationary Probability Density: Single DOF Case

3.1 The Stationary Probability Density in a Particular Case

To determine $m_{i,j}$, it is desirable to have $p_s(x,y)$ in explicit form. However, to obtain $p_s(x,y)$ explicitly is itself a very difficult task which has attracted large amount of literature. Many researchers have tried to clarify the largest class of nonlinear damping models for which an explicit stationary density can be obtained. The most recent and most inclusive account of this subject can be found in [22] and [23], which include parametric excitation as well as external excitation cases. However, the only useful result so far for the linear stiffness, nonlinear damping model with only external excitations is the following [37]: if the nonlinear damping model is of

the form

$$\ddot{x} + \mu(\frac{\omega_0^2 x^2 + \dot{x}^2}{2})\dot{x} + \omega_0^2 x = \sigma n(t)$$

then the stationary first order density is given by

$$p_s(x, y) = C \exp[-\frac{2}{\sigma^2} \int_0^{\frac{\omega_0^2 x^2 + y^2}{2}} \mu(z)dz]$$

$$\frac{1}{C} = \frac{2\pi}{\omega_0} \int_0^\infty \exp[-\frac{2}{\sigma^2} \int_0^\rho \mu(z)dz]d\rho$$

Next, let us consider the case $D(x, y) = \mu(\frac{\omega_0^2 x^2 + y^2}{2})y$. In this case, the stationary density of (1) is given by

$$p_s(x, y) = C \exp[-\frac{4\xi\omega_0}{\sigma^2} \cdot \frac{\omega_0^2 x^2 + y^2}{2} - \frac{2\gamma}{\sigma^2} \int_0^{\frac{\omega_0^2 x^2 + y^2}{2}} \mu(z)dz] \tag{37}$$

From (37) one can realize the following:

1. if $\mu(\cdot)$ is a polynomial such that $\mu(x) \geq 0$ for x large, then all the assumptions (A1)-(A3) are automatically satisfied;

2. after reaching stationarity, x and $\dot{x}$ are uncorrelated, but are not independent because $p(x, \dot{x})$ is not separable unless in the trivial case $\mu(E) = $ const., which reduces to a linear damping problem.

For later convenience, let us introduce the notation

$$m(k) \stackrel{\text{def}}{=} \int_0^\infty x^k \exp[-\frac{4\xi\omega_0}{\sigma^2}x - \frac{2\gamma}{\sigma^2} \int_0^x \mu(z)dz]dx$$

$$k = 0, 1, 2, \cdots$$

Then the normalizing constant C can be written as

$$1/C = \frac{2\pi}{\omega_0}m(0)$$

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Through substitution of variables

$$\begin{cases} x = 1/\omega_0 \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$$

we can write

$$m_{i,j}$$

$$= \frac{C}{\omega_0^{i+1}} \int_0^{2\pi} \cos^i\theta \sin^j\theta \, d\theta \int_0^\infty (2x)^{\frac{i+j}{2}} \exp[-\frac{4\zeta\omega_0}{\sigma^2}x - \frac{2\gamma}{\sigma^2}\int_0^x \mu(z)dz]dx$$

$$= \begin{cases} 0 & \text{either } i \text{ or } j \text{ is odd} \\ \frac{2^{\frac{i+j}{2}}\Gamma(\frac{i+1}{2})\Gamma(\frac{j+1}{2})m(\frac{i+j}{2})}{\pi\omega_0^i(\frac{i+j}{2})!m(0)} & \text{both } i \text{ and } j \text{ are even} \end{cases} \tag{38}$$

In (38), we have used the following identity

$$\int_0^{2\pi} \cos^i\theta \sin^j\theta \, d\theta$$

$$= \begin{cases} 0 & \text{either } i \text{ or } j \text{ is odd} \\ 2B(\frac{i+1}{2}, \frac{j+1}{2}) & \text{both } i \text{ and } j \text{ are even} \end{cases}$$

where $B(\cdot, \cdot)$ is a Beta function which has the following relation with Gamma function

$$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$$

from which (38) follows.

In addition, (38) immediately implies that

$$\omega_0^i m_{i,j} = \omega_0^j m_{j,i} \quad i,j = 0,1,2,\cdots$$

from which one can be obtained from the other.

Therefore, the problem of computing $m_{i,j}$ becomes that of evaluating the integrals $m(0)$ and those necessary $m(\frac{i+j}{2})$ for i, j both even.

Notice that (32) is not exactly a perturbation expression because the first term contains $m_{2,0}$ which depends on γ. In this particular case that $D(x,y)$ being of the form $\mu(\frac{\omega_0^2 x^2 + y^2}{2})y$, we can obtain the perturbation expression of $m_{2,0}$ and hence obtain the perturbation expression of $\Phi_{xx}(\omega)$.

Corollary 2 *Suppose $D(x,y)$ is of the form $\mu(\frac{\omega_0^2 x^2 + y^2}{2})y$, then*

$$m_{2,0} = \frac{\sigma^2 - 2\gamma\psi_y}{4\xi\omega_0^3} + O(\gamma^2) \tag{39}$$

$$\Phi_{xx}(\omega) = \frac{\sigma^2}{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2}$$
$$-\gamma\frac{\psi_y(4\xi\omega_0\omega)^2}{[(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega_0^2\omega^2]^2} + O(\gamma^2) \tag{40}$$

Proof: First, computation gives

$$\frac{m(k)}{m(0)} = (\frac{\sigma^2}{4\xi\omega_0})^k\Gamma(k+1) + \frac{8\xi\omega_0}{\sigma^4}[(\frac{\sigma^2}{4\xi\omega_0})^k k! I_0 - I_k]\gamma + O(\gamma^2) \tag{41}$$

where

$$I_j = \int_0^\infty x^j \int_0^x \mu(z)dz \exp(-\frac{4\xi\omega_0}{\sigma^2}x)dx \quad j = 0, 1, 2, \cdots$$

Successive integration by part gives

$$I_k = (\frac{\sigma^2}{4\xi\omega_0})^k k! I_0$$
$$+ \frac{\sigma^2}{4\xi\omega_0}\int_0^\infty [\sum_{j=0}^{k-1}(\frac{\sigma^2}{4\xi\omega_0})^j \frac{d^j}{dx^j}(x^k)]\mu(x)\exp(-\frac{4\xi\omega_0}{\sigma^2}x)dx \tag{42}$$

Therefore, we have

$$\frac{m(k)}{m(0)} = (\frac{\sigma^2}{4\xi\omega_0})^k k! - \frac{2\gamma}{\sigma^2}$$
$$\times \int_0^\infty [\sum_{j=0}^{k-1}(\frac{\sigma^2}{4\xi\omega_0})^j \frac{d^j}{dx^j}(x^k)]\mu(x)\exp(-\frac{4\xi\omega_0}{\sigma^2}x)dx + O(\gamma^2)$$

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In particular,

$$
\begin{aligned}
\frac{m(1)}{m(0)} &= \frac{\sigma^2}{4\xi\omega_0} - \frac{2\gamma}{4\xi\omega_0} \\
&\quad \times \frac{4\xi\omega_0}{\sigma^2} \int_0^\infty x\mu(x)\exp(-\frac{4\xi\omega_0}{\sigma^2}x)dx + O(\gamma^2) \\
&= \frac{\sigma^2 - 2\gamma\psi_y}{4\xi\omega_0} + O(\gamma^2)
\end{aligned}
$$

where we have used the relation

$$
\psi_y = \frac{4\xi\omega_0}{\sigma^2} \int_0^\infty x\mu(x)\exp(-\frac{4\xi\omega_0}{\sigma^2}x)dx
$$

which will be proved in *Theorem 4*.

Then, by (38), we obtain (39). (40) is easily seen from (39) and (32). □

In the case $D(x,y)$ is not of the form $\mu(\frac{\omega_0^2 x^2 + y^2}{2})y$, approximation approach has to be made.

3.2 The Stationary Probability Density in General Cases

For simplicity of notation, we surpress the damping term $2\xi\omega_0\dot{x} + \gamma D(x,\dot{x})$ to, say, simply $D(x,\dot{x})$. Then the nonlinear damping model (1) is rewritten as

$$
\ddot{x} + D(x,\dot{x}) + \omega_0^2 x = \sigma n(t) \tag{43}
$$

What we propose is to approximate the exact model (43) by the following modified model

$$
\ddot{x} + \mu\left(\frac{\omega_0^2 x^2 + y^2}{2}\right)\dot{x} + \omega_0^2 x = \sigma n(t) \tag{44}
$$

where $\mu(E)$ minimizes

$$
\int_0^{2\pi} [D(\frac{\sqrt{2E}}{\omega_0}\sin\psi, \sqrt{2E}\cos\psi) - \mu(E)\sqrt{2E}\cos\psi]^2 d\psi
$$

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and is given by

$$\mu(E) = \frac{4}{\pi\sqrt{2E}} \int_0^{\frac{\pi}{2}} D(\frac{\sqrt{2E}}{\omega_0} \sin\psi, \sqrt{2E}\cos\psi)\cos\psi d\psi \tag{45}$$

Presently, the problem of giving a precise estimate of the error of the above proposed approximation is still open. But we do have the following side evidences.

Theorem 3 *If $D(x,y)$ is even with respect to x and odd with respect to y, then both the exact model (43) and the modified model (44) have the same Krylov-Bogoliubov approximation [48].*

Proof: For the exact model (43),

$$\frac{da(t)}{dt} = -\frac{1}{2\pi\omega_0} \int_0^{2\pi} D(a\sin\psi, a\omega_0\cos\psi)\cos\psi d\psi \tag{46}$$

By the assumption on $D(x,y)$,

$$\begin{aligned}
\frac{d\phi(t)}{dt} &= \frac{1}{2\pi\omega_0 a} \int_0^{2\pi} D(a\sin\psi, a\omega_0\cos\psi)\sin\psi d\psi \\
&= -\frac{1}{\pi\omega_0 a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} D(a\cos\psi, a\omega_0\sin\psi)\cos\psi d\psi \\
&= 0
\end{aligned}$$

For the modified model (44), the Krylov-Bogoliubov approximation is given by

$$\begin{aligned}
\frac{da(t)}{dt} &= -\frac{1}{2\pi\omega_0} \int_0^{2\pi} \mu(\frac{\omega_0^2 a^2}{2})\omega_0 a \cos^2\psi d\psi \\
&= -\frac{a}{2}\mu(\frac{\omega_0^2 a^2}{2}) \\
&= -\frac{1}{2\pi\omega_0} \int_0^{2\pi} D(a\sin\psi, \omega_0 a\cos\psi)\cos\psi d\psi
\end{aligned}$$

$$\begin{aligned}
\frac{d\phi(t)}{dt} &= \frac{1}{2\pi\omega_0 a} \int_0^{2\pi} \mu(\frac{\omega_0^2 a^2}{2})\omega_0 a \cos\psi \sin\psi d\psi \\
&= 0
\end{aligned}$$

Therefore, (43) and (44) have the same Krylov-Bogoliubov approximation. $\square$

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Theorem 4 *If $D(x,y)$ is even with respect to x and odd with respect to y, and let ψ_x, ψ_y correspond to the exact model (43), ψ_x^μ, ψ_y^μ correspond to the modified model (44), then*

$$\psi_x = \psi_x^\mu = 0 \tag{47}$$

$$\psi_y = \psi_y^\mu = \frac{4\xi\omega_0}{\sigma^2}\int_0^\infty x\mu(x)\exp(-\frac{4\xi\omega_0}{\sigma^2}x)dx \tag{48}$$

Consequently,

$$E\left(\begin{array}{c} x(t) \\ y(t) \end{array}\right)D(x(t+\tau),y(t+\tau)) = E\left(\begin{array}{c} x(t) \\ y(t) \end{array}\right)\mu(\frac{\omega_0^2 x^2(t+\tau)+y^2(t+\tau)}{2})y(t+\tau) \tag{49}$$

Proof: (47) is obvious because both $D(x,y)$ and the corresponding $\mu(\frac{\omega_0^2 x^2+y^2}{2})y$ are even with respect to x.

For (48), we have the following

$$
\begin{aligned}
\psi_y^\mu &= \iint_{\mathbf{R}^2} \mu(\frac{\omega_0^2 x^2+y^2}{2})y^2\frac{2\xi\omega_0^2}{\pi\sigma^2}\exp[-\frac{2\xi\omega_0}{\sigma^2}(\omega_0^2 x^2+y^2)]dxdy \\
&= \int_0^\infty \int_0^{2\pi} \mu(\frac{r^2}{2})r^2\sin^2\theta\frac{2\xi\omega_0}{\pi\sigma^2}\exp(-\frac{2\xi\omega_0}{\sigma^2}r^2)rd\theta dr \\
&= \int_0^\infty \int_0^{2\pi} \frac{1}{\pi}\int_0^{2\pi} D(\frac{r}{\omega_0}\sin\psi, r\cos\psi)\cos\psi d\psi r\sin^2\theta \\
&\qquad \times \frac{2\xi\omega_0}{\pi\sigma^2}\exp(-\frac{2\xi\omega_0}{\sigma^2}r^2)rd\theta dr \\
&= \int_0^\infty \int_0^{2\pi} D(\frac{r}{\omega_0}\sin\psi, r\cos\psi)r\cos\psi d\psi \\
&\qquad \times \frac{2\xi\omega_0}{\pi\sigma^2}\exp(-\frac{2\xi\omega_0}{\sigma^2}r^2)rd\theta dr \\
&= \iint_{\mathbf{R}^2} D(x,y)y\frac{2\xi\omega_0^2}{\pi\sigma^2}\exp[-\frac{2\xi\omega_0}{\sigma^2}(\omega_0^2 x^2+y^2)]dxdy \\
&= \psi_y
\end{aligned} \tag{50}
$$

$$\tag{51}$$

The second equality in (48) is easily seen from (50). $\qquad\Box$

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The solution of the stationary Fokker-Planck equation corresponding to (44) is given by

$$p_s(x,y) = C exp[-\frac{2}{\sigma^2} \int_0^{\frac{\omega_0^2 x^2 + y^2}{2}} \mu(z)dz]$$ (52)

where

$$\frac{1}{C} = \frac{2\pi}{\omega_0} \int_0^\infty exp[-\frac{2}{\sigma^2} \int_0^\rho \mu(z)dz]d\rho$$ (53)

Example 3: Again, consider the saturation type active damping model (35).

By the identity

$$\int_0^{\pi/2} \tan^{-1}(b\cos x)\cos x dx = \frac{\pi}{2} \frac{\sqrt{1+b^2}-1}{b} \quad b \in I\!R^1$$

one can compute

$$
\begin{aligned}
&\mu(E) \\
&= \frac{4}{\pi\sqrt{2E}} \int_0^{\frac{\pi}{2}} [2\xi\omega_0\sqrt{2E}\cos\psi + \lambda\tan^{-1}(b\sqrt{2E}\cos\psi)]\cos\psi d\psi \\
&= 2\xi\omega_0 + \lambda\frac{\sqrt{1+2b^2E}-1}{bE}
\end{aligned}
$$ (54)

And consequently, one has

$$
\begin{aligned}
&-\frac{2}{\sigma^2} \int_0^E \mu(z)dz \\
&= -\frac{2\xi\omega_0}{\sigma^2}(2E) - \frac{4\lambda}{\sigma^2 b}\sqrt{1+2b^2E} \\
&\quad +\frac{4\lambda}{\sigma^2 b} \ln[1 + \sqrt{1+2b^2E}] + const.
\end{aligned}
$$

The solution of the stationary Fokker-Planck equation:

$$p(x,y) = C[1 + \sqrt{1 + b^2(\omega_0^2 x^2 + y^2)}]^{\frac{4\lambda}{\sigma^2 b}}$$

$$\times exp[-\frac{2\xi\omega_0}{\sigma^2}(\omega_0^2 x^2 + y^2) - \frac{4\lambda}{\sigma^2 b}\sqrt{1 + b^2(\omega_0^2 x^2 + y^2)}]$$

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where

$$\frac{1}{C} = \frac{\pi}{\omega_0 b^2} exp(\frac{2\xi\omega_0}{\sigma^2 b^2})$$

$$\times \int_1^\infty (1 + \sqrt{x})^{\frac{4\lambda}{\sigma^2 b}} exp[-\frac{2\xi\omega_0}{\sigma^2 b^2}x - \frac{4\lambda}{\sigma^2 b}\sqrt{x}]dx$$

It is easy to realize that $p(x,y)$ achieves maximum at the origin.

(48) provides a simpler means of computing ψ_y. From what follows, one can see ψ_y can be easily obtained without employing the integral identity (36) as in *Example 2*.

By (48) and (54), we have

$$\begin{aligned}
\psi_y &= \frac{4\xi\omega_0}{\sigma^2} \int_0^\infty \frac{\sqrt{1 + 2b^2 x} - 1}{b} \exp(-\frac{4\xi\omega_0}{\sigma^2}x)dx \\
&= \frac{2}{b} \int_{\frac{\sqrt{2\xi\omega_0}}{\sigma b}}^\infty y(\frac{\sigma b}{\sqrt{2\xi\omega_0}}y - 1)e^{-y^2}dy \exp(\frac{2\xi\omega_0}{b^2\sigma^2}) \\
&\qquad (\text{substitution of variable } 1 + 2b^2 x = \frac{b^2\sigma^2}{2\xi\omega_0}y^2) \\
&= \frac{\sigma}{\sqrt{2\xi\omega_0}} \int_{\frac{\sqrt{2\xi\omega_0}}{\sigma b}}^\infty e^{-y^2}dy \exp(\frac{2\xi\omega_0}{b^2\sigma^2}) \quad (\text{integration by part}) \\
&= \frac{\sqrt{\pi}\sigma}{2\sqrt{2\xi\omega_0}}[1 - \Phi(\frac{\sqrt{2\xi\omega_0}}{\sigma b})] \exp(\frac{2\xi\omega_0}{b^2\sigma^2})
\end{aligned}$$

which is the same as obtained in *Example 2*.

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Chapter 4

The Spectral Density of Nonlinear Damping Model: Multi-DOF Case

4.1 Introduction

As indicated in the first part of this work [70], the problem of characterizing the damping mechanism in flexible structures has received renewed attention in recent years in connection with the need to stablize flexible flight structures such as antennas deployed in space. Experimental evidence [8] suggests the need for nonlinear damping model and the need to consider the effect of random disturbances due to the uncertainties in system parameters and the environment. One of the most important subjects in nonlinear random vibration is to obtain the second order statistics, i.e., correlation function and spectral density of the stationary response, because

they provide average amplitude and frequency information about the sample histories. Unfortunately, up to now, the only practical method available is Monte Carlo simulation and there is no analytical technique for the second order statistics of nonlinear systems [32]. This paper presents an analytical technique for computing correlation function and spectral density of the stationary response of $n-$DOF nonlinear damping model subject to white noise excitation. The single DOF case was fully investigated in the first part of this work [70].

The basic nonlinear damping model we consider is, for $x(t) \in I\!\!R^n$,

$$M\ddot{x} + D_0\dot{x} + \gamma D(x, \dot{x}) + Kx = \sigma n(t) \tag{55}$$

where $\gamma > 0$ is a small constant because the damping in flexible space structures, whatever its nature, is small. Here, $M > 0$, $D_0 \geq 0$, and $n(t)$ is m dimensional white noise. σ is an $n \times m$ matrix. Since we are only interested in oscillation problem, we assume that (55) has no rigid body mode, i.e. $K > 0$.

The corresponding Fokker-Planck equation is easily seen to be

$$
\begin{aligned}
\frac{\partial p}{\partial t} &= -y^T \nabla_x p + \nabla_y \cdot [(M^{-1}Kx + M^{-1}D_0 y)p] \\
&\quad + \gamma \nabla_y \cdot (M^{-1}D(x, y)p) + \frac{1}{2}\text{tr}(M^{-1}\sigma\sigma^T M^{-1}\nabla_y^2 p) \\
&= \mathbf{L}_0 p + \gamma \nabla_y \cdot (M^{-1}D(x, y)p) \\
&\lim_{t \to 0} p(t, x, y|x_0, y_0) = \delta(x - x_0)\delta(y - y_0)
\end{aligned}
\tag{56}
$$

Notations

$z(t) \stackrel{\text{def}}{=} D(x(t), y(t))$;

$p_s(x, y)$: stationary density of $(x(t), y(t))$, i.e., the invariant measure;

$p(t, x, y|x_0, y_0)$: the fundamental solution of (2);

$p_0(t, x, y|x_0, y_0)$: the fundamental solution of (2) with $\gamma = 0$;

$$\mathbf{T}(t) \stackrel{\text{def}}{=} exp(\mathbf{L}_0 t);$$

$$q(t, s, x, y | x_0, y_0) \stackrel{\text{def}}{=} \iint_{R^{2n}} D(u, v) p_0(t - s, x, y | u, v) p(s, u, v | x_0, y_0) d|u| d|v|.$$

It is well-known that $p_0(t, x, y | x_0, y_0)$ is a $2n$-dimensional Gaussian density function. Its mean vector and covariance matrix are given by

$$\begin{pmatrix} m_x(t) \\ m_y(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{57}$$

where the matrix A is defined by

$$A \stackrel{\text{def}}{=} \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D_0 \end{pmatrix}_{2n \times 2n}$$

and

$$\Sigma(t) \stackrel{\text{def}}{=} \int_0^t e^{A(t-s)} \begin{pmatrix} 0 & 0 \\ 0 & M^{-1}\sigma\sigma^T M^{-1} \end{pmatrix} e^{A^T(t-s)} ds \tag{58}$$

Later on, we will need the notation

$$\Sigma_\infty \stackrel{\text{def}}{=} \lim_{t \to \infty} \Sigma(t) = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix}_{2n \times 2n} \tag{59}$$

and without ambiguity we will often denote $\Sigma(t)$ by Σ.

Assumptions on $D(x, y)$

(A1) Each component of $D(x, y)$ is differentiable with respect to y;

(A2) $\exists K > 0, k > 0$ such that $\|D(x, y)\| \le K[1 + (\|x\|^2 + \|y\|^2)^k]$ for $(x, y) \in I\!\!R^{2n}$;

(A3) $\int_{R^{2n}} (\|x\|^2 + \|y\|^2)^k p_s(x, y) d|x| d|y| < \infty$ for all nonnegative integers k.

Of course, to satisfy the energy nonincrease requirement, we also need

$$y^T D_0 y + \gamma y^T D(x, y) \ge 0 \quad (x, y) \in I\!\!R^{2n}$$

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4.2 An Equation of Spectral Density

Lemma 4 *Under assumptions (A2) and (A3), it holds*

$$\lim_{\|x\|^2+\|y\|^2\to\infty} q(t,s,x,y|x_0,y_0) = 0 \quad \forall\, 0 \le s \le t,\ (x_0,y_0) \in I\!\!R^{2n} \tag{60}$$

provided

$$\lim_{\|x\|^2+\|y\|^2\to\infty} p(t,x,y|x_0,y_0) = 0 \quad \forall t > 0,\ (x_0,y_0) \in I\!\!R^{2n} \tag{61}$$

PROOF: First, by Schwarz inequality, we have

$$\|q(t,s,x,y|x_0,y_0)\|$$

$$\le \iint_{I\!\!R^{2n}} \|D(u,v)\| p_0(t-s,x,y|u,v)p(s,u,v|x_0,y_0)dudv$$

$$\le [\iint_{I\!\!R^{2n}} \|D(u,v)\|^2 p_0(t-s,x,y|u,v)p(s,u,v|x_0,y_0)dudv]^{1/2}$$

$$\times [\iint_{I\!\!R^{2n}} p_0(t-s,x,y|u,v)p(s,u,v|x_0,y_0)dudv]^{1/2}$$

The second square root term goes to 0 as $\|x\|^2 + \|y\|^2 \longrightarrow \infty$ by the assumption (61). Therefore it is sufficient to show that the first square root term is bounded for all $(x,y) \in I\!\!R^{2n}$.

In fact, we have

$$\iint_{I\!\!R^{2n}} \|D(u,v)\|^2 p_0(t-s,x,y|u,v)p(s,u,v|x_0,y_0)dudv$$

$$\le \frac{1}{(2\pi)^n\sqrt{det\Sigma(t-s)}} \iint_{I\!\!R^{2n}} \|D(u,v)\|^2 p(s,u,v|x_0,y_0)dudv \tag{62}$$

In addition, by assumptions (A2) and (A3), we have

$$\iint_{I\!\!R^{2n}}\iint_{I\!\!R^{2n}} \|D(x,y)\|^2 p(t,x,y|x_0,y_0)p_s(x_0,y_0)dx_0dy_0dxdy$$

$$= \iint_{I\!\!R^{2n}} \|D(x,y)\|^2 p_s(x,y)dxdy$$

$$\le \iint_{I\!\!R^{2n}} K^2[1 + (\|x\|^2 + \|y\|^2)k]^2 p_s(x,y)dxdy$$

$$< \infty$$

By *Tonelli*'s lemma, we know that

$$||D(x,y)||^2 p(t,x,y|x_0,y_0)p_s(x_0,y_0) \in L^1(I\!\!R^{2n} \otimes I\!\!R^{2n})$$

Then, by *Fubini*'s theorem, we know that

$$\iint_{I\!\!R^{2n}} ||D(x,y)||^2 p(t,x,y|\cdot,\cdot)dxdy p_s(\cdot,\cdot) \in L^1(I\!\!R^{2n})$$

which implies

$$||D(\cdot,\cdot)||^2 p(t,\cdot,\cdot|x_0,y_0) \in L^1(I\!\!R^{2n})$$

Therefore, by (62), the first square root term is bounded for all $(x,y) \in I\!\!R^{2n}$. Thus the claim is proved. □

In the following theorem, we establish an equation for the correlation matrices which plays a fundamental role for later development.

Theorem 5 *Under the assumptions (A1)-(A3), the following equation holds:*

$$R_{xx}(\tau) = R_{xx}(0)\Phi_{11}^T(\tau) + R_{xy}(0)\Phi_{12}^T(\tau)$$
$$-\gamma \int_0^\tau R_{xz}(s)M^{-1}\Phi_{12}^T(\tau - s)ds \qquad (63)$$

for $\tau > 0$.

PROOF: (56) is equivalent to the following integral equation

$$p(t,x,y|x_0,y_0) = p_0(t,x,y|x_0,y_0)$$
$$+\gamma \int_0^t \mathbf{T}(t-s)\nabla_y \cdot [M^{-1}D(x,y)p(s,x,y|x_0,y_0)]ds$$

or more explicitly,

$$p(t,x,y|x_0,y_0) = p_0(t,x,y|x_0,y_0) + \gamma \int_0^t \int_{I\!\!R^{2n}} p_0(t-s,x,y|u,v)$$
$$\times \nabla_v [M^{-1}D(u,v)p(s,u,v|x_0,y_0)]d|u|d|v|ds \qquad (64)$$

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Performing integration by part in the second term and noticing the relation

$$\nabla_v p_0(t-s,x,y|u,v) = -\Phi_{12}^T(t-s)\nabla_x p_0(t-s,x,y|u,v)$$
$$-\Phi_{22}^T(t-s)\nabla_y p_0(t-s,x,y|u,v)$$

we obtain the following

$$\int_{R^{2n}} p_0(t-s,x,y|u,v)\nabla_v \cdot [M^{-1}D(u,v)p(s,u,v|x_0,y_0)]d|u|d|v|$$
$$= -\int_{R^{2n}} [\nabla_v p_0(t-s,x,y|u,v)]^T M^{-1}D(u,v)p(s,u,v|x_0,y_0)d|u|d|v|$$
$$= \int_{R^{2n}} [\nabla_x p_0(t-s,x,y|u,v)]^T \Phi_{12}(t-s)M^{-1}D(u,v)p(s,u,v|x_0,y_0)d|u|d|v|$$
$$+ \int_{R^{2n}} [\nabla_y p_0(t-s,x,y|u,v)]^T \Phi_{22}(t-s)M^{-1}D(u,v)p(s,u,v|x_0,y_0)d|u|d|v|$$
$$= \nabla_x \cdot \int_{R^{2n}} \Phi_{12}(t-s)M^{-1}D(u,v)p_0(t-s,x,y|u,v)p(s,u,v|x_0,y_0)d|u|d|v|$$
$$+ \nabla_y \cdot \int_{R^{2n}} \Phi_{22}(t-s)M^{-1}D(u,v)p_0(t-s,x,y|u,v)p(s,u,v|x_0,y_0)d|u|d|v|$$
$$= \nabla_x \cdot [\Phi_{12}(t-s)M^{-1}q(t,s,x,y|x_0,y_0)$$
$$+ \nabla_y \cdot [\Phi_{22}(t-s)M^{-1}q(t,s,x,y|x_0,y_0)$$

Therefore, we obtain the following integro-differential equation for $p(t,x,y|x_0,y_0)$:

$$p(t,x,y|x_0,y_0) = p_0(t,x,y|x_0,y_0)$$
$$+\gamma\nabla_x \cdot \int_0^t \Phi_{12}(t-s)M^{-1}q(t,s,x,y|x_0,y_0)ds$$
$$+\gamma\nabla_y \cdot \int_0^t \Phi_{22}(t-s)M^{-1}q(t,s,x,y|x_0,y_0)ds \qquad (65)$$

Based upon (A2) and hence *Lemma 1*, we find that the marginal transition probability density $p(t,x|x_0,y_0)$ satisfies the following equation, by integrating (65) with respect to y over $I\!\!R^n$,

$$p(t,x|x_0,y_0) = p_0(t,x|x_0,y_0) + \gamma\nabla_x \cdot \int_0^t \Phi_{12}(t-s)M^{-1}$$
$$\times \int_{R^{2n}} D(u,v)p_0(t-s,x|u,v)p(s,u,v|x_0,y_0)d|u|d|v|ds \qquad (66)$$

After integration, the third term vanishes by *Lemma 1*. And in (66), $p_0(t, x|x_0, y_0)$ is an n-dimensional Gaussian density with mean

$$\Phi_{11}(t)x_0 + \Phi_{12}(t)y_0$$

Multiplying (66) by $x_0 x^T p_s(x_0, y_0)$ and integrating with respect to (x, x_0, y_0) over $I\!\!R^{3n}$, we obtain, for $t > 0$,

$$
\begin{aligned}
R_{xx}(t) &= \int_{R^{3n}} x_0 x^T p_0(t, x|x_0, y_0) p_s(x_0, y_0) d|x| d|x_0| d|y_0| \\
&\quad + \gamma \int_{R^{3n}} x_0 x^T \nabla_x \cdot \int_0^t \Phi_{12}(t-s) M^{-1} \int_{R^{2n}} D(u, v) p_0(t-s, x|u, v) \\
&\quad \times p(s, u, v|x_0, y_0) d|u| d|v| ds\, p_s(x_0, y_0) d|x| d|x_0| d|y_0| \\
&= \int_{R^{2n}} x_0 [\Phi_{11}(t)x_0 + \Phi_{12}(t)y_0]^T p_s(x_0, y_0) d|x_0| d|y_0| \\
&\quad - \gamma \int_{R^{3n}} x_0 \int_0^t \int_{R^{2n}} D^T(u, v) M^{-1} \Phi_{12}^T(t-s) p_0(t-s, x|u, v) \\
&\quad \times p(s, u, v|x_0, y_0) d|u| d|v| p_s(x_0, y_0) d|x| d|x_0| d|y_0| ds \\
&= R_{xx}(0) \Phi_{11}^T(t) + R_{xy}(0) \Phi_{12}^T(t) \\
&\quad - \gamma \int_0^t \int_{R^{2n}} \int_{R^{2n}} x_0 D^T(u, v) M^{-1} \Phi_{12}^T(t-s) \\
&\quad \times p(s, u, v|x_0, y_0) p_s(x_0, y_0) d|u| d|v| d|x_0| d|y_0| ds \\
&= R_{xx}(0) \Phi_{11}^T(t) + R_{xy}(0) \Phi_{12}^T(t) \\
&\quad - \gamma \int_0^t R_{xz}(s) M^{-1} \Phi_{12}^T(t-s) ds
\end{aligned}
\tag{67}
$$

□

The significance of this theorem is that $R_{xx}(\tau)$ is expressed in terms of $R_{xz}(\tau)$, while the unknown matrix $R_{xz}(\tau)$ appears only in the first order coefficient of γ. Therefore, if one obtains the kth order perturbation of $R_{xz}(\tau)$, then $R_{xx}(\tau)$ is immediately given by (63) with error in the order of $O(\gamma^{k+2})$.

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4.3 The Spectral Density

In this section, we establish the formula of the spectral density of (55) with $O(\gamma^2)$ accuracy. First, we need two lemmas.

