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STATISTICAL COMPUTATION OF TOLERANCE LIMITS

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INTRODUCTION

The need for improved, accurate computation of tolerance factors is accomplished by a new theory associated with the numerical integration of the equations to yield exact solutions. The computer codes have been developed specifically to integrate the equations of some statistical properties for a normal (or Gaussian) distribution by application to the particular one- and two-sided tolerance limits.

For the normal variate X with known mean μ and known standard deviation σ, a specified proportion of the normal population (1) is contained above the lower tolerance limit, $L = \mu - K_p\sigma$, or below the upper tolerance limit, $L = \mu + K_p\sigma$, for the one-sided case, and (2) falls within the lower and upper tolerance limits for the two-sided case. K_p is the deviate corresponding to the proportion for the inverse normal probability distribution.

However, in most situations, the mean μ and standard deviation σ frequently are unknown. Therefore, the objective of the tolerance-limit analysis is to obtain the numerical solutions to the problems involving the probability distribution for the sampled population with the normal distribution with unknown mean μ and unknown standard deviation σ. Furthermore, it results in the calculation of the tolerance limits based on the mean $\bar{x}$ and the standard deviation s of the random sample. Thus, by the statistical theory, the tolerance limits can be calculated out by the following equations:

$$\begin{aligned} L &= \bar{x} - ks \\ U &= \bar{x} + ks \, , \end{aligned} \tag{1}$$

where $\bar{x}$ is an estimate of μ and s is an estimate of σ of the normal distribution computed from a sample of size n. The quantity k is the tolerance factor. Since $\bar{x}$ and s are considered random variables, the tolerance limit statement is applied to the given probability. Now, to ensure stability of the numerical solutions, it is necessary to determine k such that the probability is γ that at least a proportion p of the population is above $\bar{x}-ks$ or below $\bar{x}+ks$ for the one-sided case and lying between the limits for the two-sided case.

The statistical tolerance interval $\bar{x} \pm ks$ covers the prespecified proportion p of the sampled distribution with 100 γ percent confidence for the cases where a sample $x_1, x_2, x_3, ..., x_n$ represents the random sample from the normal distribution with mean μ and standard deviation σ. The ordinary statistical methods assume the following equations:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \, , \tag{2}$$

as a sample mean and

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2} \; , \tag{3}$$

as a sample standard deviation.

In the one- and two-sided cases, tolerance limits can be derived using noncentral t-distribution. Mathematically, the problem is to find k such that

$$Pr\{Pr(X \le \bar{x} \pm ks) > p\} = \gamma \, , \tag{4}$$

where X has the normal distribution with mean μ and standard deviation σ. γ is specified probability. Some tables for the tolerance factor k have been generated using inputs of sample size n, proportion, and probability. The values of k pertain to percentage points of the noncentral t-distribution. Specifically,

$$Pr\{\text{noncentral } t \le k\sqrt{n} \mid \delta = K_p\sqrt{n}\} = \gamma \, , \tag{5}$$

where the noncentral t-distribution has ν degrees of freedom.

Normal distributions for one- and two-sided tolerance limits are depicted in figure 1.

Wald and Wolfowitz, in their technical paper, had developed an approximate formula, which shows that k is approximated by

$$k = ru \; , \tag{6}$$

where r is determined from the normal distribution by

$$\Phi\left(\frac{1}{n} + r\right) - \Phi\left(\frac{1}{n} - r\right) , \tag{7}$$

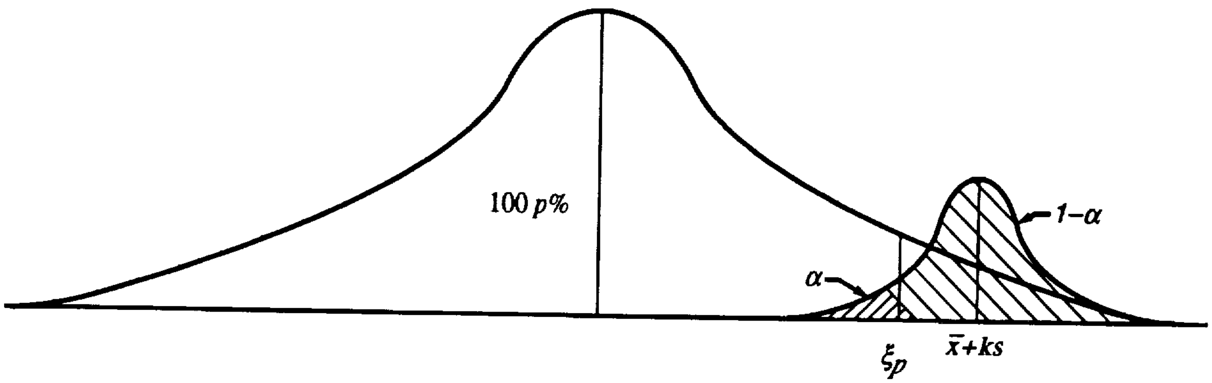
with

$$\Phi(t) = \frac{1}{2\pi} \int_{-\infty}^{t} \exp\left(-\frac{z^2}{2}\right) dz \, ,$$

and u is defined by

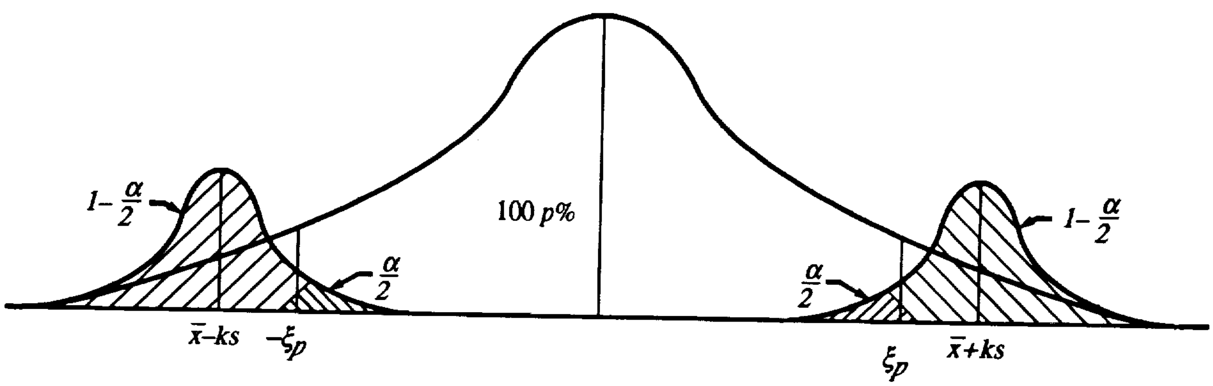
$$u = \sqrt{\frac{\nu}{\chi^2_{\gamma}(\nu)}} \; , \tag{8}$$

where $\chi^2_{\gamma}(\nu)$ is the upper 100 γ percentage point of the chi-square distribution with $\nu = n-1$ degrees of freedom. For $\nu = n-1$, the approximation converges to the true limits as n approaches infinity.



One-sided tolerance limits

The probability is $(1-\alpha)$ that at least 100 p percent of the population lies below (or above) the tolerance limit $F = \bar{x}+k_1s$ (or $F = \bar{x}-k_1s$).



Two-sided tolerance limits

The probability is $\left(1-\frac{\alpha}{2}\right)$ that at least 100 p percent of the population is contained between the tolerance limits $F_1 = \bar{x}+k_2s$ and $F_2 = \bar{x}-k_2s$.

Figure 1. Normal distributions for one- and two-sided tolerance limits.

MATHEMATICAL PROCEDURES

The mathematical procedures are given for determination of exact tolerance limits for the one- and two-sided cases.

One-Sided Tolerance Limits

Method A

For the noncentral t-distribution, let w denote a random variable having a normal distribution with mean zero and variance one; let v denote a random variable that has a chi-square distribution with ν degrees of freedom; let the quantity δ be any constant; and let w and v be stochastically independent. Then $T(\delta) = (w+\delta)/\sqrt{v/\nu}$ has a noncentral t-distribution with noncentrality parameter δ and ν degrees of freedom.

Thus, a standardized normal variate is given

$$g(w) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{w^2}{2}} , \qquad -\infty < w < \infty \tag{9}$$

and an independent chi-square variate based on ν degrees of freedom is

$$h(v) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}}}\, v^{\left(\frac{\nu-2}{2}\right)}\, e^{-\frac{v}{2}} . \qquad 0 < v < \infty \tag{10}$$

Then the joint distribution of w and v is the product of equations (9) and (10):

$$Q(w,v) = g(w)h(v) \;,$$

$$Q(w,v) = \left(\frac{1}{\sqrt{2\pi}}\, e^{-\frac{w^2}{2}}\right)\left(\frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}}}\, v^{\left(\frac{\nu-2}{2}\right)}\, e^{-\frac{v}{2}}\right) . \tag{11}$$

The new random variables are introduced, namely,

$$t = \frac{w+\delta}{\sqrt{\frac{v}{\nu}}} \;, \quad u = \sqrt{v} \;. \tag{12}$$

Applying the change of variables technique, the new distribution is obtained:

$$f(t,u) = \varphi(w,v)\,|J| \;, \tag{13}$$

where J is the Jacobian determinant of the transformation,

$$J = \begin{vmatrix} \dfrac{\partial w}{\partial t} & \dfrac{\partial w}{\partial u} \\ \dfrac{\partial v}{\partial t} & \dfrac{\partial v}{\partial u} \end{vmatrix}$$

Consider the transformation $w = tu/\sqrt{V} - \delta, v = u^2$

$$\frac{\partial w}{\partial t} = \frac{u}{\sqrt{V}} \qquad \frac{\partial w}{\partial u} = \frac{t}{\sqrt{V}}$$

$$\frac{\partial v}{\partial t} = 0 \qquad \frac{\partial v}{\partial u} = 2u \ .$$

The Jacobian is obtained

$$|J| = \frac{2u^2}{\sqrt{V}} \ . \tag{14}$$

Therefore,

$$f(t,u) = \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}\right)\left(\frac{1}{\Gamma\left(\frac{V}{2}\right)2^{\frac{V}{2}}} v^{\left(\frac{V-2}{2}\right)} e^{-\frac{v}{2}}\right)\left(\frac{2u^2}{\sqrt{V}}\right) . \tag{15}$$

The noncentral t-distribution is obtained by integrating over u from 0 to ∞.

$$f(t) = \frac{1}{\Gamma\left(\frac{V}{2}\right)2^{\frac{V}{2}-1}} \int_0^\infty e^{-\left(\frac{tu}{\sqrt{V}} - \delta\right)^2/2} u^{V-1} e^{-\frac{u^2}{2}} \frac{u}{\sqrt{V}} du \ . \tag{16}$$

Equation (16), the density function, is integrated with respect to t from minus infinity to t to give the cumulative distribution function:

$$F(t;V,\delta) = Pr\{T_V \le t\} = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{V}{2}\right)2^{\frac{(V-2)}{2}}} \int_0^\infty G\left(\frac{tu}{\sqrt{V}} - \delta\right) u^{V-1} G'(u)du \ , \tag{17}$$

where

$$G'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} ,$$

and

$$G(x) = \int_{-\infty}^{x} G'(t)dt \ .$$

The distribution has the following properties:

$$Pr\{T_\nu(\delta) \le t_0\} = 1 - Pr\{T_\nu(-\delta) \le t_0\} \ ,$$

$$Pr\{T_\nu(\delta) \le 0\} = G(-\delta) \ .$$

Equation (17) has been formulated in one of the computer codes to evaluate the one-sided k factors, but the listing of the code for this method is not included.

Method B (New Theory)

For the one-sided tolerance limits, the probability equation involves the form

$$Pr\left\{\left(\frac{L-\mu}{\sigma}\right) < -K_p , \left(\frac{U-\mu}{\sigma}\right) > K_p\right\} = \gamma \ . \tag{18}$$

The joint probability is calculated about the standard normally distributed random variable $z = \sqrt{n}\,(\bar{x}-\mu)/\sigma$ and the random variable $w = s/\sigma$. One can approximate σ, the true standard deviation, by s, the sample standard deviation. The joint probability density is integrated over the region defined by

$$z = \sqrt{n}\,kw - \sqrt{n}\,K_p \ . \tag{19}$$

Equation (19) is designated as the shaded area of the strip as shown in figure 2.

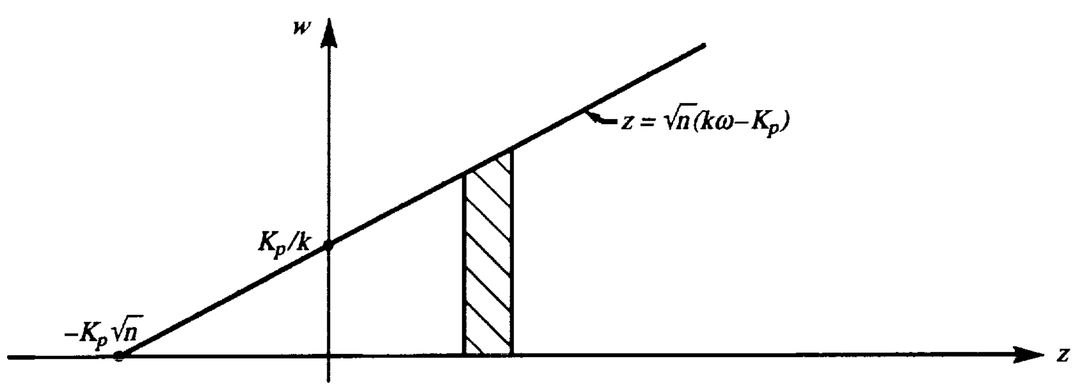


Figure 2. Area of integration for one-sided tolerance limits.

K_p/k is located at the vertex on the w-axis and $-K_p\sqrt{n}$ is at $w = 0$ on the z-axis. The shaded area is integrated to give the summation of the elements. W is the chi-square distribution with ν degrees of freedom, and z is a standard normal variable. In the one-sided case, the joint distribution of w and z is derived in the following form:

$$I = 2\int_0^\infty \frac{\left(\frac{\nu}{2}\right)^{\frac{\nu}{2}} w^{(\nu-1)} e^{-\frac{\nu w^2}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \Phi\left[\sqrt{n}\,(kw - K_p)\right] dw \ . \tag{20}$$

Let $w = x/\sqrt{\nu}$. Then $dw = dx/\sqrt{\nu}$.

