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On l^1 - Optimal Decentralized Performance

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Abstract: In this paper, the Manousiouthakis parametrization of all decentralized stabilizing controllers is employed in mathematically formulating the l^1 optimal decentralized controller synthesis problem. The resulting optimization problem is infinite dimensional and therefore not directly amenable to computations. It is shown that finite dimensional optimization problems that have value arbitrarily close to the infinite dimensional one can be constructed. Based on this result, an algorithm that solves the l^1 decentralized performance problems is presented. A global optimization approach to the solution of the finite dimensional approximating problems is also discussed.

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1. Introduction

Consider a feedback control loop with its inputs and its outputs partitioned in a compatible way: $u_1=(u_{11}^T,u_{12}^T, \cdots ,u_{1n}^T)^T$, $u_2=(u_{21}^T,u_{22}^T, \cdots ,u_{2n}^T)^T$, $y_1=(y_{11}^T,y_{12}^T, \cdots ,y_{1n}^T)^T$, and $y_2=(y_{21}^T,y_{22}^T, \cdots ,y_{2n}^T)^T$. The controller C is decentralized iff it is block diagonal i.e. the $i-th$ subvector, y_{1i}, of the manipulated variable vector is only affected by the $i-th$ subvector, y_{2i}, of the measured output:

$$y_{1i}=C_{ii}\,(u_{1i}-y_{2i})$$

Since the early 70's significant research effort has been expended on the subject of decentralized control. Nevertheless, two unanswered questions remain :

(a) given the set of measurements and manipulations, how does one select the appropriate pairings ?

(b) How can one assess fundamental limitations to decentralized control system performance ?

The first question is referred to as the decentralized control structure synthesis problem while the second can be unequivocally addressed only through the optimal decentralized controller synthesis problem. Given the set of measurements and manipulations, the solution of the decentralized controller structure synthesis problem determines the flow of information in the control loop, or equivalently the pairings between the measurements and the manipulations. The solution to the second problem determines the best achievable closed-loop dynamic performance for the given decentralized control structure.

It has been established, that given a plant and a decentralized structure there may not exist any decentralized stabilizing controllers with that structure. Aoki (1972) [2], demonstrated that there may exist decentralized control structures that prevent stabilization of the closed loop. Wang & Davison (1973) [16], introduced the notion of *decentralized fixed eigenvalues* also called as *fixed modes* of a given system. Algebraic characterizations of the notion of decentralized controllability, which is related to the fixed mode concept, for the two input vector case are given in Morse (1973) [11], Corfmat & Morse (1976) [3], [4], and Potter, Anderson & Morse (1979) [12]. Anderson & Clements (1981) [1], employed algebraic concepts and characterized the decentralized fixed eigenvalues of a system and presented computational tests for the existence of fixed modes.

Recently, the issue of stability of decentralized control systems has been addressed within the fractional representation approach to control theory. For linear time invariant processes, Manousiouthakis (1989) [9] presented a parametrization of all decentralized stabilizing controllers for a given process and a fixed decentralized control structure. Within this framework, any decentralized stabilizing controller is parametrized in terms of a stable transfer function matrix that has to satisfy a finite number of quadratic equality constraints. For the same class of processes (LTI plants) Desoer and Gundes presented an equivalent parametrization where the stable parameter satisfies a unimodularity condition (Desoer & Gundes, 1990, p.122, 165) [7].

In this paper, the Manousiouthakis parametrization is employed in mathematically formulating the optimal controller synthesis problem. The decentralized performance problem is formulated as an infinite dimensional l^1 optimization problem. Performing appropriate truncations a finite dimensional optimization problem is obtained. Theorems that establish the connection between the two problems are presented. It is shown that iterative solution of the finite dimensional problem creates a sequence of values that converges to the values of the infinite dimensional problem. Based on these conergence results a computational procedure that yields ε-optimal solutions to l^1 optimization problem is outlined. Locally optimal solutions to the intermediate finite dimensional problems can be obtained through existing nonlinear optimization algorithms (MINOS, GINO etc.). Global solution of the intermediate finite dimensional approximations guarantees that the limit of the sequence that is being created corresponds to the best performance that can be obtained by the given decentralized structure. Feasibility (or infeasibility) of the optimization problem is equivalent to existence (or nonexistence) of decentralized controllers with the given structure.

2. Mathematical Preliminaries

2.1. Fractional Representations of Linear systems

Let G be the set of all proper, rational transfer functions and $M(G)$ be the set of matrices with entries that belong to G. Also let S be the set that includes only the stable members of G and let $M(S)$ be the set of matrices with entries that belong to S.

In this work, theoretical results related to the notion of *doubly coprime fractional representations* and the *parametrization of all stabilizing compensators* are used. These results and a complete exposition of the underlying theory can be found in Vidyasagar (1985; pp. 79, 83, 108, 110) [14]. The notation used in the present work is compatible with the notation in the aforementioned reference.