Lemma 5 *Assume the pair*

$$\{A, \begin{pmatrix} 0 \\ M^{-1}\sigma \end{pmatrix} \}$$

being completely controllable, then the following matrix relations hold

$$\Sigma(t) = \Sigma_\infty - e^{At}\Sigma_\infty e^{A^T t} \tag{68}$$

$$\Sigma^{-1} - \Sigma^{-1}e^{At}(\Sigma_\infty^{-1} + e^{A^T t}\Sigma^{-1}e^{At})^{-1}e^{A^T t}\Sigma^{-1} = \Sigma_\infty^{-1} \tag{69}$$

$$(\Sigma_\infty^{-1} + e^{A^T t}\Sigma^{-1}e^{At})^{-1}e^{A^T t} = \Sigma_\infty e^{A^T t}\Sigma_\infty^{-1}\Sigma \tag{70}$$

PROOF: Under the controllability assumption we can know that both $\Sigma(t)$ and Σ_∞ are positive definite for all $t > 0$.

Let us define

$$\tilde{\Sigma}(t) \stackrel{\text{def}}{=} \Sigma_\infty - e^{At}\Sigma_\infty e^{A^T t}$$

To show (68) is equivalent to showing $\Sigma(t) \equiv \tilde{\Sigma}(t)$.

In fact, since

$$\Sigma(t) = \int_0^t e^{A(t-s)} \begin{pmatrix} 0 & 0 \\ 0 & M^{-1}\sigma\sigma^T M^{-1} \end{pmatrix} e^{A^T(t-s)}ds$$

we know that $\Sigma(t)$ satisfies the following linear differential equation

$$\frac{d}{dt}\Sigma(t) = A\Sigma(t) + \Sigma(t)A^T + \begin{pmatrix} 0 & 0 \\ 0 & M^{-1}\sigma\sigma^T M^{-1} \end{pmatrix} \tag{71}$$

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Letting $t \to \infty$, we obtain the relation

$$-A\Sigma_\infty - \Sigma_\infty A^T = \begin{pmatrix} 0 & 0 \\ 0 & M^{-1}\sigma\sigma^T M^{-1} \end{pmatrix}$$

by which we also have

$$\begin{aligned} \frac{d}{dt}\hat{\Sigma}(t) &= A(\hat{\Sigma}(t) - \Sigma_\infty) + (\hat{\Sigma}(t) - \Sigma_0)A^T \\ &= A\hat{\Sigma}(t) + \hat{\Sigma}(t)A^T + \begin{pmatrix} 0 & 0 \\ 0 & M^{-1}\sigma\sigma^T M^{-1} \end{pmatrix} \end{aligned} \tag{72}$$

Obviously, $\Sigma(t)$ and $\hat{\Sigma}(t)$ have the same initial condition

$$\Sigma(0) = \hat{\Sigma}(0) = 0$$

Therefore, the uniqueness of the solutions of linear differential equations implies $\Sigma(t) \equiv \hat{\Sigma}(t)$.

By the matrix inversion formula

$$(M_1 + M_2 M_3 M_4)^{-1} = M_1^{-1} - M_1^{-1}M_2(M_3^{-1} + M_4 M_1^{-1} M_2)^{-1}M_4 M_1^{-1} \tag{73}$$

we have, noticing (24),

$$\begin{aligned} &\Sigma^{-1} - \Sigma^{-1}e^{At}(\Sigma_\infty^{-1} + e^{A^Tt}\Sigma^{-1}e^{At})^{-1}e^{A^Tt}\Sigma^{-1} \\ &= (\Sigma + e^{At}\Sigma_\infty e^{A^Tt})^{-1} \\ &= \Sigma_\infty^{-1} \end{aligned}$$

For the proof of (70), we again use (73) and (68) to obtain

$$\begin{aligned} &(\Sigma_\infty^{-1} + e^{A^Tt}\Sigma^{-1}e^{At})^{-1}e^{A^Tt} \\ &= [\Sigma_\infty - \Sigma_\infty e^{A^Tt}(\Sigma + e^{At}\Sigma_\infty e^{A^Tt})^{-1}e^{At}\Sigma_\infty]e^{A^Tt} \end{aligned}$$

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$$= \Sigma_\infty e^{A^T t}(I - \Sigma_\infty^{-1} e^{At} \Sigma_\infty e^{A^T t})$$

$$= \Sigma_\infty e^{A^T t}[I - \Sigma_\infty^{-1}(\Sigma_\infty - \Sigma)]$$

$$= \Sigma_\infty e^{A^T t} \Sigma_\infty^{-1} \Sigma$$

$\square$

Lemma 6 *Assume the matrix pair*

$$\{A, \begin{pmatrix} 0 \\ M^{-1}\sigma \end{pmatrix}\}$$

is completely controllable. Then for the linear damping model

$$M\ddot{x} + D\dot{x} + Kx = \sigma n(t)$$

we have

$$E \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} D^T(x(t+\tau), y(t+\tau)) = \Sigma_\infty e^{A^T \tau} \Sigma_\infty^{-1} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}_{2n \times n} \tag{74}$$

where

$$\begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} = \iint_{I\!R^{2n}} \begin{pmatrix} x \\ y \end{pmatrix} D^T(x,y) \frac{1}{(2\pi)^n \sqrt{|\Sigma_\infty|}}$$

$$\times \exp[-1/2 \begin{pmatrix} x \\ y \end{pmatrix}^T \Sigma_\infty^{-1} \begin{pmatrix} x \\ y \end{pmatrix}]d|x|d|y| \tag{75}$$

PROOF: Let us use the following notations:

$$X \overset{\text{def}}{=} \begin{pmatrix} x \\ y \end{pmatrix} \quad X_0 \overset{\text{def}}{=} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

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By (69) and (70), we have

$$(X - e^{At}X_0)^T \Sigma^{-1}(X - e^{At}X_0) + X_0^T \Sigma_\infty^{-1} X_0$$

$$= X^T[\Sigma^{-1} - \Sigma^{-1}e^{At}(\Sigma_\infty^{-1} + e^{A^T t}\Sigma^{-1}e^{At})^{-1}e^{A^T t}\Sigma^{-1}]X$$

$$+[X_0 - (e^{A^T t}\Sigma^{-1}e^{At} + \Sigma_\infty^{-1})^{-1}e^{A^T t}\Sigma^{-1}X]^T(e^{A^T t}\Sigma^{-1}e^{At} + \Sigma_\infty^{-1})$$

$$\times[X_0 - (e^{A^T t}\Sigma^{-1}e^{At} + \Sigma_\infty^{-1})^{-1}e^{A^T t}\Sigma^{-1}X]$$

$$= X^T\Sigma_\infty^{-1}X$$

$$+(X_0 - \Sigma_\infty e^{A^T t}\Sigma_\infty^{-1}X)^T(e^{A^T t}\Sigma^{-1}e^{At} + \Sigma_\infty^{-1})(X_0 - \Sigma_\infty e^{A^T t}\Sigma_\infty^{-1}X)$$

Then

$$E \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} D^T(x(t+\tau), y(t+\tau))$$

$$= \iint_{I\!R^{2n}} \iint_{I\!R^{2n}} X_0 D^T(x,y) \frac{1}{2\pi\sqrt{|\Sigma(\tau)|}}$$

$$\times \exp[-1/2(X - e^{A\tau}X_0)^T \Sigma^{-1}(\tau)(X - e^{A\tau}X_0)]$$

$$\times \frac{1}{2\pi\sqrt{|\Sigma_\infty|}} \exp(-1/2 X_0^T \Sigma_\infty^{-1} X_0) d|X| d|X_0|$$

$$= \{\iint_{I\!R^{2n}} D(x,y) \frac{1}{2\pi\sqrt{|\Sigma_\infty|}} \exp(-1/2 X^T\Sigma_\infty^{-1}X) \iint_{I\!R^{2n}} X_0^T$$

$$\times \frac{1}{2\pi\sqrt{|\Sigma(\tau)|}} \exp[-1/2(X_0 - \Sigma_\infty e^{A^T\tau}\Sigma_\infty^{-1}X)^T(\Sigma_\infty^{-1} + e^{A^T\tau}\Sigma^{-1}e^{A\tau})$$

$$\times (X_0 - \Sigma_\infty e^{A^T\tau}\Sigma_\infty^{-1}X)]d|X_0|d|X|\}^T$$

$$= [\iint_{I\!R^{2n}} D(x,y) \frac{1}{2\pi\sqrt{|\Sigma_\infty|}} \exp(-1/2 X^T\Sigma_\infty^{-1}X)$$

$$\times \frac{(\Sigma_\infty e^{A^T\tau}\Sigma_\infty^{-1}X)^T}{\sqrt{|\Sigma(\tau)||\Sigma_\infty^{-1} + e^{A^T\tau}\Sigma^{-1}(\tau)e^{A\tau}|}} d|x|d|y|]^T$$

$$= \Sigma_\infty e^{A^T\tau}\Sigma_\infty^{-1} \iint_{I\!R^{2n}} X D^T(x,y) \frac{1}{2\pi\sqrt{|\Sigma_\infty|}} \exp(-1/2 X^T\Sigma_\infty^{-1}X)d|x|d|y|$$

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$$= \Sigma_\infty e^{A^T \tau} \Sigma_\infty^{-1} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

In the above, we used the fact that

$$|\Sigma(t)||\Sigma_\infty^{-1} + e^{A^T t}\Sigma^{-1}(t)e^{At}| = 1 \qquad \forall t > 0$$

which could be easily verified by using (70). □

Theorem 6 *Under the assumption of the complete controllability of*

$$\{A, \begin{pmatrix} 0 \\ M^{-1}\sigma \end{pmatrix}\}$$

the spectral density matrix of (55) is given by

$$\Psi_{xx}(\omega) = \hat{\Psi}_{xx}(\omega) + \hat{\Psi}_{xx}^*(\omega) \tag{76}$$

with

$$\begin{aligned}
\hat{\Psi}_{xx}(\omega) &= R_{xx}(0)(D - i\omega M)G(i\omega) + R_{xy}(0)MG(i\omega) \\
&\quad -\gamma \begin{pmatrix} I_{n \times n} \\ 0 \end{pmatrix}^T \Sigma_\infty \begin{pmatrix} D - i\omega M & -K \\ M & -i\omega M \end{pmatrix} \begin{pmatrix} G(i\omega) & 0 \\ 0 & G(i\omega) \end{pmatrix} \\
&\quad \times \Sigma_\infty^{-1} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} G(i\omega) + O(\gamma^2)
\end{aligned} \tag{77}$$

where

$$G(i\omega) \stackrel{\text{def}}{=} (-\omega^2 M - i\omega D + K)^{-1}$$

PROOF: First of all, it should be noted that (63) is valid only for $\tau \geq 0$. However, by the basic property of correlation function matrix

$$R_{xx}(-\tau) = R_{xx}^*(\tau)$$

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we can easily obtain $R_{xx}(\tau)$ for $\tau < 0$ as

$$R_{xx}(\tau) = R_{xx}^*(-\tau)$$

Therefore. if we define

$$\hat{\Psi}_{xx}(\omega) \overset{\text{def}}{=} \int_0^\infty R_{xx}(\tau)e^{i\omega\tau}d\tau$$

then we have the following expression for the spectral density matrix

$$\Psi_{xx}(\omega) = \hat{\Psi}_{xx}(\omega) + \hat{\Psi}_{xx}^*(\omega)$$

Next, to find the expression of $\hat{\Psi}_{xx}(\omega)$, we need to evaluate

$$\int_0^\infty e^{At}e^{i\omega t}dt$$

To this end. we perform integration on both sides of

$$e^{i\omega t}\frac{d}{dt}e^{At} = Ae^{At}e^{i\omega t}$$

to obtain

$$
\begin{aligned}
\Phi(i\omega) &= (-i\omega I_{2n\times 2n} - A)^{-1} \\
&= \begin{pmatrix} G(i\omega) & 0 \\ 0 & G(i\omega) \end{pmatrix} \begin{pmatrix} D - i\omega M & M \\ -K & -i\omega M \end{pmatrix}
\end{aligned} \tag{78}
$$

where the last equality can be easily verified.

If one writes

$$\begin{pmatrix} R_{xz}(s) \\ R_{yz}(s) \end{pmatrix} = \begin{pmatrix} R_{xz}^{(0)}(s) \\ R_{yz}^{(0)}(s) \end{pmatrix} + \gamma \begin{pmatrix} R_{xz}^{(1)}(s) \\ R_{yz}^{(1)}(s) \end{pmatrix} + \cdots \tag{79}$$

then, by *Lemma 3*,

$$\begin{pmatrix} R_{xz}^{(0)}(s) \\ R_{yz}^{(0)}(s) \end{pmatrix} = \Sigma_\infty e^{A^T s}\Sigma_\infty^{-1} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

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which implies

$$R_{xz}^{(0)}(s) = \begin{pmatrix} I_{n \times n} \\ 0 \end{pmatrix}^T \Sigma_\infty e^{A^T s} \Sigma_\infty^{-1} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}$$

And correspondingly, we have

$$\int_0^\infty R_{xz}^{(0)}(s) e^{i\omega s} ds = \begin{pmatrix} I_{n \times n} \\ 0 \end{pmatrix}^T \Sigma_\infty \begin{pmatrix} D - i\omega M & -K \\ M & -i\omega M \end{pmatrix}$$
$$\times \begin{pmatrix} G(i\omega) & 0 \\ 0 & G(i\omega) \end{pmatrix} \Sigma_\infty^{-1} \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} \qquad (80)$$

Therefore (77) becomes obvious based upon (63), (78) and (80). □

Chapter 5

On the Stationary Probability Density: Multi-DOF Case

5.1 A Necessary and Sufficient Condition of Uncorrelatedness

A natural but nontrivial question to ask is in the stationary state, whether x and $\dot{x}$ are uncorrelated as they always are in single DOF case. The answer is *no*, in general. Such an example will be given after the following theorem. Then the next immediate question is under what conditions x and $\dot{x}$ are uncorrelated. Next, we concentrate on only linear model

$$M\ddot{x} + D\dot{x} + Kx = \sigma n(t) \tag{81}$$

for which we have the following conclusion

Theorem 7 *Assume the matrix $A_{2n \times 2n}$ is stable. Then, in stationary state, x and $\dot{x}$ are uncorrelated if and only if there exists a self-adjoint, nonnegative matrix $P_{n \times n}$*

satisfying

$$\begin{cases} KPM = MPK \\ DPK + KPD = \sigma\sigma^T \end{cases} \tag{82}$$

And in this case, the stationary density is 2n-dimensional Gaussian with mean 0 and variance

$$\Sigma_\infty = \begin{pmatrix} P & 0 \\ 0 & M^{-1}KP \end{pmatrix} \geq 0$$

PROOF: ($\Longrightarrow$) First we establish the following equation concerning the structure of the variance matrix Σ_∞

$$\Sigma_\infty = \begin{pmatrix} P_{11} & P_{12} \\ -P_{12} & M^{-1}(KP_{11} - DP_{12}) \end{pmatrix} \tag{83}$$

where we use the notation

$$\Sigma_\infty = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix}$$

In fact, by the equation,

$$\frac{d}{dt} \begin{pmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \begin{pmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{pmatrix}$$

we have the following relations

$$\dot{\Phi}_{12}(t) = \Phi_{22}(t) \tag{84}$$

$$\dot{\Phi}_{22}(t) = -M^{-1}K\Phi_{12}(t) - M^{-1}D\Phi_{22}(t) \tag{85}$$

Also, by the equation

$$\begin{aligned} \Sigma_\infty &= \int_0^\infty e^{At} \begin{pmatrix} 0 & 0 \\ 0 & M^{-1}\sigma\sigma^T M^{-1} \end{pmatrix} e^{A^T t} dt \\ &= \int_0^\infty \begin{pmatrix} \Phi_{12}(t)Q\Phi_{12}^T(t) & \Phi_{12}(t)Q\Phi_{22}^T(t) \\ \Phi_{22}(t)Q\Phi_{12}^T(t) & \Phi_{22}(t)Q\Phi_{22}^T(t) \end{pmatrix} dt \end{aligned}$$

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where $Q = M^{-1}\sigma\sigma^T M^{-1}$, we obtain, in the light of (84),

$$
\begin{aligned}
P_{12} &= \int_0^\infty \Phi_{12}(t) Q \Phi_{22}^T(t) dt \\
&= \int_0^\infty \Phi_{12}(t) Q \dot{\Phi}_{12}^T(t) dt \\
&= -\int_0^\infty \dot{\Phi}_{12}(t) Q \Phi_{12}^T(t) dt \\
&= -P_{12}^T
\end{aligned}
$$

i.e., P_{12} is skrew-symmetric.

Similarly, by (84) and (85), we have

$$
\begin{aligned}
P_{22} &= \int_0^\infty \Phi_{22}(t) Q \Phi_{22}^T(t) dt \\
&= \int_0^\infty \Phi_{22}(t) Q \dot{\Phi}_{12}^T(t) dt \\
&= -\int_0^\infty \dot{\Phi}_{22}(t) Q \Phi_{12}^T(t) dt \\
&= M^{-1} K P_{11} + M^{-1} D P_{12}^T \\
&= M^{-1}(K P_{11} - D P_{12})
\end{aligned}
$$

Noticing that, by (83),

$$x \text{ and } y \text{ being uncorrelated}$$

$$\Longleftrightarrow \quad P_{12} = 0$$

$$\Longleftrightarrow \quad \Sigma_\infty = \begin{pmatrix} P_{11} & 0 \\ 0 & M^{-1} K P_{11} \end{pmatrix}$$

and that Σ_∞ satisfying

$$A\Sigma_\infty + \Sigma_\infty A^T + \begin{pmatrix} 0 & 0 \\ 0 & M^{-1}\sigma\sigma^T M^{-1} \end{pmatrix} = 0 \qquad (86)$$

then it must be true that

$$M^{-1} K P_{11} = (M^{-1} K P_{11})^T$$

$$M^{-1} D M^{-1} K P_{11} + (M^{-1} D M^{-1} K P_{11})^T = M^{-1}\sigma\sigma^T M^{-1}$$

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which can be easily reduced to (82).

($\Longleftarrow$) Suppose $P \geq 0$ satisfies (82), then we know

$$\Sigma_\infty \stackrel{\text{def}}{=} \begin{pmatrix} P & 0 \\ 0 & M^{-1}KP \end{pmatrix}$$

satisfies (86). On the other hand, the solution of (86) is unique, based upon the following general fact [45, p527]: *if $A_{m \times m}$ and $B_{n \times n}$ have no common eigenvalues, then the matrix algebraic equation*

$$T_{n \times m} A_{m \times m} - B_{n \times n} T_{n \times m} = C_{n \times m}$$

has a unique solution $T_{n \times m}$. Therefore, the Σ_∞ defined above is indeed the stationary covariance matrix, which is block diagonal. $\qquad\square$

Now let us consider the following example in which $p_s(x, y)$ is not separable, i.e. x and y are not uncorrelated.

Take, for $x \in I\!\!R^2$,

$$
\begin{aligned}
M &= I_{2 \times 2} \\
D &= \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \quad \text{with } d_1, d_2 > 0 \\
K &= \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix} \quad \text{with } \omega_1^2 \neq \omega_2^2 \\
\sigma &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\end{aligned}
$$

Obviously the corresponding matrix A is stable. And it is easy to verify that

$$\left\{ A, \begin{pmatrix} 0 \\ \sigma \end{pmatrix} \right\}$$

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is completely controllable, i.e. $\Sigma_\infty > 0$.

Next, we observe that for this example,

$$KPM = MPK$$

$$\implies \quad PK = KP$$

$$\implies \quad P = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$$

However, on the other hand,

$$DPK + KPD = 2 \begin{pmatrix} d_1 p_1 \omega_1^2 & 0 \\ 0 & d_2 p_2 \omega_2^2 \end{pmatrix} \neq \sigma\sigma^T$$

i.e., there is no such $P_{2\times 2}$ satisfying (82). This, by *Theorem 3*, implies that x and $\dot{x}$ are not uncorrelated in steady state.

Next, we make the following observations:

1. (82) is equivalent to

$$\begin{cases} KPM = MPK \\ (-DM^{-1})(MPK) + (KPM)(-DM^{-1})^T + \sigma\sigma^T = 0 \end{cases} \tag{87}$$

If $D > 0$, then $-DM^{-1}$ is stable because $-DM^{-1}$ has the same eigenvalues as $-M^{-1/2}DM^{-1/2}$ which is negative definite. Then the unique solution of (87) is given by

$$MPK = KPM = \int_0^\infty e^{-DM^{-1}t}\sigma\sigma^T e^{-M^{-1}Dt} dt \tag{88}$$

Therefore, the necessary and sufficient condition for x and $\dot{x}$ to be uncorrelated in stationary state is that

$$P = M^{-1} \int_0^\infty e^{-DM^{-1}t}\sigma\sigma^T e^{-M^{-1}Dt} dt K^{-1}$$

is self-adjoint and nonnegative.

2. Let P be the self-adjoint, nonnegative matrix satisfying (82), then $P > 0 \Longleftrightarrow (DM^{-1}, \sigma)$ is completelly controllable.

3. In the particular case, $D = k_0 \sigma \sigma^T$, where k_0 is a scalar constant, we can take $P = \frac{1}{2k_0} K^{-1}$ in (82). In this case, the stationary probability density is a function of the energy $E = 1/2(x^T K x + y^T M y)$,

$$p_s(x,y) = (\frac{k_0}{\pi})^n (|M||K|)^{-1/2} \exp[-k_0(x^T K x + y^T M y)]$$

5.2 Energy Type Nonlinear Damping Model

In fact, this conclusion can be generalized to the following type nonlinear damping model.

Theorem 8 *In the following type nonlinear damping model*

$$M\ddot{x} + \mu(\frac{x^T K D x + y^T M D y}{2})\sigma\sigma^T D\dot{x} + Kx = \sigma n(t) \qquad (89)$$

we assume D is positive definite and commutative with K and M, and (89) is stable. Then the stationary probability density is given by

$$p_s(x,y) = p_s(E_D) = C \exp(-2\int_0^{E_D} \mu(z)dz) \qquad (90)$$

where

$$E_D = 1/2(x^T K D x + y^T M D y)$$
$$1/C = 2^n(|K||M||D|^2)^{-1/2}(\prod_{j=0}^{2(n-1)} I_j)\int_0^\infty x^{n-1}\exp(-2\int_0^x \mu(z)dz)dx \qquad (91)$$

with $I_j \stackrel{\text{def}}{=} \int_0^\pi \sin^j \theta d\theta$ which has the following iteration relation

$$I_0 = \pi$$

$$I_1 = 2$$

$$I_j = \frac{j-1}{j} I_{j-2} \quad j = 2, 3, \cdots$$

PROOF: Assume that $p_s(x, y)$ is a function of E_D, then

$$\nabla_x p_s(x, y) = \frac{dp_s}{dE_D} K D x$$

$$\nabla_y p_s(x, y) = \frac{dp_s}{dE_D} M D y$$

Also, we have the relation:

$$\text{tr}[M^{-1}\sigma\sigma^T M^{-1} \nabla_y^2 p_s(x, y)] = \nabla_y \cdot [M^{-1}\sigma\sigma^T M^{-1} \nabla_y p_s(x, y)]$$

Therefore, by noticing the relation

$$-y^T \nabla_x p_s + \nabla_y \cdot (M^{-1} K x p_s)$$

$$= -y^T \nabla_x p_s + x^T K M^{-1} \nabla_y p_s$$

$$= 0$$

the stationary Fokker-Planck equation

$$0 = -y^T \nabla_x p_s + \nabla_y \cdot [\mu(E_D) M^{-1}\sigma\sigma^T D y p_s + M^{-1} K x p_s] + \frac{1}{2}\text{tr}(M^{-1}\sigma\sigma^T M^{-1} \nabla_y^2 p_s)$$

$$(92)$$

is easily reduced to the following form

$$0 = \nabla_y \cdot [\mu(E_D) p_s M^{-1}\sigma\sigma^T D y + 1/2 M^{-1}\sigma\sigma^T D y \frac{dp_s}{dE_D}]$$

or

$$0 = \nabla_y \cdot [M^{-1}\sigma\sigma^T D y (\mu(E_D) p_s + 1/2 \frac{dp_s}{dE_D})]$$

Then it is sufficient for $p_s(x, y) = p_s(E_D)$ to satisfy

$$1/2 \frac{dp_s}{dE_D} + \mu(E_D)p_s = 0$$

or, equivalently,

$$p_s(x, y) = p_s(E_D) = C \exp(-2 \int_0^{E_D} \mu(z)dz)$$

The integrability of $p_s(x, y)$ on $I\!R^{2n}$ is guaranteed by the fact that both KD and MD are positive definite under the assumptions on D.

The normalizing constant C can be obtained by first making the following variable transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sqrt{KD} & 0 \\ 0 & \sqrt{MD} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

then changing (u, v) into $2n$-dimensional polar coordinate $(r, \phi_1, \cdots, \phi_{2n-1})$. $\quad\Box$

REMARK: It is quite natural that in muti-DOF case the stationary probability density is a function of E_D instead of E. Consider a simple example,

$$m_0 \ddot{x} + D\dot{x} + \omega_0^2 x = \sigma_0 n(t) \tag{93}$$

where m_0, ω_0^2, σ_0 are positive constants, while D is a positive definite $n \times n$ matrix.

Obviously, the corresponding undamped system is uncoupled and it is easy to realize that $D > 0$ implies (93) is stable.

In this case, (82) reduces to

$$DP + PD = \frac{\sigma_0^2}{\omega_0^2} I$$

therefore,

$$P = \frac{\sigma_0^2}{2\omega_0^2} D^{-1}$$

and

$$p_s(x, y) = \frac{(m_0 \omega_0^2)^{n/2}}{(\pi \sigma_0^2)^n} |D| \exp[-1/\sigma_0^2 (\omega_0^2 x^T D x + m_0 y^T D y)]$$

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5.3 An Illustrative Example

In this subsection, we consider the following special type of nonlinear damping model

$$M\ddot{x} + (d_0 + \gamma\mu(E))D\dot{x} + Kx = \sigma n(t) \qquad (94)$$

where $D = \sigma\sigma^T$, $d_0 > 0$ and $\mu(E) \geq 0$ for large E.

We need to find the four $n \times n$ matrices ψ_x, ψ_y, $R_{xx}(0)$ and $R_{xy}(0)$.

First, by *Theorem 4* we know that the covariance matrix of the corresponding linear system is given by

$$\Sigma_\infty = 1/2d_0 \begin{pmatrix} K^{-1} & 0 \\ 0 & M^{-1} \end{pmatrix}$$

and therefore, by definition,

$$
\begin{aligned}
\psi_y &= \int_{\mathbf{R}^{2n}} \mu(E) yy^T \sigma\sigma^T (\frac{d_0}{\pi})^n \sqrt{|K||M|}\exp[-d_0(x^T Kx + y^T My)]d|x|d|y| \\
&= \int_{\mathbf{R}^{2n}} \mu(\frac{||u||^2 + ||v||^2}{2})M^{-1/2}vv^T M^{-1/2}\sigma\sigma^T(\frac{d_0}{\pi})^n \\
&\qquad \times \exp[-d_0(||u||^2 + ||v||^2)]d|u|d|v| \\
&= M^{-1}\sigma\sigma^T \int_{\mathbf{R}^{2n}} v_1^2 \mu(\frac{||u||^2 + ||v||^2}{2})(\frac{d_0}{\pi})^n \\
&\qquad \times \exp[-d_0(||u||^2 + ||v||^2)]d|u|d|v| \\
&= M^{-1}\sigma\sigma^T \int_0^\infty dr \int_0^{2\pi} d\phi_{2n-1}\int_0^{2\pi}d\phi_{2n-2}\cdots\int_0^{2\pi}d\phi_1 r^2\cos^2\phi_1\mu(r^2/2) \\
&\qquad \times(\frac{d_0}{\pi})^n e^{-d_0 r^2} r^{2n-1}\sin^{2n-2}\phi_1 \cdots \sin\phi_{2n-2} \\
&= M^{-1}\sigma\sigma^T \int_0^\infty (\frac{2d_0}{\pi})^n x^n e^{-2d_0 x}\mu(x)dx \\
&\qquad \times 2I_0 I_1 \cdots I_{2n-3}(I_{2n-2} - I_{2n}) \\
&= M^{-1}\sigma\sigma^T \frac{\kappa_n}{n\pi^n}\prod_{j=0}^{2(n-1)} I_j
\end{aligned}
$$

where

$$\kappa_n \stackrel{\text{def}}{=} (2d_0)^n \int_0^\infty x^n \mu(x) e^{-2d_0 x} dx$$

Similarly, we obtain

$$\psi_x = 0$$

$$R_{xy}(0) = 0$$

By noticing that the corresponding stationary density $p_s(x, y)$ is given by (90) and (91), we obtain

$$R_{xx}(0) = m_\gamma K^{-1} \tag{95}$$

where

$$m_\gamma = \frac{1}{n} \frac{\int_0^\infty x^n \exp(-2d_0 x - 2\gamma \int_0^x \mu(z) dz) dx}{\int_0^\infty x^{n-1} \exp(-2d_0 x - 2\gamma \int_0^x \mu(z) dz) dx}$$

By substituting the computed matrices ψ_x, ψ_y, $R_{xx}(0)$ and $R_{xy}(0)$ into (77), we obtain

$$\begin{aligned}
\hat{\Psi}_{xx}(\omega) &= m_\gamma K^{-1}(D - i\omega M)G(i\omega) \\
&\quad + \gamma \frac{\kappa_n}{n\pi^n} \left(\prod_{j=0}^{2(n-1)} I_j \right) G(i\omega) \sigma \sigma^T G(i\omega) + O(\gamma^2)
\end{aligned}$$

Consequently, the spectral density matrix is given by

$$\begin{aligned}
\Phi_{xx}(\omega) &= \hat{\Psi}_{xx}(\omega) + \hat{\Psi}_{xx}^*(\omega) \\
&= 2d_0 m_\gamma G(-i\omega) \sigma \sigma^T G(i\omega) + \\
&\quad + 2\gamma \frac{\kappa_n}{n\pi^n} \left(\prod_{j=0}^{2(n-1)} I_j \right) \Re\{G(i\omega) \sigma \sigma^T G(i\omega)\} + O(\gamma^2) \tag{96}
\end{aligned}$$

By straightforward calculation, we know

$$m_\gamma = \frac{1}{2d_0} - \gamma \frac{\kappa_n}{d_0 n!} + O(\gamma^2)$$

Therefore, the perturbation expression of the spectral density matrix $\Phi_{xx}(\omega)$ is given by

$$
\begin{aligned}
\Phi_{xx}(\omega) \quad = \quad & G(-i\omega)\sigma\sigma^T G(i\omega) + 2\gamma\kappa_n[\frac{1}{n\pi^n}(\prod_{j=0}^{2(n-1)} I_j)\Re\{G(i\omega)\sigma\sigma^T G(i\omega)\} \\
& -\frac{1}{n!}G(-i\omega)\sigma\sigma^T G(i\omega)] + O(\gamma^2) \qquad\qquad (97)
\end{aligned}
$$

Chapter 6

Infinite Dimensional Nonlinear Damping Models

In this chapter, we study energy type nonlinear damping model in an infinite dimensional setting. According to the geometry of the structures considered, energy type nonlinear damping model is divided into two types. The following two examples are considered to be representatives of the two types of models, which, later on will be called *TYPE I* and *TYPE II* model, respectively.

6.1   Nonlinear damping models - the formulations

TYPE I Model:

We consider a uniform Bernoulli beam with length L, and both ends hinged. Let

$u(t,s)$ denote the small deflection. Then the undamped model is given by

$$\begin{cases} \ddot{u}(t,s) + a^2 u'''''(t,s) = 0 & 0 \le s \le L \\ u(t,0) = u''(t,0) = 0 \\ u(t,L) = u''(t,L) = 0 \end{cases} \tag{98}$$

where super-dots represent derivatives with respect to time t, and the primes derivatives with respect to s.