Substituting the above relations into equation (20) results in

$$I = 2\int_0^\infty \frac{\left(\frac{v}{2}\right)^{\frac{v}{2}}\left(\frac{x}{\sqrt{v}}\right)^{(v-1)} e^{-\frac{vx^2}{2v}}}{\Gamma\left(\frac{v}{2}\right)} \frac{1}{\sqrt{v}}\, \Phi\left[\sqrt{n}\,(kw-K_p)\right] dx \ , \tag{21}$$

where

$$\Phi(y) = \int_{-\infty}^{y} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz \ .$$

After those manipulations, equation (21) can be expressed in the final form

$$I = \frac{1}{\Gamma\left(\frac{v}{2}\right) 2^{\left(\frac{v}{2}-1\right)}} \int_0^\infty x^{(v-1)} e^{-\frac{x^2}{2}}\, \Phi\left[\sqrt{n}\,(kw-K_p)\right] dx \ . \tag{22}$$

Equation (22) is programmed in the program code which is listed in appendix C1.

Two-Sided Tolerance Limits

Method A

From method A described for the one-sided tolerance limits, equation (17) is readily implemented as a summation for a set of lower and upper limits of the integrals to represent the two-sided tolerance limits. In terms of the tolerance interval, a proportion of a distribution is constructed as a fraction from which the sample is drawn. Let w_1 and w_2 constitute a two-sided normal distribution with zero means and unit variances. Also, let v be distributed independent of the w's and let v obtain a square root of a chi-square distribution with v degrees of freedom. Consequently, it follows that the expressions $(w_1+\delta_1)/v$ and $(w_2+\delta_2)/v$ have noncentral t-distributions with v degrees of freedom and noncentrality parameters δ_1 and δ_2.

A two-sided noncentral t-distribution is, therefore, defined as the joint distribution of $(w_1+\delta_1)/v$ and $(w_2+\delta_2)/v$. A random sample $x_1, x_2, x_3, ..., x_n$ from a normal distribution is assumed. Values of k factor are assigned from which tolerance limits $\bar{x}-ks$ (lower limit) and $\bar{x}+ks$ (upper limit) are computed, specifying with probability that no more than the proportion p_1 of the normal population is below $\bar{x}-ks$ and no more than the proportion p_2 is above $\bar{x}+ks$.

Accordingly, equation (17) is written in the following form for a set of lower $(0,R)$ and upper (R,∞) limits for two-sided noncentral t cumulative distribution function:

$$F(t,\delta;0,R) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{v}{2}\right) 2^{\frac{1}{2}(v-2)}} \int_0^R G\left[\frac{tu}{\sqrt{v}} - \delta\right] u^{v-1}\, G'(u)\, du \ , \tag{23}$$

and

$$F(t,\delta;R,\infty) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{v}{2}\right) 2^{\frac{1}{2}(v-2)}} \int_R^\infty G\left[\frac{tu}{\sqrt{v}} - \delta\right] u^{v-1}\, G'(u)\, du \ . \tag{24}$$

Thus, summation of equations (23) and (24) yields

$$Pr\{T_v \le t|\delta\} = F(t,\delta;0,R) = F(t,\delta;R,\infty) \ . \tag{25}$$

Method B (New Theory)

Dealing with a problem of solving the tolerance factor k based on the given probability and proportion after the iterative process for the two-sided tolerance limits has the following form,

$$\Phi(k) = \int_0^\infty \varphi(z)Q[w(z)]dz \ , \tag{26}$$

where

$$Q[w)z] = \frac{4\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{w(z)}^\infty x^{v-1} e^{-\frac{vx^2}{2}} dx \ , \tag{27}$$

is the chi distribution and

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \ , \tag{28}$$

is the standard normal probability density (zero mean and unit standard deviation). Derivation of the chi distribution is described in appendix A.

In equation (27), the lower limit of the integral, $w(z)$, needs to be determined. The probability is such that the quantities $(L-\mu)/\sigma$ and $(U-\mu)/\sigma$ exceed K_p with the proportion P being at least p, namely,

$$\text{Prob}\ \{G[(U-\mu)/\sigma]-G[(L-\mu)/\sigma] > p\} \ . \tag{29}$$

Equation (29) is evaluated by integrating the joint density of z and w over the shaded area in figure 3.

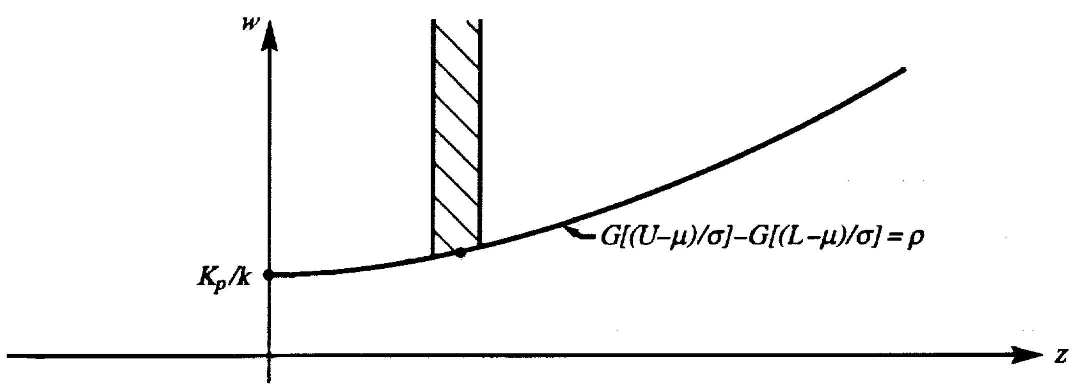


Figure 3. Area of integration for two-sided lower limits.

K_p/k is represented at the vertex on w-axis for the lower limit (parabolic curve). The area is integrated with respect to z, thus summing the area elements into a vertical strip extending from the bounding curve of the lower limit upwards. For any value of z, a strip is defined along the line $z = \sqrt{n}\, K_p - \sqrt{n}\, kw$ and another line $z = \sqrt{n}\, K_p + \sqrt{n}\, kw$. So, let

$$(U-\mu)/\sigma = K_p = \frac{z}{\sqrt{n}} + kw \ , \tag{30}$$

and

$$(L-\mu)/\sigma = K_p = \frac{z}{\sqrt{n}} - kw \ . \tag{31}$$

K_p is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{-K_p}^{K_p} \exp\left(-\frac{x^2}{2}\right) dx = p \ ,$$

and the values of K_p for one- and two-sided cases are tabulated in appendices B1 and B2.

Using equations (30) and (31), the function for the lower limit becomes

$$f(z,w) = G[(U-\mu)/\sigma] - G[(L-\mu)/\sigma] - p = 0 \ ,$$

or

$$f(z,w) = G\left(\frac{z}{\sqrt{n}} + kw\right) - G\left(\frac{z}{\sqrt{n}} - kw\right) - p = 0 \ . \tag{32}$$

Equation (32) is, for some function $f(z,w)$, the total or exact differential

$$df = \frac{\partial f}{\partial z}\, \mathrm{dz} + \frac{\partial f}{\partial w}\, \mathrm{dw} \ , \tag{33}$$

where $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial w}$ are the partial derivatives of f with respect to z and w, respectively. Then equation (32) may be written $df = 0$. Rearranging equation (33) to obtain a differential equation

$$\frac{dw}{dz} = -\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial w}} \ . \tag{34}$$

Thus,

$$\frac{\partial f}{\partial z} = \frac{1}{\sqrt{n}} \left[\frac{e^{-\left(\frac{z}{\sqrt{n}} + kw\right)^2/2} - e^{-\left(\frac{z}{\sqrt{n}} - kw\right)^2/2}}{\sqrt{2\pi}} \right] , \tag{35}$$

$$\frac{\partial f}{\partial w} = k \left[\frac{e^{-\left(\frac{z}{\sqrt{n}} + kw\right)^2/2} + e^{-\left(\frac{z}{\sqrt{n}} - kw\right)^2/2}}{\sqrt{2\pi}} \right] . \tag{36}$$

Substituting these values of $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial w}$ into equation (34) gives a final expression

$$\frac{dw}{dz} = \frac{1}{k\sqrt{n}}\left[\frac{e^{-\left(\frac{z}{\sqrt{n}}+kw\right)^2/2} - e^{-\left(\frac{z}{\sqrt{n}}-kw\right)^2/2}}{e^{-\left(\frac{z}{\sqrt{n}}+kw\right)^2/2} + e^{-\left(\frac{z}{\sqrt{n}}-kw\right)^2/2}}\right] . \tag{37}$$

The hyperbolic tangent function of u is defined by the formula

$$\tanh u = \frac{e^{-u} - e^{-u}}{e^{-u} + e^{-u}} . \tag{38}$$

Substitution of equation (38) yields the differential expression for the lower limit

$$\frac{dw}{dz} = \frac{1}{k\sqrt{n}} \tanh\left(\frac{z}{\sqrt{n}} kw\right) . \tag{39}$$

The methods for obtaining numerical solutions to differential equations and for computing the integrals by numerical methods in the computer codes reduce equation (39) to

$$w = w(z) . \tag{40}$$

The values of $w(z)$ are applied to the lower limit of the integral of equation (27) for computation. The computer code for this method is listed in appendix C2.

NUMERICAL RESULTS

A method for one-sided tolerance limits and another method for two-sided tolerance limits have been devised and enhanced, using a new theory, for the exact computation of tolerance factors k. Some tables of tolerance factors k have been generated to illustrate the exact results.

The k data are provided, for one-sided case, with the following parameters of all combinations:

Tables 1 through 3

$\gamma =$ 0.50, 0.75, 0.90

$p =$ 0.50, 0.55, 0.60, 0.65, 0.70, 0.75, 0.80, 0.8413, 0.85, 0.90, 0.95, 0.975, 0.9773, 0.99, 0.9987, 0.999, 0.9999, 0.999968, 0.99999

$n =$ 2(1)50

Tables 4 and 5

$\gamma =$ 0.95, 0.99

$p =$ 0.50, 0.75, 0.90, 0.95, 0.99, 0.999, 0.9999, 0.99999

$n =$ 2(1)10

and for two-sided case:

Tables 6 through 8

$\gamma =$ 0.50, 0.75, 0.90

$p =$ 0.50, 0.55, 0.60, 0.65, 0.6827, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 0.9545, 0.975, 0.99, 0.9973, 0.999, 0.9999, 0.999937, 0.99999

$n =$ 2(1)50

Tables 9 and 10

$\gamma =$ 0.95, 0.99

$p =$ 0.50, 0.75, 0.90, 0.95, 0.99, 0.999, 0.9999, 0.99999

$n = 2(1)10$.

Methods A and B for the one- and two-sided cases have involved several different algorithms needed for numerically integrating, differentiating, and iteratively root-solving the equations to obtain the exact solutions for the tolerance factors k. In method A, the Pegasus method with an estimated order of convergence superior to a secant method and the more accurate 20-point Gaussian quadrature procedure have been employed along with some other algorithms for numerical solutions.

A procedure for method B uses the functions of normal density, normal distribution and inverse normal distribution, calculation of factorial, secant method, Simpson method for numerical integration, and the third-order Runge-Kutta method for numerical solutions to differential equations. The Runge-Kutta method requires three derivative evaluations per step, which has an accumulated truncation error proportional to $(\Delta x)^3$. Those Pegasus and secant methods require input of good initial estimates in order to solve a root-finding problem $f(x) = 0$.

A comparison of approximate and exact results for the two-sided tolerance factor k for $n = 2$ and 10 is given in table 11 with confidence level γ and proportion p.

Based on table 11, the approximate data for $n = 2$ differ from the exact data by at most 3.18 percent. But as n increases, the error percentage decreases to less than 1 percent, as the $n = 10$ data can verify.

Table 1. One-sided *k*.