2.2. Input - Output Linear Operators

One of the frameworks developed to describe the stability and performance of dynamical systems is the input - output approach. Although the theory has been developed for both continuous and discrete systems, in this work the focus is on discrete systems.

In the sequel the fact that every linear BIBO operator can be represented by an l^1 sequence will be utilized. For such operators the $l^\infty - l^\infty$ induced norm is equal to the l^1 norm of the corresponding l^1 sequence. The results that are used can be found in Desoer and Vidyasagar (1975; pp. 23-24, 100, 239) [5].

2.3. Elements from Real and Functional Analysis

The notion of denseness will be used in the proofs in Section 4. The fact that the set ϕ_o of all sequences with finitely many nonzero elements is dense in l^1 will also be used. Properties of the compact sets will be used in Lemma 2 in Section 4. The related theory is given by Wheeden & Zygmud (1977, pp. 4, 8-9, 134) [17].

The properties of point-to-set mappings are also used. All relevant results can be found in Fiacco (1983; pp. 12, 14) [6].

3. Parametrization of Decentralized Stabilizing Controllers

In this section, the results presented by Manousiouthakis (1989) [9] are outlined. The 2-channel case is outlined in section 3.1. In section 3.2, the result corresponing to the general l-channel case is presented.

3.1. 2-Channel Decentralized Control

Consider a feedback control system shown with plant P and controller C:

$$C=\begin{bmatrix} C_{1,1} & C_{1,1} \\ C_{2,1} & C_{2,1} \end{bmatrix} \in G^{n \times m}\ ,\ C_{1,1} \in G^{n_1 \times m_1}\ ,\ C_{2,2} \in G^{(n-n_1)\times(m-m_1)}\ ,$$

$$P=\begin{bmatrix} P_{1,1} & P_{1,1} \\ P_{2,1} & P_{2,1} \end{bmatrix} \in G^{m \times n}\ ,\ P_{1,1} \in G^{m_1 \times n_1}\ ,\ P_{2,2} \in G^{(m-m_1)\times(n-n_1)}$$

Manousiouthakis (1989) [9], demonstrated that based on the YJB parametrization of all stabilizing compensators for a given plant, the set of all 2-channel decentralized stabilizing compensators for the given plant can be described as:

$$S_d(P)=\left\{ C=(\tilde{X}+D_P Q)(\tilde{Y}-N_P Q)^{-1}\ ,\ \det(\tilde{Y}-N_P Q)\neq 0\ ,\ Q \in M(S)\ ; \right.$$
$$\left. S_1+Q S_2+S_3 Q+Q S_4 Q=0 \right\} \tag{1}$$

where,

$$\left.\begin{matrix} S_1=Y L_{n_1}\tilde{X}-X L_{m_1}\tilde{Y} & S_3=Y L_{n_1} D_P+X L_{m_1} N_P \\ S_2=\tilde{N}_P L_{n_1}\tilde{X}+\tilde{D}_P L_{m_1}\tilde{Y} & S_4=\tilde{N}_P L_{n_1} D_P-\tilde{D}_P L_{m_1} N_P \end{matrix}\right\} \tag{1a}$$

and

$$L_{n_1}=\begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n-n_1} \end{bmatrix}=L_{n_1}^{-1}\quad ,\quad L_{m_1}=\begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m-m_1} \end{bmatrix}=L_{m_1}^{-1}$$

3.2. l-Channel Decentralized Control

The parametrization of all $l \times l$ block diagonal stabilizing controllers is based on the results of the previous section, namely relations (1), and (1a). It has been established that the set of all l-channel decentralized stabilizing controllers can be parametrized as (Manousiouthakis, 1989) [9]:

$$S_d(P)=\left\{ C=(\tilde{X}+D_P Q)(\tilde{Y}-N_P Q)^{-1}\ ,\ \det(\tilde{Y}-N_P Q)\neq 0\ ,\ Q \in M(S)\ ; \right.$$
$$\left. S_{1j}+Q S_{2j}+S_{3j} Q+Q S_{4j} Q=0\ ,\ j=1,\cdots,l-1 \right\} \tag{4}$$

The transfer matrices $S_{ij}\ ,\ i=1,2,3,4\ ,\ j=1,\cdots,l-1$ are given by relations similar to (1a).

4. Optimal Decentralized Control System Performance

4.1. l^1 Performance Results

The performance problem is often posed as follows : *determine whether the output of the system remains within specified bounds for all bounded external disturbances and for all times.* The idea of considering disturbances bounded in magnituded was introduced by Vidyasagar (1986) [15] and led to the l^1-optimal control problem.

