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Abstract

An elastic-plastic algorithm based on Von Mises and associative flow criteria is implemented in MHOST—a mixed-iterative finite element analysis computer program developed by NASA Lewis Research Center. The performance of the resulting elastic-plastic mixed-iterative analysis is examined through a set of convergence studies. Membrane and bending behaviors of 4-node quadrilateral shell finite elements are tested for elastic-plastic performance. Generally, the membrane results are excellent, indicating the implementation of elastic-plastic mixed-iterative analysis is appropriate.

Nomenclature

$\mathbf{A}$	Mixed constitutive matrix
$\mathbf{B}$	Strain-displacement matrix
$\mathbf{b}$	Body force
$\mathbf{C}$	Projection matrix
$\mathbf{D}$	Elastic constitutive matrix
$\mathbf{E}$	Mixed strain-displacement matrix
$\mathbf{f}$	Nodal force
G	Shear modulus
H'	Plastic modulus
k	Yield surface parameter
$\mathbf{N}$	Shape function
$\mathbf{N}_u$	Shape function for displacement
$\mathbf{N}_\sigma$	Shape function for stress
$\mathbf{N}_\varepsilon$	Shape function for strain
$\mathbf{n}$	Normal to yield surface
R	Radius of yield surface
$\mathbf{r}$	Residual force
$\mathbf{t}$	Traction force
$\mathbf{u}$	Nodal displacement
$\boldsymbol{\alpha}$	Back stress
$\boldsymbol{\sigma}$	Nodal stress
$\boldsymbol{\sigma}'$	Nodal deviatoric stress
$\boldsymbol{\sigma}_{tr}$	Nodal trial stress
$\boldsymbol{\varepsilon}$	Nodal strain
$\boldsymbol{\varepsilon}^P$	Nodal plastic strain
$\bar{\epsilon}^P$	Effective plastic strain
λ	Plastic consistency parameter
$\dot{(\ )}$	Rate of change

Introduction

Since its inception, many attempts to improve the performance of displacement-based finite element procedures have been made. Here, the primary focus is a mixed-iterative finite element approach, particularly for the elastic-plastic analysis.

An economical, and easily adaptable, iterative stress-smoothing algorithm was initially proposed by Loubignac et al. [Ref. 1 and 2], which enhanced the existing displacement-based computer codes without requiring fundamental modifications. More recently, Zienkiewicz et al. [Ref. 3 and 4] presented an iterative approach to solve mixed finite element equations derived from a Hu-Washizu variational principle. This procedure bypasses the complexity related to the direct solution of mixed equations. In many ways, the resulting iterative method is equivalent to the previously mentioned algorithm. It is certainly reassuring to realize that Loubignac's algorithm, initially developed based on engineering ingenuity, does have a formalized variational basis. Throughout this paper, the term mixed-iterative is used. This is the concept on which MHOST—a mixed-iterative finite element code developed under the sponsorship of NASA Lewis Research Center—is based.

As shown in the literature, the advantage of the mixed-iterative method is its accuracy in the stress and strain solutions; sometimes improved results in displacement may be obtained as well. Therefore, it seems appropriate to fully utilize the dominating feature of the iterative method in material nonlinear analysis. To a certain extent, material nonlinear computations are governed by stress intensity more than geometrical nonlinear analysis. Here, material nonlinearity due to rate-independent plasticity is considered.

Following the description of the mixed-iterative procedure, the implementation of an elastic-plastic algorithm in MHOST is described. The algorithm is based on Von Mises and associative flow criteria, and includes both isotropic as well as kinematic hardening options. This implementation is made for a 4-node quadrilateral shell finite element. Furthermore, the promising quality and economy of the mixed iterative method in elastic-plastic analysis are measured against the performance of the corresponding "traditional" displacement-based method. Convergence studies are conducted to quantitatively assess the speed-up provided by the iterative algorithm, i.e., the reductions of discretization requirements and computational expenses.

In addition to the existing mixed-iterative procedure, the option for the displacement-based analysis is made available in MHOST. The procedure to activate this option is given in Appendix A. This added capability allows the same code, i.e., MHOST, to be used for judging the performance of mixed-iterative versus displacement-based methods. Hence, the objectivity of the outcome of the present study is achieved, where the results obtained reflect the true nature of the methods considered rather than the variations of programming techniques. The elastic-plastic algorithm implemented is first verified against a well-established displacement-based code in order to demonstrate its validity. This is shown in Appendix B.

The present study does not claim to be exhaustive, but it provides indications on both the strengths and weaknesses of the mixed-iterative method. This information seems to be lacking in the existing literature.

Mixed-Iterative Finite Element Method

Based on a Hu-Washizu variational principle, a three-field mixed finite element formulation can be established, with displacement, stress, and strain as the essential variables [Ref. 5]. The resulting finite element equations are as follows:

$$\begin{bmatrix} \mathbf{A} & -\mathbf{C} & \mathbf{0} \\ -\mathbf{C}^T & \mathbf{0} & \mathbf{E} \\ \mathbf{0} & \mathbf{E}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\sigma} \\ \mathbf{u} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{f} \end{Bmatrix} \tag{1}$$

where

$$\mathbf{A} = \int_\Omega \mathbf{N}_\varepsilon^T \mathbf{D} \mathbf{N}_\varepsilon \, d\Omega \tag{2}$$

$$\mathbf{C} = \int_\Omega \mathbf{N}_\varepsilon^T \mathbf{N}_\sigma \, d\Omega \tag{3}$$

$$\mathbf{E} = \int_\Omega \mathbf{N}_\sigma^T \mathbf{B} \, d\Omega \tag{4}$$

$$\mathbf{f} = \int_\Omega \mathbf{N}_u^T \mathbf{b} \, d\Omega + \int_\Gamma \mathbf{N}_u^T \mathbf{t} \, d\Gamma \tag{5}$$

Some diagonal terms of the above linear algebraic equations are zero, indicating that an equation solver capable of handling indefinite matrices is needed. Furthermore, the additional nodal variables, i.e., stress and strain, enlarge the size of the equations, hence requiring more computations to solve the problem. An iterative approach to solve the mixed equations, known as mixed-iterative method [Ref. 3, 4 and 5], circumvents the above mentioned obstacles. This method is described below.

In its practical form, the iterative procedure adopts the following modifications:

1. The same shape functions are used for displacement, stress, and strain.

$$\mathbf{N}_u = \mathbf{N}_\sigma = \mathbf{N}_\varepsilon = \mathbf{N}$$

2. To save computations, the off-diagonal coefficients of the $\mathbf{C}$ matrix are neglected (lumped).

$$\mathcal{C} = \text{diagonalized } \mathbf{C}$$

3. Constitutive equations are evaluated at the nodes.

The iteration can be briefly described in the following steps:

1. Initiate iteration with $\mathbf{u}_0 = \mathbf{0}$, $\boldsymbol{\sigma}_0 = \mathbf{0}$, $\boldsymbol{\varepsilon}_0 = \mathbf{0}$, and $\mathbf{r}_0 = -\mathbf{f}$.

2. Solve for nodal displacements:

$$\mathbf{u}_{n+1} = \mathbf{u}_n - \left(\int_\Omega \mathbf{B}^T\mathbf{D}\mathbf{B}\, d\Omega\right)^{-1} \mathbf{r}_n$$

3. Calculate smoothed nodal strain by projection, and evaluate stress at the nodes:

$$\begin{aligned} \boldsymbol{\varepsilon}_{n+1} &= \mathcal{C}^{-1}\mathbf{E}\,\mathbf{u}_{n+1} \\ \boldsymbol{\sigma}_{n+1} &= [\mathbf{D}\,\boldsymbol{\varepsilon}_{n+1}]_{node} \end{aligned}$$

4. Compute residuals:

$$\mathbf{r}_{n+1} = \mathbf{E}^T\boldsymbol{\sigma}_{n+1} - \mathbf{f}$$

5. Go back to step 2 until convergence to a specified tolerance is obtained.

In other words, the process starts with a displacement-based solution and proceeds to smooth-out inter-element strain discontinuity. Stresses obtained following this procedure violate nodal force equilibrium. These unbalanced forces are treated as residuals, which are iteratively reduced, and applied to improve the solution obtained from the previous step. This procedure is similar to the constant-stiffness iterations used in nonlinear analysis. In this way, the method can be seen as a means of iteratively improving displacement-based solutions, which is equivalent to the iterative stress-smoothing procedure proposed by Loubignac et al. [Ref. 1 and 2].