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma, \ \gamma = 0.50$$

n	*P* 0.50	*P* 0.55	*P* 0.60	*P* 0.65	*P* 0.70	*P* 0.75	*P* 0.80	*P* 0.8413	*P* 0.85	*P* 0.90
2	0.00001	0.15790	0.32039	0.49221	0.67900	0.88737	1.12701	1.36022	1.41447	1.78430
3	0.00000	0.14194	0.28673	0.43743	0.59798	0.77325	0.97066	1.15962	1.20325	1.49854
4	0.00000	0.13649	0.27546	0.41959	0.57246	0.73848	0.92454	1.10191	1.14279	1.41893
5	0.00000	0.13376	0.26985	0.41081	0.56002	0.72174	0.90260	1.07472	1.11435	1.38187
6	0.00000	0.13213	0.26650	0.40559	0.55268	0.71192	0.88981	1.05895	1.09787	1.36051
7	0.00000	0.13104	0.26429	0.40214	0.54784	0.70548	0.88145	1.04866	1.08713	1.34664
8	0.00000	0.13027	0.26271	0.39969	0.54441	0.70092	0.87555	1.04142	1.07958	1.33690
9	0.00000	0.12969	0.26153	0.39786	0.54186	0.69753	0.87117	1.03606	1.07398	1.32970
10	0.00000	0.12924	0.26062	0.39644	0.53988	0.69491	0.86779	1.03192	1.06966	1.32415
11	0.00000	0.12889	0.25988	0.39531	0.53830	0.69282	0.86510	1.02863	1.06624	1.31975
12	0.00000	0.12860	0.25929	0.39439	0.53702	0.69112	0.86291	1.02596	1.06345	1.31618
13	0.00000	0.12835	0.25879	0.39362	0.53595	0.68971	0.86110	1.02374	1.06114	1.31321
14	0.00000	0.12815	0.25838	0.39298	0.53505	0.68852	0.85957	1.02187	1.05919	1.31072
15	0.00000	0.12797	0.25802	0.39242	0.53428	0.68750	0.85826	1.02028	1.05753	1.30859
16	0.00000	0.12782	0.25771	0.39194	0.53361	0.68662	0.85713	1.01890	1.05609	1.30675
17	0.00000	0.12768	0.25744	0.39152	0.53303	0.68586	0.85614	1.01770	1.05484	1.30514
18	0.00000	0.12757	0.25720	0.39115	0.53252	0.68518	0.85527	1.01664	1.05373	1.30373
19	0.00000	0.12746	0.25698	0.39082	0.53206	0.68458	0.85450	1.01570	1.05275	1.30248
20	0.00000	0.12737	0.25679	0.39053	0.53165	0.68404	0.85381	1.01486	1.05188	1.30136
21	0.00000	0.12728	0.25662	0.39027	0.53129	0.68356	0.85320	1.01410	1.05109	1.30035
22	0.00000	0.12721	0.25647	0.39003	0.53096	0.68312	0.85264	1.01342	1.05038	1.29944
23	0.00000	0.12714	0.25633	0.38981	0.53066	0.68273	0.85213	1.01280	1.04974	1.29862
24	0.00000	0.12708	0.25620	0.38962	0.53038	0.68236	0.85166	1.01224	1.04915	1.29787
25	0.00000	0.12702	0.25608	0.38943	0.53013	0.68203	0.85124	1.01172	1.04861	1.29718
26	0.00000	0.12697	0.25598	0.38927	0.52990	0.68173	0.85085	1.01125	1.04812	1.29655
27	0.00000	0.12692	0.25588	0.38911	0.52969	0.68145	0.85049	1.01081	1.04766	1.29597
28	0.00000	0.12687	0.25578	0.38897	0.52949	0.68119	0.85016	1.01041	1.04724	1.29543
29	0.00000	0.12683	0.25570	0.38884	0.52931	0.68095	0.84985	1.01003	1.04685	1.29493
30	0.00000	0.12679	0.25562	0.38872	0.52914	0.68072	0.84956	1.00968	1.04648	1.29446
31	0.00000	0.12676	0.25554	0.38860	0.52898	0.68052	0.84930	1.00936	1.04614	1.29403
32	0.00000	0.12672	0.25547	0.38850	0.52883	0.68032	0.84904	1.00905	1.04583	1.29362
33	0.00000	0.12669	0.25541	0.38840	0.52869	0.68014	0.84881	1.00876	1.04553	1.29324
34	0.00000	0.12666	0.25535	0.38830	0.52856	0.67996	0.84859	1.00850	1.04525	1.29289
35	0.00000	0.12663	0.25529	0.38821	0.52844	0.67980	0.84838	1.00824	1.04499	1.29255
36	0.00000	0.12660	0.25524	0.38813	0.52832	0.67965	0.84819	1.00801	1.04474	1.29224
37	0.00000	0.12658	0.25518	0.38805	0.52821	0.67951	0.84800	1.00778	1.04451	1.29194
38	0.00000	0.12655	0.25514	0.38798	0.52811	0.67937	0.84783	1.00757	1.04428	1.29166
39	0.00000	0.12653	0.25509	0.38791	0.52801	0.67924	0.84766	1.00737	1.04408	1.29139
40	0.00000	0.12651	0.25505	0.38784	0.52792	0.67912	0.84751	1.00718	1.04388	1.29114
41	0.00000	0.12649	0.25501	0.38777	0.52783	0.67900	0.84736	1.00700	1.04369	1.29090
42	0.00000	0.12647	0.25497	0.38771	0.52774	0.67889	0.84722	1.00683	1.04351	1.29067
43	0.00000	0.12645	0.25493	0.38766	0.52766	0.67879	0.84708	1.00666	1.04334	1.29045
44	0.00000	0.12644	0.25489	0.38760	0.52759	0.67869	0.84696	1.00651	1.04318	1.29024
45	0.00000	0.12642	0.25486	0.38755	0.52752	0.67859	0.84683	1.00636	1.04302	1.29004
46	0.00000	0.12640	0.25483	0.38750	0.52745	0.67850	0.84672	1.00622	1.04288	1.28986
47	0.00000	0.12639	0.25480	0.38741	0.52738	0.67842	0.84661	1.00608	1.04274	1.28968
48	0.00000	0.12638	0.25477	0.38740	0.52732	0.67833	0.84650	1.00595	1.04260	1.28951
49	0.00000	0.12636	0.25474	0.38736	0.52726	0.67825	0.84640	1.00582	1.04247	1.28934
50	0.00000	0.12635	0.25471	0.38732	0.52720	0.67817	0.84630	1.00571	1.04234	1.28919

Table 1. One-sided k (continued).

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma, \ \gamma = 0.50$$

n	P 0.95	P 0.975	P 0.9773	P 0.99	P 0.9987	P 0.999	P 0.9999	P 0.999968	P 0.99999
2	2.33864	2.81972	2.88062	3.37605	4.39124	4.52664	5.46827	5.88811	6.28349
3	1.93835	2.32057	2.36906	2.76454	3.57922	3.68817	4.44703	4.78583	5.10513
4	1.82945	2.18620	2.23148	2.60088	3.36261	3.46454	4.17478	4.49202	4.79097
5	1.77922	2.12446	2.16828	2.52583	3.26338	3.36211	4.05008	4.35743	4.64708
6	1.75041	2.08911	2.13210	2.48290	3.20666	3.30354	3.97880	4.28047	4.56469
7	1.73174	2.06623	2.10869	2.45514	3.16998	3.26568	3.93271	4.23072	4.51181
8	1.71866	2.05023	2.09231	2.43572	3.14434	3.23922	3.90047	4.19592	4.47429
9	1.70900	2.03841	2.08022	2.42139	3.12541	3.21967	3.87669	4.17024	4.44703
10	1.70157	2.02932	2.07092	2.41038	3.11087	3.20466	3.85840	4.15050	4.42577
11	1.69569	2.02212	2.06356	2.40165	3.09934	3.19276	3.84392	4.13486	4.40926
12	1.69090	2.01628	2.05757	2.39456	3.08999	3.18310	3.83217	4.12216	4.39532
13	1.68694	2.01144	2.05262	2.38870	3.08224	3.17510	3.82246	4.11165	4.38459
14	1.68361	2.00736	2.04845	2.38376	3.07572	3.16837	3.81417	4.10280	4.37461
15	1.68076	2.00388	2.04489	2.37955	3.07016	3.16263	3.80723	4.09525	4.36681
16	1.67830	2.00088	2.04182	2.37591	3.06536	3.15767	3.80122	4.08873	4.36002
17	1.67616	1.99826	2.03914	2.37274	3.06117	3.15335	3.79587	4.08306	4.35342
18	1.67427	1.99596	2.03679	2.36995	3.05749	3.14955	3.79131	4.07805	4.34844
19	1.67260	1.99392	2.03470	2.36748	3.05422	3.14618	3.78717	4.07362	4.34389
20	1.67111	1.99210	2.03284	2.36527	3.05131	3.14317	3.78352	4.06967	4.33941
21	1.66977	1.99046	2.03116	2.36329	3.04869	3.14047	3.78022	4.06611	4.33551
22	1.66856	1.98899	2.02965	2.36150	3.04633	3.13803	3.77731	4.06304	4.33248
23	1.66746	1.98764	2.02828	2.35987	3.04419	3.13582	3.77456	4.06001	4.32903
24	1.66646	1.98642	2.02703	2.35839	3.04223	3.13380	3.77206	4.05736	4.32629
25	1.66554	1.98530	2.02588	2.35704	3.04044	3.13195	3.76988	4.05494	4.32373
26	1.66470	1.98428	2.02483	2.35579	3.03880	3.13026	3.76783	4.05273	4.32149
27	1.66392	1.98333	2.02386	2.35464	3.03728	3.12869	3.76587	4.05070	4.31892
28	1.66321	1.98245	2.02297	2.35358	3.03588	3.12724	3.76408	4.04884	4.31678
29	1.66254	1.98164	2.02214	2.35260	3.03458	3.12590	3.76251	4.04714	4.31523
30	1.66192	1.98088	2.02136	2.35168	3.03338	3.12466	3.76097	4.04559	4.31358
31	1.66134	1.98018	2.02064	2.35083	3.03225	3.12349	3.75954	4.04420	4.31166
32	1.66080	1.97952	2.01997	2.35003	3.03119	3.12240	3.75822	4.04297	4.30999
33	1.66030	1.97890	2.01933	2.34928	3.03021	3.12138	3.75699	4.04191	4.30879
34	1.65982	1.97832	2.01874	2.34858	3.02928	3.12043	3.75585	4.04104	4.30761
35	1.65937	1.97778	2.01818	2.34792	3.02841	3.11954	3.75474	4.04040	4.30619
36	1.65895	1.97726	2.01766	2.34730	3.02759	3.11869	3.75367	4.04001	4.30480
37	1.65856	1.97678	2.01716	2.34671	3.02681	3.11788	3.75270	4.03993	4.30375
38	1.65818	1.97632	2.01669	2.34616	3.02608	3.11712	3.75181	4.03546	4.30288
39	1.65782	1.97588	2.01625	2.34563	3.02539	3.11641	3.75095	4.03455	4.30189
40	1.65749	1.97548	2.01583	2.34514	3.02473	3.11573	3.75010	4.03369	4.30077
41	1.65717	1.97508	2.01543	2.34466	3.02410	3.11509	3.74930	4.03289	4.29978
42	1.65686	1.97471	2.01505	2.34421	3.02351	3.11447	3.74856	4.03216	4.29902
43	1.65657	1.97436	2.01469	2.34378	3.02295	3.11389	3.74787	4.03149	4.29832
44	1.65630	1.97402	2.01434	2.34338	3.02241	3.11333	3.74719	4.03090	4.29751
45	1.65604	1.97370	2.01402	2.34299	3.02190	3.11280	3.74653	4.03038	4.29666
46	1.65578	1.97340	2.01370	2.34262	3.02141	3.11230	3.74591	4.02996	4.29591
47	1.65554	1.97310	2.01340	2.34226	3.02094	3.11182	3.74533	4.02966	4.29530
48	1.65532	1.97282	2.01311	2.34193	3.02049	3.11135	3.74478	4.02948	4.29472
49	1.65509	1.97256	2.01285	2.34160	3.02007	3.11091	3.74424	4.02946	4.29408
50	1.65488	1.97229	2.01259	2.34128	3.01966	3.11057	3.74368	4.02963	4.29322

Table 2. One-sided k.

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma, \ \gamma = 0.75$$

n	P 0.50	P 0.55	P 0.60	P 0.65	P 0.70	P 0.75	P 0.80	P 0.8413	P 0.85
2	0.70712	0.94130	1.20291	1.49794	1.83480	2.22476	2.68605	3.14413	3.25169
3	0.47140	0.63929	0.81829	1.01191	1.22508	1.46434	1.74027	2.00940	2.07212
4	0.38245	0.53421	0.69375	0.86408	1.04941	1.25531	1.49074	1.71892	1.77194
5	0.33125	0.47584	0.62682	0.78696	0.96020	1.15166	1.36960	1.58010	1.62893
6	0.29667	0.43722	0.58338	0.73781	0.90428	1.08766	1.29581	1.49641	1.54290
7	0.27121	0.40917	0.55223	0.70301	0.86513	1.04333	1.24520	1.43944	1.48441
8	0.25143	0.38757	0.52849	0.67671	0.83580	1.01038	1.20785	1.39763	1.44155
9	0.23546	0.37028	0.50960	0.65594	0.81278	0.98468	1.17889	1.36536	1.40849
10	0.22222	0.35602	0.49412	0.63900	0.79411	0.96394	1.15562	1.33952	1.38205
11	0.21100	0.34399	0.48112	0.62484	0.77858	0.94675	1.13642	1.31827	1.36030
12	0.20134	0.33366	0.47000	0.61278	0.76539	0.93221	1.12023	1.30040	1.34203
13	0.19289	0.32468	0.46036	0.60235	0.75402	0.91971	1.10636	1.28511	1.32641
14	0.18543	0.31676	0.45189	0.59322	0.74409	0.90882	1.09429	1.27185	1.31286
15	0.17878	0.30972	0.44437	0.58513	0.73532	0.89922	1.08369	1.26021	1.30098
16	0.17280	0.30340	0.43764	0.57790	0.72750	0.89069	1.07427	1.24989	1.29044
17	0.16738	0.29768	0.43156	0.57140	0.72047	0.88302	1.06583	1.24066	1.28102
18	0.16244	0.29249	0.42605	0.56550	0.71411	0.87610	1.05822	1.23234	1.27254
19	0.15792	0.28773	0.42101	0.56012	0.70831	0.86981	1.05131	1.22480	1.26484
20	0.15376	0.28336	0.41639	0.55518	0.70301	0.86406	1.04500	1.21792	1.25783
21	0.14991	0.27933	0.41212	0.55064	0.69813	0.85877	1.03921	1.21162	1.25140
22	0.14633	0.27558	0.40817	0.54644	0.69362	0.85389	1.03388	1.20581	1.24548
23	0.14300	0.27210	0.40450	0.54254	0.68944	0.84937	1.02894	1.20044	1.24000
24	0.13989	0.26885	0.40108	0.53890	0.68555	0.84516	1.02434	1.19545	1.23492
25	0.13697	0.26580	0.39787	0.53550	0.68191	0.84124	1.02007	1.19080	1.23019
26	0.13423	0.26294	0.39487	0.53232	0.67851	0.83757	1.01607	1.18646	1.22577
27	0.13164	0.26025	0.39204	0.52932	0.67531	0.83413	1.01232	1.18240	1.22162
28	0.12920	0.25771	0.38937	0.52650	0.67231	0.83089	1.00879	1.17858	1.21773
29	0.12690	0.25531	0.38685	0.52384	0.66947	0.82783	1.00547	1.17498	1.21407
30	0.12471	0.25303	0.38447	0.52132	0.66678	0.82495	1.00233	1.17158	1.21061
31	0.12263	0.25087	0.38221	0.51893	0.66424	0.82221	0.99936	1.16837	1.20734
32	0.12065	0.24882	0.38006	0.51666	0.66182	0.81962	0.99655	1.16533	1.20424
33	0.11876	0.24686	0.37801	0.51450	0.65953	0.81716	0.99388	1.16244	1.20130
34	0.11696	0.24499	0.37606	0.51245	0.65734	0.81481	0.99134	1.15969	1.19851
35	0.11524	0.24321	0.37419	0.51048	0.65525	0.81258	0.98891	1.15708	1.19584
36	0.11359	0.24150	0.37241	0.50861	0.65326	0.81044	0.98660	1.15458	1.19330
37	0.11202	0.23987	0.37070	0.50681	0.65136	0.80840	0.98439	1.15220	1.19088
38	0.11050	0.23830	0.36907	0.50509	0.64953	0.80645	0.98228	1.14992	1.18856
39	0.10905	0.23680	0.36750	0.50344	0.64778	0.80458	0.98025	1.14773	1.18633
40	0.10765	0.23535	0.36599	0.50186	0.64610	0.80278	0.97831	1.14564	1.18420
41	0.10630	0.23396	0.36454	0.50033	0.64449	0.80105	0.97644	1.14363	1.18216
42	0.10501	0.23262	0.36315	0.49887	0.64294	0.79940	0.97465	1.14170	1.18019
43	0.10376	0.23133	0.36180	0.49746	0.64144	0.79780	0.97293	1.13984	1.17830
44	0.10255	0.23008	0.36050	0.49609	0.64000	0.79626	0.97127	1.13805	1.17648
45	0.10138	0.22888	0.35925	0.49478	0.63861	0.79478	0.96967	1.13633	1.17473
46	0.10026	0.22772	0.35804	0.49351	0.63727	079334	0.96812	1.13466	1.17304
47	0.09917	0.22659	0.35687	0.49229	0.63598	0.79196	0.96663	1.13306	1.17141
48	0.09811	0.22551	0.35574	0.49110	0.63472	0.79063	0.96519	1.13151	1.16983
49	0.09709	0.22445	0.35465	0.48996	0.63351	0.78933	0.96380	1.13001	1.16831
50	0.09610	0.22343	0.35359	0.48884	0.63234	0.78808	0.96245	1.12856	1.16684

Table 2. One-sided k (continued).