We introduce the Hilbert space $H = L^2[0,L]$ and the inner product defined on it

$$[u,v] = \int_0^L u(s)v(s)ds$$

Let the operator A be defined by

$$Au = a^2 u'''''(t,s)$$

with domain

$$\mathcal{D}(A) = \{u \in H \mid u''''' \in L^2[0,L]; u(0) = u''(0) = u(L) = u''(L) = 0\}$$

Then, it is easy to see that

$$A^{1/2}u = -au''(t,s)$$

for

$$u \in \mathcal{D}(A^{1/2}) = \{u \in H \mid u'' \in L^2[0,L]; u(0) = u(L) = 0\}$$

And the total energy, the sum of strain energy and kinetic energy, is given by

$$\begin{aligned} E(t) &= 1/2([Au,u] + ||\dot{u}||^2) \\ &= 1/2 \int_0^L [a^2 u''(t,s)^2 + \dot{u}(t,s)^2]ds \end{aligned}$$

If we use proportional damping $A^{1/2}$ as linear damping and $E^q(t)$, $q > 0$ as nonlinear damping, then the total damping is

$$2\xi A^{1/2}\dot{u} + \gamma E^q(t)\dot{u}$$
$$= -2\xi a \dot{u}''(t,s) + \gamma\{1/2 \int_0^L [a^2 u''(t,s\prime)^2 + \dot{u}(t,s\prime)^2]ds\prime\}^q$$

Using $x(t)$ to denote $u(t,\cdot)$, we can write the complete model in the following abstract form

$$\ddot{x}(t) + 2\xi A^{1/2}\dot{x}(t) + \gamma E(t)^q \dot{x}(t) + Ax(t) = 0 \tag{99}$$

TYPE II Model:

We consider a flexible beam with a tip mass m at one end and with the other end clamped. We may assume that either the motion occurs only in the horizontal plane or the structure is in micro-g state. Therefore we do not have to take into account the effect of gravity.

The undamped model is given by

$$\ddot{u}(t,r) + \frac{EI}{\rho}u''''(t,r) = 0 \tag{100}$$

$$\ddot{u}(t,L) - \frac{EI}{m}u'''(t,L) = 0 \tag{101}$$

$$u(t,0) = u\prime(t,0) = u''(t,L) = 0 \tag{102}$$

Notations: $u(t,r)$ transverse displacement at a spatial point $0 \leq r \leq L$ at time $t \geq 0$;

E	Young's modulus of the arm material;
I	area moment of inertia of the arm material;
ρ	density of the arm material;
m	mass of the payload including the end effector;
L	length of the arm material.

Now, in order to give a complete formulation of the problem, let us introduce the Hilbert space

$$\mathcal{H} = L^2[0, L] \times I\!R^1$$

and define the operator A by

$$Aw = \begin{pmatrix} a^2 u''''(\cdot) \\ -b^2 u'''(L) \end{pmatrix} \quad \text{for} \quad \forall\, w = \begin{pmatrix} u(\cdot) \\ u(L) \end{pmatrix} \in \mathcal{D}(A)$$

with

$$\mathcal{D}(A) = \{ w = \begin{pmatrix} u(\cdot) \\ u(L) \end{pmatrix} \in \mathcal{H} | u''''(\cdot) \in L^2[0, L], u(0) = u'(0) = u''(L) = 0 \}$$

where

$$a^2 = EI/\rho; \quad b^2 = EI/m$$

The inner product on $\mathcal{H}$ is defined by

$$[w_1, w_2] = \frac{1}{a^2} \int_0^L u_1(r) u_2(r) dr + \frac{1}{b^2} u_1(L) u_2(L)$$

for

$$w_j = \begin{pmatrix} u_j(\cdot) \\ u_j(L) \end{pmatrix} \in \mathcal{H}, \quad j = 1, 2$$

Then, by integration by parts, it is very easy to verify that

$$[Aw, w] = \int_0^L |u''(r)|^2 dr \geq 0 \quad \forall w \in \mathcal{D}(A)$$

and $[Aw, w] = 0$ if and only if $w = 0$, i.e., the operator A is self-adjoint, positive definite and A^{-1} exists and is compact. Then by the spectral theorem of positive self-adjoint operators with compact resolvent, there is a sequence of increasing eigenvalues

$$\omega_1^2 \leq \omega_2^2 \leq \cdots$$

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associated with the corresponding eigenfunctions $\{\psi_n\}$ such that

$$A\psi_n = \omega_n^2 \psi_n$$

and $\{\psi_n\}$ form an orthonormal basis in $\mathcal{H}$.

With the above preparation, (100)-(102) can be rewritten as

$$\ddot{w}(t) + Aw(t) = 0 \qquad\qquad (103)$$

For linear damping part, we choose the following asymptotically proportional damping

$$Dw = \begin{pmatrix} -au''(\cdot) \\ b^2/au'(L) \end{pmatrix}$$

with domain

$$\mathcal{D}(D) = \{w = \begin{pmatrix} u(\cdot) \\ u(L) \end{pmatrix} \in H \mid u'' \in L^2[0,L]; u(0) = 0\}$$

And it is easy to verify that

$$[Dw, w] = 1/a \int_0^L |u'(s)|^2 ds \geq 0$$

i.e., D is positive definite on $\mathcal{D}(D)$.

For nonlinear damping part, we consider the energy possessed by the beam alone

$$
\begin{aligned}
E_b(t) &= E(t) - \frac{1}{2b^2}\dot{u}(t,L)^2 \\
&= 1/2(\int_0^L |u''(t,s)||^2 ds + 1/a^2 \int_0^L |\dot{u}(t,s)|^2 ds)
\end{aligned}
$$

$E_b(t)$ is the energy of the beam itself, the sum of strain energy and kinetic energy.

Then we use $\gamma E_b^q(t)\dot{w}(t)$ as the energy type damping. We notice that $1/(2b^2)\dot{u}(t,L)^2$ is essentially the kinetic energy of the tip mass. The reason of excluding the kinetic energy of the tip mass in damping is that internal damping mechanism should be dependent upon the beam material, not on any attatchments.

energy of the tip mass in damping is that internal damping mechanism should be dependent upon the beam material, not on any attatchments.

Then the abstract form of the damped model writes

$$\ddot{w}(t) + 2\xi D\dot{w}(t) + \gamma E_b^q(t)\dot{w}(t) + Ax(t) = 0 \tag{104}$$

Next, let us consider the basic modes and frequencies of the model (103). We will explicitly solve the eigenvalue problem

$$A\psi_n = \omega_n^2\psi_n \quad \psi_n \in \mathcal{D}(A) \tag{105}$$

Let

$$\psi_n = \begin{pmatrix} \phi_n(\cdot) \\ \phi_n(L) \end{pmatrix} \in \mathcal{D}(A)$$

then (105) is equivalent to the following two point boundary value problem (TPBVP)

$$\begin{cases} a^2 \phi_n{''''}(r) = \omega_n^2 \phi_n(r) \\ b^2 \phi_n{'''}(L) + \omega_n^2 \phi_n(L) = 0 \\ \phi_n(0) = \phi_n{'}(0) = \phi_n{''}(L) = 0 \end{cases} \tag{106}$$

Let $\omega_n \stackrel{\text{def}}{=} a(\frac{\beta_n}{L})^2$. Then (106) can be rewritten as

$$\phi_n{''''}(r) = (\frac{\beta_n}{L})^4 \phi_n(r) \tag{107}$$

$$(\frac{b}{a})^2 \phi_n{'''}(L) + (\frac{\beta_n}{L})^4 \phi_n(L) = 0 \tag{108}$$

$$\phi_n(0) = \phi_n{'}(0) = \phi_n{''}(L) = 0 \tag{109}$$

The general solution of (107) is given by

$$\phi_n(r) = c_{1,n} \cos\frac{\beta_n}{L}r + c_{2,n} \cosh\frac{\beta_n}{L}r + c_{3,n} \sin\frac{\beta_n}{L}r + c_{4,n} \sinh\frac{\beta_n}{L}r$$

where $c_{i,n}$, $i = 1, 2, 3, 4$ and β_n are to be determined by (108) and (109).

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(109) implies that

$$\phi_n(r) = \frac{1}{c_n}[(\cosh\frac{\beta_n}{L}r - \cos\frac{\beta_n}{L}r) - \gamma_n(\sinh\frac{\beta_n}{L}r - \sin\frac{\beta_n}{L}r)]$$

where c_n is a normalizing constant and

$$\gamma_n = \frac{\cosh\beta_n + \cos\beta_n}{\sinh\beta_n + \sin\beta_n} > 0$$

In order for ϕ_n to satisfy (108), β_n must satisfy the following transcendental equation

$$\sinh\beta_n - \sin\beta_n - \gamma_n(\cosh\beta_n + \cos\beta_n)$$
$$+\frac{m}{\rho}\frac{\beta_n}{L}[\cosh\beta_n - \cos\beta_n - \gamma_n(\sinh\beta_n - \sin\beta_n)] = 0 \qquad (110)$$

which is equivalent to

$$1 + \cos\beta_n\cosh\beta_n + \frac{m}{\rho}\frac{\beta_n}{L}(\cos\beta_n\sinh\beta_n - \sin\beta_n\cosh\beta_n) = 0 \qquad (111)$$

It is easy to realize that

$$\beta_n \longrightarrow n\pi + \pi/4 \quad \text{as } n \to \infty$$

i.e.,

$$\omega_n^2 = a^2(\frac{\beta_n}{L})^4 \longrightarrow a^2(\frac{\pi}{L})^4(n+1/4)^4 \quad \text{as } n \to \infty$$

Finally, the normalizing constants c_n should be chosen such that $||\psi_n|| = 1$. By tedious but straightforword calculation, we obtain

$$c_n = a[L + \frac{\rho}{m}(\frac{L}{\beta_n})^2(\frac{1 + \cos\beta_n\cosh\beta_n}{\sin\beta_n + \sinh\beta_n})^2]^{1/2} \quad n = 1, 2, \cdots$$

Therefore, the eigenvalue problem (105) has been solved.

6.2 Some basic results

Notations:

$$E(t) = 1/2([A^{1/2}x(t), A^{1/2}x(t)] + [\dot{x}(t), \dot{x}(t)]); \qquad \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \in \mathcal{H}_E = \mathcal{D}(A^{1/2}) \otimes H$$

$$E_D(t) = 1/2([Ax(t), Dx(t)] + [D\dot{x}(t), \dot{x}(t)]); \qquad \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \in \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \otimes \mathcal{D}(D)$$

$$\Gamma(w) = \begin{pmatrix} 0 \\ -\gamma E^q(t)v \end{pmatrix}; \qquad w = \begin{pmatrix} u \\ v \end{pmatrix}$$

Then the nonlinear damping model

$$\ddot{x}(t) + 2\xi D\dot{x}(t) + \gamma E^q(t)\dot{x}(t) + Ax(t) = 0$$

can be recast into the form

$$\frac{dw(t)}{dt} = \mathcal{A}w(t) + \Gamma(w(t)) \tag{112}$$

The other notations are just as usual.

6.2.1 Existence and uniqueness

A. Lunardi [53] established the existence and uniqueness of solutions of the nonlinear infinite dimensional system

$$\dot{w}(t) = \mathcal{L}w(t) + G(w(t))$$

by assuming

1. $\mathcal{L}$ generates an analytic semigroup, which is asymptotically stable, i.e.,

$$\sup\{\Re(\lambda); \quad \lambda \in \sigma(\mathcal{L})\} = -\omega_0 < 0$$

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2. $G(0) = 0$, and $G'(0) = 0$;

3. If $w(\cdot) \in C([t_0, t_1]; \; \mathcal{D}(\mathcal{L}))$, then $G(w(\cdot)) \in C([t_0, t_1]; \; D_L(\theta, \infty))$, where $D_L(\theta, \infty)$ is the interpolation space defined by

$$
\begin{aligned}
D_L(\theta, \infty) &= \{x | x \in X, [x]_\theta = \sup_{0 < t \le 1} ||t^{1-\theta} \mathcal{L} e^{t\mathcal{L}} x|| < \infty\} \\
||x||_{D_L(\theta, \infty)} &= ||x|| + [x]_\theta
\end{aligned}
$$

The proof is essentially the application of the fixed point theorem of contraction mapping and interpolation space theory [18][64].

Therefore, to establish the existence and uniqueness of the solution of (112), it suffices to verify the third assumption.

In fact, for any

$$
w(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \in C([t_0, t_1]; \; \mathcal{D}(\mathcal{A}))
$$

$$
\begin{aligned}
[\Gamma(w(\cdot))]_\theta &= \sup_{0 < t \le 1} ||t^{1-\theta} \mathcal{A} T(t) \Gamma(w(\cdot))|| \\
&\le \sup_{0 < t \le 1} ||T(t) \mathcal{A} \begin{pmatrix} 0 \\ -\gamma ||w(\cdot)||^{2q} v(\cdot) \end{pmatrix} || \\
&= \gamma ||w(\cdot)||^{2q} \sup_{0 < t \le 1} ||T(t) \begin{pmatrix} -v(\cdot) \\ 2\xi Dv(\cdot) \end{pmatrix} || \\
&< \infty
\end{aligned}
$$

Hence,

$$
\begin{aligned}
||\Gamma(w(\cdot))||_{D_L(\theta, \infty)} &= ||\Gamma(w(\cdot))|| + [\Gamma(w(\cdot))]_\theta \\
&\le \gamma ||w(\cdot)||^{2q+2} + [\Gamma(w(\cdot))]_\theta \\
&< \infty
\end{aligned}
$$

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Theorem 9 *For $\forall w_0$ such that $||w_0|| \leq r$ for some $r > 0$, there exists a unique solution $w(\cdot) \in C^1([0,\infty); \mathcal{H}_E) \cap C([0,\infty); \mathcal{D}(\mathcal{A}))$ which satisfies (112).*

6.2.2 Asymptotic Stability

Theorem 10 *Assume that $T(t)$, the semigroup generated by $\mathcal{A}$, is asymptotically stable, i.e.*

$$||T(t)|| \leq e^{-\omega t}; \quad \omega \in (0, \omega_0)$$

for some $\omega_0 > 0$. Then, $\forall \eta \in (0, \omega_0)$, there exists $r = r(\eta) > 0$ such that for any $w_0 \in \mathcal{D}(\mathcal{A})$ with $||w_0|| \leq r$, we have

$$||w(t)|| \leq ||w_0|| e^{-\eta t}$$

Proof: For any $\eta \in (0, \omega_0)$, choose $\beta > 0$ such that $\eta_1 = \eta + \beta < \omega_0$.

Let $r = (\frac{\beta}{\gamma})^{\frac{1}{2q+1}}$. Then, for $\forall w_0 \in \mathcal{D}(\mathcal{A})$ with $||w_0|| \leq r$, we have

$$
\begin{aligned}
||w(t)|| &= ||T(t)w_0 + \gamma \int_0^t T(t-s)\Gamma(w(s))ds|| \\
&\leq e^{-\eta_1 t}||w_0|| + \gamma \int_0^t e^{-\eta_1(t-s)}||w(s)||^{2q+2}ds
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
||w(t)||e^{\eta_1 t} &\leq ||w_0|| + \gamma \int_0^t e^{\eta_1 s}||w(s)||^{2q+2}ds \\
&\leq ||w_0|| + \gamma||w_0||^{2q+1} \int_0^t e^{\eta_1 s}||w(s)||ds
\end{aligned}
$$

Then, by Gronwall's inequality, we obtain

$$
\begin{aligned}
||w(t)|| &\leq ||w_0|| \exp[-(\eta_1 - \gamma||w_0||^{2q+1})t] \\
&\leq ||w_0|| e^{-\eta t}
\end{aligned}
$$

$\square$

6.3 An infinite dimensional Krylov-Bogoliubov approximation

6.3.1 Preliminary results

Lemma 7 *Suppose A generates an analytic semigroup $T(t)$, which is also asymptotically stable, i.e., there exists $\eta > 0$ such that*

$$\|T(t)\| \leq e^{-\eta t}; \quad t \geq 0$$

Then we have the following estimates for $[D\dot{x}(t), \dot{x}(t)]$:

1. *For linear damping model*

$$\ddot{x}(t) + 2\xi D\dot{x}(t) + Ax(t) = 0$$

we have

$$2\xi[D\dot{x}(t), \dot{x}(t)] \leq \|w_0\|\frac{M_1}{t}e^{-\eta t}(2E(t))^{1/2}, \quad t > 0$$

2. *For nonlinear damping model*

$$\ddot{x}(t) + 2\xi D\dot{x}(t) + \gamma E^q(t)\dot{x}(t) + Ax(t) = 0$$

if

$$A^{1/2}D = DA^{1/2} \text{ on } \mathcal{D}(A) \tag{113}$$

$$A^{1/2} \leq \rho D \text{ on } \mathcal{D}(D) \tag{114}$$

for some $\rho > 0$, then we have

$$2\xi[D\dot{x}(t), \dot{x}(t)] \leq M_2(2E(t))^{1/2}, \quad t \geq 0$$

where M_2 depends on ξ, γ, η, ρ and the initial state $x(0), \dot{x}(0)$.

Proof: (1) Since $T(\cdot)$ is analytic, we have [15][58]

$$\|\mathcal{A}T(t)\| \leq \frac{M_1}{t}e^{-\eta t}, \quad t > 0$$

Then,

$$
\begin{aligned}
2\xi[D\dot{x}(t), \dot{x}(t)] &= -[\frac{\mathcal{A} + \mathcal{A}^*}{2}w(t), w(t)] \\
&= -[\mathcal{A}w(t), w(t)] \\
&\leq \|\mathcal{A}w(t)\| \, \|w(t)\| \\
&= \|\mathcal{A}T(t)w_0\| \, \|w(t)\| \\
&\leq \frac{M_1}{t}e^{-\eta t}\|w_0\|(2E(t))^{1/2}
\end{aligned}
$$

(2) To prove the nonlinear version, we consider the quantity $E_D(t)$ defined by

$$E_D(t) = 1/2([Ax(t), Dx(t)] + [D\dot{x}(t), \dot{x}(t)])$$

By (113), we can show that $[Ax, Dx] \geq 0$ for $x \in \mathcal{D}(A)$, and therefore

$$E_D(t) \geq 0, \quad t \geq 0$$

And, we further have

$$
\begin{aligned}
\frac{dE_D(t)}{dt} &= -2\xi\|D\dot{x}(t)\|^2 - \gamma E^q(t)[D\dot{x}(t), \dot{x}(t)] \\
&\leq 0, \quad t \geq 0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_D(0) - E_D(t) &= 2\xi \int_0^t \|D\dot{x}(s)\|^2 ds + \gamma \int_0^t E^q(s)[D\dot{x}(s), \dot{x}(s)]ds \\
&\leq E_D(0)
\end{aligned}
$$

In particular,

$$2\xi \int_0^\infty \|D\dot{x}(t)\|^2 dt \leq E_D(0)$$

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i.e., $D\dot{x}(\cdot) \in L^2[0, \infty; H]$.

In addition, by (113) and (114), we can have that for any $v \in \mathcal{D}(A^{1/2})$,

$$
\begin{aligned}
\|A^{1/2}v\|^2 &= \sum_{n=0}^{\infty} \omega_n^2 |[v, \phi_n]|^2 \\
&\leq \rho^2 \sum_{n=0}^{\infty} [D\phi_n, \phi_n]^2 |[v, \phi_n]|^2 \\
&= \rho^2 \|Dv\|^2
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
&\| \int_0^t \mathcal{A}T(t-s)\Gamma(w(s))ds\| \\
&= \| \int_0^t T(t-s)\mathcal{A}\Gamma(w(s))ds\| \\
&= \| \int_0^t T(t-s)\gamma E^q(s) \begin{pmatrix} \dot{x}(s) \\ -2\xi D\dot{x}(s) \end{pmatrix} ds\| \\
&\leq \gamma E^q(0) \int_0^t e^{\eta(t-s)}[\|A^{1/2}\dot{x}(s)\|^2 + 4\xi^2\|D\dot{x}(s)\|^2]^{1/2}ds \\
&\leq \gamma E^q(0)(4\xi^2 + \rho^2)^{1/2} \int_0^t e^{-\eta(t-s)}\|D\dot{x}(s)\|ds \\
&\leq \gamma E^q(0)(4\xi^2 + \rho^2)^{1/2}[\int_0^t e^{-2\eta(t-s)}ds]^{1/2}\|D\dot{x}\|_{L^2[0,\infty; H]} \\
&\leq \frac{\gamma}{2}(\frac{4\xi}{\eta} + \frac{\rho^2}{\xi\eta})^{1/2}E^q(0)E_D^{1/2}(0) \\
&= \gamma C_0
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
2\xi[D\dot{x}(t), \dot{x}(t)] &= -[\mathcal{A}w(t), w(t)] \\
&\leq \|\mathcal{A}w(t)\| \|w(t)\| \\
&\leq [\|\mathcal{A}T(t)w_0\| + \| \int_0^t \mathcal{A}T(t-s)\Gamma(w(s))ds\|]\|w(t)\| \\
&\leq [\frac{M_1}{t}\|w_0\|e^{-\eta t} + \gamma C_0](2E(t))^{1/2} \\
&\leq M_2(2E(t))^{1/2}
\end{aligned}
$$

which completes the proof. □

S. Chen and R. Triggiani [31] have given the sufficient conditions on D which guarantee the analyticity of $T(t)$ generated by $\mathcal{A}$. That is, if for $1/2 \leq \alpha \leq 1$, it holds

$$\rho_1 A^\alpha \leq D \leq \rho_2 A^\alpha$$

for some constants $0 < \rho_1 < \rho_2 < \infty$, then $T(t)$ is analytic. And it is further proved that for $0 \leq \alpha < 1/2$, the semigroup is not analytic.

Theorem 11 *For the following nonlinear damping model,*

$$\ddot{x}(t) + 2\xi A^{1/2}\dot{x}(t) + \gamma E^q(t)\dot{x}(t) + Ax(t) = 0$$

its solution satisfies

$$x(t) = \exp(-\frac{\gamma}{2}\int_0^t E^q(s)ds)S(t)y(t)$$

where $S(t) = e^{-\xi\sqrt{A}t}$, and $y(t)$ satisfies the following undamped equation with exponentially decaying parametric excitation

$$\ddot{y}(t) + [(1 - \xi^2)A - \xi\gamma E^q(t)A^{1/2} + \gamma^2\theta(t)]y(t) = 0 \tag{115}$$

where $\theta(t)$ is a function of t which goes to zero exponentially as $t \to \infty$, if $q > 1/2$.

Proof: If we let

$$x(t) = \exp(-\frac{\gamma}{2}\int_0^t E^q(s)ds)S(t)y(t)$$

then it is not hard to verify that $y(t)$ satisfies (115) with

$$\theta(t) = -1/4E^{2q}(t) + qE^{2q-1}(t)||\dot{x}(t)||^2 + \frac{q}{\gamma}2\xi[D\dot{x}(t), \dot{x}(t)]E^{q-1}(t)$$

Then using *Lemma 1*, we have

$$|\theta(t)| \leq (1/4 + 2q)E^{2q}(t) + \sqrt{2}qE^{q-1/2}(t)[\frac{M_1}{\gamma t}||w_0||e^{-\eta t} + C_0]$$

Therefore, if $q > 1/2$, $|\theta(t)|$ exponentially decays to zero as long as $E(t)$ does. □

6.3.2 The Krylov-Bogoliubov approximation

Since we are interested in the effect of nonlinear damping, we now neglect the linear damping part, i.e., let $\zeta = 0$. In this case, we desire to establish a Krylov-Bogoliubov type approximation in our infinite dimensional setting. As we know, Krylov-Bogoliubov approximation technique has been widely used and its accuracy is often satisfactory. However, it has not been generalized to multi-DOF models. Krylov-Bogoliubov approximation is an averaging method. In multi-DOF case, average method does not seem to make sense because more than one natural frequencies exist. What we are going to do is to make use of the special form of nonlinear damping - energy type damping.

We first consider *TYPE I* model,

$$\ddot{x}(t) + \gamma E^q(t)\dot{x}(t) + Ax(t) = 0 \tag{116}$$

By *Theorem 3*, we already know that

$$x(t) = a(t)y(t)$$

where

$$a(t) = \exp(-\gamma/2 \int_0^t E^q(s)ds)$$

and $y(t)$ solves

$$\begin{cases} \ddot{y}(t) + (A + \gamma^2\theta(t))y(t) = 0 \\ y(0) = x(0) \\ \dot{y}(0) = \dot{x}(0) + \gamma/2E^q(0)x(0) \end{cases} \tag{117}$$

in which $\theta(\cdot)$ is uniformly bounded,

$$|\theta(t)| \le (1/4 + 2q)E^{2q}(t) \le (1/4 + 2q)E^{2q}(0)$$

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Next, we need to find appropriate $a_0(t)$ and $y_0(t)$ such that $x(t)$ is approximated by

$$x_0(t) = a_0(t)y_0(t)$$

First, since $a(t)$ is slowly varying, we make our first approximation

$$\dot{x}(t) \approx a(t)\dot{y}(t)$$

Consequently,

$$\dot{x}(0) = \dot{y}(0) \tag{118}$$

Next, we make our second approximation by letting $y(t) \approx y_0(t)$, where $y_0(t)$ is obtained by dropping $\gamma^2\theta(t)$ in (117) and replacing the second initial condition in (117) by (118). That is, y_0 satisfies

$$\begin{cases} \ddot{y}_0(t) + Ay_0(t) = 0 \\ y_0(0) = x_0(0) \\ \dot{y}_0(0) = \dot{x}(0) \end{cases} \tag{119}$$

Based upon the above two approximations we made, we obtain

$$\begin{aligned} E(t) &= a^2(t)/2([Ay_0(t), y_0(t)] + \|\dot{y}_0(t)\|^2) \\ &= a^2(t)\frac{\|Y_0(t)\|_E^2}{2} \\ &= a^2(t)E(0) \end{aligned}$$

where

$$Y_0(t) = \begin{pmatrix} y_0(t) \\ \dot{y}_0(t) \end{pmatrix}$$

Then, by definition,

$$a(t) = \exp(-\gamma/2E^q(0)\int_0^t a^{2q}(s)ds) \tag{120}$$

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We naturally let the solution of the above integral equation be $a_0(t)$.

Therefore, differentiating (120), we realize that $a_0(t)$ solves the following initial value problem

$$\begin{cases} \dot{a}_0(t) = -\frac{\gamma}{2}E^q(0)a_0^{2q+1}(t) \\ a_0(0) = 1 \end{cases} \tag{121}$$

One can easily find the solution as

$$a_0(t) = (1 + \gamma q E^q(0)t)^{-\frac{1}{2q}}$$

Therefore, the Krylov-Bogoliubov type approximation of *TYPE I* model is given by

$$x_0(t) = (1 + \gamma q E^q(0)t)^{-\frac{1}{2q}} y_0(t)$$

From this Krylov-Bogoliubov approximation, we can see, without linear damping, the free response of *TYPE I* model still goes to zero, and the decay is in the order of $t^{-\frac{1}{2q}}$. Furthermore, larger value of q implies slower decay.

For *TYPE II* model, similar result can be obtained.

In fact, through the same procedure as above, we can see that the $y_0(t)$ we choose should also satisfy (119) in this case, while the integral equation for $a_0(t)$ is slightly different.

Under the same approximations, we have

$$\begin{aligned} E_b(t) &= E(t) - \frac{1}{2b^2}\dot{u}^2(t,L) \\ &= a_0^2(t)[E(0) - \frac{1}{2b^2}\dot{u}_0^2(t,L)] \end{aligned}$$

where

$$y_0(t) = \begin{pmatrix} u_0(t,s) \\ u_0(t,L) \end{pmatrix} \qquad 0 \leq s \leq L$$

Then, the integral equation for $a_0(t)$ is given by

$$a_0(t) = \exp(-\gamma/2 \int_0^t E_b^q(t\prime)dt\prime)$$
$$= \exp[-\gamma/2 \int_0^t a_0^{2q}(t\prime)(E(0) - \frac{1}{2b^2}\dot{u}_0^2(t\prime, L))^q dt\prime]$$

from which, one can easily find,

$$a_0(t) = [1 + \gamma q \int_0^t (E(0) - \frac{1}{2b^2}\dot{u}_0^2(t\prime, L))^q dt\prime]^{-\frac{1}{2q}}$$

Thus, the Krylov-Bogoliubov approximation for *TYPE II* model is obtained.

Notice that since $\dot{u}_0(t.L)$ is periodic in t and

$$E(0) - \frac{1}{2b^2}\dot{u}_0^2(t, L) = 1/2\|Y_0(t)\|_E^2 - \frac{1}{2b^2}\dot{u}_0^2(t, L)$$
$$= 1/2[Ay_0(t), y_0(t)] + \frac{1}{2a^2} \int_0^L |\dot{u}_0(t, s)|^2 ds$$
$$\geq 0$$

we can realize that as $t \longrightarrow \infty$,

$$\int_0^t [E(0) - \frac{1}{2b^2}\dot{u}_0^2(t\prime, L)]^q dt\prime = O(t)$$

Therefore, the Krylov-Bogoliubov approximation indicates that the free response of *TYPE II* model also goes to zero at the rate of $O(t^{-\frac{1}{2q}})$ as $t \longrightarrow \infty$.

In the above, we have used the simple fact that if $f(t) \geq 0$ and is periodic with period T, then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(t\prime)dt\prime = \frac{1}{T} \int_0^T f(t\prime)dt\prime$$

6.3.3 The Error Estimate

Next, the natural question we ask is what the error of the Krylov-Bogoliubov approximation is. For simplicity, we only study the error in *TYPE I* model.

Before the estimation of the error, we first introduce a correcttion term $\tilde{y}_0$ in the initial condition of (119) so that $y_0(t)$ now satisfies

$$\begin{cases} \ddot{y}_0(t) + Ay_0(t) = 0 \\ y_0(0) = x(0) \\ \dot{y}_0(0) = \dot{x}(0) + \tilde{y}_0 \end{cases} \tag{122}$$

The purpose of introducing the correction term $\tilde{y}_0$ in the initial condition will be explained later.

Now our result is

Theorem 12 *For TYPE I model, the error of the Krylov-Bogoliubov approximation is given by*

$$\|x(t) - x_0(t)\| \le a(t)\|A^{-1/2}(\gamma/2E^q(0)x(0) - \tilde{y}_0)\| + \gamma^2 B_0(t,\gamma) \tag{123}$$

where

$$\lim_{\gamma \to 0} B_0(t,\gamma) \le \frac{\sqrt{2}}{\omega_1} E^{2q+1/2}(0)[\frac{1/4 + 2q}{\omega_1} + q/2]t$$

The error contains two parts, one part is of the form $C\gamma^2 t$. Therefore if $t \le 1/\gamma$, then this part of the error is still of the order of $O(\gamma)$. The other part of the error, $a(t)\|A^{-1/2}(\gamma/2E^q(0)x(0) - \tilde{y}_0)\|$, is decaying with time because $a(t)$ is. In addition, by choosing

$$\tilde{y}_0 = \gamma/2E^q(0)x(0) + O(\gamma^2)$$

we can keep the first part of the error in the order of $O(\gamma^2)$, hence the total error is of the form $O(\gamma^2) \times (t+1)$.

To further understand the reason of introducing the correction term $\tilde{y}_0$, recall that the first approximation we made in the derivation of the Krylov-Bogoliubov approximation is

$$\dot{x}(t) = a(t)\dot{y}(t)$$

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since $a(t)$ is slowly varying. Based upon this approximation, the initial condition

$$\dot{y}(0) = \dot{x}(0) + \gamma/2E^q(0)x(0)$$

is replaced by

$$\dot{y}_0(0) = \dot{x}(0)$$

This change of intial condition, on one hand, made the following derivation possible, on the other hand, introduces an error (in the order of $O(\gamma)$).

Through error estimate, we realize that this error can be minimized by choosing

$$\tilde{y}_0 = \gamma/2E^q(0)x(0)$$

Therefore, the correction term $\tilde{y}_0$ plays the role of compensating for the error introduced by the "slow varying" approximation.