The mixed-iterative solution is capable of capturing some of the effects produced by mesh refinements, i.e., continuity of stress and strain as well as increased accuracy, with less computational resources. On the other hand, displacements produced by the iterative method may or may not be improved, because the procedure does not increase the actual number of degrees of freedom. The resulting adjustment on displacements is solely due to residual iterations. The mixed-iterative result is usually more flexible than the solution obtained by a displacement-based method. The former tends to compensate the inherently over-stiff behavior of the latter. However, the increased flexibility may sometimes overestimate the solutions.

By the nature of the method, significant improvements on stress and strain solutions can usually be expected when coarse finite element meshes are used. Despite the advantages of the mixed-iterative method, however, it must be understood that the discontinuities to be smoothed are those originating from finite element discretizations, not from the physics of the problem.

Elastic-Plastic Mixed-Iterative Analysis

The accuracy of stress and strain solutions has a direct effect on the quality of elastic-plastic analysis. This is because the incremental computation of elastic-plastic constitutive relations is governed by the stress and strain at each element, which indicates that elastic-plastic mixed-iterative analysis may give better results than the displacement-based procedure.

Another advantage of the iterative method in elastic-plastic analysis is the evaluation of constitutive relations at the nodes. Due to the nature of finite element interpolations, nodal points contain the extreme values of the given element. Hence, a more accurate assessment of the on-set of yielding, as well as the progress of plastic flow, can be made at the nodes. From this point of view, evaluation of yield criteria at the integration points constitutes an approximation, where the values at the integration points tend to converge to those of the nodes with mesh refinements.

Implementation of an elastic-plastic algorithm in the mixed-iterative code MHOST is direct. The additional ingredient required is the elastic-plastic constitutive relations. The formulations presented in this paper are for the rate-independent plasticity based on Von Mises yield surface, associative flow rule, isotropic and kinematic hardening, and small deformation assumptions.

The mathematics of solving elastic-plastic constitutive equations falls into the category of initial-boundary value problem [Ref. 6]. Basically, given the initial values of stress and radius of yield surface, and with the time-history of strain provided, the evolution of stress and radius of yield surface are to be determined by integrations. In the mixed-iterative formulation, the above mentioned process is evaluated at the nodes. The mixed-iterative formulation of the elastic-plastic analysis satisfies the following conditions:

- Incremental constitutive equations.

$$\dot{\boldsymbol{\sigma}} = [\mathbf{D}\,(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^P)]_{node} \tag{6}$$

- Associative flow rule.

$$\dot{\boldsymbol{\varepsilon}}^P = \begin{cases} 0 & \text{If Elastic} \\ \lambda\,\mathbf{n} & \text{If Plastic} \end{cases} \tag{7}$$

- Unit normal.

$$\mathbf{n} = \frac{\boldsymbol{\sigma}'}{R} \tag{8}$$

- Consistency condition.

$$\lambda = \frac{\mathbf{n}^T\ \dot{\boldsymbol{\varepsilon}}}{1 + \frac{H'}{3G}} \tag{9}$$

- Yield criteria.

$$\left.\begin{aligned} f(\boldsymbol{\sigma}) &\leq k^2 \\ \mathbf{n}^T\ \dot{\boldsymbol{\sigma}}_{tr} &\leq 0 \end{aligned}\right\}\ \text{Elastic} \tag{10}$$

$$\left.\begin{aligned} f(\boldsymbol{\sigma}) &= k^2 \\ \mathbf{n}^T\ \dot{\boldsymbol{\sigma}}_{tr} &> 0 \end{aligned}\right\}\ \text{Plastic} \tag{11}$$

- Trial rate of stress.

$$\dot{\boldsymbol{\sigma}}_{tr} = [\,\mathbf{D}\,\dot{\boldsymbol{\varepsilon}}\,]_{node} \tag{12}$$

- Isotropic hardening: Evolution equation for the radius of yield surface.

$$\dot{R} = \sqrt{\frac{2}{3}}\,H'\,\dot{\bar{\epsilon}}^P \tag{13}$$

- Kinematic hardening: Evolution equation for the back stress.

$$\dot{\boldsymbol{\alpha}} = \frac{2}{3}\,H'\,\dot{\boldsymbol{\varepsilon}}^P \tag{14}$$

A widely-used numerical algorithm known as the radial-return method is implemented to integrate the above equations. This is an elastic-predictor-plastic-corrector procedure initially proposed by Wilkins [Ref. 7]. Krieg and Key [Ref. 8] developed the algorithm further to account for strain hardening. In essence, the plastic stress state defined by the radial-return algorithm is given by the intersection of the yield surface with the line connecting the center of the yield surface and the trial stress.

Numerical Performance

The purpose of this numerical study is to examine, in the context of elastic-plastic analysis, the capability of the mixed-iterative method in providing accurate solutions using less computational resources.

To accomplish the stated objective, convergence studies using increasingly refined finite element meshes are performed. Displacement-based solutions are used as the benchmark to measure the performance of the mixed-iterative method. This is done for the following reason: At the limits of continuous mesh refinements, the solutions obtained using the mixed-iterative approach will be identical to those given by the displacement-based method. This is because the resulting strain field will already be continuous, and therefore smoothing will be obviated. However, the goal is to determine how much the mixed-iterative procedure is capable of enhancing convergence, i.e., using coarser meshes to provide solutions with comparable accuracy as those given by the displacement-based methods. In this way, the effectiveness of the mixed-iterative method in accelerating convergence, and reducing total computation time, is objectively assessed.

Uniform finite element meshes are used throughout this study. Mesh refinements are made by bisecting each of the existing elements into four new elements. The three problems used in this study, as depicted in Figures 1 to 3, are modeled using 4-node quadrilateral finite elements. Four increasing levels of refinement are used, thus providing five finite element meshes for each problem.

Problem 1:

Cook's skewed cantilever is widely used as a benchmark for the performance of membrane behavior with a distorted mesh. The cantilever of unit thickness is subjected to a load of 3.0 Newtons, uniformly distributed along the right edge as shown in Figure 1. Isotropic hardening elastic-plastic material is considered for this problem, with modulus of elasticity of 1.0 MPa, Poisson's ratio of 0.33, yield stress of 0.25 MPa, and the hardening slope of 30.0 % of the elastic slope. The number of elements used for the convergence study are tabulated as follows:

Mesh	Number of Elements	Number of Nodes
A1	$3 \times 1 = 3$	$4 \times 2 = 8$
A2	$6 \times 2 = 12$	$7 \times 3 = 21$
A3	$12 \times 4 = 48$	$13 \times 5 = 65$
A4	$24 \times 8 = 192$	$25 \times 9 = 225$
A5	$48 \times 16 = 768$	$49 \times 17 = 833$

Excellent performance of the mixed-iterative method is shown by this problem. Figures 4 to 7 demonstrate the accelerated convergence of the displacement as well as the stress produced by this method. Using mesh A3, for example, the mixed-iterative solution achieved the same or better accuracy as displacement-based results given by the A4 mesh. Furthermore, for the same accuracy the iterative procedure requires only 52 % of the computer time consumed by the displacement-based solution.

The results shown in Figure 4 and 5 indicate an increase in the mixed-iterative convergence rate for the elastic-plastic analysis, as compared to the elastic case. For instance, to obtain less than 5.0 % error in the elastic solution requires mesh A3, while the same level of accuracy for the elastic-plastic result can be obtained using the A2 mesh. This clearly demonstrates the increased effectiveness of the mixed-iterative procedure when it is applied to this elastic-plastic analysis problem.