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma,\ \ \gamma = 0.75$$

n	P 0.90	P 0.95	P 0.975	P 0.9773	P 0.99	P 0.9987	P 0.999	P 0.9999	P 0.999968	P 0.99999
2	3.99264	5.12134	6.11237	6.23838	7.26896	9.38852	9.67251	11.65011	12.53334	13.36588
3	2.50123	3.15174	3.72465	3.79771	4.39598	5.63814	5.80496	6.96974	7.49107	7.98285
4	2.13373	2.68052	3.16171	3.22308	3.72580	4.77046	4.91083	5.89154	6.33036	6.74453
5	1.96161	2.46331	2.90438	2.96062	3.42128	4.37855	4.50719	5.40584	5.80826	6.18812
6	1.85923	2.33552	2.75387	2.80721	3.24395	4.15134	4.27327	5.12505	5.50647	5.86642
7	1.79021	2.25006	2.65367	2.70511	3.12630	4.00114	4.11868	4.93971	5.30749	5.65444
8	1.73994	2.18823	2.58143	2.63153	3.04171	3.89346	4.00788	4.80722	5.16516	5.50282
9	1.70138	2.14103	2.52645	2.57555	2.97748	3.81190	3.92398	4.70688	5.05736	5.38815
10	1.67067	2.10360	2.48296	2.53128	2.92676	3.74763	3.85788	4.62796	4.97275	5.29817
11	1.64550	2.07305	2.44752	2.49522	2.88551	3.69546	3.80424	4.56396	4.90406	5.22497
12	1.62442	2.04753	2.41799	2.46516	2.85118	3.65210	3.75964	4.51059	4.84703	5.16426
13	1.60644	2.02583	2.39291	2.43965	2.82207	3.61540	3.72194	4.46593	4.79886	5.11303
14	1.59089	2.00711	2.37130	2.41767	2.79701	3.58383	3.68947	4.42725	4.75630	5.06932
15	1.57727	1.99074	2.35244	2.39849	2.77516	3.55637	3.66122	4.39351	4.72138	5.03061
16	1.56522	1.97629	2.33581	2.38157	2.75591	3.53217	3.63637	4.36427	4.69084	4.99734
17	1.55446	1.96342	2.32100	2.36652	2.73879	3.51065	3.61426	4.33766	4.66147	4.96687
18	1.54479	1.95186	2.30772	2.35301	2.72344	3.49142	3.59448	4.31421	4.63628	4.94003
19	1.53603	1.94141	2.29572	2.34081	2.70959	3.47405	3.57665	4.29328	4.61384	4.91651
20	1.52806	1.93190	2.28482	2.32973	2.69701	3.45824	3.56045	4.27376	4.59305	4.89420
21	1.52076	1.92321	2.27486	2.31960	2.68552	3.44388	3.54566	4.25605	4.57422	4.87411
22	1.51404	1.91522	2.26571	2.31030	2.67498	3.43075	3.53220	4.24067	4.55736	4.85649
23	1.50783	1.90785	2.25727	2.30173	2.66526	3.41856	3.51966	4.22553	4.54156	4.83984
24	1.50208	1.90103	2.24946	2.29379	2.65626	3.40727	3.50804	4.21151	4.52612	4.82273
25	1.49673	1.89468	2.24221	2.28642	2.64792	3.39689	3.49739	4.19891	4.51306	4.80917
26	1.49173	1.88876	2.23544	2.27955	2.64014	3.38722	3.48744	4.18764	4.50072	4.79647
27	1.48705	1.88323	2.22912	2.27312	2.63286	3.37811	3.47811	4.17639	4.48862	4.78341
28	1.48266	1.87803	2.22319	2.26710	2.62605	3.36957	3.46932	4.16542	4.47698	4.77051
29	1.47853	1.87315	2.21762	2.26144	2.61964	3.36159	3.46113	4.15560	4.46686	4.75997
30	1.47463	1.86855	2.21237	2.25610	2.61362	3.35413	3.45348	4.14713	4.45756	4.75041
31	1.47095	1.86420	2.20742	2.25107	2.60793	3.34702	3.44624	4.13884	4.44916	4.74225
32	1.46746	1.86009	2.20273	2.24631	2.60254	3.34009	3.43927	4.13017	4.43940	4.73112
33	1.46415	1.85619	2.19828	2.24179	2.59744	3.33384	3.43260	4.12131	4.43005	4.72035
34	1.46101	1.85249	2.19406	2.23751	2.59261	3.32803	3.42658	4.11486	4.42326	4.71264
35	1.45801	1.84896	2.19005	2.23343	2.58800	3.32245	3.42082	4.10847	4.41631	4.70680
36	1.45516	1.84561	2.18623	2.22954	2.58362	3.31690	3.41530	4.10270	4.40956	4.69994
37	1.45243	1.84240	2.18259	2.22584	2.57944	3.31146	3.40978	4.09523	4.40192	4.69058
38	1.44983	1.83934	2.17910	2.22231	2.57544	3.30630	3.40465	4.08847	4.39463	4.68282
39	1.44734	1.83641	2.17577	2.21893	2.57163	3.30176	3.39968	4.08243	4.38873	4.67672
40	1.44495	1.83360	2.17258	2.21569	2.56798	3.29742	3.39519	4.07813	4.38369	4.67177
41	1.44265	1.83091	2.16952	2.21258	2.56449	3.29327	3.39083	4.07344	4.37854	4.66693
42	1.44045	1.82833	2.16659	2.20959	2.56111	3.28890	3.38655	4.06854	4.37260	4.66026
43	1.43834	1.82585	2.16377	2.20672	2.55788	3.28477	3.38227	4.06285	4.36750	4.65494
44	1.43630	1.82346	2.16105	2.20397	2.55476	3.28069	3.37822	4.05732	4.36147	4.64668
45	1.43434	1.82116	2.15844	2.20132	2.55178	3.27704	3.37440	4.05268	4.35725	4.64366
46	1.43244	1.81894	2.15592	2.19877	2.54891	3.27365	3.37085	4.04902	4.35258	4.63809
47	1.43062	1.81680	2.15350	2.19631	2.54614	3.27039	3.36743	4.04564	4.34874	4.63588
48	1.42886	1.81474	2.15116	2.19393	2.54346	3.26716	3.36404	4.04185	4.34392	4.62958
49	1.42715	1.81275	2.14889	2.19163	2.54086	3.26371	3.36061	4.03745	4.33988	4.62541
50	1.42551	1.81082	2.14670	2.18941	2.53836	3.26051	3.35739	4.03308	4.33533	4.61956

Table 3. One-sided k.

$$Pr\left\{T_v \leq k\sqrt{n} \mid K_p\sqrt{n}\right\} = \gamma,\ \ \gamma = 0.90$$

n	P 0.50	P 0.55	P 0.60	P 0.65	P 0.70	P 0.75	P 0.80	P 0.8413	P 0.85
2	2.17634	2.72056	3.34308	4.05662	4.88067	5.84256	6.98786	8.13722	8.40337
3	1.08866	1.33436	1.60253	1.89797	2.22801	2.60283	3.03934	3.46834	3.56868
4	0.81887	1.01215	1.21950	1.44461	1.69301	1.97226	2.29483	2.61005	2.68360
5	0.68567	0.85823	1.04166	1.23923	1.45576	1.69781	1.97609	2.24708	2.31021
6	0.60253	0.76411	0.93487	1.11786	1.31754	1.53989	1.79472	2.04227	2.09987
7	0.54418	0.69898	0.86192	1.03589	1.22512	1.43527	1.67554	1.90852	1.96269
8	0.50025	0.65046	0.80808	0.97592	1.15803	1.35986	1.59018	1.81319	1.86500
9	0.46561	0.61250	0.76626	0.92965	1.10659	1.30235	1.52543	1.74116	1.79125
10	0.43735	0.58173	0.73258	0.89258	1.06559	1.25673	1.47427	1.68444	1.73321
11	0.41373	0.55614	0.70470	0.86204	1.03195	1.21944	1.43261	1.63837	1.68610
12	0.39359	0.53442	0.68113	0.83632	1.00372	1.18825	1.39786	1.60005	1.64693
13	0.37615	0.51568	0.66086	0.81428	0.97960	1.16168	1.36835	1.56756	1.61373
14	0.36085	0.49930	0.64320	0.79512	0.95869	1.13871	1.34288	1.53958	1.58516
15	0.34728	0.48481	0.62762	0.77827	0.94035	1.11860	1.32064	1.51518	1.56024
16	0.33515	0.47188	0.61376	0.76330	0.92409	1.10081	1.30100	1.49366	1.53828
17	0.32421	0.46025	0.60131	0.74989	0.90955	1.08492	1.28349	1.47451	1.51874
18	0.31428	0.44971	0.59005	0.73779	0.89645	1.07064	1.26777	1.45733	1.50122
19	0.30521	0.44011	0.57981	0.72680	0.88457	1.05770	1.25355	1.44181	1.48539
20	0.29689	0.43131	0.57044	0.71676	0.87373	1.04591	1.24061	1.42770	1.47100
21	0.28921	0.42321	0.56183	0.70753	0.86379	1.03511	1.22877	1.41481	1.45786
22	0.28211	0.41571	0.55387	0.69902	0.85463	1.02518	1.21789	1.40296	1.44578
23	0.27550	0.40875	0.54649	0.69115	0.84616	1.01600	1.20785	1.39204	1.43465
24	0.26933	0.40227	0.53962	0.68382	0.83829	1.00748	1.19854	1.38192	1.42434
25	0.26357	0.39621	0.53321	0.67699	0.83096	0.99955	1.18988	1.37252	1.41476
26	0.25816	0.39053	0.52721	0.67060	0.82411	0.99215	1.18180	1.36375	1.40583
27	0.25307	0.38520	0.52157	0.66461	0.81769	0.98521	1.17424	1.35555	1.39748
28	0.24827	0.38017	0.51627	0.65897	0.81165	0.97870	1.16714	1.34786	1.38964
29	0.24373	0.37542	0.51126	0.65366	0.80596	0.97257	1.16047	1.34063	1.38228
30	0.23943	0.37093	0.50653	0.64863	0.80060	0.96678	1.15418	1.33382	1.37534
31	0.23536	0.36667	0.50204	0.64388	0.79552	0.96131	1.14823	1.32738	1.36879
32	0.23148	0.36262	0.49779	0.63937	0.79070	0.95613	1.14260	1.32129	1.36259
33	0.22779	0.35877	0.49374	0.63508	0.78613	0.95112	1.13726	1.31551	1.35671
34	0.22428	0.35510	0.48988	0.63100	0.78178	0.94654	1.13218	1.31003	1.35113
35	0.22092	0.35160	0.48621	0.62711	0.77764	0.94209	1.12735	1.30481	1.34581
36	0.21770	0.34825	0.48270	0.62340	0.77368	0.93784	1.12274	1.29984	1.34075
37	0.21462	0.34505	0.47934	0.61985	0.76991	0.93379	1.11835	1.29509	1.33592
38	0.21168	0.34198	0.47612	0.61645	0.76629	0.92991	1.11415	1.29056	1.33131
39	0.20884	0.33904	0.47303	0.61320	0.76283	0.92620	1.11013	1.28622	1.32690
40	0.20612	0.33621	0.47007	0.61007	0.75951	0.92263	1.10627	1.28206	1.32267
41	0.20351	0.33349	0.46722	0.60707	0.75632	0.91922	1.10257	1.27808	1.31861
42	0.20099	0.33087	0.46449	0.60418	0.75325	0.91594	1.09902	1.27425	1.31472
43	0.19856	0.32835	0.46185	0.60141	0.75030	0.91278	1.09561	1.27057	1.31098
44	0.19622	0.32592	0.45931	0.59873	0.74746	0.90974	1.09232	1.26703	1.30738
45	0.19396	0.32357	0.45686	0.59615	0.74472	0.90681	1.08916	1.26362	1.30391
46	0.19177	0.32131	0.45449	0.59366	0.74208	0.90398	1.08611	1.26034	1.30057
47	0.18966	0.31912	0.45221	0.59126	0.73953	0.90126	1.08316	1.25717	1.29735
48	0.18761	0.31700	0.44999	0.58893	0.73707	0.89862	1.08032	1.25411	1.29424
49	0.18563	0.31495	0.44786	0.58668	0.73469	0.89608	1.07757	1.25115	1.29123
50	0.18372	0.31297	0.44578	0.58450	0.73238	0.89362	1.07492	1.24830	1.28832

Table 3. One-sided k (continued).