The mathematical formulation of the problem is performed as follows:

(a) *Disturbance, d, is bounded*

$$0 \le |d(k)| \le w_1 , \ \forall k \ge 0 <=> \|w_1^{-1} d\|_\infty \le 1.$$

(b) *Output, y, is bounded*

$$0 \le |y(k)| \le w_2^{-1} , \ \forall k \ge 0 <=> \|w_2 y\|_\infty \le 1.$$

(c) *Satisfy bounds on y for all allowable d*

$$\Phi(P,C) = \sup_{\|w_1^{-1} d\|_\infty \le 1} \|w_2 y\|_\infty \le 1.$$

Let $H(P,C)$ be the closed loop map between the disturbance (d) and the output (y). Then, $y = H(P,C)\,d$. According to (c) the output of the system remains within the desirable bounds iff:

$$\Phi(P,C) \le 1 <=> \sup_{\|w_1^{-1} d\|_\infty \le 1} \|w_2 y\|_\infty = \sup_{\|w_1^{-1} d\|_\infty \le 1} \|w_2 H(P,C)\, w_1 (w_1^{-1} d)\|_\infty =$$
$$= \|w_2 H(P,C)\, w_1\|_{\infty\infty} \le 1$$

To determine what is the best performance that a decentralized controller can deliver, the value of the following optimization problem should be identified :

$$\nu = \inf_{C \in S_d(P)} \|w_2 H(P,C) w_1\|_{\infty\infty}$$

If the value of this problem is less than 1 then there exist controllers with the given structure such that the output of the system satisfies the performance requirements. For simplicity in notation, w_1 and w_2 will be augmented in the map $H(P,C)$.

Employing the parametrization of all stabilizing decentralized controllers the last optimization problem is expressed in terms of the stable map $Q \in M(S)$:

$$\nu = \inf_{Q \in M(S)} \|H(P,Q)\|_{\infty\infty} \inf_{Q \in M(S)} \|T_1 - T_2\, Q\, T_3\|_{\infty\infty} \qquad \text{(DPP)}$$

subject to,

$$S_1 + Q\, S_2 + S_3\, Q + Q\, S_4\, Q = 0$$

In the last problem, the optimization variable belongs to an infinite dimensional linear space. In addition, the closed loop map $H(P,Q)$ is affine in Q : $H(P,Q) = T_1 - T_2 Q T_3$ where $T_1, T_2, T_3 \in M(S)$, are known and depend on the factorization of P that is employed (Vidyasagar,1985, p.110) [14].

Let $h(p,q)$ be the impulse response sequence that corresponds to $H(P,Q)$. Let also $t_i = \{ t_i(k) \}_{k=0}^{\infty}$, $i = 1, 2, 3$ and $q = \{ q(k) \}_{k=0}^{\infty}$, where $t_i(k)$ and $q(k)$ are real matrices of appropriate dimensions. Then, the sequence $h(p,q) = \{ h(k) \}_{k=0}^{\infty}$ is given by :

$$h(k) \hat{=} \begin{bmatrix} h_{11}(k) & \cdots & h_{1n}(k) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ h_{m1}(k) & \cdots & h_{mn}(k) \end{bmatrix} = t_1(k) - \sum_{j=0}^{k} \left[t_2(k-j) \sum_{\lambda=0}^{j} q(\lambda) t_3(j-\lambda) \right] \tag{1}$$

Similarly, let s_1, s_2, s_3, s_4 be the impulse response sequences, members of $M(l^1)$, that correspond to S_1, S_2, S_3, S_4 respectively and $s_i = \{ s_i(k) \}_{k=0}^{\infty}$ $i = 1,2,3,4$. Then, the quadratic constraint is satisfied iff all the elements of the impulse response sequence that coresponds to the LHS are equal to zero. Thus the following infinite set of quadratic equality constraints is obtained:

$$f_k(q) \hat{=} s_1(k) + \sum_{j=0}^{k} \left\{ q(j) s_2(k-j) + s_3(k-j) q(j) + q(k-j) \left[\sum_{\lambda=0}^{j} s_4(j-\lambda) q(j) \right] \right\} = 0 \,, \; \forall\, k \geq 0 \tag{2}$$

The objective function of (DPP) becomes:

$$\|H(P,Q)\|_{\infty\infty} = \|h(p,q)\|_{l^1} = \max_{i=1,\cdots,m} \sum_{j=1}^{n} \sum_{k=0}^{\infty} \left| \left\{ t_1(k) - \sum_{j=0}^{k} \left[t_2(k-j) \sum_{\lambda=0}^{j} q(\lambda) t_3(j-\lambda) \right] \right\}_{i,j} \right|$$

where

$$h_{i,j}(k) \hat{=} \left\{ t_1(k) - \sum_{j=0}^{k} \left[t_2(k-j) \sum_{\lambda=0}^{j} q(\lambda) t_3(j-\lambda) \right] \right\}_{i,j}$$

Based on (1) and (2) (DPP) is transformed into the following constrained l^1-optimization problem:

$$\nu = \inf_{q \in l^1_{m \times n}} \|h(p,q)\|_{l^1} \tag{DPPs}$$

subject to,

$$f_k(q) = 0 \;,\; \forall\, k \geq 0$$

For a given sequence q and a value of k, $f_k(q)$ is an $m \times n$ matrix with entries: $f_k^{i,j}(q)$, $i=1,...,m$, $j=1,...,n$. Then (DPPs) can be reformulated as:

$$\nu = \inf_{q \in l^1_{m \times n}} \|h(p,q)\|_{l^1} \qquad \text{(DPPs-a)}$$

subject to,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=0}^{\infty} | f_k^{i,j}(q) | = 0$$

These optimization problem are infinite dimensional; the optimization variable (q) lies in an infinite dimensional linear space and the constraint is also infinite dimensional. In the following, it will be shown that one can obtain solutions, arbitrarily close to the solution of these problems, by solving appropriately constructed finite dimensional optimization problems.