Problem 2:

Figure 2 shows a 90^0 V-notched bar, with a notch-depth to half-width ratio of 1 to 2. The unit-thickness notched bar is under axial load of 6.0 Newtons, uniformly distributed at each end. The material is elastic-perfectly-plastic with a yield stress of 0.15 MPa, a modulus of elasticity of 1.0 MPa, and a Poisson's ratio of 0.33. Due to symmetry, only a quarter of the notched bar is modeled in the finite element mesh. This problem considers the performance of the mixed-iterative method specifically for the effect of stress concentration. The number of elements used in each level of mesh refinements are as shown in the table below.

Mesh	Number of Elements	Number of Nodes
B1	$4 \times 2 = 8$	$5 \times 3 = 15$
B2	$8 \times 4 = 32$	$9 \times 5 = 45$
B3	$16 \times 8 = 128$	$17 \times 9 = 153$
B4	$32 \times 16 = 512$	$33 \times 17 = 561$
B5	$64 \times 32 = 2048$	$65 \times 33 = 2145$

As displayed in Figures 8 to 12, excellent mixed-iterative results are given by this problem. Axial displacement (Figure 8) produced by the mixed-iterative method is on the flexible side as compared to the "stiff" displacement-based solution. However, the difference is less than 1.0 % and, arguably, the displacement-based solution has yet to be fully converged. So, it is conceivable that the mixed-iterative result might be more accurate. In any case, the overall performance of the mixed-iterative method shows improvements in the rate of convergence, for the displacement as well as the stress.

Stresses at the critical region, i.e., in the vicinity of the notch, are accurately represented by the mixed-iterative solutions. At the tip of the notch, the iterative solution is able to better represent the uniaxial stress state than the displacement-based approach. In the lower stressed region closer to the center of the specimen, however, the stress distributions produced by the mixed-iterative method using the B3 mesh (Figures 9 and 11) tend to oscillate about the converged solution. The oscillation vanishes when a finer mesh, i.e., B4, is used. Further investigations to examine the oscillating solutions will be necessary to determine the underlying cause.

The mixed-iterative solutions shown in Figures 10 and 12, i.e., using mesh B4, are practically indistinguishable from the displacement-based results computed using the B5 mesh. In this case, the mixed-iterative procedure takes 88.1 % of the computer time required by the displacement-based solution. Considering the critical region, the mixed-iterative solution obtained using the B3 mesh has comparable accuracy with the displacement-based result provided by mesh B4. The former requires 81.2 % of the computer time consumed by the latter. These results show that the mixed-iterative procedure is capable of producing solutions with the same level of accuracy using coarser meshes and less computation time.

Problem 3:

A clamped square plate with a center load of 2.875 Newtons is shown in Figure 3. This plate bending problem has elastic-perfectly-plastic material with modulus of elasticity of 1092.0 GPa, Poisson's ratio of 0.3, and yield stress of 300.0 MPa. The thickness of the plate is 0.1 mm. Due to symmetry, only a quarter of the plate is modeled in the finite element mesh. The number of elements used in each level of mesh refinements are as follows:

Mesh	Number of Elements	Number of Nodes
C1	$2 \times 2 = 4$	$3 \times 3 = 9$
C2	$4 \times 4 = 16$	$5 \times 5 = 25$
C3	$8 \times 8 = 64$	$9 \times 9 = 81$
C4	$16 \times 16 = 256$	$17 \times 17 = 289$
C5	$32 \times 32 = 1024$	$33 \times 33 = 1089$

As depicted by Figures 13 to 15, the mixed-iterative solutions tend to be less accurate than the displacement-based results, both in stress and displacement. Stress distributions produced by the mixed-iterative method using both the C3 and C4 meshes tend to oscillate about the converged solution. The oscillation diminishes with increasing mesh refinements, and vanishes when mesh C5 is used (not shown in the figures).

In this problem, the mixed-iterative process does not produce any improvement, but degrades the displacement-based solutions. It might be that the strain-smoothing strategy of the mixed-iterative analysis is unable to fully capture the more complex kinematic relationships of plate bending. Further investigations on the method are necessary.

Conclusion

An elastic-plastic analysis algorithm has been implemented in a mixed-iterative finite element code—MHOST. Membrane and bending behaviors of 4-node quadrilateral shell finite element models are studied in the context of mixed-iterative elastic-plastic analysis. Using displacement-based solutions as references, the effectiveness of the mixed-iterative method is assessed.

The promising features of the mixed-iterative elastic-plastic analysis are clearly demonstrated in membrane cases. Convergence is accelerated by approximately an order of magnitude for the given problems and mesh-refinement schemes. This is accomplished with a considerable saving of computer time. The time efficiency will be even higher if the time required for mesh generation is taken into consideration. Based on membrane performance, it can be said that the mixed-iterative method presents a viable alternative to the conventional displacement-based analyses. This is especially true when limited computational resources are available.

On the other hand, the limited test shows that the performance of the mixed-iterative method for plate bending is not encouraging. The algorithm seems to produce adverse effects on the solution. A more comprehensive study is definitely necessary before any conclusive judgement about the limitation of the mixed-iterative method can be made.

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Appendix A: Displacement-Based Procedure in MHOST

As part of its mixed-iterative scheme, MHOST employs a distinctive node-based data structure designed to accommodate the mixed-iterative solution algorithm [Ref. 9]. Computer memory is allocated to store nodal values of strain, stress, and constitutive coefficient matrix, among other variables. Displacement-based codes, on the other hand, usually store the corresponding element quantities. Utilizing the existing code organization, the option for displacement-based analysis is made available by implementing the necessary code modifications as described below.

First of all, it should be mentioned that the existing code does provide a rudimentary version of a displacement-based solver. This is achieved by simply terminating the mixed-iterative procedure before any residual iteration is performed. The displacement, strain, and stress obtained in this fashion are not erroneous, but often the nodal forces and support reactions contain residuals that are not as negligibly small as those obtained using a true displacement-based procedure. This can be explained further as follows. In the mixed-iterative procedure, an intermediate value of strain at the element level is calculated (at integration points), this is then projected to the nodes. The nodal values are stored in the allocated memory. The evaluations of constitutive equations for stress are made and stored at the nodes, using the smoothed (projected) values of the nodal strain. Element level forces are computed using stress values obtained by interpolating its nodal values—the smoothed stress—to the integration points. In general, the smoothed stress violates equilibrium conditions, hence the resulting element forces tend to produce quite significant residuals. This residual error may sometimes convey the wrong impression that the entire solutions are inaccurate.

From the perspective of the "traditional" displacement-based procedures, another difference of the code is that any computations beyond the stress recovery is automatically of a mixed-iterative nature. Therefore, in the original MHOST, true displacement-based solutions were obtained only for linear-elastic analysis, with strain-smoothing iterations and load increments being prevented. For the above reasons, it seems appropriate to make available a more generally applicable displacement-based procedure. It should be emphasized, however, the intention is not to make a displacement-based finite element code out of MHOST. Rather, the purpose was to facilitate a true displacement-based analysis capability such that a fair comparison with regard to computer time requirement can be made for this study. The existing data structures and algorithms are utilized as much as possible. The modifications implemented do not fundamentally alter the code—the mixed-iterative character remains intact.

Basically, the set-up for the displacement-based analysis capability in MHOST involves the modifications of the code as well as the users' input data. This is given as follows:

- Whenever the displacement-based option is activated, strain projection, nodal evaluation of stress, and the interpolation of nodal stress to integration points are avoided. Strain and stress are computed at the integration points, and stored in the computer memory allocated as described next.

- In the finite element mesh, duplicate nodes must be used at all nodes connected to adjacent elements. This should be such that each element has its own unique set of nodes. In this manner, each element is provided with the necessary memory to store integration-point quantities similar to the displacement-based codes. Hence, users are required to bear the burden of modifying their mesh-generation procedures in order to utilize this added capability.

These modifications allow the displacement-based procedure to be performed without restriction to linear-elastic analysis only. It should be mentioned, however, that the use of duplicate nodes increases computer time and memory requirements.

The displacement-based procedure is used to verify the elastic-plastic algorithm implemented. The result is given in Appendix B. Furthermore, this facility allows the performance of the mixed-iterative method to be assessed using the displacement-based analysis option given by the same code. Hence, the comparison of the two methods is made with minimized differences related to variations in programming techniques. It should be noted that the additional computations incurred due to the use of duplicate nodes are not included in the comparison.