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma,\ \gamma = 0.90$$

n	P 0.90	P 0.95	P 0.975	P 0.9773	P 0.99	P 0.9987	P 0.999	P 0.9999	P 0.999968	P 0.99999
2	10.25263	13.08970	15.58624	15.90419	18.50009	23.86296	24.58138	29.58651	31.82294	33.93159
3	4.25832	5.31132	6.24382	6.36300	7.34060	9.38023	9.65315	11.61123	12.57066	13.54674
4	3.18796	3.95645	4.63705	4.72408	5.43835	6.92852	7.12924	8.53312	9.16226	9.75668
5	2.74244	3.39974	3.98138	4.05574	4.66607	5.93969	6.11121	7.31101	7.84992	8.35726
6	2.49377	3.09179	3.62049	3.68806	4.24262	5.39963	5.55546	6.64551	7.13408	7.59641
7	2.33272	2.89372	3.38925	3.45257	3.97211	5.05574	5.20167	6.22244	6.68002	7.11184
8	2.21866	2.75421	3.22689	3.28728	3.78263	4.81548	4.95454	5.92730	6.36368	6.77410
9	2.13294	2.64982	3.10573	3.16396	3.64152	4.63696	4.77097	5.70841	6.12882	6.52611
10	2.06574	2.56830	3.01130	3.06787	3.53173	4.49835	4.62847	5.53861	5.94779	6.33201
11	2.01135	2.50254	2.93529	2.99053	3.44349	4.38710	4.51409	5.40210	5.80033	6.17552
12	1.96626	2.44818	2.87254	2.92670	3.37074	4.29554	4.42000	5.29022	5.68076	6.04943
13	1.92814	2.40233	2.81970	2.87296	3.30955	4.21861	4.34093	5.19587	5.57857	5.94073
14	1.89540	2.36304	2.77447	2.82697	3.25723	4.15295	4.27346	5.11610	5.49393	5.85073
15	1.86690	2.32891	2.73523	2.78707	3.21188	4.09597	4.21494	5.04642	5.41893	5.77042
16	1.84183	2.29893	2.70080	2.75206	3.17212	4.04629	4.16390	4.98587	5.35425	5.70191
17	1.81955	2.27234	2.67029	2.72104	3.13692	4.00213	4.11850	4.93202	5.29682	5.64149
18	1.79960	2.24856	2.64303	2.69333	3.10549	3.96280	4.07809	4.88375	5.24515	5.58591
19	1.78160	2.22714	2.61849	2.66839	3.07721	3.92748	4.04182	4.84158	5.19963	5.53804
20	1.76526	2.20771	2.59626	2.64580	3.05161	3.89547	4.00892	4.80210	5.15740	5.49293
21	1.75035	2.19001	2.57600	2.62521	3.02830	3.86635	3.97898	4.76665	5.11887	5.44959
22	1.73667	2.17378	2.55745	2.60636	3.00696	3.83976	3.95178	4.73441	5.08522	5.41644
23	1.72407	2.15884	2.54039	2.58902	2.98734	3.81530	3.92661	4.70445	5.05404	5.38395
24	1.71241	2.14504	2.52462	2.57301	2.96922	3.79264	3.90331	4.67636	5.02057	5.34950
25	1.70158	2.13223	2.51001	2.55816	2.95243	3.77177	3.88190	4.65123	4.99556	5.32192
26	1.69149	2.12031	2.49641	2.54434	2.93681	3.75237	3.86201	4.62802	4.97368	5.29677
27	1.68207	2.10918	2.48371	2.53144	2.92224	3.73418	3.84331	4.60601	4.94785	5.27071
28	1.67324	2.09875	2.47183	2.51937	2.90860	3.71718	3.82583	4.58465	4.92513	5.24427
29	1.66494	2.08896	2.46068	2.50805	2.89582	3.70132	3.80956	4.56575	4.90513	5.22500
30	1.65712	2.07975	2.45020	2.49739	2.88379	3.68640	3.79427	4.54825	4.88706	5.20546
31	1.64974	2.07107	2.44031	2.48735	2.87245	3.67227	3.77975	4.53141	4.86871	5.18785
32	1.64277	2.06285	2.43096	2.47786	2.86174	3.65892	3.76603	4.51384	4.84877	5.16441
33	1.63616	2.05508	2.42212	2.46887	2.85160	3.64636	3.75314	4.49798	4.83359	5.14914
34	1.62988	2.04770	2.41372	2.46035	2.84199	3.63446	3.74094	4.48502	4.81785	5.13313
35	1.62392	2.04069	2.40575	2.45225	2.83287	3.62312	3.72930	4.47207	4.80727	5.11794
36	1.61824	2.03401	2.39817	2.44455	2.82418	3.61236	3.71827	4.45854	4.79051	5.10119
37	1.61282	2.02765	2.39094	2.43721	2.81590	3.60203	3.70763	4.44478	4.77468	5.08605
38	1.60764	2.02158	2.38404	2.43020	2.80800	3.59217	3.69762	4.43156	4.76329	5.07383
39	1.60269	2.01577	2.37744	2.42350	2.80046	3.58293	3.68803	4.42211	4.75102	5.06045
40	1.59795	2.01021	2.37113	2.41710	2.79324	3.57396	3.67896	4.41152	4.73975	5.04962
41	1.59341	2.00488	2.36509	2.41096	2.78633	3.56537	3.67036	4.40170	4.72783	5.03632
42	1.58904	1.99977	2.35929	2.40507	2.77970	3.55715	3.66156	4.39075	4.71836	5.02528
43	1.58485	1.99487	2.35372	2.39942	2.77333	3.54929	3.65338	4.38058	4.70742	5.01493
44	1.58082	1.99015	2.34837	2.39398	2.76722	3.54172	3.64575	4.37180	4.69809	5.00274
45	1.57695	1.98561	2.34322	2.38876	2.76134	3.53443	3.63826	4.36384	4.68879	4.99546
46	1.57321	1.98124	2.33827	2.38373	2.75568	3.52740	3.63115	4.35571	4.68171	4.98802
47	1.56961	1.97702	2.33349	2.37887	2.75022	3.52065	3.62414	4.34706	4.67348	4.98060
48	1.56613	1.97296	2.32888	2.37419	2.74494	3.51415	3.61745	4.33942	4.66184	4.96883
49	1.56277	1.96903	2.32443	2.36967	2.73986	3.50788	3.61088	4.33144	4.65238	4.95529
50	1.55952	1.96523	2.32013	2.36531	2.73494	3.50182	3.60480	4.32221	4.64034	4.94138

Table 4. One-sided k.

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma,\ \gamma = 0.95$$

n	P 0.50	P 0.75	P 0.90	P 0.95	P 0.99	P 0.999	P 0.9999	P 0.99999
2	4.46459	11.76357	20.58168	26.25874	37.09454	49.27437	59.30125	68.00715
3	1.68586	3.80619	6.15526	7.65595	10.55273	13.85715	16.59796	18.98555
4	1.17668	2.61761	4.16193	5.14389	7.04235	9.21413	11.01895	12.59298
5	0.95339	2.14969	3.40663	4.20268	5.74110	7.50187	8.96599	10.24328
6	0.82264	1.89503	3.00626	3.70768	5.06197	6.61179	7.90051	9.02489
7	0.73445	1.73225	2.75543	3.39947	4.64172	6.06283	7.24412	8.27490
8	0.66984	1.61782	2.58191	3.18729	4.35386	5.68754	6.79623	7.76348
9	0.61985	1.53220	2.45376	3.03124	4.14302	5.41335	6.46926	7.39032
10	0.57968	1.46524	2.35464	2.91096	3.98112	5.20320	6.21886	7.10472

Table 5. One-sided k.

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma,\ \gamma = 0.99$$

n	P 0.50	P 0.75	P 0.90	P 0.95	P 0.99	P 0.999	P 0.9999	P 0.99999
2	22.50910	58.93765	103.00707	131.44022	185.62868	246.60432	296.71629	340.22161
3	4.02105	8.72819	13.99520	17.37020	23.89604	31.34765	37.53276	42.92185
4	2.27035	4.71519	7.37995	9.08343	12.38733	16.17570	19.32742	22.07792
5	1.67569	3.45411	5.36175	6.57833	8.93907	11.64932	13.90653	15.87733
6	1.37373	2.84809	4.41108	5.40554	7.33456	9.54992	11.39546	13.00732
7	1.18782	2.49072	3.85913	4.72786	6.41193	8.34573	9.95691	11.36407
8	1.05994	2.25337	3.49721	4.28525	5.81177	7.56416	9.02403	10.29907
9	0.96549	2.08314	3.24041	3.97226	5.38890	7.01440	8.36842	9.55092
10	0.89222	1.95433	3.04791	3.73832	5.07374	6.60539	7.88102	8.99499

Table 6. Two-sided *k*.

$$Pr\{T_v \leq k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma, \ \gamma = 0.50$$

n	*P* 0.50	*P* 0.55	*P* 0.60	*P* 0.65	*P* 0.6827	*P* 0.70	*P* 0.75	*P* 0.80	*P* 0.85
2	1.24272	1.38411	1.53324	1.69250	1.80367	1.86529	2.05670	2.27497	2.53508
3	0.94201	1.05228	1.16905	1.29422	1.38184	1.43049	1.58191	1.75510	1.96210
4	0.85225	0.95308	1.06009	1.17502	1.25561	1.30040	1.44000	1.59999	1.79158
5	0.80826	0.90436	1.00647	1.11628	1.19335	1.23621	1.36991	1.52332	1.70727
6	0.78196	0.87518	0.97431	1.08098	1.15590	1.19758	1.32766	1.47706	1.65635
7	0.76439	0.85568	0.95278	1.05733	1.13078	1.17165	1.29928	1.44593	1.62205
8	0.75181	0.84169	0.93732	1.04033	1.11272	1.15301	1.27884	1.42350	1.59729
9	0.74235	0.83116	0.92568	1.02751	1.09908	1.13893	1.26339	1.40652	1.57854
10	0.73496	0.82293	0.91658	1.01748	1.08842	1.12791	1.25129	1.39321	1.56383
11	0.72903	0.81633	0.90927	1.00942	1.07984	1.11905	1.24156	1.38250	1.55197
12	0.72417	0.81091	0.90327	1.00281	1.07280	1.11177	1.23355	1.37367	1.54219
13	0.72011	0.80639	0.89825	0.99727	1.06690	1.10567	1.22684	1.36628	1.53400
14	0.71666	0.80254	0.89399	0.99257	1.06189	1.10049	1.22114	1.35999	1.52702
15	0.71370	0.79924	0.89033	0.98852	1.05758	1.09604	1.21624	1.35458	1.52102
16	0.71114	0.79638	0.88715	0.98501	1.05384	1.09217	1.21197	1.34987	1.51579
17	0.70888	0.79386	0.88436	0.98193	1.05056	1.08877	1.20822	1.34573	1.51119
18	0.70689	0.79164	0.88190	0.97920	1.04765	1.08577	1.20491	1.34207	1.50712
19	0.70512	0.78966	0.87970	0.97678	1.04506	1.08309	1.20196	1.33881	1.50349
20	0.70353	0.78788	0.87773	0.97460	1.04274	1.08069	1.19931	1.33588	1.50024
21	0.70210	0.78629	0.87596	0.97264	1.04065	1.07852	1.19692	1.33324	1.49730
22	0.70080	0.78484	0.87435	0.97086	1.03875	1.07656	1.19475	1.33084	1.49463
23	0.69962	0.78353	0.87289	0.96924	1.03702	1.07477	1.19278	1.32866	1.49220
24	0.69854	0.78232	0.87155	0.96776	1.03544	1.07314	1.19098	1.32666	1.48997
25	0.69756	0.78122	0.87032	0.96640	1.03399	1.07164	1.18932	1.32482	1.48793
26	0.69664	0.78020	0.86919	0.96515	1.03266	1.07025	1.18779	1.32313	1.48604
27	0.69580	0.77926	0.86815	0.96399	1.03142	1.06898	1.18638	1.32157	1.48430
28	0.69502	0.77838	0.86718	0.96292	1.03028	1.06779	1.18507	1.32012	1.48268
29	0.69430	0.77757	0.86628	0.96192	1.02921	1.06669	1.18386	1.31877	1.48118
30	0.69362	0.77682	0.86544	0.96100	1.02822	1.06566	1.18272	1.31751	1.47978
31	0.69299	0.77612	0.86465	0.96013	1.02730	1.06471	1.18166	1.31634	1.47847
32	0.69240	0.77546	0.86392	0.95932	1.02643	1.06381	1.18067	1.31524	1.47724
33	0.69185	0.77484	0.86323	0.95855	1.02562	1.06297	1.17974	1.31421	1.47609
34	0.69133	0.77425	0.86259	0.95784	1.02485	1.06217	1.17886	1.31324	1.47500
35	0.69084	0.77371	0.86198	0.95716	1.02413	1.06143	1.17804	1.31232	1.47398
36	0.69037	0.77319	0.86140	0.95653	1.02345	1.06072	1.17726	1.31146	1.47302
37	0.68994	0.77270	0.86086	0.95592	1.02281	1.06006	1.17652	1.31064	1.47211
38	0.68952	0.77224	0.86034	0.95536	1.02220	1.05943	1.17583	1.30987	1.47124
39	0.68913	0.77180	0.85986	0.95482	1.02162	1.05883	1.17516	1.30914	1.47043
40	0.68876	0.77138	0.85939	0.95430	1.02107	1.05826	1.17454	1.30844	1.46965
41	0.68841	0.77099	0.85895	0.95382	1.02055	1.05772	1.17394	1.30778	1.46891
42	0.68807	0.77061	0.85854	0.95335	1.02006	1.05721	1.17338	1.30715	1.46821
43	0.68775	0.77025	0.85814	0.95291	1.01959	1.05672	1.17283	1.30655	1.46754
44	0.68744	0.76991	0.85776	0.95249	1.01914	1.05626	1.17232	1.30598	1.46690
45	0.68715	0.76958	0.85739	0.95209	1.01871	1.05581	1.17182	1.30543	1.46628
46	0.68687	0.76927	0.85705	0.95170	1.01830	1.05539	1.17135	1.30491	1.46570
47	0.68660	0.76897	0.85671	0.95133	1.01790	1.05498	1.17090	1.30441	1.46514
48	0.68635	0.76869	0.85640	0.95098	1.01752	1.05459	1.17047	1.30393	1.46461
49	0.68610	0.76841	0.85609	0.95064	1.01716	1.05422	1.17006	1.30347	1.46409
50	0.68587	0.76815	0.85580	0.95032	1.01682	1.05386	1.16966	1.30303	1.46360