To reduce (DPPs) to a finite dimensional optimization problem two types of truncations are performed:

Truncate the constraint

$$\nu^M = \inf_{q \in l^1_{m \times n}} \|h(p,q)\|_{l^1} \qquad \text{(DT1)}$$

subject to,

$$f_k(q) = 0 \ , \quad k = 0, \cdots, M$$

and

Truncate the variable q and the constraint

$$\nu_N^M = \inf_{q \in \phi_{o,N}} \|h(p,q)\|_{l^1} \qquad \text{(DT2)}$$

subject to,

$$f_k(q) = 0 \ , \quad k = 0, \cdots, M$$

where $\phi_{o,N} \subset \phi_o$ is the set of all $l^1_{m \times n}$ sequences with their first $N+1$ elements nonzero.

In the following, it is assumed that (DPPs) is feasible, thus feasibility of (DT1) is guaranteed. Under this condition, feasibility of (DT2) is guaranteed provided that $N > M$. The following theorem establishes the relationship between (DT1) and (DT2).

Theorem 1

$$\lim_{N \to \infty} \nu_N^M = \nu^M$$

Before proceeding with the proof of the theorem, let us consider the following sets:

$$G_M = \left\{ q \in l^1_{m \times n} \mid f_i(q) = 0 \ , \ \ i = 0, \cdots, M \right\}$$

and

$$G_{o,M} = \left\{ q_1 \in \phi_o \mid f_i(q_1) = 0 \ , \ \ i = 0, \cdots, M \right\} \subset G_M .$$

The first set is the feasible set of (DT1). The feasible set of (DT2) is a subset of the second set. Then the following lemma establishes a result that will be used in the proof of theorem 1.

Lemma 1

The set $G_{o,M}$ is dense in G_M.

Proof of Lemma 1

To prove that the first set is dense in the second it suffices to show that : given $\delta > 0$ and $q \in G_M$ there exists $q_1 \in G_{o,M}$ such that $\|q - q_1\|_{l^1} < \delta$.

The set of $M+1$ constraints $f_i(q) = 0$, $i=0, \cdots, M$ involves only the first $M+1$ elements of the sequence q. Any other sequence with the same $M+1$ first elements satisfies the constraints. Therefore, q can be partitioned as : $q = [\, q_a \mid q_b \,] \in G_M$, where $q_a \in \mathrm{R}^{M+1}$ is the vector of elements of q that appear in the constraints. Clearly, $q_b \in l^1$ and there exists a sequence $q_{1,b} \in \phi_o$ such that for given $\delta > 0 \Rightarrow \|q_b - q_{1,b}\|_{l^1} < \delta$. The sequence $q_1 = [\, q_a \mid q_{1,b} \,]$ is a member of $G_{o,M}$ and :

$$\|q - q_1\|_{l^1} = \|q_a - q_a\|_1 + \|q_b - q_{1,b}\|_{l^1} < \delta.$$

□

Now, the proof of theorem 1 follows.

Proof of Theorem 1

We want to prove that $\lim_{N \to \infty} \nu_N^M = \nu^M$.

Equivalently, for a given $\varepsilon > 0$ we want to prove that there exists N such that $|\nu^M - \nu_N^M| < \varepsilon$. Since, $\nu^M = \inf_{q \in G_M} \|h(p,q)\|_{l^1}$, for a given $\varepsilon > 0$ there exists a $q \in G_M$ such that :

$$\left| \ \nu^M - \|h(p,q)\|_{l^1} \right| < \frac{\varepsilon}{2} \tag{2}$$

Since $\|h(p, \cdot)\|_{l^1}$ is continuous on G_M then, given $q \in G_M$ there exists $\delta > 0$ such that for all $q' \in G_M$,

$$\|q - q'\|_{l^1} < \delta \ \ \Rightarrow \ \ \left| \ \|h(p,q)\|_{l^1} - \|h(p,q')\|_{l^1} \right| < \frac{\varepsilon}{2}$$

Since $G_{o,M}$ is dense in G_M => there exists $q_1 \in G_{o,M}$: $\|q - q_1\|_{l^1} < \delta$. Let N be the smallest number such that $q_1 \in \phi_{o,N}$ and q_1 satisfies the last relationship.