Appendix B: Verification Of The Elastic-Plastic Algorithm

To assess its performance, first the elastic-plastic algorithm implemented in MHOST is verified using displacement-based procedures. This benchmark testing is achieved using the solutions obtained from MARC—a commercial displacement-based finite element code [Ref. 10].

It is well known that the formulation of a particular element stiffness matrix has a direct effect on the finite element solutions. Many different techniques in the stiffness formulation have been developed to achieve better accuracy and efficiency. Arguably, depending on its applications each particular element formulation has its own merits and drawbacks. However, the intention here is not to evaluate which formulation is best—but just to state the existing differences between MHOST and MARC with regard to their element stiffness matrix formulations. This is relevant for the purpose of verifying the elastic-plastic algorithm. Since it is necessary to eliminate as much of the differences pertaining to the codes as possible—except of course the elastic-plastic algorithm of interest—in order to achieve a meaningful comparison. For this reason, in the benchmark test for the elastic-plastic algorithm, the element stiffness matrix of MHOST is modified to be identical to the one used in MARC. Basically, this involves the modifications of the **B** matrix and the quadrature rules used in the stiffness calculations. It should be emphasized, however, this modification is only made for the stated purpose. And, its application is restricted to the following benchmark problem only.

Problem A:

A cantilever plate of 200.0 mm thickness is subjected to uniformly distributed transverse and inplane loads, q_1 and q_2, respectively, along the right edge as shown in Figure 16. These loads are independently applied. The incremental sequence of the transverse load applications is as follows: Increase the load to 32.0 Newtons, decrease it to -33.6 Newtons, and re-load it to 35.0 Newtons. This sequence models loading, unloading, reverse loading and re-loading. The inplane load is incrementally applied in the similar way: 180.0 Newtons, -186.0 Newtons, and 190.0 Newtons. Both isotropic and kinematic hardening elastic-plastic materials are considered for this problem, with the modulus of elasticity = 1.0 MPa, Poisson's ratio = 0.0, yield stress = 3.0 kPa, and the hardening slope = 1.0 % of the elastic slope. Sixteen finite elements are used to model this problem.

The resulting displacement histories at point C, i.e., at the center of the right edge, are shown in Figures 17 to 20. The isotropic hardening results show stiffening responses during re-loading process, since Bauschinger effect is not considered. The kinematic hardening models provide more realistic solutions. For all the cases, MHOST produces practically identical solutions as those given by MARC, both in bending (Figures 17 and 18) and membrane (Figures 19 and 20) solutions. Although stress is usually the main interest to engineers, however, comparing displacement results for the purpose of assessing the elastic-plastic algorithm is considered sufficient. This is because in the incremental elastic-plastic analysis, stresses significantly affect displacement results, i.e., small variations in stress produce noticeable cumulative effects on displacement. Therefore, given the excellent agreement on the displacement solutions of MHOST and MARC, it is conservative to say that the stresses produced by the two codes should be in good agreement as well.

This verification problem serves to demonstrate the performance of the elastic-plastic algorithm implemented in MHOST, for both bending and membrane behaviors. It should be mentioned that the two codes use comparable computer times to solve the problem.

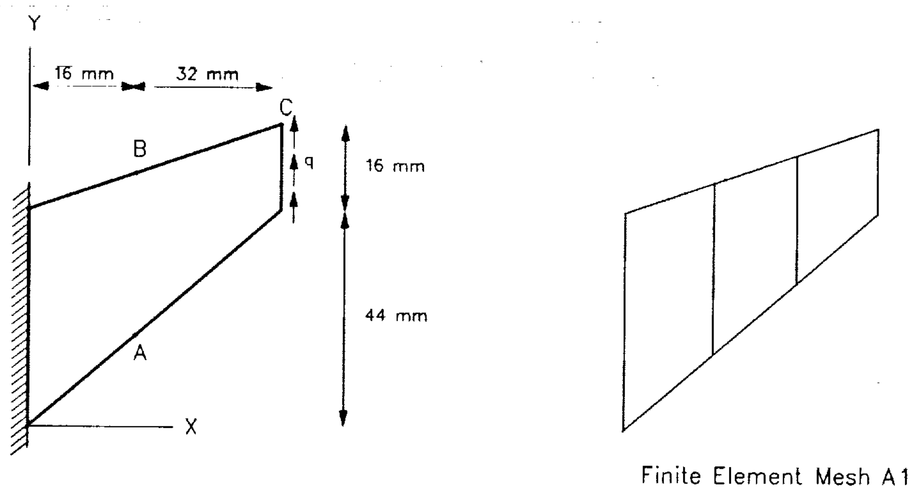


Figure 1. The skewed cantilever used in problem 1.

Y

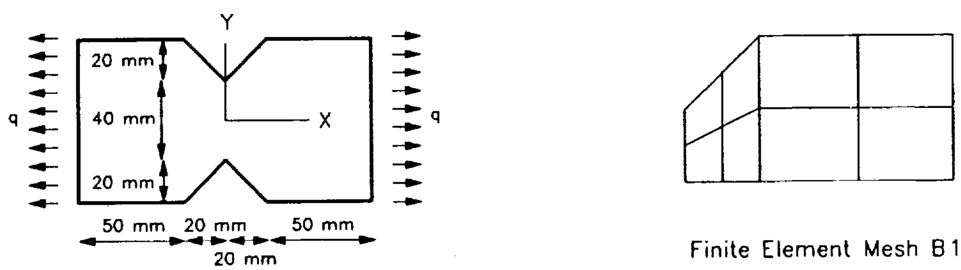
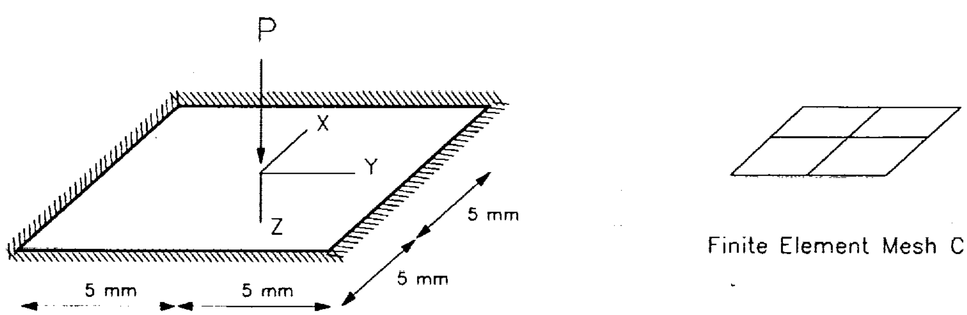


Figure 2. The notched bar used in problem 2.



Finite Element Mesh C1

Figure 3. The clamped plate used in problem 3.