Table 6. Two-sided k (continued).

$$Pr\{T_v \leq k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma, \ \ \gamma = 0.50$$

n	P 0.90	P 0.95	P 0.9545	P 0.975	P 0.99	P 0.9973	P 0.999	P 0.9999	P 0.999937	P 0.99999
2	2.86935	3.37560	3.43942	3.82222	4.34762	5.00759	5.45637	6.37697	6.54406	7.17947
3	2.22891	2.63423	2.68542	2.99276	3.41535	3.94709	4.30910	5.05242	5.18743	5.70098
4	2.03901	2.41574	2.46338	2.74961	3.14369	3.64018	3.97847	4.67364	4.79995	5.28058
5	1.94517	2.30796	2.35388	2.62992	3.01036	3.49011	3.81723	4.48982	4.61208	5.07736
6	1.88846	2.24282	2.28769	2.55762	2.92991	3.39975	3.72028	4.37964	4.49953	4.95588
7	1.85021	2.19884	2.24301	2.50881	2.87562	3.33882	3.65497	4.30557	4.42389	4.87436
8	1.82257	2.16702	2.21068	2.47347	2.83631	3.29472	3.60771	4.25203	4.36924	4.81551
9	1.80161	2.14286	2.18613	2.44662	2.80642	3.26119	3.57178	4.21137	4.32774	4.77085
10	1.78514	2.12386	2.16682	2.42548	2.78288	3.23476	3.54348	4.17934	4.29504	4.73569
11	1.77185	2.10850	2.15121	2.40838	2.76382	3.21337	3.52055	4.15339	4.26857	4.70723
12	1.76089	2.09582	2.13831	2.39425	2.74806	3.19566	3.50157	4.13192	4.24666	4.68367
13	1.75170	2.08516	2.12748	2.38237	2.73479	3.18075	3.48558	4.11383	4.22819	4.66382
14	1.74386	2.07608	2.11824	2.37223	2.72347	3.16801	3.47192	4.09836	4.21241	4.64686
15	1.73711	2.06824	2.11026	2.36347	2.71368	3.15699	3.46010	4.08497	4.19875	4.63217
16	1.73123	2.06140	2.10331	2.35583	2.70513	3.14736	3.44977	4.07326	4.18680	4.61933
17	1.72605	2.05538	2.09719	2.34910	2.69760	3.13887	3.44066	4.06293	4.17625	4.60799
18	1.72147	2.05005	2.09176	2.34313	2.69091	3.13132	3.43255	4.05374	4.16687	4.59790
19	1.71738	2.04528	2.08692	2.33779	2.68493	3.12457	3.42530	4.04551	4.15847	4.58886
20	1.71371	2.04100	2.08256	2.33299	2.67954	3.11849	3.41877	4.03810	4.15090	4.58072
21	1.71039	2.03713	2.07862	2.32865	2.67468	3.11299	3.41285	4.03138	4.14404	4.57334
22	1.70738	2.03362	2.07504	2.32471	2.67025	3.10798	3.40747	4.02526	4.13780	4.56662
23	1.70464	2.03041	2.07178	2.32111	2.66621	3.10341	3.40255	4.01967	4.13208	4.56047
24	1.70212	2.02748	2.06879	2.31781	2.66250	3.09921	3.39803	4.01453	4.12684	4.55482
25	1.69981	2.02477	2.06604	2.31478	2.65909	3.09534	3.39387	4.00979	4.12200	4.54961
26	1.69768	2.02228	2.06351	2.31198	2.65594	3.09177	3.39003	4.00542	4.11753	4.54479
27	1.69571	2.01998	2.06116	2.30938	2.65302	3.08846	3.38646	4.00136	4.11338	4.54032
28	1.69389	2.01784	2.05898	2.30698	2.65031	3.08539	3.38315	3.99758	4.10952	4.53616
29	1.69218	2.01584	2.05695	2.30473	2.64778	3.08252	3.38006	3.99405	4.10592	4.53228
30	1.69060	2.01398	2.05506	2.30264	2.64542	3.07984	3.37717	3.99076	4.10256	4.52866
31	1.68912	2.01225	2.05329	2.30068	2.64321	3.07733	3.37447	3.98767	4.09941	4.52525
32	1.68773	2.01062	2.05163	2.29884	2.64114	3.07498	3.37193	3.98477	4.09644	4.52206
33	1.68642	2.00908	2.05007	2.29712	2.63920	3.07277	3.36954	3.98205	4.09366	4.51905
34	1.68519	2.00764	2.04860	2.29550	2.63736	3.07068	3.36729	3.97947	4.09103	4.51621
35	1.68404	2.00629	2.04722	2.29396	2.63563	3.06871	3.36517	3.97704	4.08855	4.51353
36	1.68294	2.00500	2.04591	2.29252	2.63400	3.06685	3.36316	3.97475	4.08620	4.51100
37	1.68191	2.00379	2.04468	2.29115	2.63245	3.06509	3.36126	3.97257	4.08397	4.50860
38	1.68093	2.00264	2.04350	2.28985	2.63098	3.06342	3.35946	3.97051	4.08186	4.50632
39	1.68001	2.00155	2.04239	2.28862	2.62959	3.06184	3.35774	3.96854	4.07986	4.50415
40	1.67913	2.00051	2.04134	2.28745	2.62827	3.06033	3.35611	3.96668	4.07795	4.50209
41	1.67829	1.99953	2.04033	2.28634	2.62701	3.05889	3.35456	3.96490	4.07613	4.50012
42	1.67749	1.99859	2.03938	2.28528	2.62581	3.05752	3.35308	3.96320	4.07440	4.49825
43	1.67673	1.99769	2.03846	2.28426	2.62467	3.05622	3.35167	3.96158	4.07274	4.49646
44	1.67600	1.99684	2.03760	2.28330	2.62358	3.05497	3.35032	3.96004	4.07116	4.49474
45	1.67531	1.99602	2.03676	2.28238	2.62253	3.05378	3.34903	3.95856	4.06964	4.49311
46	1.67465	1.99524	2.03597	2.28149	2.62153	3.05264	3.34780	3.95714	4.06819	4.49154
47	1.67401	1.99449	2.03521	2.28065	2.62057	3.05154	3.34661	3.95578	4.06680	4.49003
48	1.67340	1.99378	2.03448	2.27984	2.61965	3.05049	3.34548	3.95447	4.06547	4.48859
49	1.67282	1.99309	2.03378	2.27906	2.61877	3.04949	3.34439	3.95322	4.06419	4.48720
50	1.67226	1.99243	2.03310	2.27831	2.61793	3.04852	3.34334	3.95202	4.06296	4.48586

Table 7. Two-sided k.

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma,\ \ \gamma = 0.75$$

n	P 0.50	P 0.55	P 0.60	P 0.65	P 0.6827	P 0.70	P 0.75	P 0.80	P 0.85
2	2.67326	2.97234	3.28759	3.62408	3.85888	3.98902	4.39315	4.85395	5.40305
3	1.49123	1.66318	1.84499	2.03960	2.17568	2.25121	2.48613	2.75463	3.07535
4	1.21074	1.35258	1.50287	1.66407	1.77698	1.83969	2.03501	2.25863	2.52619
5	1.08297	1.21089	1.34664	1.49246	1.59470	1.65152	1.82866	2.03173	2.27500
6	1.00899	1.12874	1.25595	1.39271	1.48868	1.54205	1.70851	1.89952	2.12855
7	0.96038	1.07469	1.19620	1.32693	1.41872	1.46977	1.62911	1.81206	2.03160
8	0.92580	1.03621	1.15362	1.28000	1.36877	1.41816	1.57235	1.74949	1.96218
9	0.89984	1.00729	1.12160	1.24468	1.33116	1.37929	1.52957	1.70228	1.90975
10	0.87957	0.98470	1.09656	1.21705	1.30172	1.34885	1.49604	1.66526	1.86860
11	0.86326	0.96651	1.07639	1.19477	1.27798	1.32429	1.46898	1.63536	1.83533
12	0.84982	0.95152	1.05976	1.17639	1.25838	1.30402	1.44662	1.61064	1.80782
13	0.83854	0.93892	1.04578	1.16094	1.24190	1.28698	1.42781	1.58983	1.78464
14	0.82891	0.92817	1.03385	1.14774	1.22782	1.27241	1.41173	1.57203	1.76481
15	0.82059	0.91888	1.02353	1.13633	1.21564	1.25981	1.39782	1.55662	1.74763
16	0.81332	0.91076	1.01451	1.12634	1.20499	1.24878	1.38563	1.54313	1.73258
17	0.80690	0.90359	1.00654	1.11752	1.19558	1.23904	1.37487	1.53120	1.71926
18	0.80119	0.89720	0.99945	1.10967	1.18719	1.23036	1.36528	1.52056	1.70740
19	0.79607	0.89148	0.99309	1.10263	1.17967	1.22258	1.35667	1.51102	1.69674
20	0.79145	0.88632	0.98735	1.09627	1.17288	1.21555	1.34890	1.50240	1.68711
21	0.78725	0.88163	0.98214	1.09050	1.16672	1.20917	1.34184	1.49457	1.67836
22	0.78343	0.87736	0.97739	1.08524	1.16110	1.20334	1.33540	1.48742	1.67037
23	0.77992	0.87344	0.97303	1.08041	1.15594	1.19800	1.32949	1.48086	1.66304
24	0.77670	0.86983	0.96902	1.07596	1.15119	1.19308	1.32404	1.47482	1.65628
25	0.77372	0.86650	0.96531	1.07185	1.14680	1.18854	1.31901	1.46923	1.65003
26	0.77096	0.86341	0.96187	1.06804	1.14273	1.18432	1.31435	1.46405	1.64424
27	0.76839	0.86053	0.95868	1.06450	1.13894	1.18040	1.31001	1.45923	1.63884
28	0.76599	0.85785	0.95569	1.06119	1.13541	1.17674	1.30595	1.45473	1.63381
29	0.76375	0.85534	0.95290	1.05810	1.13210	1.17331	1.30216	1.45052	1.62909
30	0.76164	0.85299	0.95028	1.05520	1.12900	1.17010	1.29860	1.44656	1.62466
31	0.75967	0.85078	0.94782	1.05247	1.12608	1.16708	1.29526	1.44284	1.62050
32	0.75780	0.84869	0.94551	1.04990	1.12333	1.16424	1.29210	1.43934	1.61658
33	0.75605	0.84673	0.94332	1.04747	1.12074	1.16155	1.28913	1.43603	1.61288
34	0.75438	0.84487	0.94125	1.04517	1.11828	1.15901	1.28631	1.43290	1.60937
35	0.75281	0.84310	0.93928	1.04300	1.11596	1.15660	1.28364	1.42994	1.60605
36	0.75131	0.84143	0.93742	1.04093	1.11375	1.15431	1.28111	1.42712	1.60289
37	0.74989	0.83984	0.93565	1.03897	1.11165	1.15214	1.27870	1.42444	1.59989
38	0.74854	0.83833	0.93397	1.03710	1.10965	1.15006	1.27640	1.42189	1.59704
39	0.74725	0.83688	0.93236	1.03532	1.10775	1.14809	1.27422	1.41946	1.59431
40	0.74602	0.83551	0.93083	1.03362	1.10593	1.14621	1.27213	1.41714	1.59171
41	0.74484	0.83419	0.92936	1.03199	1.10419	1.14441	1.27013	1.41492	1.58922
42	0.74371	0.83293	0.92796	1.03043	1.10253	1.14268	1.26822	1.41279	1.58684
43	0.74264	0.83172	0.92661	1.02894	1.10093	1.14103	1.26639	1.41076	1.58456
44	0.74160	0.83056	0.92532	1.02751	1.09940	1.13944	1.26463	1.40880	1.58237
45	0.74061	0.82945	0.92408	1.02614	1.09793	1.13792	1.26295	1.40693	1.58027
46	0.73965	0.82838	0.92290	1.02482	1.09652	1.13646	1.26133	1.40512	1.57825
47	0.73873	0.82735	0.92175	1.02355	1.09517	1.13506	1.25977	1.40339	1.57630
48	0.73785	0.82636	0.92065	1.02233	1.09386	1.13370	1.25827	1.40172	1.57443
49	0.73700	0.82541	0.91959	1.02115	1.09260	1.13240	1.25682	1.40011	1.57263
50	0.73618	0.82449	0.91856	1.02001	1.09138	1.13114	1.25542	1.39856	1.57089

Table 7. Two-sided k (continued).