PROOF: First we estimate $\|y(t) - y_0(t)\|$. We know that

$$
\begin{aligned}
y(t) &= C(t)x(0) + S(t)(\dot{x}(0) + \gamma/2E^q(0)x(0)) - \gamma^2 \int_0^t \theta(\tau)S(\tau)y(\tau)d\tau \\
y_0(t) &= C(t)x(0) + S(t)(\dot{x}(0) + \tilde{y}_0)
\end{aligned}
$$

where $C(t) : H \longrightarrow H$ and $S(t) : H \longrightarrow \mathcal{D}(A^{1/2})$ are cosine and sine operators defined by, $\forall x \in H$,

$$
\begin{aligned}
C(t)x &= \sum_{n=1}^{\infty} \cos \omega_n t [x, \psi_n]\psi_n \\
S(t)x &= \sum_{n=1}^{\infty} \frac{\sin \omega_n t}{\omega_n} [x, \psi_n]\psi_n
\end{aligned}
$$

Obviously, we have

$$\|S(t)x\| \le \|A^{-1/2}x\| \quad \text{for} \quad x \in H$$

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Then,

$$\|y(t) - y_0(t)\|$$

$$\leq \quad \|S(t)(\gamma/2E^q(0)x(0) - \tilde{y}_0)\| + \gamma^2(1/4 + 2q)E^{2q}(0)\int_0^t \|S(\tau)y(\tau)\|d\tau$$

$$\leq \quad \|A^{-1/2}(\gamma/2E^q(0)x(0) - \tilde{y}_0)\| + \gamma^2(1/4 + 2q)E^{2q}(0)\int_0^t \|A^{-1/2}y(\tau)\|d\tau \quad (124)$$

Secondly, we estimate $|a(t) - a_0(t)|$. For this purpose, we consider

$$Y(t) = \mathcal{U}(t)\begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} + \mathcal{U}(t)\begin{pmatrix} 0 \\ \gamma/2E^q(0)x(0) \end{pmatrix} - \gamma^2\int_0^t \mathcal{U}(t-\tau)\theta(\tau)\begin{pmatrix} 0 \\ y(\tau) \end{pmatrix}d\tau$$

where

$$Y(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} \quad \text{and} \quad \mathcal{U}(t) = \begin{pmatrix} C(t) & S(t) \\ -AS(t) & C(t) \end{pmatrix}$$

is a unitary operator from $\mathcal{H}_E$ to $\mathcal{H}_E$.

From the above equation, we can easily know that

$$\|Y(t)\|_E^2 = 2E(0) + \gamma E^q(0)[x(0), \dot{x}(0)] + O(\gamma^2) \quad (125)$$

In addition, from

$$\dot{x}(t) = -\gamma/2E^q(t)a(t)y(t) + a(t)\dot{y}(t)$$

we know that

$$\|\dot{x}(t)\|^2 \quad = \quad a^2(t)\|\dot{y}(t) - \gamma/2E^q(t)y(t)\|^2$$

$$= \quad a^2(t)\{\|\dot{y}(t)\|^2 - \gamma E^q(t)[y(t), \dot{y}(t)] + \gamma^2/4E^{2q}(t)\|y(t)\|^2\} \quad (126)$$

Then, by (125) and (126), we can rewrite the energy corresponding to *TYPE I* model (116) as

$$E(t) \quad = \quad 1/2\{[Ax(t), x(t)] + \|\dot{x}(t)\|^2\}$$

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$$= 1/2a^2(t)\{[Ay(t), y(t)] + \|\dot{y}(t)\|^2 - \gamma E^q(t)[y(t), \dot{y}(t)] + \gamma^2/4E^{2q}(t)\|y(t)\|^2\}$$

$$= a^2(t)\{1/2\|Y(t)\|_E^2 - \gamma/2E^2(t)[y(t), \dot{y}(t)] + O(\gamma^2)\}$$

$$= a^2(t)[E(0) + \gamma f(t) + O(\gamma^2)]$$

where

$$f(t) = 1/2\{E^q(0)[x(0), \dot{x}(0)] - E^q(t)[y(t), \dot{y}(t)]\}$$

Then, $a(t)$ can be rewritten as

$$
\begin{aligned}
a(t) &= \exp(-\gamma/2 \int_0^t E^q(\tau)d\tau) \\
&= \exp\{-\gamma/2 \int_0^t a^{2q}(\tau)[E(0) + \gamma f(\tau) + O(\gamma^2)]^q d\tau\}
\end{aligned}
$$

from which, $a(t)$ can be easily solved to be

$$
\begin{aligned}
a(t) &= [1 + \gamma q \int_0^t [E(0) + \gamma f(\tau) + O(\gamma^2)]^q d\tau]^{-\frac{1}{2q}} \\
&= [1 + \gamma q E^q(0)t + \gamma^2 q^2 E^{q-1}(0) \int_0^t f(\tau)d\tau + O(\gamma^3)]^{-\frac{1}{2q}}
\end{aligned}
$$

Therefore, we obtain

$$|a(t) - a_0(t)| \le \gamma^2 q/2 E^{q-1}(0)[\int_0^t f(\tau)d\tau + O(\gamma)] \tag{127}$$

Finally, using (124) and (127), we obtain the error estimate as

$$
\begin{aligned}
\|x(t) - x_0(t)\| &\le a(t)\|y(t) - y_0(t)\| + |a(t) - a_0(t)| \, \|y_0(t)\| \\
&\le a(t)\|A^{-1/2}(\gamma/2E^q(0)x(0) - \tilde{y}_0)\| \\
&\quad + \gamma^2(1/4 + 2q)a(t)E^{2q}(0) \int_0^t \|A^{-1/2}y(\tau)\|d\tau \\
&\quad + \gamma^2 q/2 E^{q-1}(0)[\int_0^t |f(\tau)|d\tau + O(\gamma)] \, \|y_0(t)\| \\
&= a(t)\|A^{-1/2}(\gamma/2E^q(0)x(0) - \tilde{y}_0)\| + \gamma^2 B_0(t, \gamma)
\end{aligned}
$$

where

$$B_0(t,\gamma) = (1/4 + 2q)a(t)E^{2q}(0)\int_0^t ||A^{-1/2}y(\tau)||d\tau$$
$$+ q/2E^{q-1}(0)[\int_0^t |f(\tau)|d\tau + O(\gamma)]||y_0(t)||$$

By noticing that for $\gamma \longrightarrow 0$,

$$|f(t)| \leq 1/2E^q(0)(||x(0)|| \; ||\dot{x}(0)|| + ||y_0(t)|| \; ||\dot{y}_0(t)||)$$

$$\leq \frac{E^q(0)}{2\omega_1}(||A^{1/2}x(0)|| \; ||\dot{x}(0)|| + ||A^{1/2}y_0(t)|| \; ||\dot{y}_0(t)||)$$

$$\leq \frac{E^q(0)}{2\omega_1}(\frac{||A^{1/2}x(0)||^2 + ||\dot{x}(0)||^2}{2} + \frac{||A^{1/2}y_0(t)||^2 + ||\dot{y}_0(t)||^2}{2})$$

$$\leq \frac{1}{\omega_1}E^{q+1}(0)$$

we can obtain by easy calculation that

$$\lim_{\gamma \to 0} B_0(t,\gamma) \leq \frac{\sqrt{2}}{\omega_1}[q/2 + \frac{1/4 + 2q}{\omega_1}]E^{2q+1/2}(0)t$$

$\square$

6.4 Frequency response - single frequency excitation case

We consider the following infinite dimensional nonlinear damping model

$$\ddot{x}(t) + 2\xi A^{1/2}\dot{x}(t) + \gamma([Ax(t), x(t)] + ||\dot{x}(t)||^2)^q\dot{x}(t) + Ax(t) = E(t) \qquad (128)$$

For the reason of simplicity, we consider strictly proportional linear damping and total energy (*TYPE I*) nonlinear damping so that we can easily obtain its modal decomposition.

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We are interested in the stationary amplitude response of each mode to a single sinusoidal input to the system. To specify the excitation input $E(t)$, let $B : \mathbb{R}^1 \longrightarrow H$ be a linear bounded operator. Then we let the excitation be of the form

$$E(t) = \epsilon B \cos \omega t$$

where $\epsilon > 0$ is a small parameter in the order of ξ and γ. That is, we let

$$\xi = \xi_0 \epsilon; \qquad \gamma = \gamma_0 \epsilon$$

We order the input amplitude as $O(\epsilon)$ for the following reasons:

1. For a weakly damped system, a small amplitude $(O(\epsilon))$ excitation at the natural frequency produces a relatively large $(O(1))$ amplitude response.

2. In the actural system, large oscillation are limited by damping. Thus to obtain a uniformly valid approximate solution of this problem, we need to order the external excitation amplitude such that, in the perturbation procedure, excitation term appears in the same equation as damping terms.

Therefore, we are considering the stationary response of

$$\ddot{x}(t) + \epsilon[2\xi_0 A^{1/2} \dot{x}(t) + \gamma_0 E^q(t)\dot{x}(t)] + Ax(t) = \epsilon B \cos \omega t \qquad (129)$$

The method we will use for the following analysis is *multiple scale method*. Define

$$T_0 = t; \qquad T_1 = \epsilon t$$

and the notations

$$D_0 f(T_0, T_1) = \frac{\partial f}{\partial T_0}; \qquad D_1 f(T_0, T_1) = \frac{\partial f}{\partial T_1}$$

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Let the solution of (129) be

$$x(t) = x_0(T_0, T_1) + \epsilon x_1(T_0, T_1) + \cdots$$

then

$$\dot{x}(t) = D_0 x_0 + \epsilon(D_1 x_0 + D_0 x_1) + O(\epsilon^2) \qquad (130)$$

$$\ddot{x}(t) = D_0^2 x_0 + \epsilon(2D_0 D_1 x_0 + D_0^2 x_1) + O(\epsilon^2) \qquad (131)$$

Then substituting (130)-(131) into (129) and equating those terms withe same order, we obtain

$$\epsilon^0 \quad : \quad D_0^2 x_0(T_0, T_1) + A x_0(T_0, T_1) = 0$$

$$\epsilon^1 \quad : \quad D_0^2 x_1(T_0, T_1) + A x_1(T_0, T_1)$$

$$= -2D_0 D_1 x_0 - 2\xi_0 A^{1/2} D_0 x_0 - \gamma_0([A x_0, x_0] + ||D_0 x_0||^2)^q D_0 x_0 + B \cos \omega t$$

Obviously, $x_0(T_0, T_1)$ and $x_1(T_0, T_1)$ can be written in the form

$$x_0(T_0, T_1) = \sum_{m=1}^{\infty} a_m(T_0, T_1)\psi_m$$

$$x_1(T_0, T_1) = \sum_{m=1}^{\infty} b_m(T_0, T_1)\psi_m$$

Then, $\{a_n(T_0, T_1); \ n = 1, 2, \cdots\}$ and $\{b_n(T_0, T_1); \ n = 1, 2, \cdots\}$ satisfy

$$D_0^2 a_n(T_0, T_1) + \omega_n^2 a_n(T_0, T_1) = 0$$

$$D_0^2 b_n(T_0, T_1) + \omega_n^2 b_n(T_0, T_1)$$

$$= -2D_0 D_1 a_n - 2\xi_0 \omega_n D_0 a_n - \gamma_0 [\sum_{m=1}^{\infty} \omega_m^2 a_m^2 + (D_0 a_m)^2]^q D_0 a_n + B^* \psi_n \cos \omega t$$

$$n = 1, 2, \cdots$$

Obviously, for $n = 1, 2, \cdots$,

$$a_n(T_0, T_1) = A_n(T_1)e^{i\omega_n T_0} + \bar{A}_n(T_1)e^{-i\omega_n T_0}$$

from which, we know

$$\sum_{m=1}^{\infty} \omega_m^2 a_m^2(T_0, T_1) + [D_0 a_m(T_0, T_1)]^2$$

$$= \sum_{m=1}^{\infty} \omega_m^2 |2A_m(T_1)|^2 \qquad (132)$$

Then the equations for $b_n(T_0, T_1)$'s become

$$D_0^2 b_n(T_0, T_1) + \omega_n^2 b_n(T_0, T_1)$$

$$= e^{i\omega_n T_0}[-2i\omega_n A_n\prime(T_1) - 2i\xi_0\omega_n^2 A_n(T_1) - \gamma_0(\sum_{m=1}^{\infty} \omega_m^2 |2A_m|^2)^q i\omega_n A_n(T_1)]$$

$$+ B^*\psi_n/2e^{i\omega t} + cc \qquad (133)$$

where cc stands for complex conjugate of the preceding terms.

Next, we assume that the excitation frequency ω is close to a particular natural frequency, say ω_1, i.e., without loss of generality,

$$\omega = \omega_1 + \epsilon\sigma$$

where σ is a detuning parameter, quantitatively describing the nearness of ω to ω_1.

Then $b_1(T_0, T_1)$ satisfies

$$D_0^2 b_1(T_0, T_1) + \omega_1^2 b_1(T_0, T_1)$$

$$= e^{i\omega_1 T_0}[-2i\omega_1 A_1\prime(T_1) - 2i\xi_0\omega_1^2 A_1(T_1) - \gamma_0(\sum_{m=1}^{\infty} \omega_m^2 |2A_m|^2)^q A_1(T_1) \qquad (134)$$

$$+ B^*\psi_1/2e^{i\sigma T_1}] + cc \qquad (135)$$

If the coefficient of $e^{i\omega_1 T_0}$ is not zero, then the right hand side of (134) is of the form $f(T_1)\cos\omega_1 T_0$, which would produce terms like $T_0^m \cos\omega_1 T_0$ and/or $T_0^m \sin\omega_1 T_0$. Such terms are called *secular terms*, which are obviously unbounded. Since a positively damped system cannot have unbounded response corresponding to a bounded input, secular terms should not appear. Therefore, in order to eliminate

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secular terms, we need to choose $A_1(T_1)$ such that the coefficient of $e^{i\omega_1 T_0}$ vanishes, i.e.,

$$-2i\omega_1 A_1\prime(T_1) - 2i\xi_0\omega_1^2 A_1(T_1) - \gamma_0(\sum_{m=1}^{\infty} \omega_m^2|2A_m|^2)^q i\omega_1 A_1(T_1)$$
$$+B^*\psi_1/2e^{i\sigma T_1} = 0 \tag{136}$$

For the same reason, for $n \geq 2$, $A_n(T_1)$ should satisfy, noticing $\omega = \omega_1 + \epsilon\sigma$ is away from ω_n,

$$- 2i\omega_n A_n\prime(T_1) - 2i\xi_0\omega_n^2 A_n(T_1) - \gamma_0(\sum_{m=1}^{\infty} \omega_m^2|2A_m|^2)^q i\omega_n A_n(T_1) = 0 \tag{137}$$

and hence, for $n \geq 2$, $b_n(T_0, T_1)$ satisfy

$$D_0^2 b_n(T_0, T_1) + \omega_n^2 b_n(T_0, T_1) = B^*\psi_n \cos\omega t \tag{138}$$

Next, in order to solve $\{A_n(T_1); \ n = 1, 2, \cdots \}$ from (136) and (137), let

$$A_n(T_1) = \frac{1}{2}\rho_n(T_1)e^{i\beta_n(T_1)}, \qquad n = 1, 2, \cdots$$

Then substituting $A_n(T_1)$ into (136) and (137), collecting the real and the imaginary parts, one obtain the following systems of differential equations for $\{\rho_n(T_1), \ \beta_n(T_1); \ n = 1, 2, \cdots\}$

$$\begin{cases} \frac{d\rho_1(T_1)}{dT_1} = -\xi_0\omega_1\rho_1(T_1) - \gamma_0/2(\sum_{m=1}^{\infty}\omega_m^2\rho_m^2(T_1))^q\rho_1(T_1) \\ \\ \qquad\qquad +\frac{B^*\psi_1}{2\omega_1}\sin\gamma_1(T_1) \\ \\ \rho_1(T_1)\frac{d\gamma_1(T_1)}{dT_1} = \sigma\rho_1(T_1) + \frac{B^*\psi_1}{2\omega_1}\cos\gamma_1(T_1) \end{cases} \tag{139}$$

$$\begin{cases} \frac{d\rho_n(T_1)}{dT_1} = -\xi_0\omega_n\rho_n(T_1) - \gamma_0/2(\sum_{m=1}^{\infty}\omega_m^2\rho_m^2(T_1))^q\rho_n(T_1) \\ \\ \rho_n(T_1)\frac{d\gamma_n(T_1)}{dT_1} = \sigma\rho_n(T_1) \qquad n = 2, 3, \cdots \end{cases} \tag{140}$$

It is obvious that for each solution of (140), it holds

$$\lim_{T_1 \to \infty} \rho_n(T_1) = 0, \quad n \geq 2$$

For (139), by setting $\frac{d\rho_1(T_1)}{dT_1} = 0$ and $\frac{d\gamma_1(T_1)}{dT_1} = 0$, we know that at steady state, ρ_1 and γ_1 satisfy

$$\begin{cases} 0 = -\xi_0 \omega_1 \rho_1 - \gamma_0/2(\omega_1 \rho_1)^{2q} \rho_1 + \frac{B^* \psi_1}{2\omega_1} \cos \gamma_1 \\ \\ 0 = \sigma \rho_1 + \frac{B^* \psi_1}{2\omega_1} \cos \gamma_1 \end{cases} \tag{141}$$

Then, eliminating γ_1 from (141), we obtain the equation for the steady state harmonic amplitude of the first mode

$$\rho_1^2 \{\sigma^2 + [\xi_0 \omega_1 + \gamma_0/2(\omega_1 \rho_1)^{2q}]^2\} = (\frac{B^* \psi_1}{2\omega_1})^2 \tag{142}$$

This is called *frequency response equation*. It is easy to see that the frequency response equation has a unique positive solution ρ_1 for each value of σ, denoted by $g(\sigma|\omega_1, B^* \psi_1)$. The plot of ρ_1 in terms of σ is called *frequency response curve*. From (142) we can see that $\rho_1(\sigma) = g(\sigma|\omega_1, B^* \psi_1)$ is an even function of σ.

In fact, the existence and uniqueness of a positive solution of (142) becomes trivial if one rewrites (142) in the form

$$\rho_1^2 = \frac{[B^* \psi_1/(2\omega_1)]^2}{\sigma^2 + [\xi_0 \omega_1 + \gamma_0/2(\omega_1 \rho_1)^{2q}]^2} \tag{143}$$

Next, let us examine the steady state response of the other modes. We already know that $\rho_n = 0$ for $n \geq 2$ at steady state. But what about the $O(\epsilon)$ order term? Solving $b_n(T_0, T_1)$ from (138) gives

$$b_n(T_0, T_1) = C_n(T_1) \cos(\omega_n T_0 + \theta_n(T_1)) + \frac{B^* \psi_n}{\omega_n^2 - \omega^2} \cos \omega T_0$$

Then it is not hard to realize that $C_n(T_1) \longrightarrow 0$ as $T_1 \longrightarrow \infty$.

Therefore, for $\omega = \omega_1 + \epsilon\sigma$, the nth $(n \geq 2)$ mode stationary amplitude is of order $O(\epsilon)$, given by

$$\rho_n = \epsilon\frac{|B^*\psi_n|}{|\omega_n^2 - \omega^2|} + O(\epsilon^2)$$

Similarly, we can infer that if the excitation frequency ω is away from ω_1, then the steady state amplitude of the first mode is given by

$$\rho_1(\omega) = \epsilon\frac{|B^*\psi_1|}{|\omega_1^2 - \omega^2|} + O(\epsilon^2)$$

In summary, the frequency response curve of any mode, corresponding to a single frequency ω excitation is given by

$$\rho_n(\omega) = \begin{cases} g(\sigma|\omega_n, B^*\psi_n) & \text{when} \quad \omega = \omega_n + \epsilon\sigma \\[2em] \epsilon\frac{|B^*\psi_n|}{|\omega_n^2-\omega^2|} & \text{when} \quad |\omega - \omega_n| = O(1) \end{cases} \tag{144}$$

6.5 Frequency response - multi-frequency excitation case

Next, we consider the case in which the external excitation $E(t)$ contains M distinct frequencies, and each of them is close to a natural frequency. Say, they are $\omega_n + \epsilon\sigma_n$, $n = 1, 2, \cdots M$. Then the excitation takes the form

$$E(t) = \epsilon\sum_{n=1}^{M} f_n B \cos(\omega_n t + \epsilon\sigma_n t + \tau_n)$$

where f_n and τ_n are certain constants.

Through similar procedures and with the same notations as in the single frequency excitation case, we can obtain

$$D_0^2 b_n(T_0, T_1) + \omega_n^2 b_n(T_0, T_1)$$

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$C\text{-}2$

$$= e^{i\omega_n T_0}[-2i\omega_n A_n\prime(T_1) - 2i\xi_0\omega_n^2 A_n(T_1) - \gamma_0(\sum_{m=1}^{\infty} \omega_m^2|2A_m|^2)^q i\omega_n A_n]$$

$$+ B^*\psi_n/2\sum_{j=1}^{M} f_j e^{i\omega_j T_0 + i\sigma_j T_1 + i\tau_j} \qquad n = 1, 2, \cdots$$

In order to eliminate secular terms, $A_n(T_1)$ should satisfy

$$-2i\omega_n A_n\prime(T_1) - 2i\xi_0\omega_n^2 A_n(T_1) - \gamma_0(\sum_{m=1}^{\infty} \omega_m^2|2A_m|^2)^q i\omega_n A_n$$

$$+ B^*\psi_n/2 f_n e^{i(\sigma_n T_1 + \tau_n)} = 0$$

$$\text{for } n = 1, 2, \cdots, M$$

$$-2i\omega_n A_n\prime(T_1) - 2i\xi_0\omega_n^2 A_n(T_1) - \gamma_0(\sum_{m=1}^{\infty} \omega_m^2|2A_m|^2)^q i\omega_n A_n(T_1) = 0$$

$$\text{for } n = M+1, M+2, \cdots$$

Similarly, by letting

$$A_n(T_1) = \rho_n(T_1)/2 e^{i\beta_n(T_1)} \qquad n = 1, 2, \cdots$$

we can know that

$$\lim_{T_1 \to \infty} \rho_n(T_1) = 0 \quad \text{for } n \geq M+1$$

and for $1 \leq n \leq M$,

$$\frac{d\rho_n(T_1)}{dT_1} = -\xi_0\omega_n\rho_n(T_1) - \gamma_0/2(\sum_{m=1}^{\infty} \omega_m^2\rho_m^2(T_1))^q \rho_n(T_1)$$

$$+ B^*\psi_n/(2\omega_n)f_n \sin\gamma_n(T_1)$$

$$\rho_n(T_1)\frac{d\gamma_n(T_1)}{dT_1} = \sigma_n\rho_n(T_1) + B^*\psi_n/(2\omega_n)f_n \cos\gamma_n(T_1)$$

where $\gamma_n(T_1) = \sigma_n T_1 + \tau_n - \beta_n(T_1)$.

Again, after reaching stationarity, ρ_n, $1 \leq n \leq M$ satisfy

$$\{[\xi_0\omega_n + \gamma_0/2(\sum_{j=1}^{M} \omega_j^2\rho_j^2)^q]^2 + \sigma_n^2\}^2\rho_n^2 = (\frac{f_n B^*\psi_n}{2\omega_n})^2 \tag{145}$$

Therefore, in the case of multi-frequency excitation, the stationary harmonic response amplitudes ρ_n, $1 \le n \le M$ satisfy the above coupled nonlinear equation.

In order to study the solutions of (145), we introduce the notation

$$G = \gamma_0/2(\sum_{j=1}^{M} \omega_j^2 \rho_j^2)^q$$

Then, from (145), we obviously have

$$\omega_n^2 \rho_n^2 = \frac{(f_n B^* \psi_n/2)^2}{(\xi_0 \omega_n + G)^2 + \sigma_n^2} \qquad n = 1, 2, \cdots, M \qquad (146)$$

On both sides of (146). summing up from 1 to M, raising to the power q and multiplying by $\gamma_0/2$, one obtains

$$G = \gamma_0/2[\sum_{n=1}^{M} \frac{(f_n B^* \psi_n/2)^2}{(\xi_0 \omega_n + G)^2 + \sigma_n^2}]^q \qquad (147)$$

It is easy to see that for each $\vec{\sigma}^2 = (\sigma_1^2, \sigma_2^2, \cdots, \sigma_M^2)$, (147) has a unique positive solution denoted by $G(\vec{\sigma}^2)$. Therefore, for $1 \le n \le M$, ρ_n satisfies

$$\rho_n^2(\vec{\sigma}^2) = \frac{(f_n B^* \psi_n/(2\omega_n))^2}{\sigma_n^2 + [\xi_0 \omega_n + G(\vec{\sigma}^2)]^2} \qquad (148)$$

$\rho_n(\vec{\sigma}^2)$ is single valued because of the uniqueness of $G(\vec{\sigma}^2)$ for each $\vec{\sigma}^2$.

To study the behavior of $\rho_n(\vec{\sigma}^2)$, we first notice that by differentiating (147) with respect to σ_j^2, we realize that

$$\frac{\partial G(\vec{\sigma}^2)}{\partial \sigma_j^2} < 0 \qquad j = 1, 2, \cdots, M$$

i.e., $G(\vec{\sigma}^2)$ is decreasing as σ_j^2 increases with the other σ_k's fixed.

Next, we restrict our attention to the case $M = 2$, the interation of the first two modes. As σ_2^2 increases, $G(\sigma_1, \sigma_2)$ decreases, and hence $\rho_1(\sigma_1, \sigma_2)$ increases. As σ_1^2 increases, $\rho_2(\sigma_1, \sigma_2)$ increases because $G(\sigma_1, \sigma_2)$ decreases. Then it must be the case that $\rho_1(\sigma_1, \sigma_2)$ decreases, since $G(\sigma_1, \sigma_2)$ decreases. Therefore the plot of $\rho_1(\sigma_1, \sigma_2)$

is a saddle surface. For fixed σ_2, the curve $\rho_1(\cdot, \sigma_2)$ is still bell shaped, reaching maximum at $\sigma_1 = 0$. As σ_2^2 decreases, the energy possessed by the second mode becomes bigger which results in bigger damping in the nonlinear energy type damping, therefore due to coupling the first mode amplitude becomes smaller. Similarly, $\rho_2(\sigma_1, \sigma_2)$ is also a saddle surface.

Numerical Results

For any fixed n, $1 \leq n \leq M$, in order to find the frequency response $\rho_n(\vec{\sigma}^2)$ from (148), one has to resort to numerical methods to find $G(\vec{\sigma}^2)$ first for each $\vec{\sigma}^2$. Here, we pointed out that if we use fixed point iteration starting from 0, then we have monotone convergence, hence avoiding the common phenomenon in solving nonlinear algebraic equation that the convergence depends on the choice of initial data.

Specifically, we define

$$\Gamma(\vec{\sigma}^2; x) = \gamma_0/2 [\sum_{n=1}^{M} \frac{(f_n B^* \psi_n/2)^2}{\sigma_n^2 + (\xi_0 \omega_n + x)^2}]^q$$

and

$$\begin{cases} \Gamma_0(\vec{\sigma}^2) &= \Gamma(\vec{\sigma}^2; 0) \\ \Gamma_n(\vec{\sigma}^2) &= \Gamma(\vec{\sigma}^2; \Gamma_{n-1}(\vec{\sigma}^2)), \quad n = 1, 2, \cdots \end{cases}$$

It can be shown that $\Gamma_{2n}(\vec{\sigma}^2)$ is monotone decreasing as n increases, and $\Gamma_{2n+1}(\vec{\sigma}^2)$ is monotone increasing as n increases. In addition, it holds

$$\Gamma_{2n+1}(\vec{\sigma}^2) \leq G(\vec{\sigma}^2) \leq \Gamma_{2n}(\vec{\sigma}^2), \quad n = 0, 1, 2, \cdots \tag{149}$$

In fact, first it is obvious that

$$\Gamma_1(\vec{\sigma}^2) \leq G(\vec{\sigma}^2) \leq \Gamma_0(\vec{\sigma}^2)$$

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Then, we can easily show that

$$G(\vec{\sigma}^2) \leq \Gamma_2(\vec{\sigma}^2) = \Gamma(\vec{\sigma}^2; \Gamma_1(\vec{\sigma}^2)) \leq \Gamma(\vec{\sigma}^2; 0) = \Gamma_0(\vec{\sigma}^2)$$

$$G(\vec{\sigma}^2) \geq \Gamma_3(\vec{\sigma}^2) = \Gamma(\vec{\sigma}^2; \Gamma_2(\vec{\sigma}^2)) \geq \Gamma(\vec{\sigma}^2; \Gamma_0(\vec{\sigma}^2)) = \Gamma_1(\vec{\sigma}^2)$$

Then, by assuming

$$G(\vec{\sigma}^2) \leq \Gamma_{2n}(\vec{\sigma}^2) \leq \Gamma_{2n-2}(\vec{\sigma}^2)$$

$$G(\vec{\sigma}^2) \geq \Gamma_{2n+1}(\vec{\sigma}^2) \geq \Gamma_{2n-1}(\vec{\sigma}^2)$$

we obtain

$$G(\vec{\sigma}^2) \leq \Gamma_{2n+2}(\vec{\sigma}^2) = \Gamma(\vec{\sigma}^2; \Gamma_{2n+1}(\vec{\sigma}^2)) \leq \Gamma(\vec{\sigma}^2; \Gamma_{2n-1}(\vec{\sigma}^2)) = \Gamma_{2n}(\vec{\sigma}^2)$$

$$G(\vec{\sigma}^2) \geq \Gamma_{2n+3}(\vec{\sigma}^2; \Gamma_{2n+2}(\vec{\sigma}^2)) \geq \Gamma(\vec{\sigma}^2; \Gamma_{2n}(\vec{\sigma}^2)) = \Gamma_{2n+1}(\vec{\sigma}^2)$$

Therefore, by induction, we have shown that $\{\Gamma_{2n}(\vec{\sigma}^2)\}$ ($\{\Gamma_{2n+1}(\vec{\sigma}^2)\}$) is monotone decreasing (increasing), and (149) holds.

A few numerical examples are made, in which we have chosen $M = 3$, $q = 3/2$, $\gamma_0 = 2$, $f_n B^* \psi_n = 2$, for $n = 1, 2, 3$. From the computer tests we realize that when ξ_0 is not small, the convergence is quite satisfactory, while the convergence is very slow for small ξ_0. Therefore, in choosing ϵ, we should let $\epsilon = \xi/k$ where $k \geq 2$ to guarantee fast convergence.

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	$\xi_0 = 2$ $\sigma_1 = 0$ $\sigma_2 = 0$ $\sigma_3 = 0$	$\xi_0 = 2$ $\sigma_1 = 1$ $\sigma_2 = 1$ $\sigma_3 = 1$	$\xi_0 = 1$ $\sigma_1 = 1$ $\sigma_2 = 2$ $\sigma_3 = 3$	$\xi_0 = 0$ $\sigma_1 = 3$ $\sigma_2 = 2$ $\sigma_3 = 1$
Solution	0.333	0.236	0.448	0.848
Γ_0	0.662	0.323	0.811	1.588
Γ_1	0.195	0.211	0.278	0.379
Γ_2	0.433	0.244	0.563	1.356
Γ_3	0.280	0.234	0.384	0.483
Γ_4	0.368	0.237	0.488	1.241
Γ_5	0.314	0.236	0.424	0.547
Γ_6	0.346	0.236	0.462	1.169
Γ_7	0.326	0.236	0.439	0.592
Γ_8	0.338	0.236	0.453	1.118
Γ_9	0.331	0.236	0.445	0.627

By ploting $\rho_1(\sigma_1^2, \sigma_2^2)$, we verified that it is indeed a saddle surface. However, the increase in σ_2 direction for fixed σ_1 is very small, which is also clear from the corresponding contour plot. The decrease of $\rho_1(\sigma_1^2, \sigma_2^2)$ in σ_1 direction is much more significant than that in σ_2 direction. This indicates that even though there exists coupling due to nonlinear damping, the coupling effect is rather weak.

Chapter 7

Active Damping of Flexible Structures via Saturating Acturators

7.1 Introduction

In the previous chapters we have studied nonlinear passive damping problem. In applications, active dampers are often used to enhance the stability of a flexible structures. Moreover, due to actuator saturations, active damping become non-linear. Therefore, it is also important to study nonlinear active damping, as well as passive damping. In this chapter, we study the active damping aspect of the SCOLE problem [4] - a recent NASA project. The primary objective of the SCOLE (Spacecraft Control Laboratory Experiment) problem includes the task of directing the line-of-sight of the shuttle/antenna configuration towards a fixed target (*Figure 1*). Due to the facts of very small passive damping in the supporting truss structure

and the micro-g environment in the orbit, structure vibration is inevitable after each slewing maneuver. In order to maintain the prescribed pointing accuracry of the antenna line-of-sight, active dampers are required to enhance the structure stability. Only active damping problem will be studied and colocated sensor/actuator arrangement will be used. The controls are force and moments actuators and the sensors are rate gyros.

A distributed parameter model is used in this investigation. In Section 2, the continuum model and problem formulation are presented.

In Section 3, a group of rather weak sufficient conditions for strong stabilizability is presented. Although other sufficient conditions for strong stabilizability has been obtained in [7], the present conditions are much weaker in the sense that internal damping is no longer required to be positive definite, and, in fact, can even be zero while strong stabilizability can still be achieved by active damping.