Consequently,

$$\left| \|h(p,q)\|_{l^1} - \|h(p,q_1)\|_{l^1} \right| < \frac{\varepsilon}{2} \tag{3}$$

From (2) and (3) the following relation is obtained :

$$\left| \nu^M - \|h(p,q_1)\|_{l^1} \right| < \varepsilon$$

Since $\nu^M \le \nu_N^M$ and $0 \le \nu_N^M \le \|h(p,q_1)\|_{l^1}$ we finally obtain : $|\nu^M - \nu_N^M| < \varepsilon$. □

This theorem establishes the first approximation result. However, its application assumes that for any given N and M the value ν_N^M is known, or can be computed. The solution of (DT2) involves the minimization of an infinite sum.

Given an $l^1_{m\times n}$ sequence ξ, its $l^1(L)$ sum is defined as :

$$\|\xi\|_{l^1(L)} = \max_{i=1,\cdots,m} \sum_{j=1}^{n} \|\xi_{i,j}\|_{l^1(L)} = \max_{i=1,\cdots,m} \sum_{j=1}^{n} \sum_{k=0}^{L} |\xi_{i,j}(k)|$$

Define the set of sequences $q \in \phi_{o,N}$ that are norm bounded by some positive number B :

$$q \in \phi_{o,N}^B = \left\{ q \in \phi_{o,N} \; ; \; \|q\|_{l^1} \le B \right\}$$

The set $\phi_{o,N}^B$ is finite dimensional (so is $\phi_{o,N}$), closed, bounded and therefore by the Heine-Borel theorem it is compact. Consider a formulation of (DT2) where the variable is norm bounded:

$$\nu_N^M(B) = \inf_{q \in \phi_{o,N}^B} \|h(p,q)\|_{l^1} \tag{DT2a}$$

subject to,

$$f_i(q) = 0 \;\; , \;\; i = 0, \cdots, M$$

Clearly, $\nu_N^M \le \nu_N^M(B)$ for all values of B.

Let $\overline{q} \in \phi_{o,N}$ be a suboptimal solution of (DT2), i.e.:

$$\forall \; \varepsilon > 0 \, , \; \exists \overline{q} \in \phi_{o,N} \, ; \;\; 0 \le \|h(p,\overline{q})\|_{l^1} - \nu_N^M < \varepsilon$$

The norm of $\overline{q}$ is finite. Let $\|\overline{q}\|_{l^1} = B(\varepsilon)$. Existence of the solution of (DT2a) is guaranteed by the compactness of its feasible set (see proof of lemma 2). Let $\overline{q}'$ be the solution of (DT2a). Then, $\overline{q}'$ satisfies the relation:

$$\nu_N^M \le \nu_N^M(B(\varepsilon)) = \|h(p,\overline{q}')\|_{l^1} \le \|h(p,\overline{q})\|_{l^1}$$

Then, combining these statements the following is obtained:

$$\forall\, \varepsilon > 0\,,\ \exists B(\varepsilon)\,;\ 0 \le \nu_N^M(B(\varepsilon)) - \nu_N^M < \varepsilon$$

If B is selected to be sufficiently large, the solution of (DT2a) is arbitrarily close to the solution of (DT2). In subsequent sections it will be demonstrated that the calculation of bounds for the optimization variables is feasible.

Based on this discussion, the following optimization problem is formulated:

$$\nu_{N,L}^M = \inf_{q \in \phi_{o,N}^B} \|h(p,q)\|_{l^1(L)} \tag{DT3}$$

subject to,

$$f_k(q) = 0\ ,\quad k = 0, \cdots, M$$

The following lemma establishes the fact that the solution to (DT2) can be obtained through iterative solution of (DT3) for increasing values of L.

Lemma 2

$$\lim_{L \to \infty} \nu_{N,L}^M = \nu_N^M(B)$$

Proof

The sequence $\{\, \nu_{N,L}^M \,\}_{L=0}^{\infty}$ is nondecreasing and bounded:

$$\forall\, L \ge 0 :\ 0 \le \nu_{N,L}^M \le \nu_N^M(B)\,.$$

As a result : $\exists \alpha > 0\,;\ \lim_{L \to \infty} \nu_{N,L}^M = \alpha.$

This limit cannot be greater than $\nu_N^M(B)$. Assume that $\alpha < \nu_N^M(B)$.

The feasible set of (DT3) is compact. Indeed, it can be written as the intersection of a finite dimensional, closed and bounded set ($\phi_{o,N}^B$) with a finite dimensional closed set:

$$\phi_{o,N}^B \cap G_{o,M} = \phi_{o,N}^B \cap \left\{\, q_1 \in \phi_{o,N} \ \mid f_i(q_1) = 0\ ,\quad i = 0, \cdots, M \right\}$$

As a result the feasible set is finite dimensional, closed and bounded.