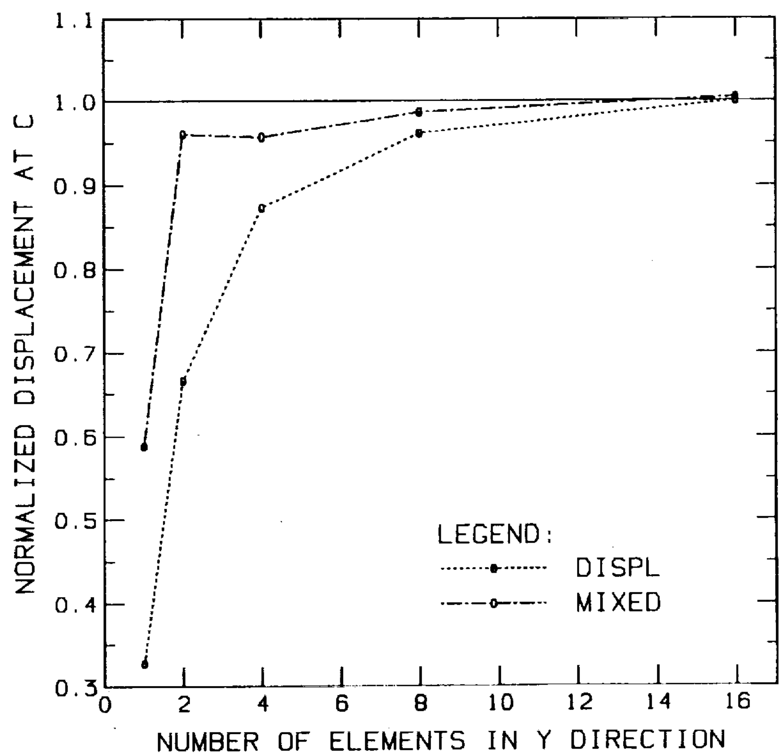
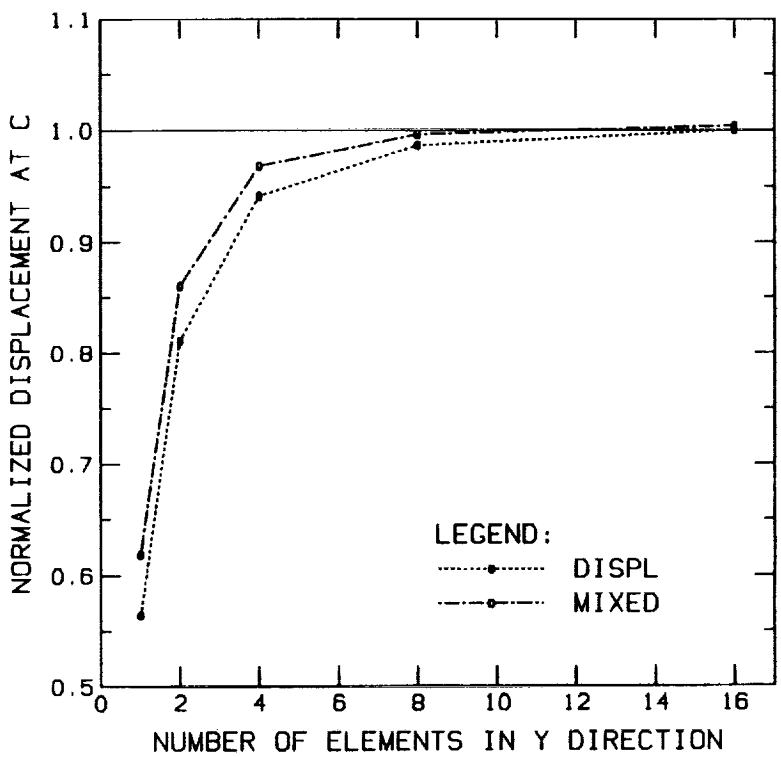


Figure 4. Cantilever: Convergence of Elastic-Plastic Displacement at C



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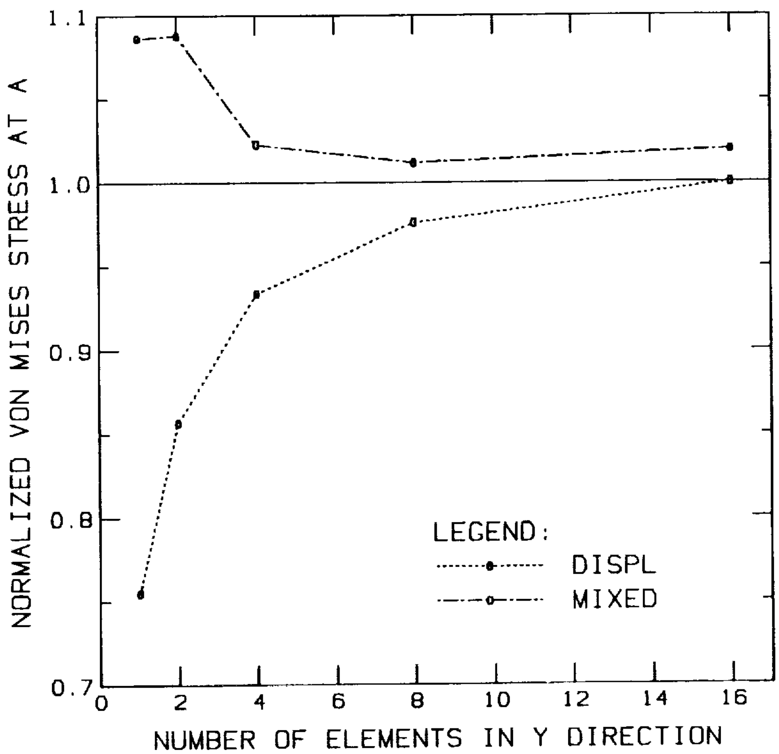