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma, \ \ \gamma = 0.75$$

n	P 0.90	P 0.95	P 0.9545	P 0.975	P 0.99	P 0.9973	P 0.999	P 0.9999	P 0.999937	P 0.99999
2	6.10874	7.17761	7.31238	8.12079	9.23062	10.62509	11.57354	13.51957	13.87284	15.21637
3	3.48853	4.11606	4.19530	4.67110	5.32538	6.14882	6.70953	7.86116	8.07036	8.86625
4	2.87148	3.39687	3.46328	3.86232	4.41166	5.10377	5.57542	6.54476	6.72091	7.39126
5	2.58936	3.06836	3.12896	3.49324	3.99516	4.62804	5.05957	5.94693	6.10824	6.72220
6	2.42481	2.87673	2.93394	3.27798	3.75234	4.35088	4.75918	5.59912	5.75185	6.33323
7	2.31579	2.74969	2.80464	3.13524	3.59132	4.16711	4.56004	5.36866	5.51573	6.07562
8	2.23766	2.65856	2.71189	3.03280	3.47573	4.03517	4.41708	5.20324	5.34625	5.89076
9	2.17860	2.58961	2.64170	2.95525	3.38817	3.93521	4.30876	5.07790	5.21784	5.75070
10	2.13220	2.53538	2.58650	2.89422	3.31924	3.85647	4.22342	4.97914	5.11666	5.64034
11	2.09467	2.49148	2.54179	2.84477	3.26335	3.79261	4.15418	4.89900	5.03455	5.55078
12	2.06360	2.45510	2.50475	2.80377	3.21699	3.73960	4.09670	4.83245	4.96636	5.47640
13	2.03740	2.42439	2.47348	2.76914	3.17782	3.69479	4.04810	4.77615	4.90868	5.41347
14	2.01498	2.39809	2.44670	2.73946	3.14422	3.65633	4.00638	4.72781	4.85914	5.35941
15	1.99554	2.37526	2.42344	2.71369	3.11502	3.62290	3.97010	4.68576	4.81605	5.31239
16	1.97849	2.35524	2.40305	2.69107	3.08939	3.59354	3.93823	4.64879	4.77817	5.27104
17	1.96341	2.33751	2.38499	2.67104	3.06667	3.56750	3.90995	4.61599	4.74455	5.23434
18	1.94996	2.32169	2.36887	2.65315	3.04638	3.54422	3.88468	4.58666	4.71449	5.20151
19	1.93788	2.30747	2.35439	2.63706	3.02812	3.52328	3.86192	4.56024	4.68741	5.17194
20	1.92696	2.29461	2.34128	2.62251	3.01160	3.50431	3.84131	4.53629	4.66287	5.14513
21	1.91704	2.28291	2.32936	2.60926	2.99655	3.48703	3.82254	4.51448	4.64050	5.12070
22	1.90797	2.27222	2.31847	2.59716	2.98279	3.47122	3.80535	4.49450	4.62002	5.09832
23	1.89964	2.26240	2.30846	2.58603	2.97015	3.45669	3.78955	4.47613	4.60119	5.07773
24	1.89197	2.25335	2.29924	2.57577	2.95848	3.44328	3.77496	4.45916	4.58379	5.05872
25	1.88488	2.24498	2.29070	2.56628	2.94768	3.43086	3.76145	4.44344	4.56767	5.04109
26	1.87829	2.23720	2.28278	2.55746	2.93765	3.41931	3.74889	4.42881	4.55268	5.02469
27	1.87216	2.22996	2.27540	2.54924	2.92830	3.40855	3.73717	4.41517	4.53869	5.00940
28	1.86643	2.22320	2.26850	2.54157	2.91956	3.39849	3.72622	4.40242	4.52561	4.99509
29	1.86107	2.21686	2.26204	2.53438	2.91137	3.38906	3.71595	4.39045	4.51334	4.98166
30	1.85604	2.21091	2.25598	2.52763	2.90368	3.38020	3.70631	4.37921	4.50181	4.96904
31	1.85131	2.20532	2.25028	2.52127	2.89644	3.37185	3.69722	4.36861	4.49094	4.95715
32	1.84685	2.20004	2.24490	2.51528	2.88960	3.36398	3.68864	4.35861	4.48068	4.94592
33	1.84263	2.19505	2.23981	2.50961	2.88314	3.35653	3.68053	4.34915	4.47098	4.93530
34	1.83864	2.19033	2.23500	2.50425	2.87703	3.34948	3.67285	4.34018	4.46178	4.92523
35	1.83486	2.18586	2.23044	2.49916	2.87123	3.34279	3.66556	4.33168	4.45305	4.91567
36	1.83127	2.18161	2.22610	2.49433	2.86572	3.33643	3.65862	4.32358	4.44475	4.90658
37	1.82786	2.17756	2.22198	2.48973	2.86047	3.33038	3.65203	4.31588	4.43685	4.89793
38	1.82460	2.17371	2.21805	2.48535	2.85547	3.32461	3.64574	4.30853	4.42931	4.88967
39	1.82150	2.17004	2.21431	2.48117	2.85070	3.31910	3.63974	4.30152	4.42212	4.88179
40	1.81854	2.16652	2.21073	2.47718	2.84615	3.31384	3.63400	4.29482	4.41524	4.87425
41	1.81571	2.16317	2.20730	2.47336	2.84179	3.30881	3.62851	4.28840	4.40866	4.86703
42	1.81300	2.15995	2.20403	2.46971	2.83761	3.30398	3.62324	4.28225	4.40234	4.86012
43	1.81040	2.15687	2.20088	2.46620	2.83361	3.29936	3.61820	4.27635	4.39629	4.85348
44	1.80790	2.15392	2.19787	2.46284	2.82977	3.29492	3.61336	4.27069	4.39048	4.84711
45	1.80551	2.15108	2.19498	2.45961	2.82608	3.29065	3.60870	4.26524	4.38489	4.84098
46	1.80321	2.14835	2.19219	2.45650	2.82252	3.28655	3.60422	4.26000	4.37952	4.83509
47	1.80099	2.14572	2.18951	2.45351	2.81911	3.28260	3.59991	4.25496	4.37434	4.82942
48	1.79886	2.14319	2.18693	2.45063	2.81582	3.27879	3.59576	4.25010	4.36935	4.82394
49	1.79680	2.14075	2.18444	2.44785	2.81264	3.27512	3.59175	4.24541	4.36454	4.81867
50	1.79482	2.13840	2.18205	2.44517	2.80958	3.27158	3.58789	4.24088	4.35990	4.81357

Table 8. Two-sided k.

$$Pr\{T_v \leq k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma, \ \gamma = 0.90$$

n	P 0.50	P 0.55	P 0.60	P 0.65	P 0.6827	P 0.70	P 0.75	P 0.80	P 0.85
2	6.80826	7.56607	8.36478	9.21730	9.81216	10.14185	11.16572	12.33317	13.72437
3	2.49185	2.77604	3.07632	3.39762	3.62223	3.74686	4.13449	4.57746	5.10651
4	1.76562	1.97041	2.18721	2.41956	2.58221	2.67253	2.95372	3.27554	3.66046
5	1.47271	1.64535	1.82837	2.02478	2.16241	2.23888	2.47715	2.75015	3.07703
6	1.31356	1.46857	1.63308	1.80981	1.93375	2.00264	2.21743	2.46374	2.75894
7	1.21299	1.35677	1.50946	1.67362	1.78882	1.85288	2.05269	2.28198	2.55697
8	1.14337	1.27929	1.42372	1.57910	1.68818	1.74885	1.93818	2.15557	2.41643
9	1.09212	1.22220	1.36051	1.50935	1.61388	1.67204	1.85358	2.06211	2.31246
10	1.05269	1.17826	1.31181	1.45559	1.55659	1.61280	1.78828	1.98993	2.23212
11	1.02133	1.14329	1.27304	1.41276	1.51093	1.56557	1.73620	1.93233	2.16796
12	0.99574	1.11474	1.24136	1.37775	1.47360	1.52695	1.69358	1.88517	2.11541
13	0.97442	1.09094	1.21495	1.34854	1.44245	1.49472	1.65800	1.84578	2.07149
14	0.95634	1.07076	1.19255	1.32376	1.41601	1.46736	1.62779	1.81231	2.03416
15	0.94080	1.05341	1.17328	1.30244	1.39325	1.44381	1.60177	1.78348	2.00198
16	0.92729	1.03831	1.15650	1.28388	1.37344	1.42330	1.57910	1.75836	1.97393
17	0.91541	1.02504	1.14175	1.26755	1.35600	1.40525	1.55915	1.73624	1.94922
18	0.90487	1.01326	1.12867	1.25306	1.34053	1.38923	1.54144	1.71659	1.92727
19	0.89545	1.00273	1.11697	1.24010	1.32669	1.37490	1.52559	1.69901	1.90762
20	0.88698	0.99326	1.10643	1.22843	1.31422	1.36200	1.51131	1.68316	1.88991
21	0.87930	0.98468	1.09689	1.21786	1.30294	1.35031	1.49838	1.66881	1.87386
22	0.87232	0.97687	1.08821	1.20823	1.29265	1.33966	1.48659	1.65572	1.85922
23	0.86593	0.96972	1.08026	1.19942	1.28324	1.32992	1.47581	1.64374	1.84582
24	0.86005	0.96316	1.07295	1.19133	1.27459	1.32096	1.46589	1.63273	1.83350
25	0.85464	0.95710	1.06621	1.18386	1.26660	1.31269	1.45673	1.62256	1.82212
26	0.84962	0.95148	1.05997	1.17694	1.25921	1.30503	1.44825	1.61314	1.81157
27	0.84496	0.94627	1.05417	1.17050	1.25234	1.29791	1.44037	1.60438	1.80176
28	0.84062	0.94141	1.04876	1.16451	1.24593	1.29127	1.43302	1.59621	1.79262
29	0.83656	0.93687	1.04371	1.15891	1.23994	1.28507	1.42614	1.58857	1.78406
30	0.83275	0.93261	1.03897	1.15365	1.23432	1.27925	1.41970	1.58141	1.77604
31	0.82918	0.92862	1.03452	1.14872	1.22905	1.27379	1.41365	1.57468	1.76851
32	0.82582	0.92485	1.03034	1.14407	1.22408	1.26865	1.40795	1.56835	1.76141
33	0.82264	0.92130	1.02638	1.13969	1.21940	1.26379	1.40257	1.56237	1.75471
34	0.81964	0.91794	1.02265	1.13555	1.21497	1.25920	1.39748	1.55671	1.74837
35	0.81680	0.91477	1.01911	1.13162	1.21077	1.25486	1.39267	1.55136	1.74237
36	0.81411	0.91175	1.01575	1.12790	1.20679	1.25073	1.38810	1.54627	1.73668
37	0.81155	0.90888	1.01256	1.12436	1.20301	1.24681	1.38375	1.54144	1.73126
38	0.80911	0.90616	1.00953	1.12099	1.19941	1.24308	1.37962	1.53684	1.72611
39	0.80679	0.90356	1.00663	1.11778	1.19598	1.23952	1.37568	1.53246	1.72119
40	0.80457	0.90108	1.00387	1.11472	1.19270	1.23613	1.37191	1.52827	1.71650
41	0.80246	0.89871	1.00124	1.11179	1.18957	1.23289	1.36832	1.52427	1.71202
42	0.80043	0.89644	0.99871	1.10899	1.18657	1.22978	1.36488	1.52045	1.70772
43	0.79849	0.89427	0.99629	1.10631	1.18370	1.22681	1.36158	1.51678	1.70361
44	0.79663	0.89219	0.99397	1.10373	1.18095	1.22396	1.35842	1.51326	1.69967
45	0.79484	0.89019	0.99175	1.10126	1.17831	1.22122	1.35539	1.50989	1.69589
46	0.79313	0.88827	0.98961	1.09889	1.17577	1.21859	1.35247	1.50664	1.69225
47	0.79148	0.88642	0.98755	1.09661	1.17333	1.21606	1.34967	1.50352	1.68875
48	0.78989	0.88464	0.98557	1.09441	1.17098	1.21363	1.34697	1.50052	1.68538
49	0.78836	0.88293	0.98366	1.09229	1.16872	1.21128	1.34436	1.49762	1.68213
50	0.78688	0.88128	0.98182	1.09025	1.16653	1.20902	1.34186	1.49483	1.67900

Table 8. Two-sided k (continued).