From the continuity of the $l^1(L)$ norm, and the compactness of the feasible set it follows that (Luenberger, 1969; p.14) [8]:

$$\forall\, L > 0\,,\ \exists q_L \in \phi_{o,N}^B \cap G_{o,M}\ ;\ \nu_{N,L}^M = \|h(p,q_L)\|_{l^1(L)}$$

Compactness of the feasible set implies that the sequence $\{\, q_L \,\}_{L=0}^{\infty}$ has a subsequence $\{\, q_{L_k} \,\}_{k=0}^{\infty}$ that converges in $\phi_{o,N}^B \cap G_{o,M}$:

$$\lim_{k \to \infty} q_{L_k} = \bar{q} \in \phi_{o,N}^B \cap G_{o,M}$$

Then,

$$\lim_{L\to\infty} \nu_{N,L}^{M} = \lim_{L\to\infty} \|h(p,q_L)\|_{l^1(L)} = \lim_{k\to\infty} \|h(p,q_{L_k})\|_{l^1(L_k)} = \|h(p,\bar{q})\|_{l^1} = \alpha$$

This contradicts the assumption that $\alpha < \nu_N^M(B)$. Hence, $\alpha = \nu_N^M(B)$.

□

In summary, lemma 2 establishes that solution of (DT3) for increasing values of L identifies the solution of (DT2a). Based on the "equivalence" of (DT2a) and (DT2), it can be said that this iterative procedure identifies ε - optimal solutions of (DT2).

Remark : In the case where the coprime factors of P have been constructed to be FIR's, for each $q \in \phi_{o,N}$ the sequence $h(p,q)$ will have a finite number of nonzero elements. This number is known and depends only on N. Therefore the norm of $h(p,q)$ can be exactly calculated by a finite sum. In this case, the exact solution of (DT2) is obtained in one step.

The relation between the two different types of truncated problems has been established through Theorem 1. In the remaining of this section the connection between (DT1) and (DPPs) is shown. For a non-negative real number δ consider the following sets:

$$G(\delta) \hat{=} \left\{ q \in l_{m\times n}^1 : \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=0}^{\infty} | f_k^{i,j}(q) | \leq \delta \right\},\ G^M(\delta) \hat{=} \left\{ q \in l_{m\times n}^1 : \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=0}^{M} | f_k^{i,j}(q) | \leq \delta \right\}$$

Then, the following theorem holds:

Theorem 2

Assume that $G(\cdot)$ and $G^M(\cdot)$ represent upper semicontinuous mappings from the non-negatives reals to subsets of l^1. Then,

$$\lim_{M\to\infty} \nu^M = \nu.$$

Before we proceed with the proof of Theorem 2, two intermediate results will be presented. Consider the two optimization problems:

$$\nu(\omega) = \inf_{q\in l_{m\times n}^1} \left\{ \|h(p,q)\|_{l^1} + \omega \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=0}^{\infty} | f_k^{i,j}(q) | \right\} \tag{P1}$$

and

$$\nu^M(\omega) = \inf_{q\in l_{m\times n}^1} \left\{ \|h(p,q)\|_{l^1} + \omega \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=0}^{M} | f_k^{i,j}(q) | \right\} \tag{P2}$$

It is easily verified that (P1) is the penalty function formulation of (DPPs-a) (Luenberger, 1969, pp.302-305) [8]. For this type of problems the following lemma can be shown to hold:

Lemma 3

Let $\{\omega^r\}_{r=0}^{\infty}$ be an icreasing sequence such that $\lim_{r\to\infty} \omega^r = \infty$. Let also $G(\cdot)$ be an upper semicontinuous mapping from the non-negative reals to subsets of $l_{m\times n}^1$. Then, the following statements hold:

1. $\nu(\omega^{r+1}) \geq \nu(\omega^r)$
2. $\nu \geq \nu(\omega^r)$
3. $\lim_{r \to \infty} \nu(\omega^r) = \nu$.

Proof

Part 1 & 2

The proof of Part 1 & 2 is performed in a similar way as the proof of statements 1 & 2 in Lemma 1 in Luenberger (1969, p.305) [8].

Part 3

From Part 1 & 2 it follows that the limit exists and is less than or equal to ν. The proof is similar to the proof of Part 3 in Lemma 4.2 in Sourlas and Manousiouthakis (1992) [13].

□ Employing the same line of arguments the following corollary can be shown to hold:

Corollary 1

Let $G^M(\cdot)$ be an upper semicontinuous mapping from the non-negative reals to subsets of $l^1_{m \times n}$. Then,

1. $\nu^M(\omega^{r+1}) \geq \nu^M(\omega^r)$
2. $\nu^M \geq \nu^M(\omega^r)$
1. $\lim_{r \to \infty} \nu^M(\omega^r) = \nu^M$.

For a given value of ω, Lemma 4 establishes the relation between $\nu^M(\omega)$ and $\nu(\omega)$.

Lemma 4

$$\lim_{M \to \infty} \nu^M(\omega) = \nu(\omega).$$

Proof

The proof is based on the fact that the sequence $\{ f_k(q) \}_{k=0}^{\infty}$ belongs to $l^1_{m \times n}$ thus allowing the approximation of the infinite sum in the penalty term in (P1) with a finite sum. The detailed proof can be found is similar to the proof of Lemma 4.3 in Sourlas and Manousiouthakis (1992) [13]. □

Now the proof of Theorem 2 follows.