In order to understand the nature of active damping, in particular, the nature of saturation type active damping, the mode excitation problem is studied in Section 4. In this study, some notions in classical feedback control such as *Characteristic Equation* and *Root Locus* are extended to our distributed parameter system, which is a feature of this work. As we will see, the root locus can provide an insight of the nature of active damping. The root locus method has been a powerful and useful approach for the analysis and design of finite dimensional control systems. However, the notions of *Characteristic equation* and its *Root locus* have not been extended to the study of distributed parameter systems, due to the difficulty caused by its infinite dimensional nature. In this work, we have taken the advantage of the fact that, although the system is infinite dimensional, the number of actuators and sensors used is always finite (here, we have excluded those applications in which
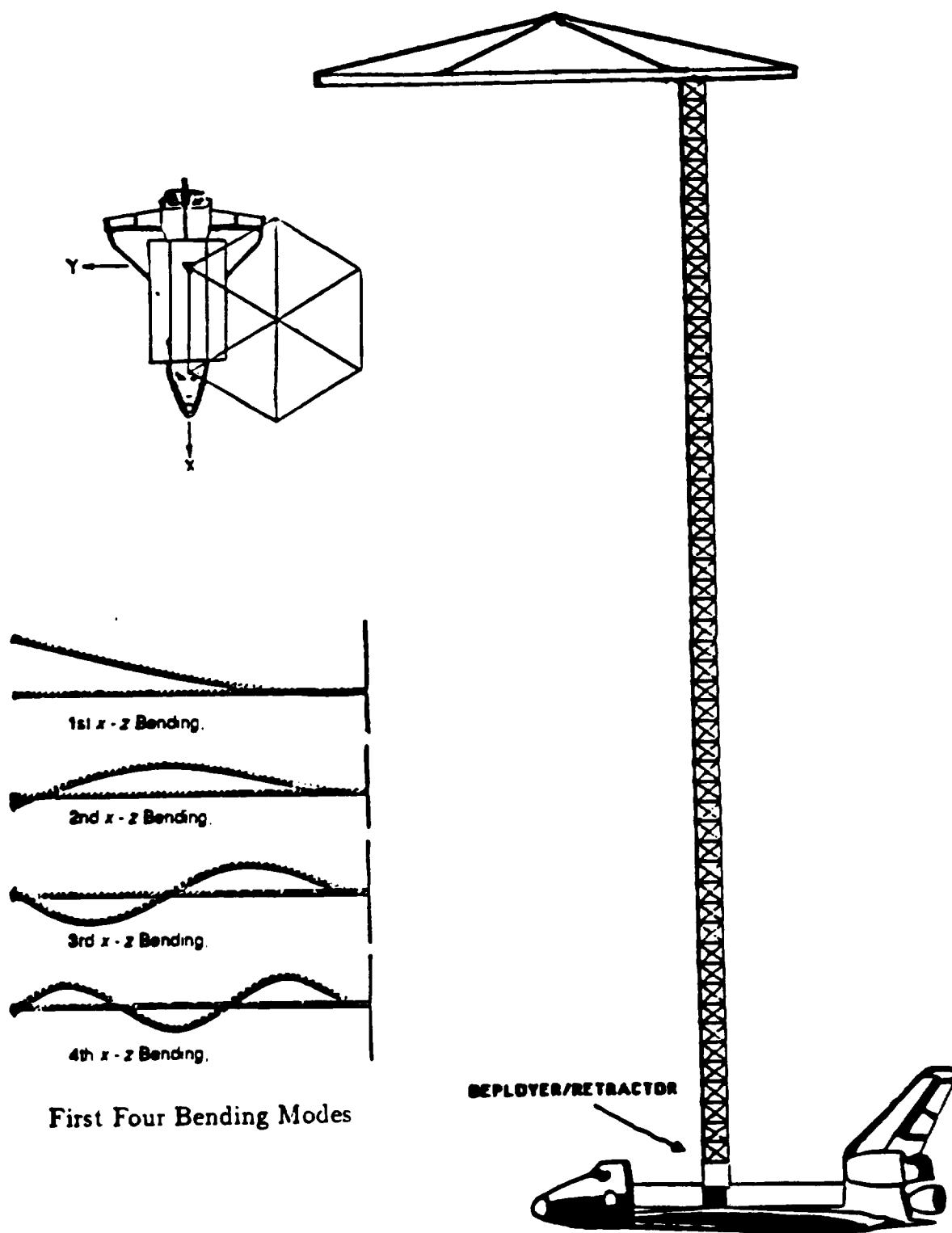
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Y

X

1st x - z Bending.

2nd x - z Bending.

3rd x - z Bending.

4th x - z Bending.

First Four Bending Modes

DEPLOYER/RETRACTOR

Figure 1. Shuttle Orbiter/Antenna Configuration

distributed actuators and/or distributed sensors are used).

While much work has been done in the active damping of distributed parameter oscillation systems by using linear feedback control [18],[39],[53], from practical point of view, actuators can be linear only in certain range (small amplitude) and become saturated for large amplitude. Therefore, we need to take the nonlinear (saturating) nature of the actuators into consideration in control design. Generally, actuator saturation brings many difficulties. Nowadays, such systems are still designed by intuition, experience, and simulation using trial and error. Their effects on the loop response are still poorly undrstood from a theoretical point of view. In Section 4, the effect of actuator saturation is studied and compared with linear damping case.

In Section 5, the notions of *Characteristic Equation* and its *Root Locus* are extended to general multi-actuators/sensors case. The generalization is based upon an operator inverse identity.

Since sensor noise is inevitable for most sensors, in Section 6, we consider sensor noise problem in active damping through direct connection. By studying the effect of sensor noise on the steady state antenna motion, a design guideline is provided for the choice of the feedback gain constant in active damping.

Section 7 summarizes the conclusions of this work.

7.2   A Simple Continuum Model

The model we will consider is based upon a simplified version of the SCOLE problem - a flexible truss, clamped at one end, with an offset antenna at the other end, has only bending motion in a plane. The truss structure is modelled by an equivalent uniform Bernoulli beam of length L along z-axis, extending from $r = 0$, the clamped

end, to $r = L$, the antenna end. The antenna is considered as an attached tip mass with mass m. With r, $0 \leq r \leq L$, denoting the spatial variable along the z-axis and t denoting time, let $u(t, r)$ denote the displacement of the truss in $y - z$ plane. For simplicity, we suppose we use only one control force actuator which is located at the antenna end. Then the beam deflection $u(t, r)$ sloves the following boundary coupled linear partial differential equations:

$$\ddot{u}(t, r) + \frac{EI}{\rho} u''''(t, r) = 0$$
$$\ddot{u}(t, L) - \frac{EI}{m} u'''(t, L) + z(t) = 0$$
$$u(t, 0) = 0 \qquad\qquad\qquad (150)$$
$$u'(t, 0) = 0$$
$$u''(t, L) = 0$$

where super-dots represent derivatives with respect to time t, and the primes derivatives with respect to r. Here we use $z(t)$ to denote control force and E, I, ρ are Young's modulus, moment of inertia and density of the beam material, respectively.

In order to study this problem systematically and rigorously, we need to reformulate this problem in a Hilbert space setting.

First we introduce a Hilbert space

$$H = L^2[0, L] \otimes I\!R^1$$

and the inner product on H,

$$[x_1, x_2] = \frac{1}{a^2} \int_0^L u_1(r) u_2(r) dr + \frac{1}{b^2} c_1 c_2$$

where

$$x_j = \begin{pmatrix} u_j(r) \\ c_j \end{pmatrix} \in H, \quad j = 1, 2$$

and

$$a^2 = EI/\rho, \quad b^2 = EI/m$$

For a general element $x \in H$, its scalar component does not have to have any relation with the $L^2[0, L]$-function component.

And, we define the stiffness operator A by

$$Ax = \begin{pmatrix} a^2 u''''(\cdot) \\ -b^2 u'''(L) \end{pmatrix}$$

for

$$\forall x = \begin{pmatrix} u(\cdot) \\ u(L) \end{pmatrix} \in \mathcal{D}(A)$$

with

$$\mathcal{D}(A) = \{ x = \begin{pmatrix} u(\cdot) \\ u(L) \end{pmatrix} \in H \mid u''''(\cdot) \in L^2[0, L], \ u(0) = u'(0) = u''(L) = 0 \}$$

Next, we define the control operator B as

$$B : \mathbb{R}^1 \longrightarrow H, \quad Bz = \begin{pmatrix} 0 \\ z \end{pmatrix}, \quad z \in \mathbb{R}^1$$

i.e., the control is applied only to the beam tip. Obviously, $\mathcal{N}(B) = \{0\}$.

Under the above notations, (150) can be written as

$$\ddot{x}(t) + Bz(t) + Ax(t) = 0 \qquad (151)$$

It can be shown that A is self-adjoint, positive definite on $\mathcal{D}(A)$ and A^{-1} exists and is compact. Then by the spectral theorem of positive self-adjoint operators with compact resolvent, there is a sequence of eigenvalues of A (natural frequencies)

$$0 < \omega_1^2 \le \omega_2^2 \le \cdots \longrightarrow \infty$$

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associated with the corresponding eigenvectors $\{\psi_n, \ n = 1, 2, \cdots\}$ such that

$$A\psi_n = \omega_n^2 \psi_n, \qquad n = 1, 2, \cdots \tag{152}$$

and, furthermore, $\{\psi_n, \ n = 1, 2, \cdots\}$ form an orthonormal basis in H.

The eigenvalue problem (152) can be solved to give

$$\psi_n = \begin{pmatrix} \phi_n(r) \\ \phi_n(L) \end{pmatrix}$$

$$\omega_n = a(\frac{\beta_n}{L})^2, \qquad n = 1, 2, \cdots$$

in which

$$\phi_n(r) = \frac{1}{c_n}[\cosh \frac{\beta_n}{L}r - \cos \frac{\beta_n}{L}r - \gamma_n(\sinh \frac{\beta_n}{L}r - \sin \frac{\beta_n}{L}r)]$$

and $\{\beta_n, \ n = 1, 2, \cdots\}$ are the solutions of

$$1 + \cos \beta_n \cosh \beta_n + \frac{m\beta_n}{\rho L}(\cos \beta_n \sinh \beta_n - \sin \beta_n \cosh \beta_n) = 0$$

In fact, we also have

$$|\beta_n - (n - 2 + 1/4)\pi| \longrightarrow 0, \quad \text{as } n \to \infty$$

In the above, the constants c_n, γ_n are defined by

$$c_n = a[L + \frac{\rho}{m}(\frac{L}{\beta_n})^2 X_n^2]^{1/2}$$

$$\gamma_n = \frac{\cosh \beta_n + \cos \beta_n}{\sinh \beta_n + \sin \beta_n}$$

$$X_n = \frac{1 + \cos \beta_n \cosh \beta_n}{\sinh \beta_n + \sin \beta_n}$$

Later on, we will need the following realtions which are not difficult to verify,

$$\phi_n(L) = \frac{2\rho L X_n}{m c_n}\frac{1}{\beta_n}$$

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$$= O(\frac{1}{\beta_n}) \qquad \text{as } n \longrightarrow \infty \tag{153}$$

$$\frac{\phi_n^2(L)}{b^2} = (\frac{2}{a^2})^2 \frac{\rho/m \frac{\chi_n^2}{\beta_n^2}}{1/L + \rho/m \frac{\chi_n^2}{\beta_n^2}}$$

$$= O(\frac{1}{\beta_n^2}) \qquad \text{as } n \longrightarrow \infty \tag{154}$$

Sometimes, we need to write (151) in the form of first order system

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -B \end{pmatrix} z(t)$$

The underlying Hilbert space is

$$\mathcal{H}_E = \mathcal{D}(A^{1/2}) \otimes H$$

equipped with the inner product

$$[w_1, w_2]_E = [A^{1/2}x_1, A^{1/2}x_2] + [y_1, y_2], \qquad w_j = \begin{pmatrix} x_j \\ y_j \end{pmatrix} \in \mathcal{H}_E, \quad j = 1, 2$$

For later convenience, we define the operators

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \text{with } \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \otimes H$$

and

$$\mathcal{B} = \begin{pmatrix} 0 \\ -B \end{pmatrix} : \mathbb{R}^1 \longrightarrow \mathcal{H}_E$$

7.3 Strong Stabilizability Conditions

In this section, we present a group of sufficient conditions for the strong stabilizability of *general* distributed parameter oscillation systems, not only for the particular

model presented in the last section. We concentrate on the following abstract wave equation version of an infinite dimensional linear oscillation system

$$\ddot{x}(t) + D\dot{x}(t) + Ax(t) = Bz(t) \quad z \in I\!\!R^m \tag{155}$$

on a Hilber space H. Last section can be considered as a simple example of reducing a concrete PDE model to the above abstract wave equation model.

In (155), A is a stiffness operator with domain $\mathcal{D}(A)$, and is generally nonnegative definite with compact resolvent. There exist a sequence of natural frequencies $\{\omega_n, \ n = 1, 2, \cdots\}$ and the corresponding linear natural modes $\{\psi_n, \ n = 1, 2, \cdots\}$ such that

1. $A\psi_n = \omega_n^2 \psi_n \quad n = 1, 2, \cdots$;

2. $\omega_1 \leq \omega_2 \leq \cdots$, and $\lim_{n\to\infty} \omega_n = \infty$;

3. $\{\psi_n, \ n = 1, 2, \cdots\}$ form an orthonormal basis on H.

From now on, we assume $\omega_1 > 0$, i.e., there is no rigid body mode.

D is the linear internal damping operator, which is nonnegative definite on its domain $\mathcal{D}(D)$. $B : I\!\!R^m \longrightarrow H$ is a finite dimensional linear operator.

The energy of the system is defined by

$$E(t) = 1/2[||A^{1/2}x(t)||^2 + ||\dot{x}(t)||^2]$$

The question we want to answer is: taking the actuator saturation into consideration, what kind of feedback stabilizing control we should use, in order to make the closed-loop system strongly stable. By strong stability, we mean, for any given initial data, $(x(0), \ \dot{x}(0))$, the corresponding closed-loop system response satisfy

$$\lim_{t\to\infty} E(t) = 0$$

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Our main result of this section is the following

Theorem 13 *Under the following assumptions:*

1. $f(\cdot)$ is Lipschitz continuous;

2. $[f(x), x]_{R^m} > 0, \quad x \neq 0;$

3. $\mathcal{N}((D + BB^*)|_{E_n}) = \{0\}$ [1]

the following rate feedback with colocated sensors/actuators

$$z(t) = -f(B^*\dot{x}(t))$$

strongly stabilizes (155), i.e., the closed-loop system

$$\ddot{x}(t) + D\dot{x}(t) + Bf(B^*\dot{x}(t)) + Ax(t) = 0 \tag{156}$$

is strongly stable.

Here E_n denotes the subspace spanned by those natural modes ψ_{n_j} corresponding to the natural frequency ω_n.

In particular, if all ω_n's are distinct, then Assumption 3 can be replaced by

Assumption $3'$: $[D\psi_n, \psi_n] + \|B\psi_n\|^2 > 0 \quad n = 1, 2, \cdots$

Before the proof, we make the following remarks:

1. In this theorem, we do not require the passive (internal) damping D to be positive definite. In fact, even we neglect the internal damping $(D = 0)$ in the modeling, (156) is still strongly stable as long as

$$\|B^*\psi_n\| > 0, \quad n = 1, 2, \cdots$$

and *Assumptions 1, 2* are satisfied.

---

[1] $\mathcal{N}(P)$ stands for the null space of the operator P, and $(D+BB^*)|_{E_n}$ stands for the restriction of the operator $D + BB^*$ on the subspace E_n.

2. *Assumption 3* is also a necessary condition. Since if $\psi \in \mathcal{N}(\; (D + BB^*)|_{E_n} \;)$, then

$$D\psi = 0; \quad B^*\psi = 0$$

Then it is easy to see that (156) has the following solution

$$x(t) = (a \cos \omega_n t + b \sin \omega_n t)\psi$$

which is obviously not strongly stable.

3. In particular, if we let $f(x) \equiv x$, then we obtain that the necessary and sufficient condition for the linear system

$$\ddot{x}(t) + D\dot{x}(t) + BB^*\dot{x}(t) + Ax(t) = 0$$

to be strongly stable is *Assumption 3* holds (or *Assumption 3/* holds if all ω_n's are distinct). .

4. *Assumption 1* not only plays the role of assuring the existence of solution of (156), but also contributes to the strong stability of (156). In other words, to guarantee the strong stability of (156), it is important to require $f(\cdot)$ being continuous at least in the neighborhood of $x = 0$. In the following example, which is although of single-DOF, we can see since $f(\cdot)$ is not continuous at $x = 0$, $x(t)$ does not always go to 0 as $t \longrightarrow \infty$, even *Assumptions 2, 3* are satisfied.

EXAMPLE: Consider the governing equation of a spring-mass system with *Coulomb damping*

$$\ddot{x}(t) + \mu \mathrm{sgn}(\dot{x}(t)) + \omega_0^2 x(t) = 0$$

in which $H = I\!R^1$.

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By considering $\dot{x}(t) > 0$ and $\dot{x}(t) < 0$ separately, we can obtain the solution

$$
\begin{cases}
\omega_0 x(t) + \frac{\mu}{\omega_0}\mathrm{sgn}(\dot{x}(t)) = \omega_0 x(0)\cos\omega_0 t + \dot{x}(0)\sin\omega_0 t \\
\qquad\qquad\qquad\qquad +\mu\,\mathrm{sgn}(\dot{x}(t))\cos\omega_0 t/\omega_0 \\
\\
\dot{x}(t) = -\omega_0 x(0)\sin\omega_0 t + \dot{x}(0)\cos\omega_0 t \\
\qquad\qquad -\mu\,\mathrm{sgn}(\dot{x}(t))\sin\omega_0 t/\omega_0
\end{cases}
\tag{157}
$$

Therefore, in the $(\omega_0 x(t),\ \dot{x}(t))$ phase plane, the trajectory is governed by the following circle

$$
[\omega_0 x(t) + \frac{\mu}{\omega_0}\mathrm{sgn}(\dot{x}(t))]^2 + [\dot{x}(t)]^2 = [\omega_0 x(0) + \frac{\mu}{\omega_0}\mathrm{sgn}(\dot{x}(t))]^2 + [\dot{x}(0)]^2
$$

When the representative point $(\omega_0 x(t), \dot{x}(t))$ is in the upper half plane, the trajectories consist of a series of circular arcs with center $(-\mu/\omega_0, 0)$ and radius depending upon the initial data or the state when the representative point enters the upper half plane from the lower half plane. Similarly, when the representative point $(\omega_0 x(t), \dot{x}(t))$ is in the lower half plane, its trajectories consist of a series of circular arcs with center $(\mu/\omega_0, 0)$, see *Figure 2*.

If the initial data satisfies $|\omega_0 x(0)| > \mu/\omega_0$, then, starting from the initial point $(\omega_0 x(0), \dot{x}(0))$, the representative point moves clockwise along various circular arcs in the upper and lower plane. The process stops when the representative point intersects the $\omega_0 x(t)$ axis between $-\mu/\omega_0$ and μ/ω_0. The motion ceases at such a point because the maximum possible friction force exceeds the force in the spring, i.e.

$$
\mu > \omega_0^2 x(t)
$$

Similarly, if the initial data is such that

$$
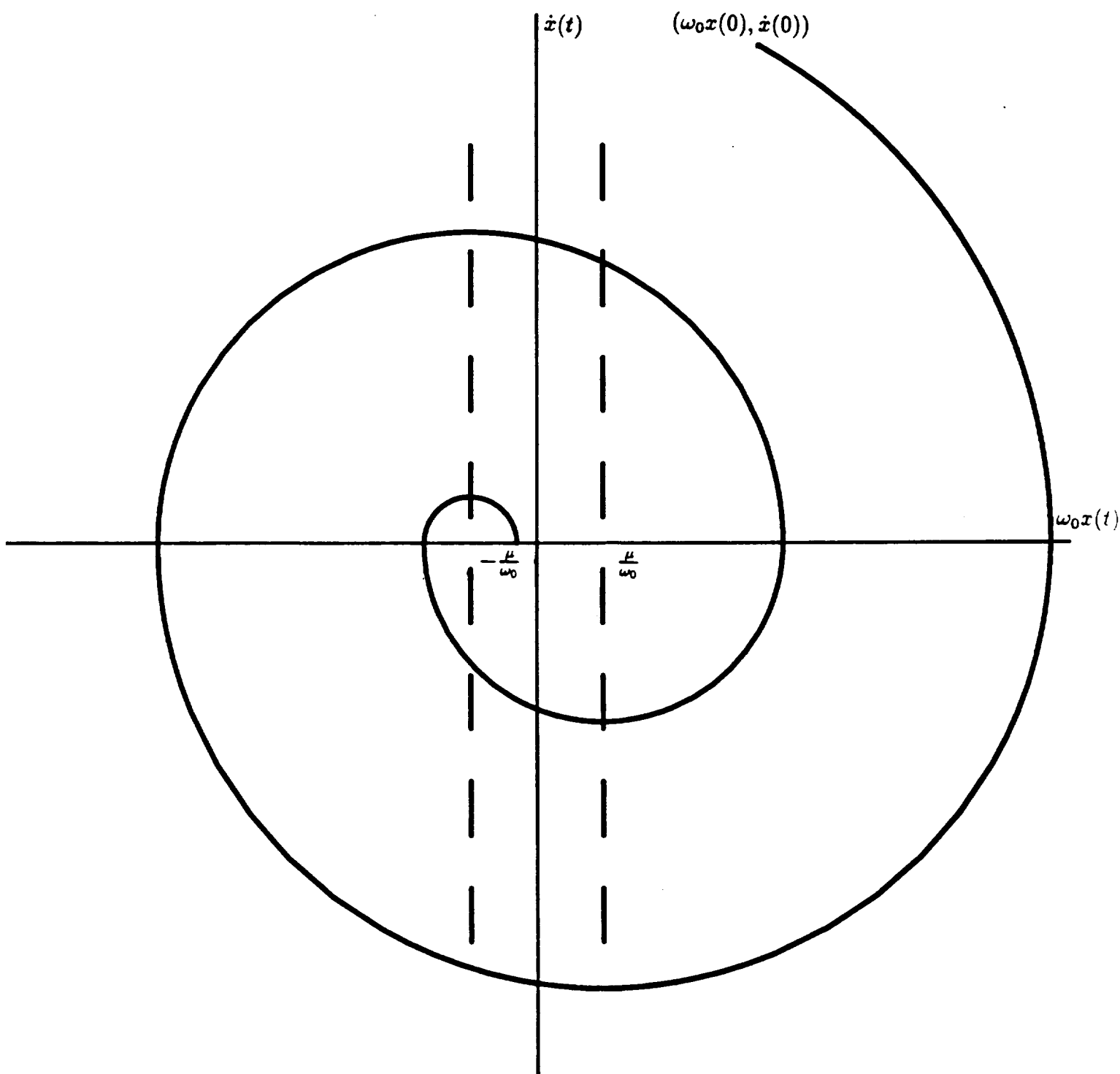\dot{x}(0) = 0; \quad |x(0)| \le \mu/\omega_0^2
$$

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Figure 2. Phase plane trajectory of the spring-mass system with *Coulomb damping*.

then there is no motion because the spring force $\omega_0^2 x(0)$ cannot overcome the friction force μ.

Therefore, from this example we can see, since

$$f(x) = \mu \mathrm{sgn}(x)$$

which is not continuous at $x = 0$, the system energy does not generally go to zero as $t \longrightarrow \infty$. After the mass stops moving, its strain energy $1/2\omega_0^2 x^2(t)$ is generally positive.

PROOF OF THEOREM 13: First of all, *Assumptions 2* and *3* imply $f(0) = 0$. In fact, let

$$f(x) = \begin{pmatrix} f_1(x_1, \cdots, x_m) \\ \vdots \\ f_m(x_1, \cdots, x_m) \end{pmatrix}$$

Suppose some $f_j(0, \cdots, 0) \neq 0$ for some j. Without loss of generality, suppose $f_1(0, \cdots, 0) > 0$. By the continuity of f at $x = 0$, there exist $\delta > 0$, $\epsilon > 0$ such that

$$f_1(x_1, 0, \cdots, 0) \geq f_1(0, \cdots, 0) - \epsilon > 0, \quad \text{for } |x_1| < \delta$$

Then,

$$f_1(-\delta/2, 0, \cdots, 0)(-\delta/2) \leq -\delta/2[f_1(0, \cdots, 0) - \epsilon] < 0$$

However, from *Assumption 2* we already know

$$f_1(-\delta/2, 0, \cdots, 0)(-\delta/2) > 0$$

This contradiction implies $f_1(0, \cdots, 0) = 0$. Similarly, we can show

$$f_j(0, \cdots, 0) = 0, \quad j = 2, \cdots, m$$

Next, for any $(x(0), \dot{x}(0)) \in \mathcal{D}(A) \otimes \mathcal{D}(D)$, by virtue of *Assumption 1*, the existence and uniqueness of solution of (156) is immediate.

Since

$$\frac{dE(t)}{dt} = -[D\dot{x}(t), \dot{x}(t)] - [f(B^*\dot{x}(t)), B^*\dot{x}(t)] \leq 0$$

$E(t)$ is monotone decreasing and is lower bounded (by 0). Therefore, there exists $E(\infty) \geq 0$ such that

$$\lim_{t \to \infty} E(t) = E(\infty)$$

It suffices for us to show that $E(\infty) = 0$. For this purpose, let us assume that $E(\infty) > 0$. Then there exists energy-preserving steady state motion, denoted by $(x_s(t), \dot{x}_s(t))$, with corresponding energy

$$E_s(t) = 1/2([Ax_s(t), x_s(t)] + ||\dot{x}_s(t)||^2) \equiv E(\infty)$$

Then,

$$\begin{aligned} \frac{dE_s(t)}{dt} &= -[D\dot{x}_s(t), \dot{x}_s(t)] - [f(B^*\dot{x}_s(t)), B^*\dot{x}_s(t)] \\ &= 0 \end{aligned}$$

which immediately implies

$$\begin{cases} D\dot{x}_s(t) \equiv 0 \\ B^*\dot{x}_s(t) \equiv 0 \end{cases} \tag{158}$$

by *Assumption 2* and $f(0) = 0$. However, (158) indicates that $x_s(t)$ solves

$$\ddot{x}_s(t) + Ax_s(t) = 0$$

or

$$\begin{aligned} x_s(t) &= \sum_{n=1}^{\infty}(a_n \cos \omega_n t + b_n \sin \omega_n t)\psi_n \\ \dot{x}_s(t) &= \sum_{n=1}^{\infty} \omega_n(b_n \cos \omega_n t - a_n \sin \omega_n t)\psi_n \end{aligned}$$

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Since we do not intend to lose the generality by excluding the repeated ω_n case, we use $\{\psi_{k_i}\}$ to denote the eigenvectors corresponding to the repeated natural frequency ω_k. Then, $\dot{x}_s(t)$ can be rewritten as

$$\dot{x}_s(t) \;=\; \sum_{k=1}^{\infty} \omega_k \cos \omega_k t (\sum_{k_i} b_{k_i} \psi_{k_i})$$
$$-\sum_{k=1}^{\infty} \omega_k \sin \omega_k t (\sum_{k_i} a_{k_i} \psi_{k_i})$$

Then, (158) is equivalent to

$$\sum_{k=1}^{\infty} \omega_k \cos \omega_k t (D + BB^*)(\sum_{k_i} b_{k_i} \psi_{k_i})$$
$$-\sum_{k=1}^{\infty} \omega_k \sin \omega_k t (D + BB^*)(\sum_{k_i} a_{k_i} \psi_{k_i}) = 0 \qquad (159)$$

Therefore, it must be the case that

$$\begin{cases} (D + BB^*)(\sum_{k_i} b_{k_i} \psi_{k_i}) = 0 \\ (D + BB^*)(\sum_{k_i} a_{k_i} \psi_{k_i}) = 0; \quad k = 1, 2, \cdots \end{cases} \qquad (160)$$

Then, by *Assumption 3*, we know that

$$\begin{cases} \sum_{k_i} b_{k_i} \psi_{k_i} = 0 \\ \sum_{k_i} a_{k_i} \psi_{k_i} = 0; \quad k = 1, 2, \cdots \end{cases} \qquad (161)$$

Therefore, the orthogonality of $\{\psi_{k_i}\}$ implies

$$a_{k_i} = 0; \quad b_{k_i} = 0, \quad \text{for all } k_i, \text{ and } k = 1, 2, \cdots$$

i.e., $x_s(t) \equiv 0$ or $E(\infty) = E_s(t) \equiv 0$. □

From the proof we can easily see that if *Assumption 3* is replaced by

$$\text{Assumption } 3a: \quad [D\psi_n, \psi_n] > 0, \quad n = 1, 2, \cdots,$$

then *Assumption 2* can be replaced by a weaker version, i.e.

$$Assumption\ 2a\ :\ [f(x), x]_{\mathbb{R}^m} \geq 0\ \text{for}\ x \in \mathbb{R}^m.$$

Under *Assumption 3a*, the uncontrolled system

$$\ddot{x}(t) + D\dot{x}(t) + Ax(t) = 0$$

is itself strongly stable. By using active damping $u(t) = -f(B^*\dot{x}(t))$, we are enhancing the system stability.

7.4 Mode Excitation by Active Damping

In this section, we study the following problem: For an undamped linear oscillation system,

$$\begin{cases} \ddot{x}(t) + Ax(t) = 0 \\ \\ x(0) = x_0 \qquad \dot{x}(0) = \dot{x}_0 \end{cases}$$

If $x(0)$, $\dot{x}(0)$ are linear combinations of finite number of modes, say,

$$x_0 = \sum_{k=1}^{K} \alpha_{n_k} \psi_{n_k}$$

$$\dot{x}_0 = \sum_{k=1}^{K} \beta_{n_k} \psi_{n_k}$$

then the corresponding solution $x(t)$ always stays on these modes and is of the form

$$x(t) = \sum_{k=1}^{K} a_{n_k}(t) \psi_{n_k}$$

where $a_{n_k}(t)$ satisfies

$$\begin{cases} \ddot{a}_{n_k}(t) + \omega_{n_k}^2 a_{n_k}(t) = 0 \\\\ a_{n_k}(0) = \alpha_{n_k} \\\\ \dot{a}_{n_k}(0) = \beta_{n_k} \end{cases}$$

However, with active damping, the system response no longer stays on the initial modes. For simplicity of notations, suppose the damped system starts from the first mode, i.e.,

$$x(0) = a_1(0)\psi_1, \quad \dot{x}(0) = 0$$

then the active damping will excite other modes. How many more modes are excited? In what magnitudes? What is the behavior of those excited higher order modes? These are some of the questions we will answer in the following analysis. The idea is to solve the feedback control effort $z(t) = f(\dot{u}(t, L))$, a finite dimensional time function, without solving the whole system - a PDE.

First, we write the damped system response in its mode decomposition form

$$x(t) = \sum_{n=1}^{\infty} a_n(t)\psi_n$$

where the mode responses $a_n(t)$, $n = 1, 2, \cdots$ solve

$$\begin{cases} \ddot{a}_n(t) + \omega_n^2 a_n(t) = -\phi_n(L)/b^2 z(t) \\\\ a_1(0) = a_1(0), \quad \dot{a}_1(0) = 0 \\\\ a_n(0) = 0, \quad\quad \dot{a}_n(0) = 0, \quad n \geq 2 \end{cases} \qquad (162)$$

where $z(t) = f(\dot{u}(t, L))$, the rate feedback control.