Figuge 6. Cantilever: Convergence of Von Mises Stress at A

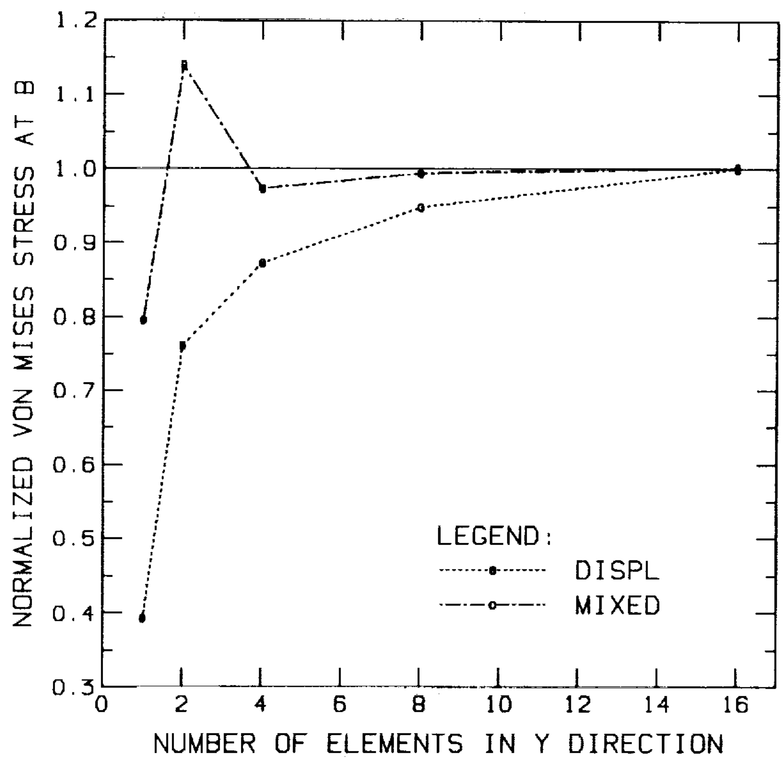

Figure 7. Cantilever: Convergence of Von Mises Stress at B

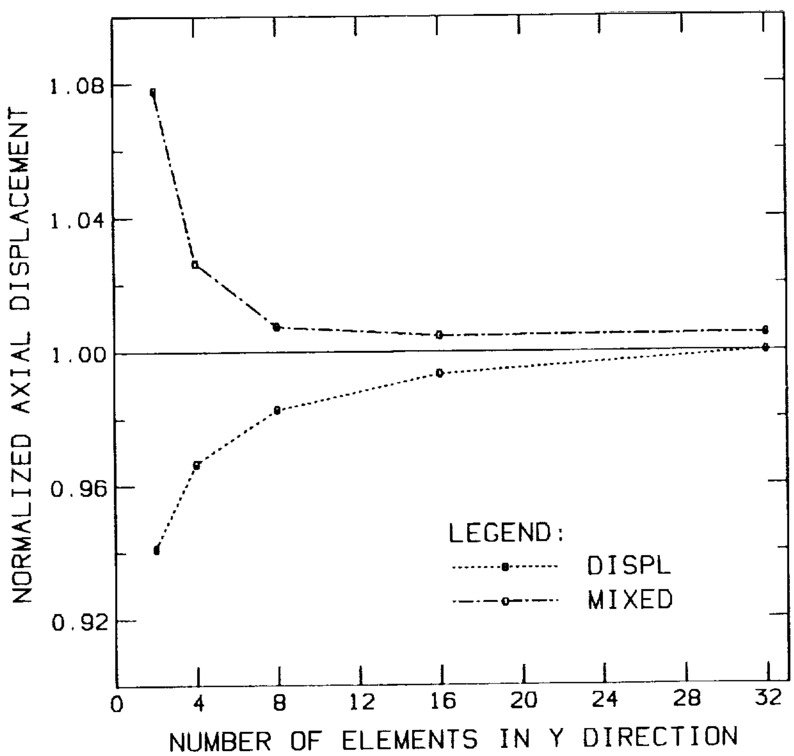

Figure 8. Notched Bar: Convergence of Axial Displacement

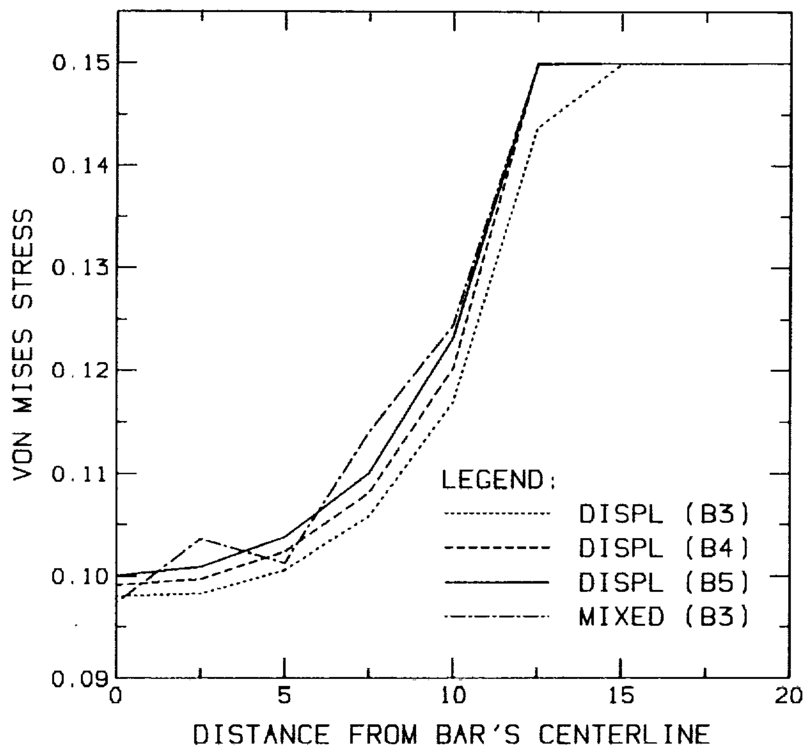

Figure 9. Notched Bar: Distributions of Von Mises Stress Across Half-Width

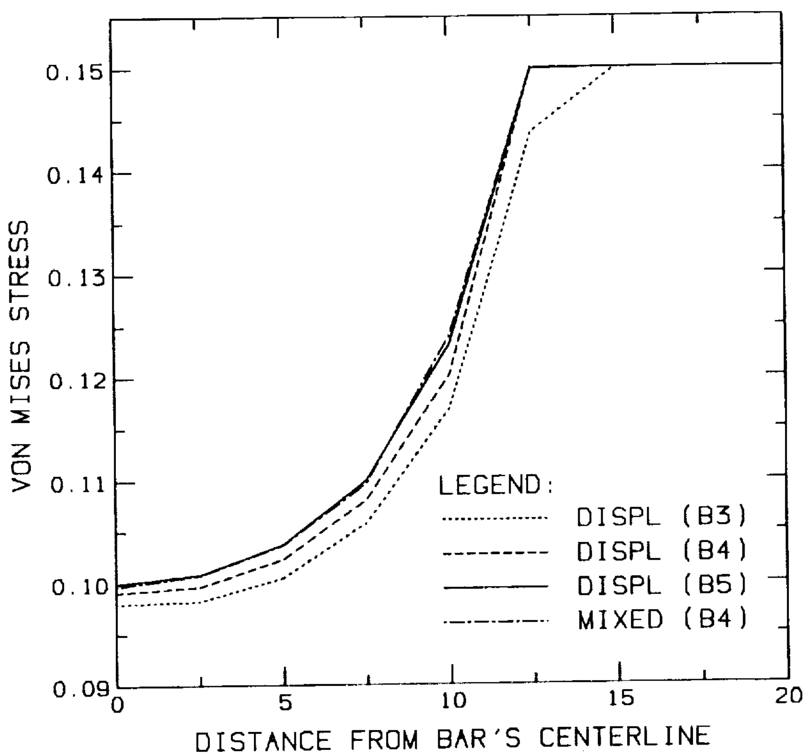

Figure 10. Notched Bar: Distributions of Von Mises Stress Across Half-Width

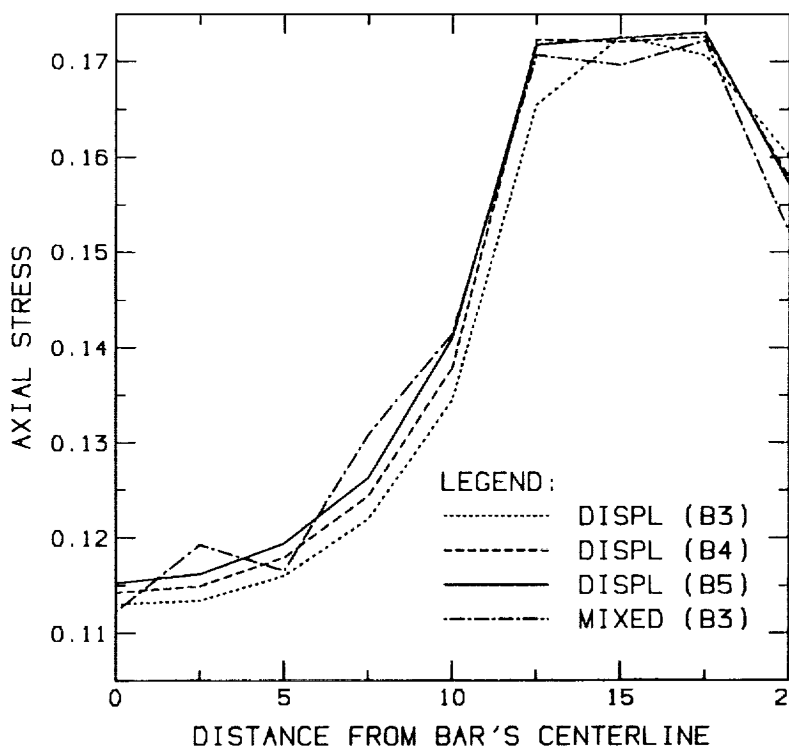