Proof of Theorem 2

The proof consists of two parts. First existence of the limit will be shown and then convergence to ν will be demonstrated. The sequence ν^M is nondecreasing, and bounded above by ν. As a result, it converges to a real number:

$$\alpha \hat{=} \lim_{M \to \infty} \nu^M \leq \nu.$$

Using Corollary 1, Lemma 3 and 4 the last relation can be rewritten as:

$$\alpha = \lim_{M\to\infty} v^M = \lim_{M\to\infty}\lim_{\omega\to\infty} v^M(\omega) \le v = \lim_{\omega\to\infty}\lim_{M\to\infty} v^M(\omega) \tag{4}$$

In the remaining of the proof it will be shown that the two iterated limits are equal.

Using the proof technique of Lemma 3 it can be established that:

$$v^M(\omega) \le \lim_{\omega\to\infty} v^M(\omega) \ , \ \forall M \ , \forall \omega$$

This implies that:

$$v(\omega) = \lim_{M\to\infty} v^M(\omega) \le \lim_{M\to\infty}\lim_{\omega\to\infty} v^M(\omega) \ , \ \forall \omega \ =>$$

$$v = \lim_{\omega\to\infty}\lim_{M\to\infty} v^M(\omega) \le \lim_{M\to\infty}\lim_{\omega\to\infty} v^M(\omega) = \alpha$$

In view of (4), the last relation implies that $\alpha = v$.
As a result, the following statement has been proven:

$$\alpha = \lim_{M\to\infty} v_M = \lim_{M\to\infty}\lim_{N\to\infty} v_{N,M} = v$$

□

4.2. Computational Procedure

In view of theorems 1,2 and lemma 2, the value v is obtained through a limiting procedure, which involves solution of (DT3) for increasing values of L, N and M. The finite dimensional optimization problem can be formulated as a nonlinear program, which can be solved by finite dimensional optimization techniques. For a particular example, the structure of the nonlinear program is given in appendix A.

The computationally intensive part of the procedure is the solution of (DT3), which is a nonlinear programming problem, for different values of L, N and M. The resulting nonlinear programs are nonconvex, due to the existence of the quadratic equality constraints. Nevertheless, global solution of these problems determines the globally optimum performance that can be achieved by a certain decentralized structure. If (DT3) is solved locally, then the limit of the resulting sequence will identify an upper bound to the l^1 optimal decentralized performance.

4.3. Global Optimization Approach

Global optimization approaches that solve the general nonconvex optimization problem have recently been developed. Manousiouthakis & Sourlas (1992) [10], presented a procedure that is based on the transformation of the original nonlinear optimization problem into one that has convex constraints and objective with an additional separable, quadratic, reverse convex constraint. Employing this transformation procedure, (DT3) becomes a convex programming problem with an additional reverse convex, quadratic and separable constraint. This problem can then be solved and its global optimum can be identified through the use of a branch and bound type of algorithm.

The implementation of this global optimization algorithm requires the existence of valid upper and lower bounds for all elements of the the sequence q that appear in the quadratic equality constraints in (DT3). Bounds on these variables can always be obtained from local minima information, namely the value a local minimum of (DT2) for fixed values of N and M. One can always obtain such information if the solution of (DT3) for fixed N, M and increasing L is performed using local optimization algorithms. The generated sequence of values converges to the value of (DT2) at a local minimum. Let δ_{local} be this value. This is an upper bound to the globally optimal value of (DT2). Thus, for $q \in \phi_{o,N}$ and for each value of L, it holds that:

$$\|h(p,q)\|_{l^1(L)} \le \delta_{local} <=> \max_{i=1,\cdots,m} \sum_{j=1}^{n} \sum_{k=0}^{L} |h_{i,j}(k)| \le \delta_{local} =>$$

$$h_{i,j}(k) \in \left[-\delta_{local}, \delta_{local}\right] \quad ,i=1,\cdots,m \ , \ j=1,\cdots,n \ , \ k=0,\cdots,L \tag{5}$$

Then, one can obtain lower (upper) bounds on all elements of the sequence q through the solution of a minimization (maximization) problem with objective the corresponding element of the sequence and constraints the inclusion realtions that appear in (5). These optimization problems are linear.

5. Example

In this section the computational procedure introduced in section 4.2 is applied to the following 2×2 example.

$$P(z) = \begin{bmatrix} \dfrac{3z+1}{z} & \dfrac{0.5z+2}{z} \\ \dfrac{z^2+2.5z+1}{z^2} & \dfrac{3z+4.5}{z} \end{bmatrix}$$

The objective of our analysis is to determine the best possible achievable performance for a decentralized control system featuring the pairings $\{(y_1,u_1);(y_2,u_2)\}$.