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Recall that (151) can be written as

$$\dot{w}(t) = \mathcal{A}w(t) + \mathcal{B}z(t)$$

where

$$w(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

The semigroup $T(t)$ generated by $\mathcal{A}$ is given by

$$T(t) = \begin{pmatrix} C(t) & S(t) \\ -AS(t) & C(t) \end{pmatrix}$$

where $C(t): H \longrightarrow H$, $S(t): H \longrightarrow \mathcal{D}(A^{1/2})$ are cosine and sine operators defined by

$$C(t)x = \sum_{n=1}^{\infty} \cos \omega_n t [x, \psi_n] \psi_n, \quad x \in H$$

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin \omega_n t}{\omega_n} [y, \psi_n] \psi_n, \quad y \in H$$

Therefore the response $(x(t), \dot{x}(t))$ is given by

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = T(t) \begin{pmatrix} x(0) \\ \dot{x}(0) \end{pmatrix} + \int_0^t T(t-\tau)\mathcal{B}z(\tau)d\tau \qquad (163)$$

from which we obtain, noticing that $\dot{x}(0) = 0$ and $x(0) = a_1(0)\psi_1$,

$$\begin{aligned}
x(t) &= C(t)x(0) + \int_0^t S(t-\tau)(-Bz(\tau))d\tau \\
&= a_1(0) \cos \omega_1 t \, \psi_1 - \int_0^t \sum_{n=1}^{\infty} \frac{\sin \omega_n(t-\tau)}{\omega_n} \frac{\phi_n(L)}{b^2} \psi_n z(\tau)d\tau
\end{aligned}$$

Then, separating the boundary component in $x(t)$ gives

$$u(t,L) = a_1(0)\phi_1(L) \cos \omega_1 t - \frac{1}{b^2} \int_0^t \sum_{n=1}^{\infty} \frac{\sin \omega_n(t-\tau)}{\omega_n} \phi_n^2(L)z(\tau)d\tau$$

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Differentiating both sides with respect to t gives

$$\dot{u}(t, L) = -a_1(0)\phi_1(L)\omega_1 \sin \omega_1 t - \int_0^t K_1(t - \tau)z(\tau)d\tau \qquad (164)$$

where

$$K_1(t) = \frac{1}{b^2} \sum_{n=1}^{\infty} \phi_n^2(L) \cos \omega_n t$$

From (154) it is obvious that the infinite series is absolutely convergent and the convergence is uniform in t.

Therefore, from (164) we conclude that the feedback control $z(t)$ is uniquely determined by the following nonlinear integral equation:

$$z(t) = f[-a_1(0)\phi_1(L)\omega_1 \sin \omega_1 t - \int_0^t K_1(t - \tau)z(\tau)d\tau] \qquad (165)$$

If we can solve $z(t)$ from (165), then we can find each mode response $a_n(t)$ from (162). We first study

(1) Linear Damping Case ($f(x) = kx$) We introduce the notation

$$G(s, r) = \frac{1}{b^2} \sum_{n=1}^{\infty} \frac{\phi_n(L)\phi_n(r)}{s^2 + \omega_n^2}, \quad 0 \le r \le L, \quad s \in \mathbf{C}$$

And, later on, we will often use the capital letters to denote the Laplace transforms of the functions denoted by the corresponding lower case letters, such as

$$Z(s) = \mathcal{L}[z(t)], \quad A_n(s) = \mathcal{L}[a_n(t)]$$

etc.

By performing Laplace transforms on both sides of (165), one can obtain

$$Z(s) = \frac{-ka_1(0)\omega_1^2\phi_1(L)}{(s^2 + \omega_1^2)(1 + ksG(s, L))}$$

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Furthermore, from (162), one can obtain the mode responses

$$
\begin{aligned}
A_1(s) &= a_1(0) \frac{s(1 + ks/b^2 \sum_{n=2}^{\infty} \frac{\phi_n^2(L)}{s^2 + \omega_n^2}) + k\phi_1^2(L)/b^2}{(s^2 + \omega_1^2)(1 + ksG(s, L))} \\
A_n(s) &= \frac{ka_1(0)\omega_1^2\phi_1(L)\phi_n(L)/b^2}{(s^2 + \omega_1^2)(s^2 + \omega_n^2)(1 + ksG(s, L))} \\
& \qquad n = 2, 3, \cdots
\end{aligned}
\tag{166}
$$

From the definition of $G(s, L)$, we can realize that $\{\pm i\omega_n, \ n = 1, 2, \cdots\}$ are not the poles of $A_n(s)$, $n = 1, 2, \cdots$ due to cancellation. In fact, all $A_n(s)$'s have the same poles and they are simply the roots of the equation

$$
1 + ksG(s, L) = 0
\tag{167}
$$

(2) **Saturating Nonlinear Damping Case.** First of all, the saturation function $f(x)$ can be written as

$$
f(x) = kx - \psi(x)
$$

where $\psi(x)$ is the following dead-zone function

$$
\psi(x) = \begin{cases} 0 & |x| \le M/k \\ k(x - \text{sgn}(x)M/k) & |x| > M/k \end{cases}
$$

Then from (164), we have

$$
\begin{aligned}
z(t) &= f(\dot{u}(t, L)) \\
&= k\dot{u}(t, L) - \psi(\dot{u}(t, L)) \\
&= k[-a_1(0)\phi_1(L)\omega_1 \sin \omega_1 t - \int_0^t K_1(t - \tau)z(\tau)d\tau] - \psi(\dot{u}(t, L))
\end{aligned}
$$

Then, performing Laplace transform on both sides and letting

$$
\Psi(s) = \mathcal{L}[\psi(\dot{u}(t, L))]
$$

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give

$$Z(s) = \frac{-ka_1(0)\omega_1^2\phi_1(L) - (s^2 + \omega_1^2)\Psi(s)}{(s^2 + \omega_1^2)(1 + ksG(s,L))}$$

Similarly as in linear damping case, we obtain

$$A_1(s) = a_1(0)\frac{s(1 + ks/b^2\sum_{n=2}^{\infty}\frac{\phi_n^2(L)}{s^2+\omega_n^2}) + k\phi_1^2(L)/b^2}{(s^2 + \omega_1^2)(1 + ksG(s,L))} + \frac{\phi_1(L)/b^2\Psi(s)}{(s^2 + \omega_1^2)(1 + ksG(s,L))}$$

$$A_n(s) = \frac{ka_1(0)\omega_1^2\phi_1(L)\phi_n(L)/b^2}{(s^2 + \omega_1^2)(s^2 + \omega_n^2)(1 + ksG(s,L))} + \frac{\phi_n(L)/b^2\Psi(s)}{(s^2 + \omega_n^2)(1 + ksG(s,L))} \quad (168)$$

$$n = 2, 3, \cdots$$

Comparing (168) with (166), we can realize that each mode response $A_n(s)$ is simply the linear damping mode response $A_n(s)$ plus a correction term $C_n(s)$, with

$$C_n(s) = \phi_n(L)/b^2\frac{\Psi(s)}{(s^2 + \omega_n^2)(1 + ksG(s,L))}$$

Furthermore, all the poles of each correction term are again the roots of (167), because we can show that $\Psi(s)$ is analytic on the complex plane.

In fact, from *Theorem 13*, we can see that under saturation type nonlinear damping, the damped system is strongly stable. Hence,

$$1/b|\dot{u}(t,L)| \leq ||\dot{x}(t)|| \leq [2E(t)]^{1/2} \longrightarrow 0, \quad \text{as } t \to \infty$$

That is, for fixed linear range slope k, and saturation level M, there exists $T = T(k, M) < \infty$ such that

$$|\dot{u}(t,L)| \leq M/k, \quad \text{for } t \geq T$$

Therefore, $\Psi(s)$ can be written as

$$\Psi(s) = \mathcal{L}[\psi(\dot{u}(t,L))] = \int_0^T \psi(\dot{u}(t,L))e^{-st}dt$$

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Since T is finite and $\psi(\dot{u}(t, L))$ is bounded and continuous with respect to t, we thus conclude that $\Psi(s)$ is analytic.

From above analysis we can see that in both linear and nonlinear damping cases, all $A_n(s)$, $n = 1, 2, \cdots$ have the same poles, which are simply the roots of (167). Therefore, we now introduce

Definition 1 *(167) is called the* <u>Characteristic Equation</u> *of the distributed parameter system (150).*

To study the behavior of the mode responses $a_n(t)$, $n = 1, 2, \cdots$, we need to study the characteristic equation in detail, in particular its root locus.

1. The closed form expression of $G(s, L)$ is given by

$$1/G(s,L) = s^2 + \frac{b^2}{\sqrt{2}}(\frac{s}{a})^{3/2}\frac{2 + \cosh\sqrt{\frac{2s}{a}}L + \cos\sqrt{\frac{2s}{a}}L}{\sinh\sqrt{\frac{2s}{a}}L - \sin\sqrt{\frac{2s}{a}}L} \qquad (169)$$

Hence, the characteristic equation (167) is reduced to

$$s = -k - \frac{b^2}{\sqrt{2}}\frac{\sqrt{s}}{a^{3/2}}\frac{2 + \cosh\sqrt{\frac{2s}{a}}L + \cos\sqrt{\frac{2s}{a}}L}{\sinh\sqrt{\frac{2s}{a}}L - \sin\sqrt{\frac{2s}{a}}L} \qquad (170)$$

In particular, we have

$$\sum_{n=1}^{\infty} \frac{\phi_n^2(L)}{\omega_n^2} = b^2 G(0, L) = L^3/3$$

The derivation of the closed form of $G(s, L)$ is given in Appendix A.

2. $G(s, L)$ is analytic on $\mathbf{C}\backslash\{\pm i\omega_n, \ n = 1, 2, \cdots\}$. $\{\pm i\omega_n, \ n = 1, 2, \cdots\}$ are the first order poles of $G(s, L)$. All the zeros of $sG(s, L)$ are on the imaginary axis, except for one at infinity, i.e., they are $\{0, \ \infty, \ \pm iz_n, \ n = 2, 3, \cdots\}$ where

$$|z_n - \omega_n| \longrightarrow 0, \quad \text{as } n \to \infty$$

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In fact, the first statement is obvious. To show the second statement, it is easy to see that 0 and ∞ are two of the zeros of $sG(s,L)$ from the definition of $G(s,L)$. The other zeros, from the closed from expression of $G(s,L)$, are simply the roots of

$$\sinh\sqrt{2s/a}L - \sin\sqrt{2s/a}L = 0 \qquad (171)$$

However, the roots of the transcendental equation

$$\sinh z = \sin z$$

are given by $\{(1 \pm i)x_k, \quad k = 0,1,2,\cdots\}$ where $\{x_k, \quad k = 0,1,2,\cdots\}$ are the roots of

$$\tan x = \tanh x$$

Obviously, we have

$$|x_k - (k+1/4)\pi| \longrightarrow 0, \text{ as } k \to \infty$$

Hence, from the relation $z = \sqrt{2s/a}L$, we know that the zeros of (171) are given by

$$\{\frac{a}{2}[\frac{(1 \pm i)x_k}{L}]^2, \quad k = 0,1,2,\cdots\}$$

or

$$\{\pm ia(x_k/L)^2, \quad k = 0,1,2,\cdots\}$$

Therefore, letting $z_n = a(x_{n-2}/L)^2, \quad n = 2,3,\cdots$, we realize that the zeros of $sG(s,L)$ are $\{\pm iz_n, \quad n = 2,3,\cdots\}$, in addition to $\{0, \infty\}$. Furthermore, from the definition of $G(s,L)$, it is not difficult to see that

$$\omega_{n-1} < z_n < \omega_n, \quad n = 2,3,\cdots$$

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3. For $0 < k < \infty$, the characteristic equation (167) has a sequence of roots $\{s_n, \overline{s_n}, \ n = 1, 2, \cdots\}$ such that

$$\Re[s_n] < 0; \quad n = 1, 2, \cdots$$

and

$$\Re[s_n] \longrightarrow 0, \quad |\Im[s_n] - \omega_n| \longrightarrow 0, \quad n \to \infty$$

In fact, first of all, the characteristic equation (167) cannot possibly have any roots on the imaginary axis, because $ksG(s, L)$ will be purely imaginary on the imaginary axis excluding $\{\pm i\omega_n, \ n = 1, 2, \cdots\}$.

To see the rest of the claim, it is sufficient to plot the root locus of (167).

As usual, the root loci start from the poles of $sG(s, L)$, i.e. $\{\pm i\omega_n, \ n = 1, 2, \cdots\}$ and end at the zeros of $sG(s, L)$, i.e., $\{0, \ \infty, \ \pm iz_n, n = 2, 3, \cdots\}$. The angles of departures from the poles and the angles of arrivals at the zeros are all 180 deg with respect to the positive real axis. Therefore, we can realize that the root loci stay in the left half plane, since it cannot cross the imaginary axis.

Since one of the zeros of $sG(s, L)$ is at infinity, we know that one branch of the root loci extends to infinity along an asymptote. The angle of the asymptote to the positive real axis can be computed to be 180 deg. Furtheremore, in general, the root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros (in our case, there is exactly one zero on the real axis, which is $s = 0$). Therefore, the root locus occupies the whole negative real axis.

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Of course, the root locus is symmetric with respect to the real axis, since

$$\overline{1 + ksG(s, L)} = 1 + k\bar{s}G(\bar{s}, L)$$

The root locus of (167) is shown in *Figure 3*.

To verify the statements on s_n, the roots of the characteristic equation, we let $w = \sqrt{2s/a}L$. Then (167) can be written as

$$(\frac{1}{L} + \frac{2kL}{aw^2})\frac{\sinh w - \sin w}{2 + \cosh w + \cos w} + (\frac{b}{a})^2\frac{1}{w} = 0$$

Since the roots $w_n = \sqrt{2s_n/a}L$ are unbounded as $n \to \infty$, we can conclude that

$$|w_{n+2} - (1 + i)(n + 1/4)\pi| \longrightarrow 0, \quad \text{as } n \to \infty$$

Hence

$$
\begin{aligned}
s_n &= a/2(\frac{w_n}{L})^2 \\
&= ia(\frac{\beta_n}{L})^2(\frac{w_n/(1 + i)}{\beta_n})^2 \\
&= i\omega_n(\frac{w_n/(1 + i)}{\beta_n})^2
\end{aligned}
$$

Therefore, we can show that

$$\Re[s_n] \longrightarrow 0, \quad |\Im[s_n] - \omega_n| \longrightarrow 0, \quad \text{as } n \to \infty$$

Based upon the above study of the characteristic equation and its root locus, we can obtain the following conclusions:

(*I*) In linear damping case, we have found the Laplace transform of each mode response $A_n(s)$, $n = 1, 2, \cdots$, supposing the beam starts from the first mode. Active damping excites all other modes. The active damping process is as follows: the
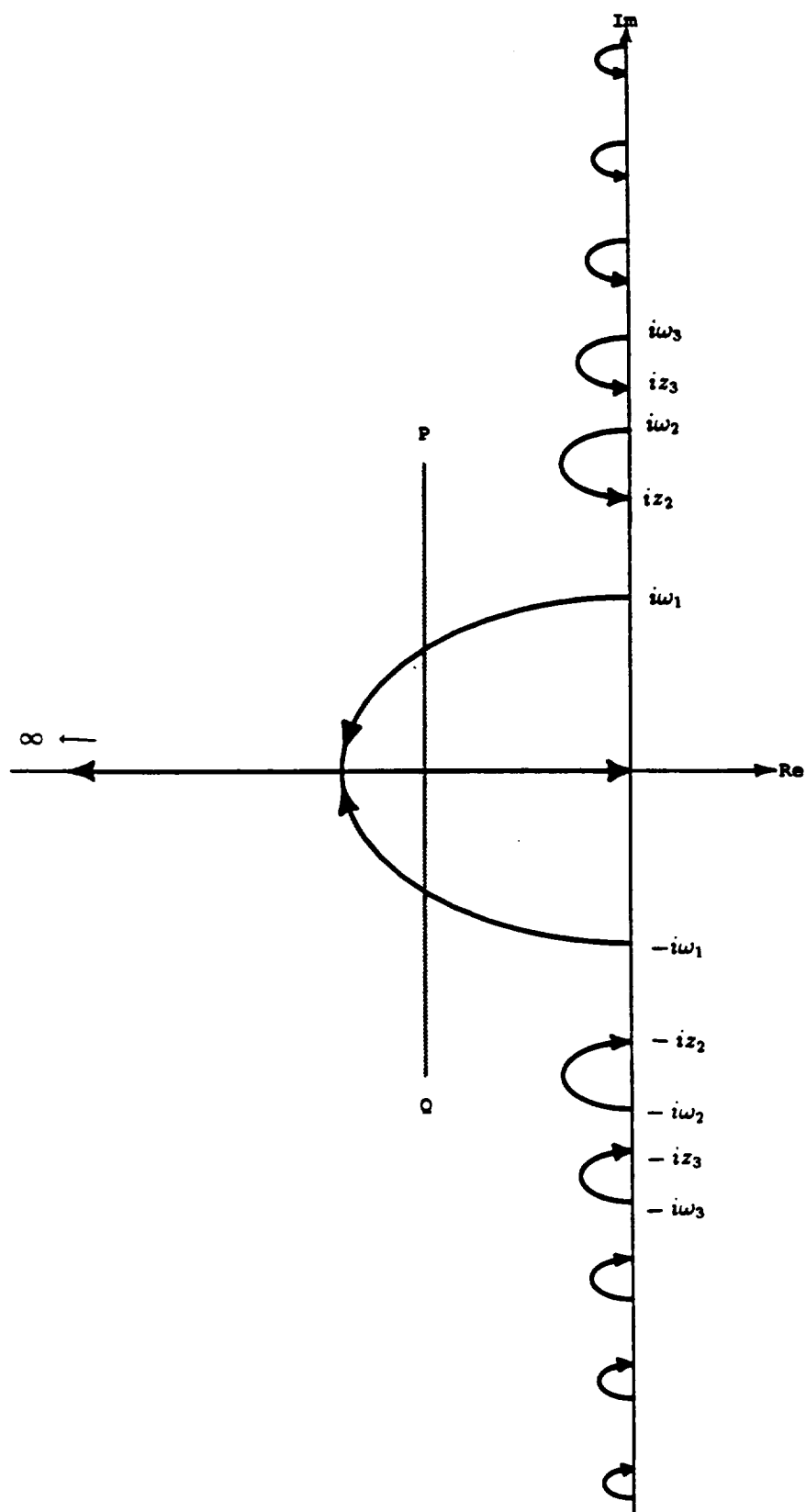
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Figure 3. The root locus of the characteristic equation $1 + ksG(s, L) = 0$, for $0 < k < \infty$.

system energy initially possessed by only the first mode, $1/2\omega_1^2 a_1^2(0)$, is first partially shifted to all other modes, and then the energy acquired by each mode is gradually absorbed by the active damper. Quantitatively, the excited mode response $a_n(t)$, $n \geq 2$, is equivalent to the output of a stable system with the following transfer function

$$H_n(s) = \frac{\omega_1^2 \phi_1(L)/b^2}{(s^2 + \omega_1^2)(s^2 + \omega_n^2)(1 + ksG(s, L))}$$

corresponding to an initial impulse excitation with magnitude $ka_1(0)\phi_n(L)$. Each of the excited mode responses is proportional to the feedback gain k.

The excited mode response in time domain can be found through inverse Laplace transform. Let Γ be a closed contour consisting of the imaginary axis and the semicircle with infinite radius on the left half plane. Then for $n \geq 2$,

$$
\begin{aligned}
a_n(t) &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} A_n(s)e^{st}ds \\
&= \frac{1}{2\pi i} \oint_{\Gamma} A_n(s)e^{st}ds \\
&= \sum_{k=1}^{\infty} [\text{Res}_{s=s_k}\{A_n(s)e^{st}\} + \text{Res}_{s=\overline{s_k}}\{A_n(s)e^{st}\}] \\
&= \sum_{k=1}^{\infty} \rho_k^{(n)} \exp(\Re[s_k]t)\cos(\Im[s_k]t + \theta_k^{(n)})
\end{aligned}
$$

(172)

where

$$
\begin{aligned}
\rho_k^{(n)} &= |\text{Res}_{s=s_k}\{A_n(s)\}| \\
\theta_k^{(n)} &= \text{Arg}[\text{Res}_{s=s_k}\{A_n(s)\}] \\
& \qquad k = 1, 2, \cdots
\end{aligned}
$$

From (172) we can see that each mode response $a_n(t)$ involves all frequencies, which are slightly smaller than the corresponding natural frequencies. As frequency goes higher, the damping effect of the active damper becomes weaker, because $\Re[s_k] \longrightarrow 0$ as $k \to \infty$.

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(II) In saturation type nonlinear damping case, we have shown that $A_n(s)$ is equivalent to the sum of the $A_n(s)$ corresponding to linear damping case and a correction term. Furthermore, all the correction terms have the same poles as in linear damping case, i.e., the poles are the roots of the characteristic equation. In other words, the $A_n(s)$ in both linear and nonlinear damping cases have the same poles, but with different residues.

In saturation type nonlinear damping case, the mode response can be similarly obtained (attention needs to be paid on the restriction $t \geq T(k,M)$ to guarantee the integrand vanishing on the infinite semicircle).

$$a_n(t) = \sum_{k=1}^{\infty} \tilde{\rho}_k^{(n)} \exp(\Re[s_k]t) \cos(\Im[s_k]t + \tilde{\theta}_k^{(n)})$$

with $\tilde{\rho}_k^{(n)}, \tilde{\theta}_k^{(n)}$ similarly defined as in linear damping case.

Therefore, the existence of saturation in active damping only causes variation of amplitudes and phases, and does not result in any qualitative change in terms of mode responses. In particular, the component frequencies remain the same and no extra frequency is introduced. These confirm that saturation type nonlinearity is amplitude sensitive but frequency insensitive.

(III) From the root locus (*Figure 9*), we can see that when the feedback gain k is chosen such that s_1 and $\bar{s}_1$ are located to the left of $\overline{PQ}$, the base frequency oscillation decays much faster than higher frequency oscillations. Then higher frequency oscillations soon become dominant in this case.

When k is sufficiently large such that the first pair of roots are on the real axis but to the left of $\overline{PQ}$, base frequency oscillation is then completely eliminated. However, it is impossible to eliminate any of the higher order frequency oscillations with only one actuator. In other words, active damping effect is significant only to

the base frequency oscillation.

(*IV*) From (172) we can see that for any $n \geq 2$, there does not exist $\sigma > 0$ such that

$$|a_n(t)| \leq \text{const.} e^{-\sigma t}$$

That is to say, the total energy $E(t)$ of the system does not possess exponential decay.

However, if we use modal truncation as an approximation, the finite dimensional truncated system energy does decay exponentially in both linear and saturation non-linear damping cases. This is a difference between the original infinite dimensional model and the finite dimensional truncated model.

Let us condiser the saturation type nonlinear damping case. Let the finite mode approximation of $x(t)$ be

$$x(t) \approx \sum_{n=1}^{N} a_n(t)\psi_n$$

Then $\{a_n(t),\ n = 1, \cdots, N\}$ satisfy

$$\ddot{a}_n(t) + \phi_n(L)/b^2 z_N(t) + \omega_n^2 a_n(t) = 0, \quad n = 1, 2, \cdots, N \tag{173}$$

where

$$z_N(t) = f(\sum_{n=1}^{N} \dot{a}_n(t)\phi_n(L))$$

By introducing the following notations

$$\Omega = \text{Diagonal}\{\omega_1^2, \omega_2^2, \cdots, \omega_N^2\}$$

$$x_N(t) = \begin{pmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{pmatrix}$$

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$$B_N = \begin{pmatrix} \phi_1(L) \\ \vdots \\ \phi_N(L) \end{pmatrix}$$

(173) can be recast into the form

$$\ddot{x}_N(t) + \frac{1}{b^2} B_N f(B_N^* \dot{x}_N(t)) + \Omega x_N(t) = 0$$

$$x_N(0) = \begin{pmatrix} a_1(0) \\ \vdots \\ a_N(0) \end{pmatrix}, \qquad \dot{x}_N(0) = \begin{pmatrix} \dot{a}_1(0) \\ \vdots \\ \dot{a}_N(0) \end{pmatrix}$$

Applying *Theorem 13*, we can conclude that for any initial data

$$\|x_N(t)\| \longrightarrow 0, \quad \text{as } t \to \infty$$

Then, through similar steps as in obtaining (164) we can obtain

$$\dot{a}_n(t) = -a_n(0)\omega_n \sin \omega_n t + \dot{a}_n(0) \cos \omega_n t - \frac{1}{b^2} \int_0^t \cos \omega_n(t - \tau)\phi_n(L)z_N(\tau)d\tau$$

$$n = 1, 2, \cdots, N$$

and, further that

$$
\begin{aligned}
z_N(t) &= f(\sum_{n=1}^{N} \dot{a}_n(t)\phi_n(L)) \\
&= k[\sum_{n=1}^{N}(\dot{a}_n(0) \cos \omega_n t - a_n(0)\omega_n \sin \omega_n t)\phi_n(L) \\
&\quad - \frac{1}{b^2} \int_0^t \sum_{n=1}^{N} \phi_n^2(L) \cos \omega_n(t - \tau)z_N(\tau)d\tau] - \psi(\sum_{n=1}^{N} \dot{a}_n(t)\phi_n(L))
\end{aligned}
$$

Therefore,

$$Z_N(s) = \frac{1}{1 + ksG_N(s, L)}[k \sum_{n=1}^{N} \frac{\dot{a}_n(0)s - a_n(0)\omega_n^2}{s^2 + \omega_n^2}\phi_n(L) - \Psi_N(s)]$$

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$$A_n(s) = \frac{1}{s^2 + \omega_n^2}[a_n(0)s + \dot{a}_n(0) - \phi_n(L)/b^2 Z_N(s)]$$

$$= \frac{P_n(s)}{\prod_{j=1}^{N}(s^2 + \omega_j^2)(1 + ksG_N(s,L))} + \frac{\phi_n(L)}{b^2}\frac{\Psi_N(s)}{(s^2 + \omega_n^2)(1 + ksG_N(s,L))}$$

where

$$G_N(s,L) = \frac{1}{b^2}\sum_{n=1}^{N}\frac{\phi_n^2(L)}{s^2 + \omega_n^2}$$

$$\Psi_N(s) = \mathcal{L}[\psi(\sum_{n=1}^{N}\dot{a}_n(t)\phi_n(L))]$$

and each $P_n(s)$ is a polynomial of s of order $2N - 1$, for $n = 1, 2, \cdots, N$.

Let

$$\sigma = \min_{1 \leq n \leq N}\{-\Re[s_n]\} > 0$$

where $\{s_n, \overline{s_n}, n = 1, 2, \cdots, N\}$ are the roots of the following rational characteristic equation

$$1 + ksG_N(s,L) = 0$$

Then, through inverse Laplace transform we can see that

$$|a_n(t)| \leq M_n e^{-\sigma t}$$

$$|\dot{a}_n(t)| \leq M_n e^{-\sigma t}$$

for some $M_n > 0$. And hence,

$$E_N(t) = 1/2\sum_{n=1}^{N}(\omega_n^2 a_n^2(t) + \dot{a}_n^2(t)) \leq K_0 e^{-2\sigma t}, \quad t \geq T(k, M)$$

for some $K_0 > 0$. Therefore, we can find $K > 0$ such that

$$E_N(t) \leq Ke^{-2\sigma t}, \quad t \geq 0$$

i.e., the energy of the modal truncation model decays exponentially.

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7.5 Sensor Noise Effect

Since in most cases, measurements are corrupted by sensor noise, we consider the effect of sensor noise in this section. We are particularly interested in the sensor noise effect on the steady state motion of the offset antenna. For simplicity, we assume that the sensor noise is a white noise process $n(t)$ with constant spectral density 1 over the whole frequency range. Then the sensor output is given by

$$
\begin{aligned}
v(t) &= B^*\dot{x}(t) - \sigma n(t) \\
&= \dot{u}(t,L) - \sigma n(t)
\end{aligned}
$$

If we use direct connection, the feedback control is

$$
z(t) = f(\dot{u}(t,L) - \sigma n(t))
$$

Again, we use the continuum model (150), which, in this case with sensor noise, can be rewritten as

$$
\begin{aligned}
&\ddot{u}(t,r) + a^2 u''''(t,r) = 0 \\
&\ddot{u}(t,L) - b^2 u'''(t,L) + f(\dot{u}(t,L) - \sigma n(t)) = 0 \\
&u(t,0) = 0 \\
&u'(t,0) = 0 \\
&u''(t,L) = 0
\end{aligned}
\qquad (174)
$$

where $f(x)$ can be either a linear function, $f(x) = kx$, or, a saturation function represented by

$$
f(x) = M \tan^{-1}(kx)
$$

for our convenience.

Linear Damping Case

In linear damping case, things are rather trivial because we have the luxury of transfer function - we can find the transfer function between the beam tip position $u(t, L)$ and the sensor noise input $N(t)$.

In fact, by setting the initial conditions to be zero, we obtain as before

$$u(t, L) = \frac{1}{b^2} \int_0^t \sum_{n=1}^\infty \frac{\sin \omega_n (t - \tau)}{\omega_n} \phi_n^2(L) z(\tau) d\tau$$

where, in this case, $z(t) = k[\dot{u}(t, L) - N(t)]$. Then, through Laplace transform one can find

$$\frac{U(s, L)}{N(s)} = \frac{kG(s, L)}{1 + ksG(s, L)}$$

In particular, if $N(t) = \sigma n(t)$, a white noise process, then we have the spectral density of $u(t, L)$ in stationarity:

$$
\begin{aligned}
\Phi_{uu}(\omega) &= \frac{\sigma^2 k^2 |G(i\omega, L)|^2}{|1 + ki\omega G(i\omega, L)|^2} \\
&= \frac{\sigma^2}{|i\omega + 1/k [G(i\omega, L)]^{-1}|^2} \qquad -\infty < \omega < \infty
\end{aligned}
$$

Then from the spectral density, we can further calaulate the stationary variance of $u(t, L)$ by integration. *Figure 4* is a typical plot of the spectral density $\Phi_{uu}(\omega)$, from which, we can see that if the sensor noise involves only those frequencies, $\{z_n, \ n = 1, 2, \cdots \}$, i.e. the zeros of $G(iz, L)$, such that

$$N(t) = \sum_n c_n \cos z_n t + d_n \sin z_n t$$

then the beam tip is motionless at steady state, while the beam itself is vibrating under the sensor noise excitation.

Instead of going further, we switch our attention to saturation type nonlinear damping case.
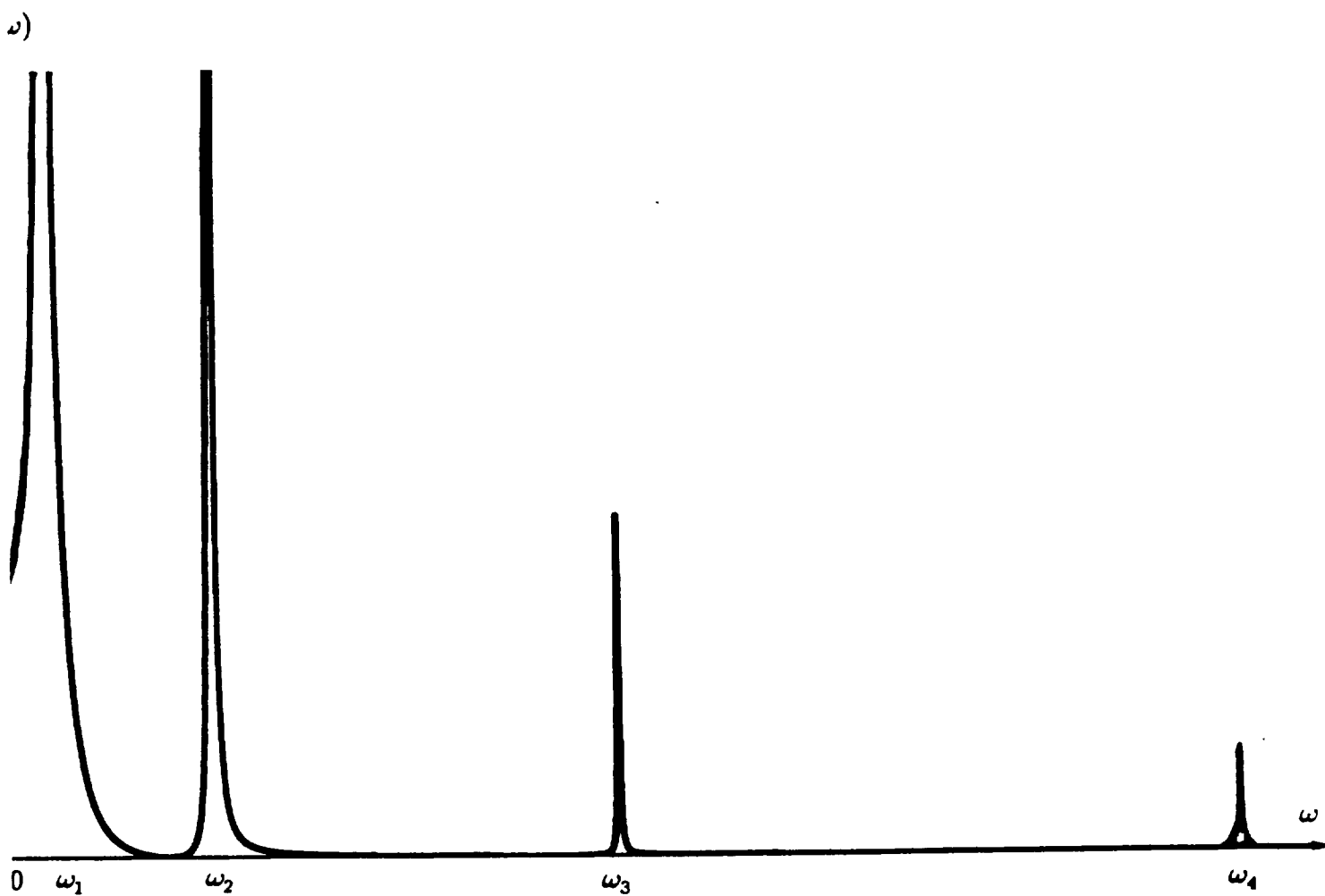
$\omega)$

Figure 4. The spectral density of the stationary beam tip motion.