Figure 11. Notched Bar: Distributions of Axial Stress Across Half-Width

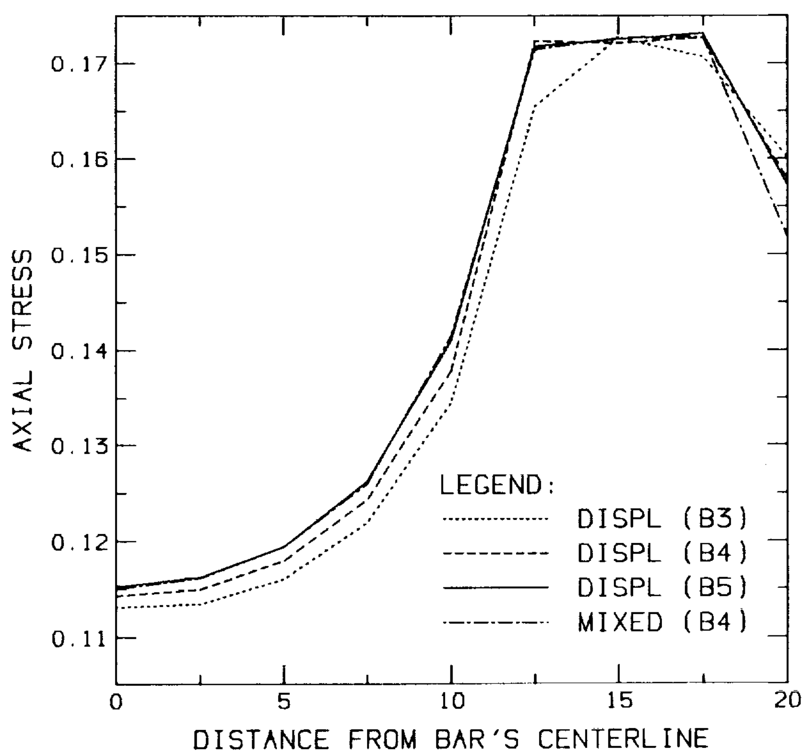

Figure 12. Notched Bar: Distributions of Axial Stress Across Half-Width

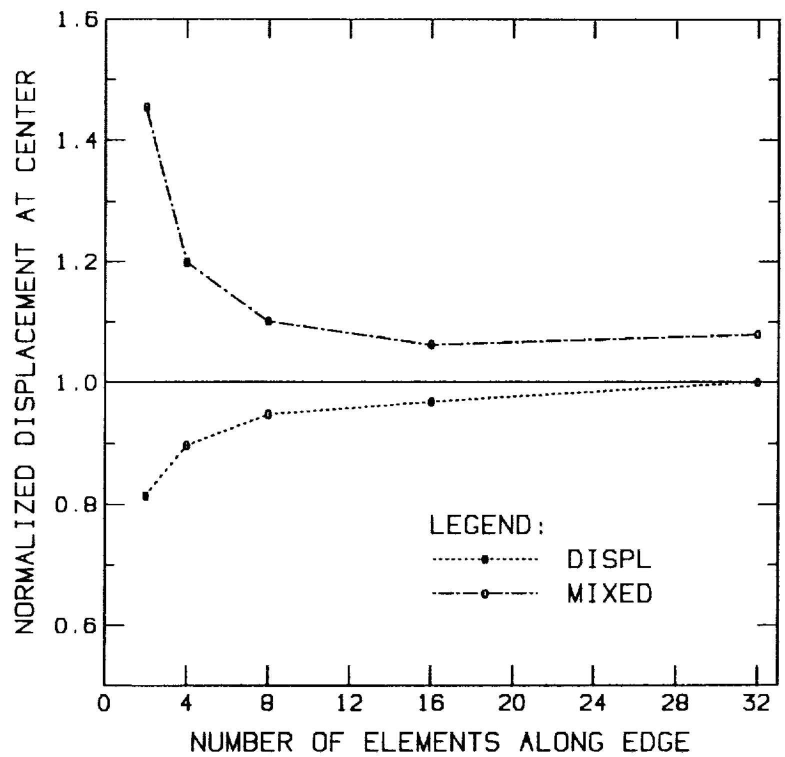

Figure 13. Plate: Convergence of Displacement at Center

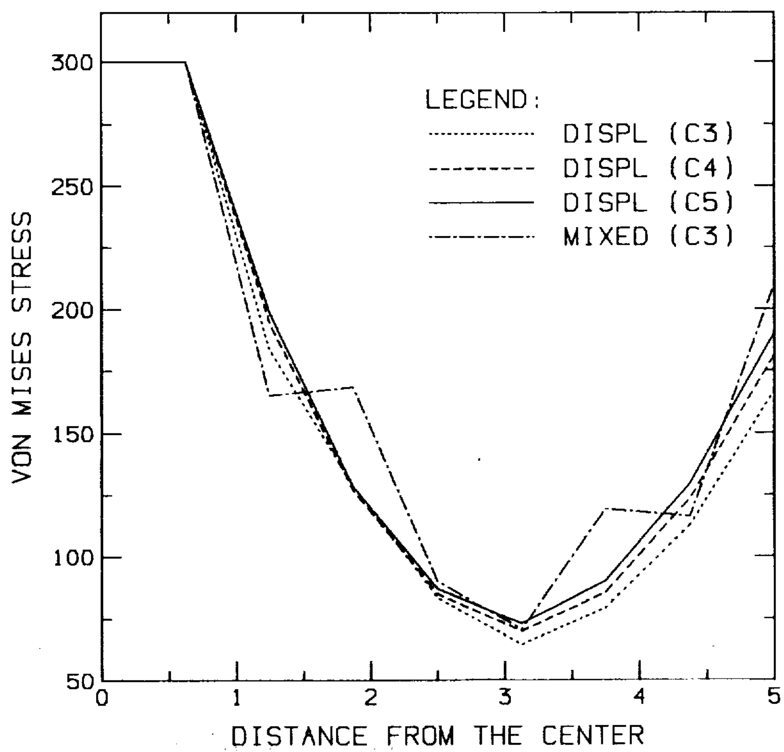

Figure 14. Plate: Distributions of Von Mises Stress From Center to Edge

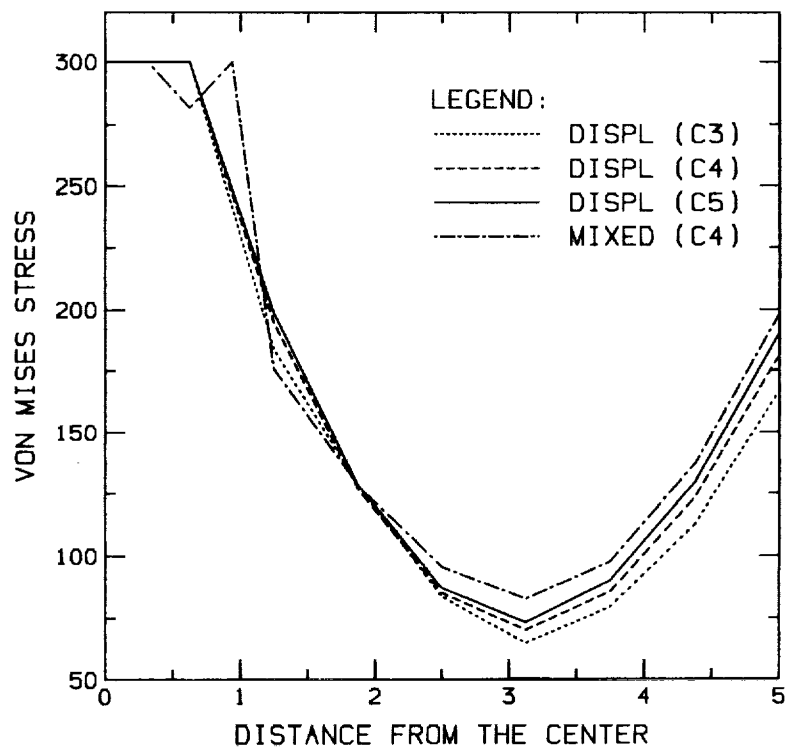

Figure 15. Plate: Distributions of Mises Stress From Center to Edge

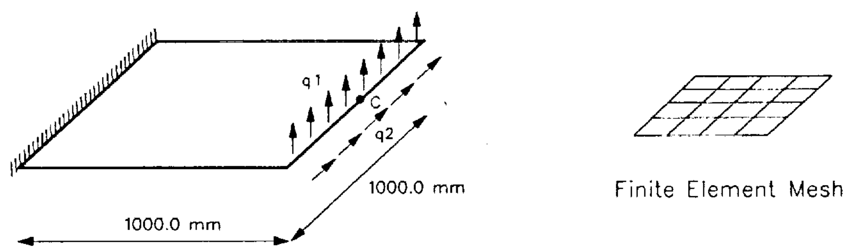

Figure 16. The square cantilever used in problem A.