Parametrization of all Decentralized Stabilizing Controllers

The process P is stable. Therefore one can select coprime factors as follows:

$$N_P = \tilde{N}_P = P \ , \ D_P = Y = I \ , \ \tilde{D}_P = \tilde{Y} = I \ , \ X = \tilde{X} = 0$$

According to (2), sec.3.1, any stabilizing controller for P is parametrized as:

$$C = Q(I-PQ)^{-1} \ , \ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in M(S) \tag{1}$$

The controller C is decentralized iff:

$$Q_{21}P_{12} = P_{21}Q_{12}$$
$$Q_{12} = -P_{12}(Q_{11}Q_{22} - Q_{12}Q_{21})$$

The Input - Output map between disturbances and output is given by:

$$H(P,C): u_2 \rightarrow y_2 \; ; \; H(P,C) = (I+PC)^{-1}P = (I-PQ)P$$

Problem Formulation

The disturbance rejection problem is formulated according to the guidelines introduced in section 4.1. First, the known bounds on the disturbance and the desired bounds on the objective are defined:

Disturbance: $u_2^T(t) \in [-1,1] \times [-1,1]$, $\forall t$

Output: $y_2^T(t) \in [-1,1] \times [-1,1]$, $\forall t$

Then, (DPPs) is readily formulated and transformed to (DT3) according to the procedure presented in Section 4. Based on Appendix A, (DT3) is in turn transformed into a nonlinear programming problem. Since the coprime factors are F.I.R. then the solution to (DT2) is obtained in one step. For $N=M=0,1,2$ the globally optimum value of the corresponding optimization problem has been identified. The complete sequence of values that converges to the value of v for this particular problem is shown in the following table:

N	M	v_N^M	N	M	v_N^M
0	0	8.00	5	5	5.20
1	1	6.70	6	6	5.16
2	2	5.85	7	7	5.13
3	3	5.51	8	8	5.13
4	4	5.32	9	9	5.13

Hence, with $\varepsilon = 10^{-2}$ the value of the l^1 - optimal decentralized performance problem was found to be 5.13. When only the linear constraints are considered the value of the global lower bound to the l^1 decentralized performance is 4.98. The l^1 - optimal centralized performance, for the same set of specifications was to found to be equal to 4.72.

6. Conclusions

Based on the Manousiouthakis parametrization of all decentralized stabilizing controllers, the l^1 optimal decentralized performance problem has been formulated as an infinite dimensional optimization problem. This problem was transformed into a finite dimensional one through the introduction of appropriate truncations. Theorems that establish the equivalence (in the limit) of the original problems to their finite dimensional approximations were proven, and a computational procedure was proposed. It has been established that solution to the optimal decentralized performance problem amounts to global solution of a series of quadratically constrained programming problems. If locally optimal solutions are identified for each of the finite dimensional problems, the limit of the corresponding sequence of values will be an upper bound to the optimal l^1 decentralized performance. Based on this work, one can actually evaluate the best performance achievable by a given decentralized structure.

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Appendix A

The optimization problem (DT3) can be formulated as a nonlinear programming problem. The steps that make this transformation feasible follow. For illustration purposes the 2×2 case, with stable plant, the same as the one in the example, is considered. From the definition of the $l^1(L)$ sum :

$$h(p,q) \in l^1_{2\times 2} \Rightarrow \|h(p,q)\|_{l^1(L)} = \delta = \max_{i=1,2} \sum_{k=0}^{L} |h_{i1}(k)| + |h_{i2}(k)|$$

Using the definition of the maximum as the least upper bound, (DT3) is finally transformed into:

$$\nu^M_{N,L} = \inf_{\substack{q_{ij}(k),\delta,\delta_{ij}(k) \\ i,j=1,2 \\ k=0,\cdots,L}} \delta \tag{A.1}$$

subject to,

$$\left.\begin{aligned} f_k(q) &= \sum_{i=0}^{k} \Big[q_{12}(i)p_{21}(k-i) - q_{21}(i)p_{12}(k-i) \Big] = 0 \\ g_k(q) &= q_{12}(k) + \sum_{i=0}^{k} p_{12}(k-i) \sum_{j=0}^{i} \Big[q_{11}(j)q_{22}(i-j) - q_{12}(j)q_{21}(i-j) \Big] = 0 \end{aligned}\right\} k=0,\cdots,M$$

$$q_{ij}(k) = 0 \ , \ i,\ j = 1,2 \ , \ N+1 \le k \le L$$

$$-\delta_{ij}(k) \le h_{ij}(k) \le \delta_{ij}(k) \ , \ i,\ j=1,2 \ , \ k=0,\cdots,L$$

$$\sum_{k=0}^{L} \Big[\delta_{i1}(k) + \delta_{i2}(k) \Big] \le \delta \ , \ i=1,2$$

where $f_k(q)$ and $g_k(q)$ are scalar constraints resulting from the application of the parametrization to this particular case (2×2 controller, stable plants).