Saturation Damping Case

In order to cast (174) into Ito form, we first employ the following approximation

$$
\begin{aligned}
f(\dot{u}(t,L)) - \sigma n(t) &\approx f(\dot{u}(t,L)) - f'(\dot{u}(t,L))\sigma n(t) + \sigma^2/2 f''(\dot{u}(t,L)) \\
&= M \tan^{-1}(k\dot{u}(t,L)) - \frac{Mk\sigma n(t)}{1 + [k\dot{u}(t,L)]^2} \\
&\quad - M(\sigma k)^2 \frac{k\dot{u}(t,L)}{[1 + (k\dot{u}(t,L))^2]^2}
\end{aligned}
$$

so that the antenna motion equation in (174) is approximated by

$$
\ddot{u}(t,L) + f(\dot{u}(t,L)) + \sigma^2/2 f''(\dot{u}(t,L)) - b^2 u'''(t,L)
$$
$$
= f'(\dot{u}(t,L))\sigma n(t) \tag{175}
$$

Next, in order to simplify the diffusion coefficient, we divide both sides of (175) by $1/(Mk)f'(\dot{u}(t,L))$, which is always positive, to obtain

$$
\ddot{u}(t,L) + MD(k\dot{u}(t,L)) + [k\dot{u}(t,L)]^2[\ddot{u}(t,L) - b^2 u'''(t,L)]
$$
$$
- b^2 u'''(t,L) = Mk\sigma n(t) \tag{176}
$$

where

$$
D(x) = (1 + x^2)\tan^{-1} x - (\sigma k)^2 \frac{x}{1 + x^2} \qquad x \in I\!R^1
$$

Before we proceed, we pause on the curve of $D(z)$ versus z. First, $D(z)$ is an odd function, hence it is sufficient to study $D(z)$ curve for positive z. We can easily have, when $\sigma k \leq 1$,

$$
D'(z) \geq [1 - (\sigma k)^2] + 2z \tan^{-1} z \geq 0 \qquad z \geq 0
$$

Therefore, $D(z)$ is positive and monotone increasing on $(0, \infty)$ when $\sigma k \leq 1$. When $\sigma k > 1$,

$$
D'(0) = 1 - (\sigma k)^2 < 0
$$

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That is, there exists $z_0 > 0$ such that $D(z) < 0$ on $(0, z_0)$ and $D(z) > 0$ on (z_0, ∞). The $D(z)$ curves corresponding to a few σk values are plotted in *Figure 5.*

Next, consider that

$$
\begin{aligned}
& [k\dot{u}(t, L)]^2[\ddot{u}(t, L) - b^2 u'''(t, L)] \\
= & (kb)^2 \dot{u}(t, L)\frac{1}{b^2}\dot{u}(t, L)[\ddot{u}(t, L) - b^2 u'''(t, L)] \\
= & (kb)^2 \dot{u}(t, L)[\dot{x}(t), \ddot{x}(t) + Ax(t)] \\
= & (kb)^2 \frac{dE(t)}{dt} B^* \dot{x}(t)
\end{aligned}
$$

Hence, (176) can be rewritten as

$$
\ddot{u}(t, L) + MD(kB^* \dot{x}(t)) + (kb)^2 \dot{E}(t) B^* \dot{x}(t)
$$

$$
-b^2 u'''(t, L) = Mk\sigma n(t)
$$

and, from which, (174) is approximated by

$$
\ddot{x}(t) + MBD(kB^* \dot{x}(t)) + (kb)^2 \dot{E}(t) BB^* \dot{x}(t)
$$

$$
+Ax(t) = Mk\sigma Bn(t) \tag{177}
$$

In what follows, we are only interested in the steady state behavior of (177). After the system reaching stationarity, the total energy $E(t)$ fluctuates around a constant energy level and $\dot{E}(t)$ becomes a zero mean stationary stochastic process. Therefore, we can intuitively see that, after reaching stationarity, among the two damping terms, the nonlinear damping term $MBD(kB^* \dot{x}(t))$ becomes dominant over the other damping term, which has a zero mean random damping coefficient. Therefore, we neglect the damping term $(kb)^2 \dot{E}(t) BB^* \dot{x}(t)$ in (177) so that the stationary structure response is approximated by the following Ito type stochastic distributed parameter system:

$$
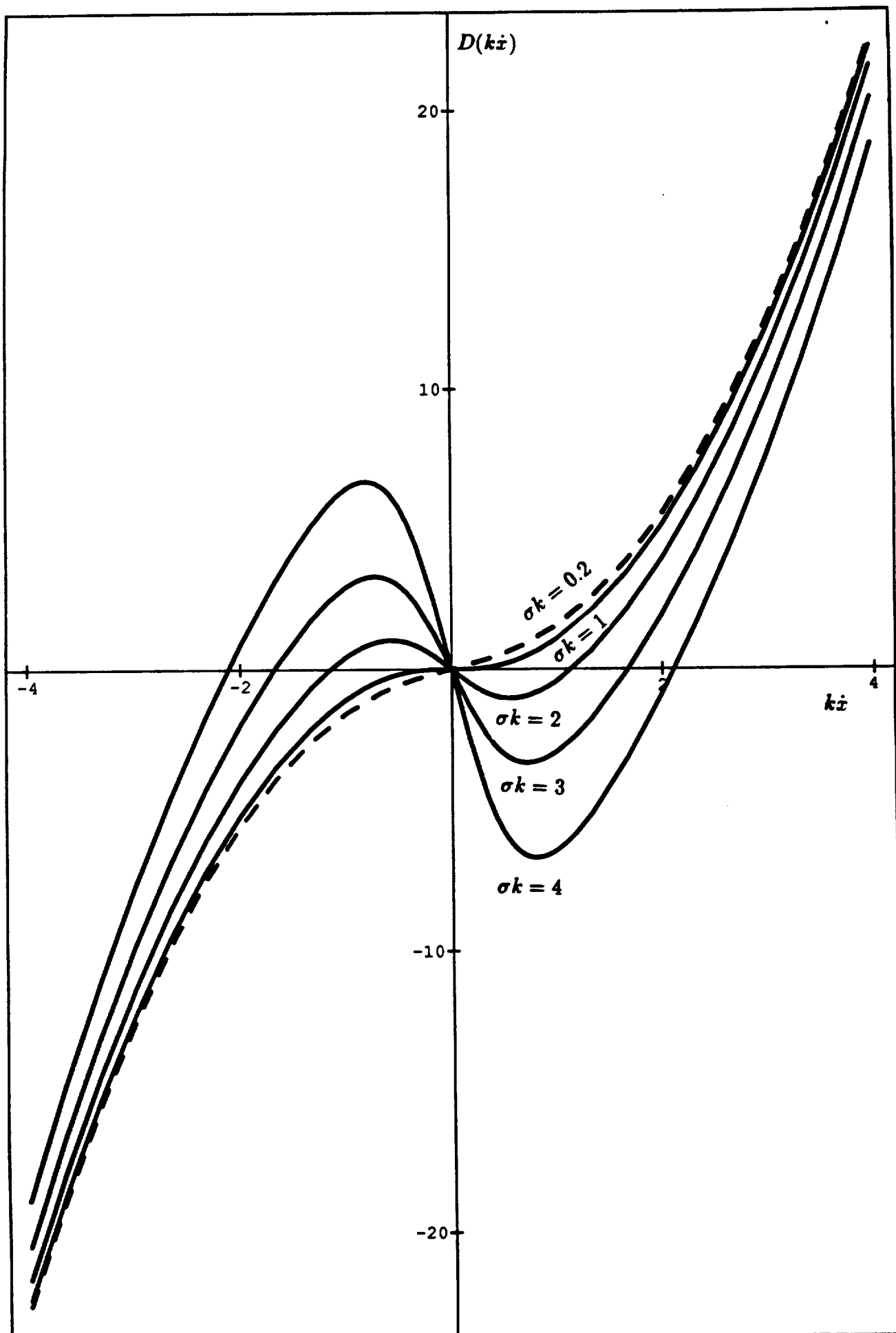\ddot{x}(t) + MBD(kB^* \dot{x}(t)) + Ax(t) = Mk\sigma Bn(t) \tag{178}
$$

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Figure 5. The curves of $D(k\dot{x})$ for various σk values.

From (178), through similar procedures as in Section 4, we can obtain

$$x(t) = \int_0^t S(t - \tau)B[-MD(k\dot{u}(\tau, L)) + Mk\sigma n(\tau)]d\tau$$

where we have set the initial conditions to be zero since we are only concerned with the stationary behavior.

By separating the boundary (scalar) component in $x(t)$, we obtain from the above equation

$$u(t, L) = M \int_0^t K_2(t - \tau)[-D(k\dot{u}(\tau, L)) + k\sigma n(\tau)]d\tau$$

where

$$K_2(t) = 1/b^2 \sum_{n=1}^{\infty} \frac{\phi_n^2(L)}{\omega_n} \sin \omega_n t$$

From (154), we have

$$1/b^2 \frac{\phi_n^2(L)}{\omega_n} = O(n^{-4})$$

Therefore, $K_2(t)$ is a fast converging series and we can reasonably use first N terms as an approximation of the infinite sum to obtain

$$u(t, L) = \frac{M}{b^2} \int_0^t \sum_{n=1}^{N} \frac{\phi_n^2(L)}{\omega_n} \sin \omega_n(t - \tau)[-D(k\dot{u}(\tau, L)) + k\sigma n(\tau)]d\tau$$

By defining

$$y_n(t) = \int_0^t \frac{M}{b^2} \frac{\phi_n^2(L)}{\omega_n} \sin \omega_n(t - \tau)[-D(k\dot{u}(\tau, L)) + k\sigma n(\tau)]d\tau$$
$$n = 2, 3, \cdots$$

we can realize that the beam tip (antenna) motion $(u(t, L), \dot{u}(t, L))$ satisfies the

following system of stochastic differential equations:

$$
\begin{cases}
\ddot{u}(t,L) + \frac{M}{b^2}[\sum_{n=1}^{N} \phi_n^2(L)]D(k\dot{u}(t,L)) \quad + \quad \omega_1^2 u(t,L) \\[2em]
\qquad + \sum_{n=1}^{N}(\omega_n^2 - \omega_1^2)y_n(t) \quad = \quad \frac{M}{b^2}[\sum_{n=1}^{N}\phi_n^2(L)]k\sigma n(t) \\[2em]
\ddot{y}_n(t) + \frac{M}{b^2}\phi_n^2(L)D(k\dot{u}(t,L)) \quad + \quad \omega_n^2 y_n(t) = \frac{M}{b^2}\phi_n^2(L)k\sigma n(t) \\[2em]
\qquad\qquad n \quad = \quad 2,3,\cdots,N
\end{cases}
\tag{179}
$$

We call (179) the N-th order approximation of the beam tip motion. Notice that in (179), the damping coefficient

$$
\frac{1}{b^2}\sum_{n=1}^{N}\phi_n^2(L) \quad \longrightarrow \quad 1 \ \text{ as } N \to \infty
$$

because

$$
\frac{1}{b^2}\sum_{n=1}^{\infty}\phi_n(r)\phi_n(L) =
\begin{cases}
1 & r = L \\[1em]
0 & 0 \le r < L
\end{cases}
$$

In fact, it can be easily seen from

$$
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sum_{n=1}^{\infty}[\begin{pmatrix} 0 \\ 1 \end{pmatrix},\psi_n]\psi_n
$$

$$
= \frac{1}{b^2}\sum_{n=1}^{\infty}\phi_n(L)\begin{pmatrix} \phi_n(r) \\ \phi_n(L) \end{pmatrix}
$$

In particular, the first order $(N = 1)$ approximation of the beam tip motion is given by

$$
\ddot{u}(t,L) + \lambda D(k\dot{u}(t,L)) + \omega_1^2 u(t,L) = \lambda k\sigma n(t)
\tag{180}
$$

where $\lambda = M/b^2\phi_1^2(L)$.

When $k\sigma > 1$, the deterministic version of (180) has a stable limit cycle (*Figure 5*).

Stationary Statistics for Single-DOF Nonlinear Vibration

In the rest of this section, we will find the approximate stationary probability density and varaince of (180). This is a central topic in nonlinear random vibration theory. Various methods have been reported by many authors. For surveys, see [25] [33].

The method we are going to use is called *Method of Energy Approximation*, developed by the author [72].

What we propose is to approximate a general nonlinear damping model

$$\ddot{x}(t) + \gamma D(x, \dot{x}) + \omega_0^2 x(t) = \sigma \dot{w}(t) \tag{181}$$

by the following energy type nonlinear damping model

$$\ddot{x}(t) + \gamma \mu(\frac{\omega_0^2 x^2 + y^2}{2})\dot{x}(t) + \omega_0^2 x(t) = \sigma \dot{w}(t) \tag{182}$$

where $\mu(E)$ is chosen in such a way that it minimizes

$$\int_0^{2\pi} [D(\sqrt{2E}/\omega_0 \sin\psi, \sqrt{2E}\cos\psi) - \mu(E)\sqrt{2E}\cos\psi]^2 d\psi$$

In the above minimization, E is considered to be fixed, because after reaching stationarity, the energy fluctuates around a constant level. In other words, at stationarity, the energy absorbed by the damping mechanism is statistically equivalent to the energy input due to the external noise.

It turns out that $\mu(E)$ is given by [72]

$$\mu(E) = \frac{4}{\pi\sqrt{2E}} \int_0^{\pi/2} D(\sqrt{2E}/\omega_0 \sin\psi, \sqrt{2E}\cos\psi)\cos\psi d\psi \tag{183}$$

The following are two facts supporting the above approximation. The first fact is [72]

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Theorem 14 *If $D(x, y)$ is even with respect to x and odd with respect to y, then both the original model (181) and the corresponding modified model (182) have the same Krylov-Bogoliubov approximation, given by*

$$\begin{cases} da(t)/dt &= -\gamma/\omega_0 L(a(t)) \\ d\phi(t)/dt &= 0 \end{cases}$$

where

$$L(a) = \frac{1}{2\pi} \int_0^{2\pi} D(a \sin \psi, a\omega_0 \cos \psi) \cos \psi d\psi$$

The second fact is concerned with limit cycle.

For the general nonlinear damping model (181) without noise input, i.e.,

$$\ddot{x}(t) + \gamma D(x, \dot{x}) + \omega_0^2 x(t) = 0$$

by Krylov-Bogoliubov approximation, we know that a limit cycle exists if and only if

$$L(a) = 0$$

has a positive solution a_p. And in this case, the limit cycle has approximate amplitude a_p and frequency ω_0.

To find the stability of the limit cycle, let us define the deviation $\Delta a(t)$ from the limit cycle amplitude, i.e.,

$$a(t) = a_p + \Delta a(t)$$

Substituting into $\dot{a}(t) = -\gamma/\omega_0 L(a(t))$, we obtain

$$\begin{aligned} \frac{d\Delta a(t)}{dt} &= -\gamma/\omega_0 L(a_p + \Delta a(t)) \\ &\approx -\gamma/\omega_0 (\frac{dL}{da}|_{a=a_p})\Delta a(t) \end{aligned}$$

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Then, it becomes obvious that the limit cycle is stable if and only if

$$\frac{dL(a)}{dt}\Big|_{a=a_p} > 0$$

In summary, for a general nonlinear damping model, the existence, amplitude and stability of a limit cycle are determined by

$$\begin{cases} L(a) & = & 0 \\ \frac{dL(a)}{da}\Big|_{a=a_p} & > & 0 \text{ or } < 0 \end{cases}$$

Since both the original model and the corresponding modified model have the same $L(a)$ function, we conclude that both of the nonlinear damping models have the same limit cycles and stability propertity. Of course, this analysis is based upon the Krylov-Bogoliubov approximation.

For the modified model (182), its stationary probability density function is given by [72]

$$p_s(x,y) = C \exp[-\frac{2}{\sigma^2}\int_0^{\frac{\omega_0^2 x^2 + y^2}{2}} \mu(z)dz]$$

$$1/C = \frac{2\pi}{\omega_0}\int_0^\infty \exp[-\frac{2}{\sigma^2}\int_0^\rho \mu(z)dz]d\rho$$

In order to compute the function $\mu(E)$ for our problem (180), we will need the following three identities

$$\int_0^{\pi/2} \tan^{-1}(k\cos x)\cos x\,dx = \frac{\pi}{2k}(\sqrt{1+k^2}-1) \tag{184}$$

$$\int_0^{\pi/2} \tan^{-1}(k\cos x)\cos^3 x\,dx = \frac{\pi}{2k}[\sqrt{1+k^2}-1/2+\frac{1-(1+k^2)^{3/2}}{3k^2}] \tag{185}$$

$$\int_0^{\pi/2} \frac{\cos^2 x}{1+k^2\cos^2 x}dx = \frac{\pi}{2k^2}(1-1/\sqrt{1+k^2}) \tag{186}$$

Using (184) - (186), we obtain

$$\mu(E) = \frac{4}{\pi\sqrt{2E}}\int_0^{\pi/2} \lambda D(k\sqrt{2E}\cos\psi)\cos\psi\,d\psi$$

$$= 2\lambda[\frac{(1+2k^2E)^{3/2}-1}{3kE} - k/2 - \frac{(\sigma k)^2}{2kE} + \frac{(\sigma k)^2}{2kE\sqrt{1+2k^2E}}]$$

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Next, we need to evaluate $\int_0^E \mu(z)dz$. For this purpose, we first calaulate

$$\int_0^E [-\frac{(\sigma k)^2}{2kz} + \frac{(\sigma k)^2}{2kz\sqrt{1+2k^2z}}]dz = -\sigma^2 k \ln(1+\sqrt{1+2k^2E}) + \sigma^2 k \ln 2$$

$$\int_0^E [\frac{(1+2k^2z)^{3/2}-1}{3kz} - k/2]dz = \frac{2}{9k}(4+2k^2E)\sqrt{1+2k^2E}$$
$$-\frac{2}{3k}\ln(1+\sqrt{1+2k^2E})$$
$$-k/2E + \frac{1}{3k}(2\ln 2 - 8/3)$$

Then, we obtain

$$-\frac{2}{(\lambda k\sigma)^2}\int_0^E \mu(z)dz = -\frac{1}{\lambda k^3\sigma^2}(4+2k^2E)(8/9\sqrt{1+2k^2E}-1)$$
$$+\frac{4}{\lambda k}(1+\frac{2}{3(k\sigma)^2})\ln(1+\sqrt{1+2k^2E}) + \text{constant}$$

Therefore, the stationary probability density is given by

$$p(x,y) = C[1+\sqrt{1+k^2(\omega_1^2x^2+y^2)}]^\alpha$$
$$\times \exp\{-\beta[4+k^2(\omega_1^2x^2+y^2)][8/9\sqrt{1+k^2(\omega_1^2x^2+y^2)}-1]\}(187)$$

where

$$\alpha = \frac{4}{\lambda k} + \frac{8}{3}\beta$$
$$\beta = \frac{1}{\lambda k(\sigma k)^2}$$

and C is the normalizing constant, which is given by

$$1/C = \frac{2\pi}{\omega_1 k^2}\int_1^\infty x(1+x)^\alpha \exp[-8/9\beta(x^2+3)(x-9/8)]dx \qquad (188)$$

If we define

$$I_n(\alpha,\beta) = \int_1^\infty x^n(1+x)^\alpha \exp[-8/9\beta(x^2+3)(x-9/8)]dx$$

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for $n = 1, 2, \cdots$, then

$$C = \frac{\omega_1 k^2}{2\pi} I_1^{-1}(\alpha, \beta)$$ (189)

It is not difficult to see that the stationary variance is given by

$$\begin{aligned}
\mathbf{E} x^2 &= \frac{1}{2\omega_1^2 k^2} [\frac{I_3(\alpha, \beta)}{I_1(\alpha, \beta)} - 1] \\
&= \frac{1}{2\omega_1^2 k^2} \frac{I_1(\alpha + 2, \beta) - 2I_1(\alpha + 1, \beta)}{I_1(\alpha, \beta)}
\end{aligned}$$

Next, we study how the stationary density $p(x, y)$ is affected by the value of the parameter σk.

For this purpose, we define

$$\begin{aligned}
\xi(E) &= \sqrt{1 + 2k^2 E} \\
q(\xi) &= C(1 + \xi)^\alpha \exp[-8/9\beta(\xi^2 + 3)(\xi - 9/8)]
\end{aligned}$$ (190)

Obviously,

$$p(x, y) = q(\sqrt{1 + k^2(\omega_1^2 x^2 + y^2)})$$ (191)

Since

$$q'(\xi(0)) = q'(1) = C \frac{2^{\alpha+1}}{\lambda k} e^{4/9\beta} [1 - (\sigma k)^{-2}]$$

we realize that, when $\sigma k \leq 1$, the stationary density $p(x, y)$ attains maximum at the origin. For a typical plot of $p(x, y)$, see *Figure 6*. When $\sigma k > 1$, the maximum is not achieved at the origin, but rather, along a parabola with center located at the origin. A typical plot of $p(x, y)$ in this case is shown in *Figure 7*. *Figure 8* is the density plot of $p(x, y)$ corresponding to *Figure 7*, clearly indicating the altitude of the stationary density.

We can actually find the equation of the parabola on which $p(x, y)$ achieves maximum in the case $\sigma k > 1$. Consider the equation
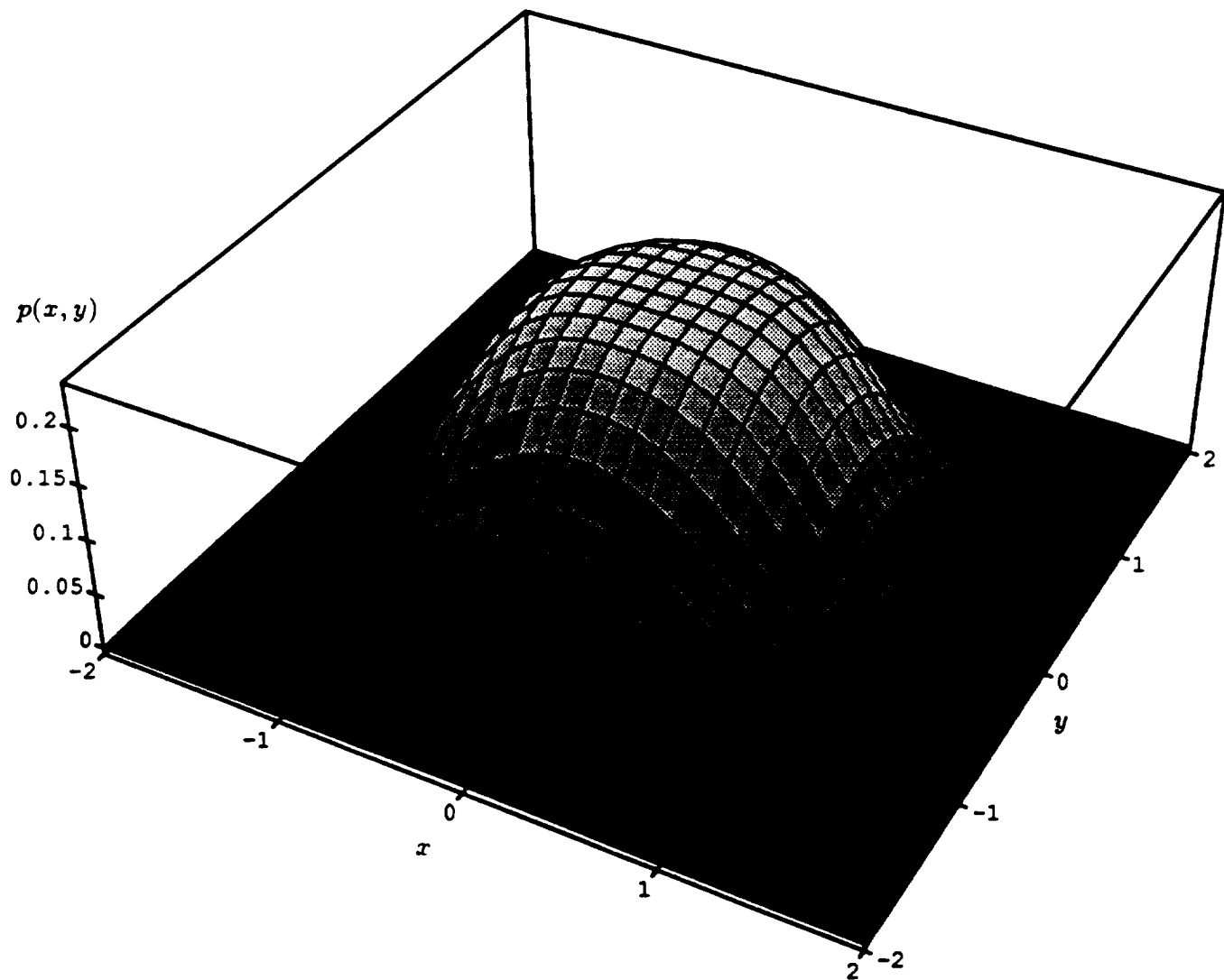
$$q'(\xi(E)) = 0$$

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Figure 6. The stationary probability density $p(x, y)$ for $\sigma = 1$, $k = 1$, $\lambda = 1$,
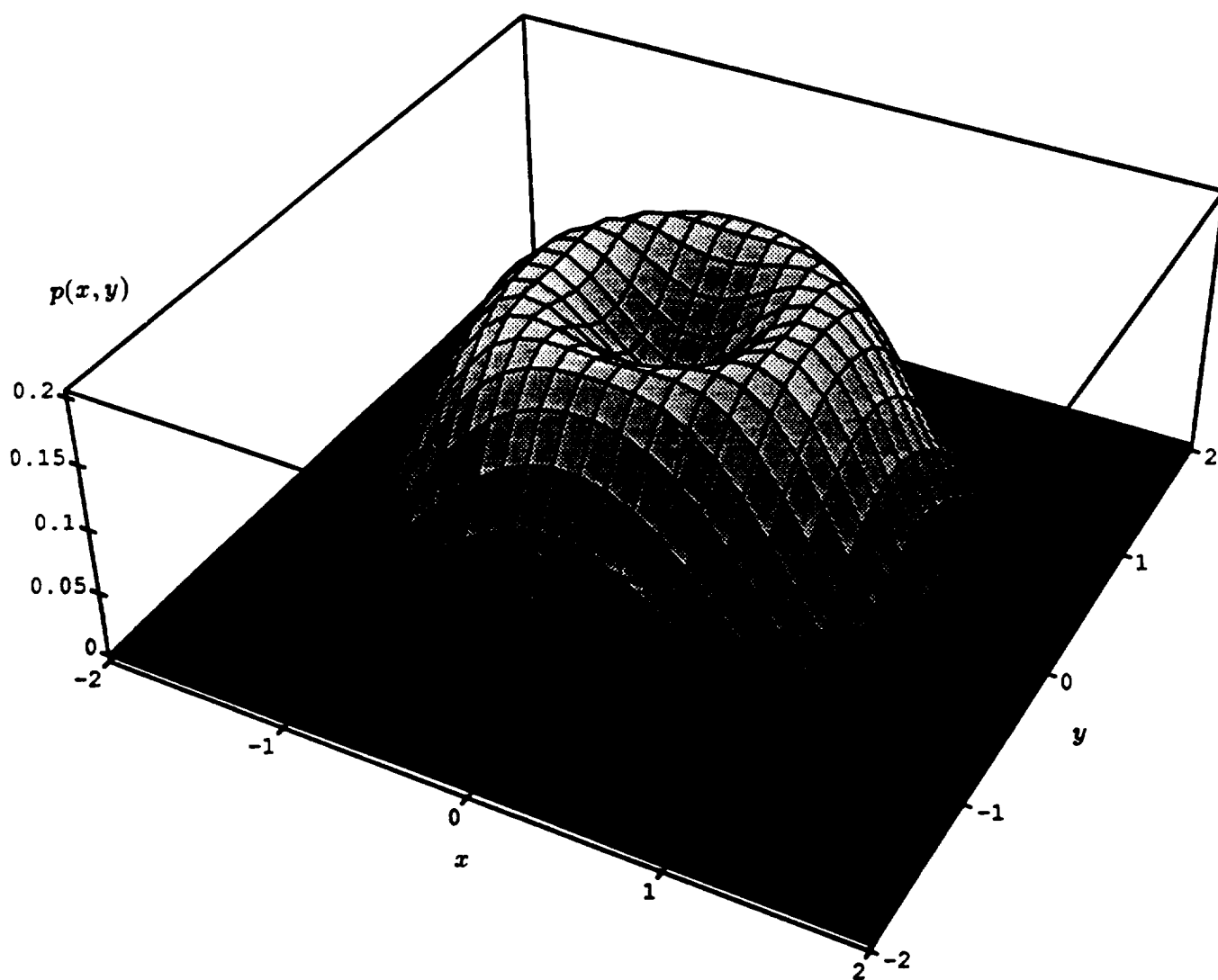
i.e. $\sigma k \leq 1$ case.

Figure 7. The stationary probability density $p(x, y)$ for $\sigma = 1$, $k = 3$, $\lambda = 1$, i.e. $\lambda k > 1$ case.
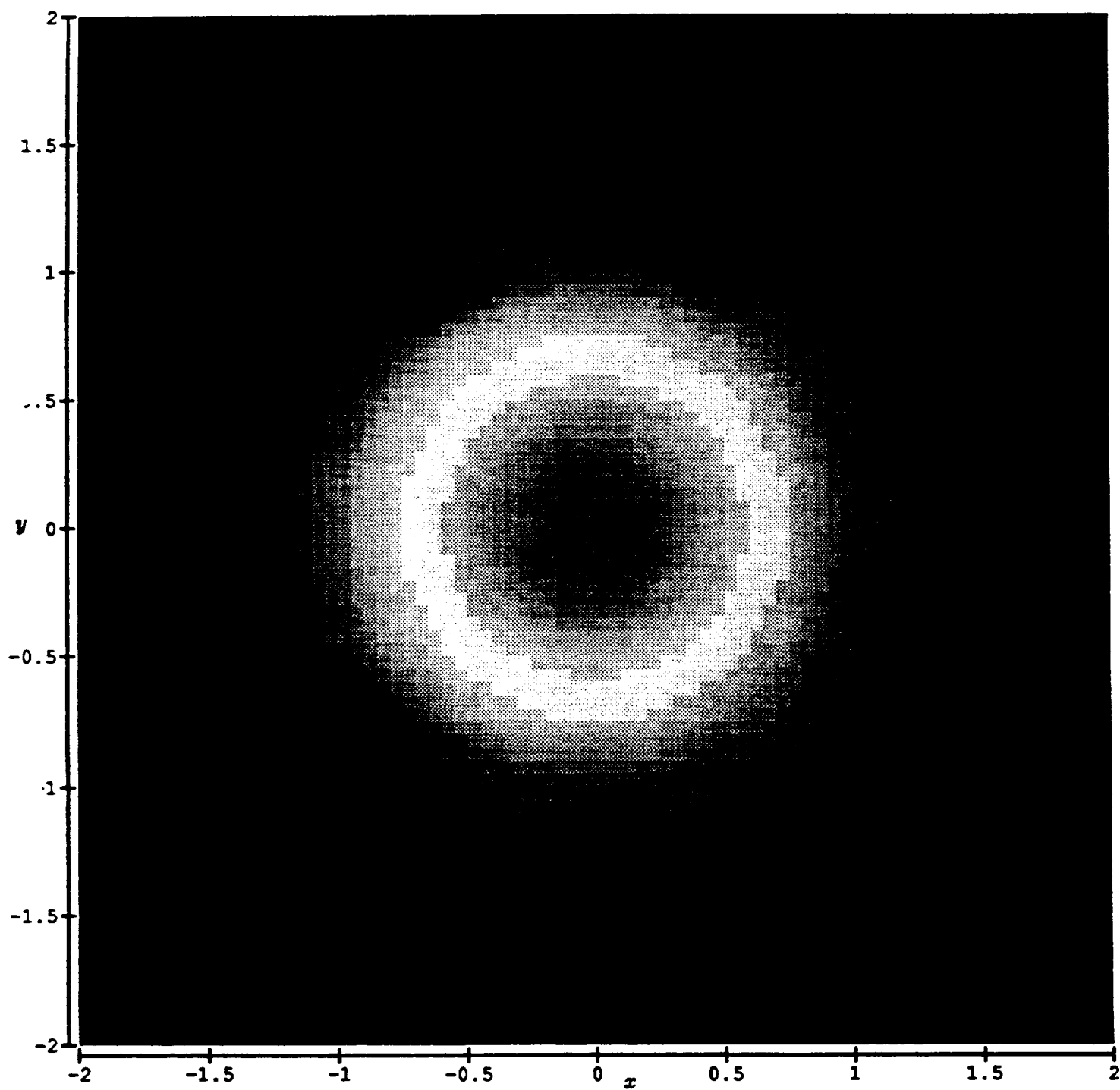
Figure 8. The density plot of $p(x, y)$ with $\sigma = 1$, $k = 3$, and $\lambda = 1$.

which is equivalent to

$$\xi^3 + 1/4(\xi^2 + \xi) = 3/2(\sigma k)^2 \tag{192}$$

The solution of (192) is given by

$$\xi_0(\sigma k) = h(3/2(\sigma k)^2) - \frac{11}{12^2}[h(3/2(\sigma k)^2)]^{-1} - 1/12$$

where

$$h(x) = \{\frac{17}{1728} + x/2 + [\frac{1331}{2985984} + (\frac{17}{1728} + x/2)^2]^{1/2}\}^{1/3}$$

Therefore, the equation of the parabola is

$$\sqrt{1 + k^2(\omega_1^2 x^2 + y^2)} = \xi_0(\sigma k)$$

or

$$\omega_1^2 x^2 + y^2 = \frac{1}{k^2}[\xi_0^2(\sigma k) - 1]$$

And, it is easy to verify that

$$\xi_0(\sigma k) > 1 \quad \Longleftrightarrow \quad \sigma k > 1$$

In *Figure 9*, the variation of the stationary statistics Ex^2 as a function of k and λ is plotted. It is obvious that the stationary variance increases more significantly in k direction than in λ direction, for fixed observation noise intensity $\sigma > 0$.