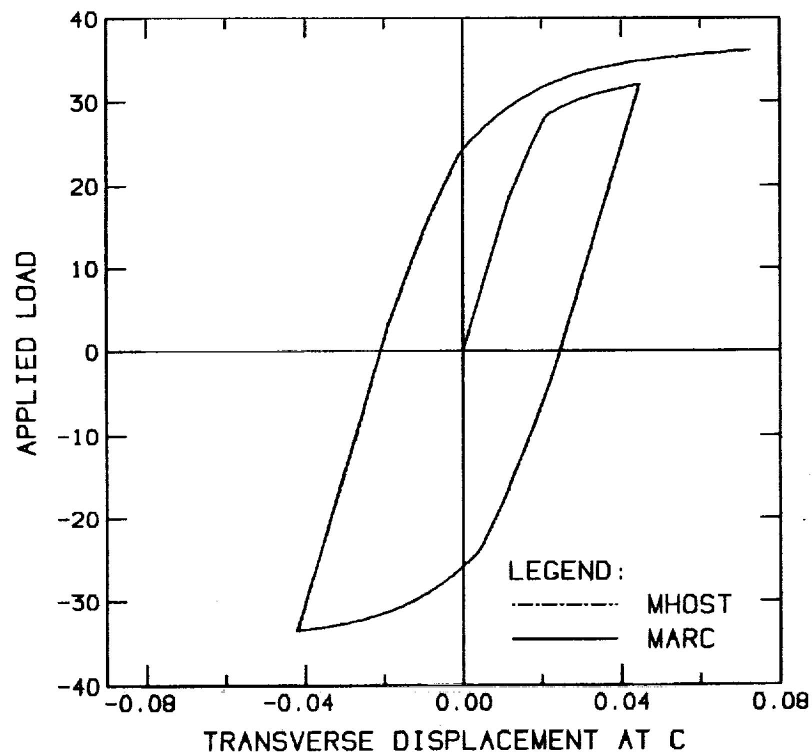

Figure 17. Isotropic Hardening Transverse Load-Displacement Curves

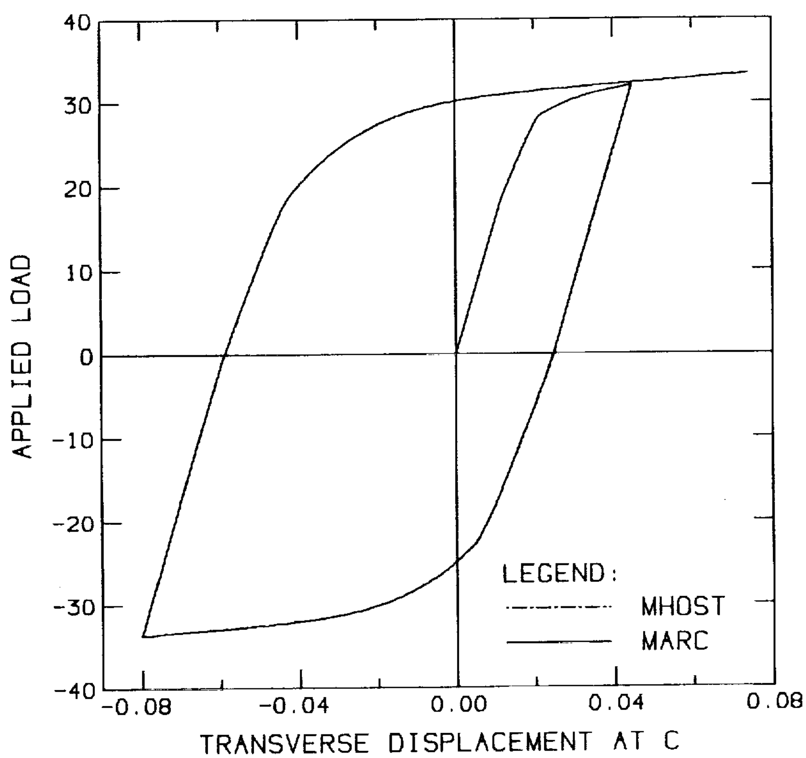

Figure 18. Kinematic Hardening Transverse Load-Displacement Curves

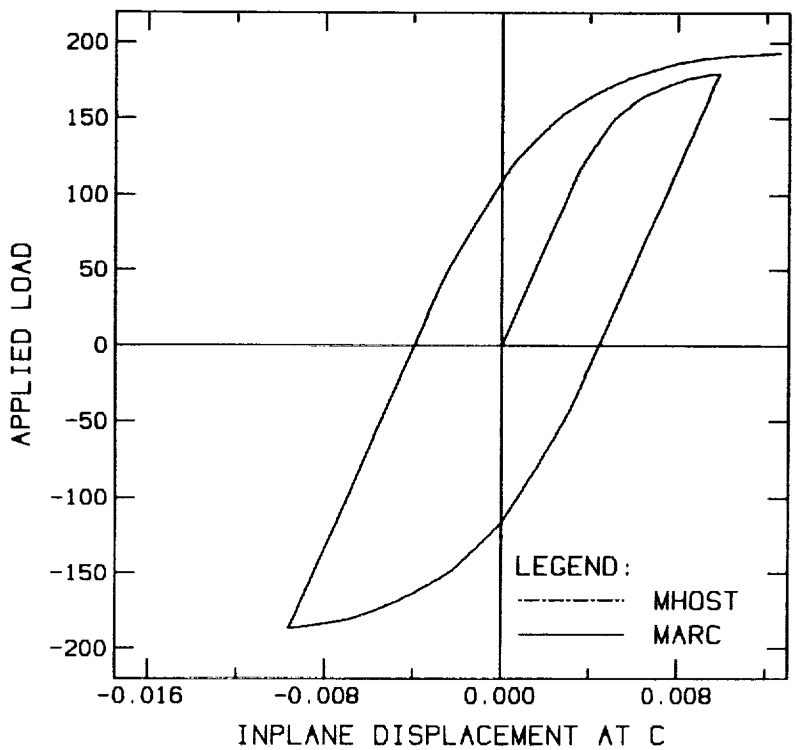

Figure 19. Isotropic Hardening Inplane Load-Displacement Curves

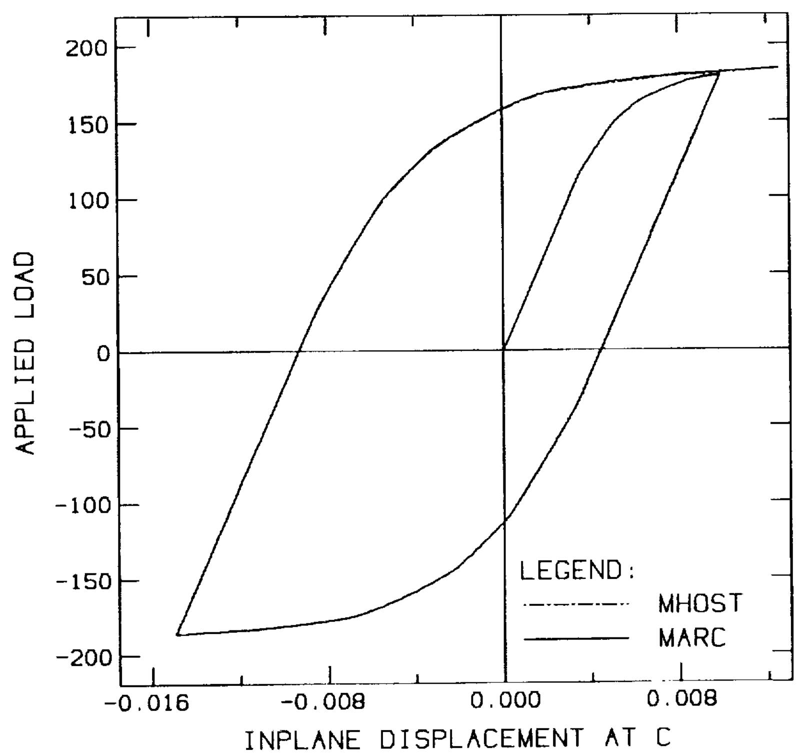

Figure 20. Kinematic Hardening Inplane Load-Displacement Curves

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13. ABSTRACT ***(Maximum 200 words)***

An elastic-plastic algorithm based on Von Mises and associative flow criteria is implemented in MHOST–a mixed-iterative finite element analysis computer program developed by NASA Lewis Research Center. The performance of the resulting elastic-plastic mixed-iterative analysis is examined through a set of convergence studies. Membrane and bending behaviors of 4-node quadrilateral shell finite elements are tested for elastic-plastic performance. Generally, the membrane results are excellent, indicating the implementation of elastic-plastic mixed-iterative analysis is appropriate.

14. SUBJECT TERMS

Associative flow; Rule; Computational mechanics; Elastic-plastic analysis; Finite element analysis; Iterative solution; Mixed formulation; Plasticity; Strain hardening; Structural mechanics

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