Among those parameters, σ - the sensor noise intensity, and λ - the maximum output of the actuator, are dependent upon the sensor and actuator used. The only parameter we can adjust is k in the linear range slope λk. Through the above analysis, we know that when k is large ($\sigma k > 1$), the stationary variance (deviation from the equilibrium) is large. Therefore, to achieve small deviation from the equilibrium, a small k is desired. However, at transient, if k is small, the active damper no longer significantly enhance the stability. Therefore we have two options:
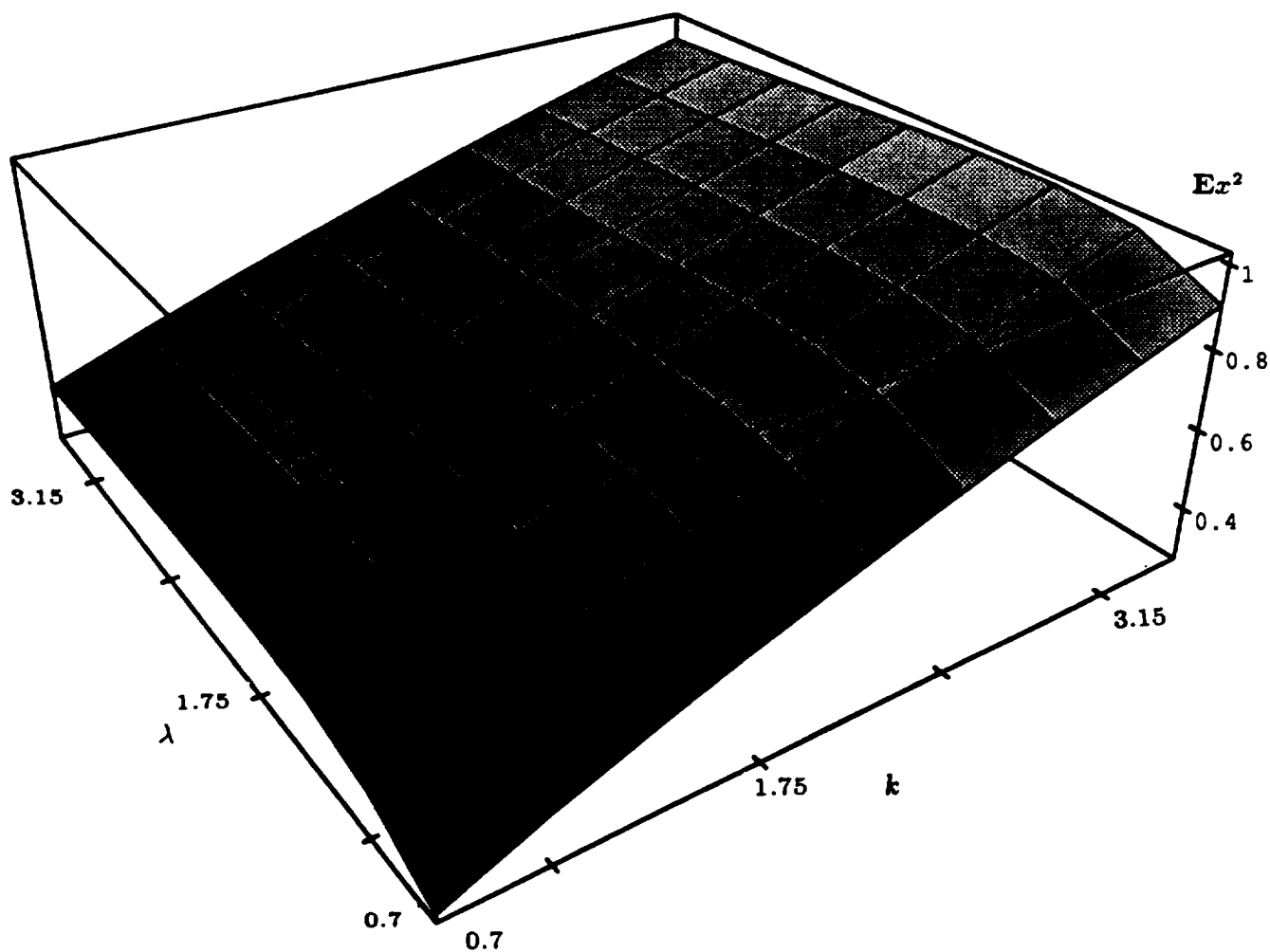
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Figure 9. The plot of the stationary variance Ex^2 versus k and λ with $\sigma = 1$.

- preset a compromized value for k such that $k \approx 1/\sigma$;

- let k vary in the following manner:

$$k(t) = \begin{cases} \text{relatively large } k_1 \text{ at transient or when signal } - \text{ noise ratio is large;} \\ \text{relatively small } k_2 \text{ after transient or when signal } - \text{ noise ratio is} \\ \text{small.} \end{cases}$$

7.6 Conclusions

A group of sufficient conditions for strong stabilizability is provided for general distributed parameter oscillation system, taking the actuator saturation into consideration. These are the weakest sufficient conditions obtained so far and it is found that the nature of internal damping is not crucial in guaranteeing the strong stabilizability.

By extending the notions of *Characteristic Equation* and *Root Locus* to our distributed parameter system, we studied the nature of active damping. Active damping excites all other modes even if only one mode is involved initially. Saturation type nonlinear damping does not generate any qualitative change in terms of mode response. In particular, each component frequency remains the same and no extra frequency is introduced. Indeed, the Laplace transform of each mode response in both linear and nonlinear damping cases have the same poles, which are the roots of the characteristic equation. Through the root locus, it is also found that using one active damper can only significantly damp or eliminate the base frequency oscillation and higher frequency oscillations become dominant in the response of the actively damped system.

We studied the effect of sensor noise on the steady state response of the beam

tip. By calculating the approximate stationary probability density of the beam tip motion $(u(t, L), \dot{u}(t, L))$, we have found that a large feedback gain (linear range slope) will result in a relatively large steady state deviation of the beam tip from the equilibrium due to sensor noise excitation. Therefore, in the design of the feedback gain, a compromise has to be made between satisfactory transient damping effect and reasonably small steady state deviation of the beam tip.

Chapter 8

Conclusions and Some Open Questions

8.1 Conclusions

Frequency Response of Nonlinear Damping Model: Single DOF Case

A method of computing the correlation function and the spectral density of nonlinear damping model is obtained. In the case that $D(x,y)$ being of the form $\mu(\frac{\omega_0^2 x^2 + y^2}{2})y$, the spectral density is given by (40) in which ψ_y is given by (48). In the case of general $D(x,y)$, the spectral density is given by (32) for which one needs to evaluate ψ_y and $m_{2,0}$. ψ_y can be obtained from (31) or from (48) by first finding the corresponding $\mu(\cdot)$ through the formula (45). Based upon (38), the approximate value of $m_{2,0}$ is given by

$$m_{2,0} = \frac{1}{\omega_0^2} \frac{m(1)}{m(0)}$$

This paper also proposes a method to obtain the approximate explicit stationary

density for nonlinear damping model with general $D(x, y)$. The idea is to replace $D(x, y)$ in the exact model by the corresponding $\mu(\frac{\omega_0^2 x^2 + y^2}{2})y$ to obtain the modified model. The approximate explicit stationary density is given by (52) and (53). It is shown that both the exact model and the modified model have the same Krylov-Bogoliubov approximation and the same ψ_x, ψ_y.

In the Spacecraft Control Laboratory Experiment (SCOLE) program, the primary control task is to rapidly slew or change the line-of-sight of an antenna attached to the space shuttle orbiter, and to settle or damp the structural vibrations to the degree required for precise pointing of the antenna. The objective will be to minimize the time required to slew and settle, untill the antenna line-of-sight remains within the prescribed angle. From practical consideration, the maximum moment and force generating capability of the controllers on both the shuttle and the antenna beam/reflector are limited (maximum moment on both shuttle and antenna reflector is 10^4 ft-lb for each axis, maximum control on the reflector is 800 lb). Therefore, saturation type of control is inevitable. To avoid significant excitation of the beam while applying the slew control, the first harmonic of the slew control versus time should stay away from the resonant frequencies of the first a few modes of the antenna beam. This consideration is important in the design of the slew control. This paper provides an analytical frame of finding the spectral density of each mode, which is basically the amplitude of the system response corresponding to the sinusoidal input with each frequency ω. Specifically, the spectral density tells the designers where the resonant frequency is.

Theorem 9 gives an interesting result concerning the modelling and identification of nonlinear internal damping in flexible space structures. In spite of the fundamental importance of the damping term, the nature of internal damping has been little

known. In [8], the following nonlinear damping model is proposed

$$\ddot{x} + 2\xi\omega_n\dot{x} + \gamma x^{2m}|x|^\alpha \dot{x}^{2n+1}|\dot{x}|^\beta + \omega_n^2 x = 0 \qquad (193)$$

The free response of (193) fits NASA flight data with great accuracy. However, if we replace the nonlinear damping term in (193) by $\mu(\frac{\omega_n^2 x^2 + \dot{x}^2}{2})\dot{x}$ where

$$
\begin{aligned}
\mu(E) &= \frac{4}{\pi\sqrt{2E}} \int_0^{\pi/2} D(\frac{\sqrt{2E}}{\omega_n}\sin\theta, \sqrt{2E}\cos\theta)\cos\theta d\theta \\
&= \frac{\mu_0}{\omega_n^{2m+\alpha}}(\omega_n^2 x^2 + \dot{x}^2)^q
\end{aligned}
$$

in which

$$
\begin{aligned}
q &= m + n + (\alpha + \beta)/2 \\
\mu_0 &= \frac{2}{\pi}\frac{\Gamma(m + \frac{\alpha+1}{2})\Gamma(n + 1 + \frac{\beta+1}{2})}{\Gamma(q + 2)}
\end{aligned}
$$

we obtain the following energy type nonlinear damping model

$$\ddot{x} + 2\xi\omega_n\dot{x} + \gamma\frac{\mu_0}{\omega_n^{2m+\alpha}}(\omega_n^2 x^2 + \dot{x}^2)^q \dot{x} + \omega_n^2 x = 0 \qquad (194)$$

By *Theorem 3*, both (193) and (194) have the same Krylov-Bogoliubov approximation, that is to say, these two nonlinear damping models can not be distinguished based upon their free responses (for details, see [72]).

Frequency Response of Nonlinear Damping Model: Multi-DOF Case

A formula for computing the spectral density matrix of nonlinear damping model with n-DOF is presented, (77). The error of the formula (77) is of the order $O(\gamma^2)$.

It is not surprising that the spectral density matrix depends on the first order statistics $R_{xx}(0)$ and $R_{xy}(0)$ for which we need the (first order) stationary probability density. However, to find the stationary density in general is itself a difficult task.

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In the case of energy type damping, i.e.

$$D(x,y) = \mu(\frac{x^T K D x + y^T M D y}{2})\sigma\sigma^T D y$$

the explicit stationary probability density can be obtained and is given by (90) and (91). In this case, the stationary density function is a function of energy E_D with

$$E_D = 1/2(x^T K D x + y^T M D y)$$

and $R_{xy}(0) = 0$, $R_{xx}(0)$ can be obtained explicitly and is proportional to K^{-1}, see (95).

As we know, in single-DOF case, x and $\dot{x}$ are always uncorrelated in stationary state. However, this luxiury does not extend to multi-DOF model. It is pointed out in this paper that even for linear multi-DOF model, x and $\dot{x}$ are in general not uncorrelated. A necessary and sufficient condition for uncorrelatedness is given by (82).

The conclusions on the infinite dimensional model

1. *The shape of the frequency-response curve*

   Let us consider any fixed mode, say the first mode. For single frequency excitation with frequency close to ω_1, i.e.,

   $$\omega = \omega_1 + \epsilon\sigma$$

   the frequency-response equation is given by

   $$\rho_1^2(\sigma^2) = \frac{(B^*\psi_1/(2\omega_1))^2}{\sigma^2 + [\xi_0\omega_1 + \gamma_0/2(\omega_1^2\rho_1^2(\sigma^2))^q]^2}$$

   The unique positive solution, denoted by $g(\sigma|\omega_1, B^*\psi_1)$, when plotted, is still bell-shaped. And there is no multi-peak phenomenon because $g(\cdot|\omega_1, B^*\psi_1)$ is an even function of σ and is monotone decreasing as σ^2 increases.

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The frequency-response curve is given by

$$
\rho_1(\omega) =
\begin{cases}
g(\sigma|\omega_1, B^*\psi_1) & \text{when } \omega = \omega_1 + \epsilon\sigma \\
\\
\epsilon\frac{|B^*\psi_1|}{|\omega^2-\omega_1^2|} & \text{when } |\omega - \omega_1| = O(1)
\end{cases}
$$

or, generally,

$$
\rho_n(\omega) =
\begin{cases}
g(\sigma|\omega_n, B^*\psi_n) & \text{when } \omega = \omega_n + \epsilon\sigma \\
\\
\epsilon\frac{|B^*\psi_n|}{|\omega^2-\omega_n^2|} & \text{when } |\omega - \omega_n| = O(1)
\end{cases}
$$

However, for nonlinear stiffness (linear damping) problem, the shape of the frequency-response curve is more complex and qualitatively different. For example, for the following *Duffing oscillator*

$$
\ddot{x} + 2\epsilon\mu\dot{x} + \omega_0^2 x + \epsilon\alpha x^3 = E(t)
$$

the corresponding frequency-response equation is given by

$$
\rho^2(\sigma) = \frac{k^2/(2\omega_0)^2}{\mu^2 + [\sigma - \frac{3}{8}\frac{\alpha}{\omega_0}\rho^2(\sigma)]^2}
$$

One can easily see that $\rho(\sigma)$ is no longer an even function of σ. In fact, in certain range of frequency, $\rho(\sigma)$ is even multi-valued. As long as $\alpha \neq 0$, the frequency-response curve is a *backbone curve*.

In this case, there are jump phenomenon and caotic behavior which will be discussed later. Of course, if $\alpha = 0$, the model becomes linear and the frequency-response curve is single-valued and takes the shape we are familiar with.

2. *Comparison with linear damping problem*

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It is well known that for a single DOF linear oscillator

$$\ddot{x} + 2\epsilon \xi_0 \omega_1 \dot{x} + \omega_1^2 x = \epsilon B^* \psi_1 \cos \omega t$$

the stationary amplitude is given by its spectral density

$$\rho_L^2(\omega) = \frac{(\epsilon B^* \psi_1)^2}{(\omega^2 - \omega_1^2)^2 + 4\xi_0^2 \epsilon^2 \omega_1^2 \omega^2}$$

When $\omega = \omega_1 + \epsilon \sigma$, i.e.,

$$\omega^2 - \omega_1^2 = 2\epsilon \sigma \omega_1 + \epsilon^2 \sigma^2$$

we have, for the linear model,

$$\rho_L^2(\omega) = \frac{(B^* \psi_1/(2\omega_1))^2}{(\sigma + \epsilon \frac{\sigma^2}{2\omega_1})^2 + (\xi_0 \omega_1 + \epsilon \xi_0 \sigma)^2}$$

While the corresponding nonlinear counterpart is

$$\rho_N^2(\omega) = \frac{(B^* \psi_1/(2\omega_1))^2}{\sigma^2 + [\xi_0 \omega_1 + \gamma_0/2(\omega_1^2 \rho_N^2)^q]^2}$$

As we can see,

$$\rho_N(\omega) < \rho_L(\omega) \quad \text{for} \quad \omega = \omega_1 + \epsilon \sigma$$

This is not surprising because inclusion of nonlinear damping makes the total damping greater. This in turn results in smaller steady state response amplitude.

When ω, the excitation frequency, is away from ω_1, we have

$$\rho_L(\omega) = \epsilon \frac{|B^* \psi_1|}{|\omega^2 - \omega_1^2|} + O(\epsilon^3)$$

$$\rho_N(\omega) = \epsilon \frac{|B^* \psi_1|}{|\omega^2 - \omega_1^2|} + O(\epsilon^2)$$

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i.e., these two frequency-response curves are very close when the excitation frequency is away from the natural frequency.

From above we can see, comparing with linear model, the frequency-response curve of nonlinear damping model has no qualitative change and the only difference is the frequency-response curve of nonlinear damping model is below the curve corresponding to the linear model, especially in the neighborhood of the natural frequency of that mode.

3. *Jump phenomenon and chaotic behavior*

For nonlinear stiffness problem such as the *Duffing oscillator*, the multivaluedness of the frequency-response curve due to the nonlinearity of stiffness has a significance from the physical point of view because it leads to jump phenomenon. To explain this, we imagine that an experiment is performed in which the amplitude of the excitation is held fixed, the frequency of the excitation, i.e. σ, is slowly varied up and down through the natural frequency ω_0. We observe the amplitude of the harmonic response. If σ starts from the left side of the peak and increases, the amplitude will jump from the peak value to the lower value. Conversely, if σ starts from the right side of the peak and decreases, the response amplitude will jump from the lower value to a higher value. This jump phenomenon is due to the presence of nonlinearity.

Then what value does the steady state amplitude take if the excitation frequency starts and stays at a point within the multivalued region? The answer is, it depends on the initial condition. In other words, if more than one steady states exist, the initial condition determine which steady state is physically realized by the system.

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This chaotic behavior is exclusively possessed by nonlinear stiffness model. For nonlinear damping (linear stiffness) problem, the frequency-response curve is always single-valued. This means that the steady state response of a nonlinear damping system is independent of the initial conditions. And for nonlinear damping problem, all the points in the frequency-response curve are physically realizable and there is no jump phenomenon or chaotic behavior. This is one of the fundamental differences between nonlinear damping problems and nonlinear stiffness problems.

4. *Internal resonance and energy exchange between modes*

What is internal resonance? For multi-DOF nonlinear systems, an important case occurs whenever two or more natural frequencies are commensurable or nearly commensurable. Examples of near-commensurability are

$$\omega_2 \approx 2\omega_1, \qquad \omega_2 \approx 3\omega_1, \qquad \omega_3 \approx \omega_2 \pm \omega_1,$$

$$\omega_3 \approx 2\omega_2 \pm \omega_1, \quad \omega_4 \approx \omega_3 \pm \omega_2 \pm \omega_1$$

Depending on the order of the nonlinearity in the system, these commensurable relationships of frequencies can cause the corresponding modes to be strongly coupled, and an *internal resonance* is said to exist. When an internal resonance exists in a free system, energy imparted initially to one of the modes involved in the internal resonance will be continuously exchanged among all the modes involved in the internal resonance.

For example, we consider the motion of a mass m attached to a spring that is swinging in a vertical plane. If we let $x(t)$ denote the stretch in the spring beyond its equilibrium and $\theta(t)$ denote the angle between the spring and the

vertical line, then the governing equations of the motion are

$$\ddot{\theta}(t) + \frac{g \sin \theta(t) + 2\dot{x}(t)\dot{\theta}(t)}{l + x(t)} = 0$$

$$\ddot{x}(t) + k/mx(t) - (l + x(t))\dot{\theta}^2 - g \cos \theta = 0$$

where k is a spring constant, l is the natural length of the spring, and g is the acceleration of gravity.

The two natural frequencies are

$$\omega_1 = (g/l)^{1/2}$$
$$\omega_2 = (k/m)^{1/2}$$

Suppose l and m are chosen such that $\omega_2 \approx \omega_1$. If one starts the motion when $\theta = \theta_0 \neq 0$, by pulling the mass m down, one finds that the mass oscillates up and down first, and that then a pendulum-type component of motion develops and grows at the expense of the spring-type motion. After a while, the pendulum-type motion starts to decrease and the spring-type motion starts to grow. Thus the energy is transferred continuously back and forth between the two modes of oscillation.

Whether commensurable or nearly commensurable frequencies can cause internal resonance depends on the degree of the nonlinearity and the geometry of the system.

For energy type nonlinear damping system, internal resonance never occurs for any commensurable or nearly commensurable frequencies. This fact can be seen from its Krylov-Bogoliubov approximation. By Krylov-Bogoliubov approximation, we know the energy possessed by the nth mode is given by

$$E_n(0)(1 + \gamma q E^q(0)t)^{-\frac{1}{2q}}$$

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where $E_n(0)$ is the initial energy of the nth mode. Therefore, the energy of each mode is continuously absorbed by the damping mechanism and there is no exchange of energy between any two modes. This is another fundamental difference between nonlinear damping problems and nonlinear stiffness problems.

5. *Steady state response to multi-frequency excitation and coupling effect*

Suppose the external excitation contains M frequencies, say they are

$$\omega_n + \epsilon \sigma_n \qquad n = 1, 2, \cdots, M$$

Then the steady state response is dominated by these M modes. And the frequency-response equation for the nth mode is given by

$$\rho_n^2(\sigma_n) = \frac{(f_n B^* \psi_n/(2\omega_n))^2}{\sigma_n^2 + [\xi_0 \omega_n + G(\vec{\sigma}^2)]^2}$$

where $G(\vec{\sigma}^2)$ solves

$$G(\vec{\sigma}^2) = \gamma_0/2 [\sum_{n=1}^{M} \frac{(f_n B^* \psi_n/2)^2}{\sigma_n^2 + (\xi_0 \omega_n + G(\vec{\sigma}^2))^2}]^q$$

and we used the notation $\vec{\sigma}^2 = (\sigma_1^2, \sigma_2^2, \cdots, \sigma_M^2)$.

Comparing with the single frequency case, the stationary amplitude corresponding to multi-frequency excitation becomes smaller due to the coupling effect of nonlinear damping. However, qualitatively, there is no change in the shape of the frequency-response curve. Therefore we see that the coupling effect due to nonlinear damping is weak. This can also be seen by examining the Krylov-Bogoliubov approximation, which tells us, to certain accuracy, if the initial data involve only finite number of modes, nonlinear damping itself will not involve modes other than those involved initially. The free response will stay on those modes initially involved.

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In one word, by using nonlinear damping, one can only quantitatively change the decay rate so that test data can be fitted. Nonlinear damping model with linear stiffness does not produce any peculiar behavior such as internal resonance, jump phenomenon or chaotic behavior. If the experiments on flexible space structures indicated the existence of any of these peculiar behaviors, nonlinear stiffness model becomes necessary.

8.2   Some Open Questions

Inspite of the above investigations, some questions remain unsolved. They are listed below:

1. What is the spectral density of the following nonlinear stiffness model with linear damping

$$\ddot{x} + 2\xi\dot{x} + g(x) = \sigma n(t) \tag{195}$$

where $\xi > 0$ is a small parameter, and $g(\cdot)$ is an odd function of $x \in I\!R^1$.

Even more challenging is the same question without assuming ξ being small. A handy example is the so-called *Duffing oscillator*

$$\ddot{x} + \eta\dot{x} + x^3 = \sigma n(t) \quad (\eta > 0) \tag{196}$$

R. N. Iyengar [45] studied this problem by first enhancing the dimension (DOF), then using *Equivalent Linearization Mehtod*(ELM) to obtain an approximating linear system of equations, thus reducing the original single DOF nonlinear stiffness model to a two-DOF linear model, for which the spectral density become trivial. To illustrate the idea, let us consider the above *Duffing oscilla'or* (196).

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Let $z = x^3$, then

$$\dot{z} = 3x^2\dot{x}$$

$$\ddot{z} = 6x\dot{x}^2 + 3x^2(\sigma n(t) - x^3 - \eta\dot{x})$$

$$= 6x\dot{x}^2 + 3\sigma x^2 n(t) - 3zx^2 - \eta\dot{z}$$

Therefore, we obtain the second equation

$$\ddot{z} + \eta\dot{z} + 3x^2 z - 6\dot{x}^2 x = 3\sigma x^2 n(t) \tag{197}$$

Then by applying *Equivalent Linearization Method* to (196) and (197), one obtains the corresponding two-DOF linear model

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ z \end{pmatrix} + \eta\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -Q & P \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} \sigma \\ 3\sigma\sigma_1^2 \end{pmatrix}n(t) \tag{198}$$

where P, Q, σ_1 are certain positive constants.

The spectral density obtained by this approach shows two peaks reflecting the existence of subharmonics in the system. The secondary resonance occurs at about three times the primary resonance frequency. Notice that the resulted linear model has non-symmetric stiffness matrix.

Only *Equivalent Linearization Method* itself cannot account for the existence of higher harmonics which is one of the very important nonlinear phenomenon. Applying ELM to a single DOF nonlinear oscillator leads to a single DOF linear oscillator which can oscillate at only one frequency. This viewpoint hints the desirability of increasing the DOF of the equivalent linear system, so as to allow more than one natural frequency to exist.

It is worthwhile to mention that the stationary probability density of nonlinear stiffness model (195) has been found, see, for example [25], which is given by

$$p_s(x,y) = C_0 \exp[-2\xi/\sigma^2(G(x) + y^2/2)]$$

where C_0 is the normalizing constant and

$$G(x) = \int_0^x g(u)du$$

2. In *Section 2* of *Chapter 3*, to find the approximate explicit stationary probability density of general nonlinear damping model in single DOF case, what the author proposed is to replace the damping term $D(x,y)$ by the corresponding energy type damping $\mu(\frac{\omega_0^2 x^2 + y^2}{2})y$, where $\mu(E)$ is given by

$$\mu(E) = \frac{4}{\pi\sqrt{2E}} \int_0^{\pi/2} D(\sqrt{2E}/\omega_0 \sin\psi, \sqrt{2E}\cos\psi)\cos\psi d\psi \qquad (199)$$

And we have seen that both

$$\ddot{x} + \xi D(x,\dot{x}) + \omega_0^2 x = \sigma n(t)$$

and

$$\ddot{x} + \xi\mu(\frac{\omega_0^2 x^2 + \dot{x}^2}{2})\dot{x} + \omega_0^2 x = \sigma n(t)$$

have the same Krylov-Bogoliubov approximation and the same ψ_x and ψ_y, which are defined in (31). What we do not know now is the difference (in an appropriate sense) between the two stationary probability densities corresponding to the above nonlinear damping models. And what are the differences between the stationary (first order) statistics, such as $R_{xx}(0)$ and $R_{yy}(0)$?

In this regard, I make the following conjecture: for the general nonlinear damping model with single DOF

$$\ddot{x} + D(x,\dot{x}) + \omega_0^2 x = \sigma n(t)$$

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the stationary probability density can be written in the following form

$$p_s(x,y) = C_0 \exp[-\frac{2\omega_0}{\pi\sigma^2}\iint_{(u,v)\in S(x,y)} \frac{D(u,v)v}{\omega_0^2 u^2 + v^2} du dv] \qquad (200)$$

where $S(x,y)$ is a (x,y)-dependent closed area.

As we already know, the stationary probability density for energy type non-linear damping model is given by

$$p_s(x,y) = C_0 \exp[-2/\sigma^2 \int_0^E \mu(z)dz]$$

If we use (199), the relation between $\mu(E)$ and $D(x,\dot{x})$, then we obtain

$$
\begin{aligned}
p_s(x,y) &= C_0 \exp[-2/\sigma^2 \int_0^E \mu(z)dz] \\
&= C_0 \exp[-\frac{2}{\pi\sigma^2} \int_0^E \int_0^{2\pi} D(\frac{\sqrt{2z}}{\omega_0}\sin\psi, \sqrt{2z}\cos\psi) \\
&\qquad\qquad \times \cos\psi d\psi d\sqrt{2z}] \\
&= C_0 \exp[-\frac{2}{\pi\sigma^2} \int_0^{\sqrt{\omega_0^2 x^2 + y^2}} \int_0^{2\pi} 1/\rho^2 \\
&\qquad\qquad \times D(\frac{\rho\sin\psi}{\omega_0}, \rho\cos\psi)\rho^2 \cos\psi d\psi d\rho] \\
&= C_0 \exp[-\frac{2\omega_0}{\pi\sigma^2}\iint_{\omega_0^2 u^2 + v^2 \leq \omega_0^2 x^2 + y^2} \frac{D(u,v)v}{\omega_0^2 u^2 + v^2} du dv]
\end{aligned}
$$

In this particular case, $S(x,y)$ is a (x,y)-dependent parabola with center at the origin and axes $(\omega_0^2 x^2 + y^2)^{1/2}/\omega_0$ and $(\omega_0^2 x^2 + y^2)^{1/2}$.

3. If the self-adjoint operator A is positive definite and has compact resolvent, with $0 \in \rho(A)$, then the eigenfunctions of $\mathcal{A}_0$

$$\mathcal{A}_0 \overset{\text{def}}{=} \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$$

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given by

$$\left\{\Phi_n^+ = \begin{pmatrix} \phi_n \\ i\omega_n\phi_n \end{pmatrix}, \quad \Phi_n^- = \begin{pmatrix} \phi_n \\ -i\omega_n\phi_n \end{pmatrix}; \quad n = 1, 2, \cdots \right\}$$

form an orthogonal basis in $\mathcal{H} \otimes \mathcal{H}$.

In fact, first it is easy to verify that $\{\Phi_n^+, \Psi_n^-; n = 1, 2, \cdots\}$ is an orthogonal sequence. To prove its completeness, let

$$w = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \otimes \mathcal{H}$$

and

$$w \quad \perp \quad \text{span}\{\Phi_n^+, \Phi_n^-; \quad n = 1, 2, \cdots\}$$

. Or equivalently, we have

$$[w, \Phi_n^+]_E = \omega_n^2[x, \phi_n] - i\omega_n[y, \phi_n] = 0$$

$$[w, \Phi_n^-]_E = \omega_n^2[x, \phi_n] + i\omega_n[y, \phi_n] = 0$$

which immediately gives

$$[y, \phi_n] = 0; \quad [x, \phi_n] = 0, \quad n = 1, 2, \cdots$$

By the assumptions upon A we know $\{\phi_n; \ n = 1, 2, \cdots\}$ is an orthonormal basis in $\mathcal{H}$, therefore, $x = 0$, $y = 0$, i.e., $w = 0$. Then the completeness is justified for the non-damped case.

Now the open question is, with damping term D, are the eigenfunctions of

$$\mathcal{A} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & I \\ -A & -D \end{pmatrix}$$

complete in $\mathcal{H} \otimes \mathcal{H}$?

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