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma,\ \gamma = 0.90$$

n	P 0.90	P 0.95	P 0.9545	P 0.975	P 0.99	P 0.9973	P 0.999	P 0.9999	P 0.999937	P 0.99999
2	15.51241	18.22084	18.56234	20.61097	23.42361	26.95790	29.36192	34.29474	35.19027	38.59612
3	5.78809	6.82328	6.95401	7.73901	8.81863	10.17758	11.10308	13.00424	13.34964	14.66376
4	4.15709	4.91267	5.00818	5.58205	6.37213	7.36767	8.04617	9.44089	9.69438	10.65908
5	3.49927	4.14247	4.22383	4.71291	5.38677	6.23652	6.81597	8.00764	8.22430	9.04894
6	3.14058	3.72253	3.79619	4.23910	4.84973	5.62020	6.14581	7.22717	7.42381	8.17239
7	2.91276	3.45574	3.52449	3.93805	4.50850	5.22863	5.72007	6.73145	6.91541	7.61576
8	2.75414	3.26988	3.33521	3.72828	4.27069	4.95572	5.42336	6.38599	6.56110	7.22788
9	2.63673	3.13223	3.19501	3.57285	4.09445	4.75343	5.20339	6.12987	6.29844	6.94032
10	2.54594	3.02571	3.08652	3.45253	3.95796	4.59673	5.03299	5.93144	6.09493	6.71752
11	2.47340	2.94054	2.99976	3.35628	3.84875	4.47129	4.89657	5.77255	5.93197	6.53911
12	2.41395	2.87068	2.92860	3.27731	3.75910	4.36830	4.78453	5.64204	5.79810	6.39254
13	2.36422	2.81222	2.86904	3.21118	3.68400	4.28199	4.69063	5.53262	5.68588	6.26965
14	2.32194	2.76247	2.81835	3.15489	3.62004	4.20846	4.61061	5.43936	5.59023	6.16490
15	2.28548	2.71955	2.77462	3.10630	3.56481	4.14494	4.54148	5.35877	5.50756	6.07435
16	2.25367	2.68209	2.73645	3.06387	3.51657	4.08944	4.48106	5.28831	5.43528	5.99518
17	2.22565	2.64906	2.70279	3.02645	3.47401	4.04045	4.42772	5.22609	5.37146	5.92526
18	2.20074	2.61969	2.67286	2.99316	3.43613	3.99684	4.38023	5.17068	5.31461	5.86297
19	2.17844	2.59338	2.64604	2.96333	3.40217	3.95772	4.33763	5.12096	5.26360	5.80707
20	2.15833	2.56965	2.62186	2.93641	3.37152	3.92241	4.29916	5.07604	5.21752	5.75657
21	2.14009	2.54812	2.59991	2.91198	3.34370	3.89034	4.26422	5.03524	5.17566	5.71068
22	2.12347	2.52848	2.57989	2.88969	3.31830	3.86107	4.23232	4.99797	5.13742	5.66877
23	2.10824	2.51048	2.56155	2.86926	3.29502	3.83422	4.20305	4.96378	5.10234	5.63030
24	2.09423	2.49392	2.54466	2.85045	3.27358	3.80948	4.17609	4.93227	5.07000	5.59484
25	2.08128	2.47862	2.52906	2.83307	3.25376	3.78662	4.15116	4.90312	5.04009	5.56204
26	2.06929	2.46443	2.51460	2.81695	3.23537	3.76540	4.12802	4.87606	5.01233	5.53159
27	2.05813	2.45123	2.50114	2.80195	3.21826	3.74564	4.10648	4.85087	4.98647	5.50322
28	2.04773	2.43891	2.48859	2.78795	3.20229	3.72720	4.08636	4.82733	4.96232	5.47673
29	2.03800	2.42739	2.47684	2.77485	3.18734	3.70993	4.06752	4.80529	4.93970	5.45191
30	2.02887	2.41659	2.46582	2.76256	3.17331	3.69373	4.04984	4.78460	4.91846	5.42861
31	2.02029	2.40643	2.45546	2.75101	3.16012	3.67848	4.03321	4.76513	4.89848	5.40667
32	2.01221	2.39686	2.44570	2.74012	3.14768	3.66411	4.01752	4.74676	4.87963	5.38598
33	2.00458	2.38782	2.43648	2.72984	3.13593	3.65053	4.00271	4.72941	4.86181	5.36643
34	1.99737	2.37927	2.42777	2.72011	3.12482	3.63768	3.98868	4.71298	4.84495	5.34792
35	1.99053	2.37116	2.41950	2.71088	3.11428	3.62550	3.97538	4.69739	4.82895	5.33035
36	1.98404	2.36347	2.41166	2.70213	3.10428	3.61392	3.96274	4.68259	4.81375	5.31366
37	1.97788	2.35616	2.40420	2.69380	3.09476	3.60292	3.95072	4.66850	4.79929	5.29778
38	1.97200	2.34919	2.39710	2.68587	3.08570	3.59243	3.93927	4.65508	4.78551	5.28265
39	1.96640	2.34255	2.39033	2.67831	3.07705	3.58242	3.92834	4.64227	4.77236	5.26821
40	1.96106	2.33621	2.38386	2.67109	3.06879	3.57287	3.91791	4.63003	4.75980	5.25441
41	1.95595	2.33015	2.37768	2.66419	3.06090	3.56373	3.90792	4.61832	4.74778	5.24120
42	1.95106	2.32435	2.37176	2.65758	3.05334	3.55498	3.89836	4.60711	4.73626	5.22856
43	1.94637	2.31879	2.36609	2.65124	3.04609	3.54659	3.88920	4.59636	4.72523	5.21643
44	1.94188	2.31345	2.36065	2.64517	3.03914	3.53854	3.88040	4.58604	4.71463	5.20479
45	1.93756	2.30833	2.35542	2.63933	3.03246	3.53080	3.87195	4.57612	4.70445	5.19360
46	1.93342	2.30341	2.35040	2.63372	3.02604	3.52337	3.86383	4.56659	4.69466	5.18284
47	1.92943	2.29867	2.34557	2.62832	3.01987	3.51621	3.85601	4.55741	4.68523	5.17248
48	1.92558	2.29411	2.34092	2.62312	3.01392	3.50932	3.84847	4.54856	4.67615	5.16250
49	1.92188	2.28971	2.33644	2.61811	3.00818	3.50267	3.84121	4.54004	4.66740	5.15288
50	1.91831	2.28547	2.33211	2.61328	3.00265	3.49626	3.83420	4.53181	4.65895	5.14359

Table 9. Two-sided *k*.

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma,\ \ \gamma = 0.95$$

n	P 0.50	P 0.75	P 0.90	P 0.95	P 0.99	P 0.999	P 0.9999	P 0.99999
2	13.65195	22.38300	31.09257	36.51962	46.94492	58.84431	68.72903	77.34849
3	3.58453	5.93751	8.30599	9.78880	12.64717	15.92005	18.64401	21.02187
4	2.28765	3.81840	5.36809	6.34110	8.22068	10.37691	12.17354	13.74289
5	1.81188	3.04096	4.29061	5.07689	6.59799	8.34530	9.80237	11.07567
6	1.56593	2.63841	3.73258	4.42216	5.75776	7.29355	8.57502	9.69524
7	1.41526	2.39114	3.38953	4.01960	5.24111	6.64689	7.82046	8.84664
8	1.31316	2.22304	3.15604	3.74551	4.88923	6.20643	7.30652	8.26867
9	1.23916	2.10084	2.98607	3.54590	4.63284	5.88545	6.93196	7.84745
10	1.18293	2.00770	2.85631	3.39343	4.43691	5.64007	6.64561	7.52540

Table 10. Two-sided *k*.

$$Pr\{T_v \le k\sqrt{n} \mid K_p\sqrt{n}\} = \gamma,\ \ \gamma = 0.99$$

n	P 0.50	P 0.75	P 0.90	P 0.95	P 0.99	P 0.999	P 0.9999	P 0.99999
2	68.31937	112.00224	155.57780	182.73050	234.89082	294.42674	343.88292	387.00878
3	8.12193	13.43495	18.78297	22.13140	28.58650	35.97834	42.13085	47.50184
4	4.02855	6.70618	9.41628	11.11801	14.40563	18.17773	21.32116	24.06716
5	2.82387	4.72420	6.65502	7.86984	10.22024	12.92062	15.17279	17.14110
6	2.26957	3.81168	5.38326	6.37355	8.29163	10.49752	12.33840	13.94780
7	1.95378	3.29122	4.65760	5.51964	7.19077	9.11420	10.72011	12.12448
8	1.75015	2.95508	4.18866	4.96771	6.47906	8.21973	9.67364	10.94536
9	1.60788	2.71977	3.86013	4.58094	5.98019	7.59265	8.93993	10.11860
10	1.50274	2.54553	3.61664	4.29420	5.61021	7.12749	8.39562	9.50523

Table 11. Comparison of approximate and exact results for two-sided tolerance factor k for $n = 2$ and 10.

$n = 2$

p	γ = 0.90 Approx.	0.90 Exact	0.95 Approx.	0.95 Exact	0.99 Approx.	0.99 Exact
0.50	6.80958	6.80826	13.64602	13.65195	68.27314	68.31937
0.75	11.40606	11.16572	22.85712	22.38300	114.35770	112.00224
0.90	15.97787	15.51241	32.01879	31.09257	160.19493	155.57780
0.95	18.79974	18.22084	37.67366	36.51962	188.48716	182.73050
0.99	24.16655	23.42361	48.42846	46.94492	242.29504	234.89082
0.999	30.22649	29.36192	60.57225	58.84431	303.05230	294.42674

$n = 10$

p	γ = 0.90 Approx.	0.90 Exact	0.95 Approx.	0.95 Exact	0.99 Approx.	0.99 Exact
0.50	1.04151	1.05269	1.16612	1.18293	1.47161	1.50274
0.75	1.77504	1.78828	1.98740	2.00770	2.50805	2.54553
0.90	2.53516	2.54594	2.83846	2.85631	3.58207	3.61664
0.95	3.01829	3.02571	3.37941	3.39343	4.26472	4.29420
0.99	3.95930	3.95796	4.43299	4.43691	5.59432	5.61021
0.999	5.04563	5.03299	5.64929	5.64007	7.12926	7.12749

Nonparametric Tolerance Limits

The nonparametrics (distribution-free) statistical analysis is applied to the one- and two-sided tolerance limits to determine the level of dependency of the interval on the sample size at given probability and proportion of the population. The nonparametrics analysis assumes a continuous distribution, but it precludes any requirement for assumptions, regarding the distribution of data. The data from tables 1 through 10 cannot be applied to the nonparametrics analysis.

One-sided tolerance limits pertain to the relative effect of the proportion: at least a proportion P of the population is greater than $x^{(r)}$ or less than $x^{(n-1-m)}$ with probability γ. In a comparison, two-sided tolerance limits primarily concern the interval within which at least a proportion P of the population occurs. A typical problem embraces a question of how large the sample size n should be so that at least a proportion P of the population is between $x^{(r)}$ and $x^{(n-1-m)}$ with probability γ or more. One interpretation of this nonparametrics problem demonstrates a situation, for example, where one has a sample of n events and determines to know how large sample size n can be obtained if one has attained a confidence level of 90 percent that at least 80 percent of the population lies between $x^{(1)}$ and $x^{(n)}$, the smallest and largest events in one's sample.

The random sample values, which are arranged in order of increasing magnitude, are represented as $x_{(1)}, x_{(2)}, x_{(3)}, \ldots, x_{(n)}$ from a distribution having a density function $f(x)$ and distribution function $F(x)$. The order statistics $x_{(r)}$ and $x_{(s)}$ consist of the nonparametric tolerance limits. Let x be a continuous random variable and one calculates the lower limit L and upper limit U. Accordingly, the probability that the value of the random variable falls between L and U is established as

$$Pr\left\{\int_{L}^{U} f(x)dx \geq P\right\} = \gamma \ , \tag{41}$$

where $f(x)$ is the unknown continuous probability density function of the random variable x. L and U will be defined to be 100 P percent tolerance limits at probability γ.

Letting $L = x_{(r)}$ (lower tolerance limit) and $U = x_{(s)}$ (upper tolerance limit) take place in equation (41), the resulting expression becomes

$$Pr\left\{\int_{x_{(r)}}^{x_{(s)}} f(x)dx \geq P\right\} = \gamma \ . \tag{42}$$

Defining

$$F(t) = \int_{-\infty}^{t} f(x)dx \ ,$$

it follows that equation (42) is now rewritten as

$$Pr\{[F(x_s) - F(x_r)] \geq P\} = \gamma \ . \tag{43}$$

A random sample $x_1, x_2, x_3, \ldots, x_n$ of size n is obtained from a cumulative distribution function $F(x)$, and their distribution is defined by

$$dB = dF(x_1)\, dF(x_2)\, dF(x_3) \ \ldots\ dF(x_n) \ . \tag{44}$$

Hence, a transformation is made to perform integration for

$$z_r = \int_{-\infty}^{x_r} dF(t) = F(x_r) \ . \tag{45}$$

The distribution of z_r is derived from a beta distribution of the first kind given by

$$dW = \frac{n!}{(r-1)!(n-r)!}\, z_r^{r-1}(1-z_r)^{n-r} dz_r \ , \quad 0 \leq z_r \leq 1 \ . \tag{46}$$

Thus, considering next a distribution for the rth order statistic x_r, one puts $F(x_r) = z_r$ to obtain the following expression:

$$dW = \frac{n!}{(r-1)!(n-r)!}\, \{F(x_r)\}^{r-1} \{1-F(x_r)\}^{n-r} f(x_r) dx_r \ . \tag{47}$$

Since the samples of size n constitute any distribution $F(x)$ with a continuous probability density function $f(x)$, the sampling distribution of $F(x_r)$ yields

$$dW = \frac{\{F(x_r)\}^{r-1}\{1-F(x_r)\}^{n-r}\,dF(x_r)}{B(r,n-r+1)} \ , \tag{48}$$

where

$$B(r,n-r+1) = \frac{n!}{(r-1)!(n-r)!} \ ,$$

from the gamma function. If x_r and x_s are independent and continuous, then the joint distribution for $F(x_r)$ and $F(x_s)$, $r < s$, is given by

$$dV = \frac{\{F(x_r)\}^{r-1}\{F(x_s)-F(x_r)\}^{s-r-1}\{1-F(x_s)\}^{n-s}\,dF(x_r)\,dF(x_s)}{B(r,s-r)B(s,n-s+1)} \ . \tag{49}$$

New variables of the transformation y,z can be introduced by setting

$$y = F(x_s)-F(x_r) \ , \ z = F(x_r) \ . \tag{50}$$

Furthermore, the Jacobian is obtained

$$J = \frac{\partial(y,z)}{\partial(s,r)} = \begin{vmatrix} \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial z}{\partial s} & \dfrac{\partial z}{\partial r} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \ .$$

Then, by the change of variable formula,

$$f_{y,z}(y,z) = f_s f_r(y,z)\,|J| \ , \tag{51}$$

is obtained. The bars $|\ |$ denote absolute value.

The process from equation (51) yields the following result

$$dV_{y,z} = \frac{z^{r-1}y^{s-r-1}(1-y-z)^{n-s}dydz}{B(r,s-r)\,B(s,n-s+1)} \ . \tag{52}$$

When the joint distribution of two random variables is given, one needs to obtain the distribution of one of the variables alone. Thus, equation (52) is integrated out the variable z over its range $(0,1-y)$, obtaining for the marginal distribution of y

$$dV_{r,s} = \frac{y^{s-r-1}dy}{B(r,s-r)\,B(s,n-s+1)} \int_0^{1-y} z^{r-1}(1-y-z)^{n-s}dz \ . \tag{53}$$

Use of $z = (1-y)t$ reduces equation (53) to

$$dV_{r,s} = \frac{y^{s-r-1}(1-y)^{n-s+r}dy}{B(r,s-r)\,B(s,n-s+1)} \int_0^1 t^{r-1}(1-t)^{n-s}dt = \frac{y^{s-r-1}(1-y)^{n-s+r}dy}{B(r,s-r)B(s,n-s+1)}\,B(r,n-s+1) \ ,$$

$$dV_{r,s} = \frac{y^{s-r-1}(1-y)^{n-s+r}dy}{B(s-r,n-s+r+1)} \ , \quad 0 \le y \le 1 \tag{54}$$

Thus, $y = F(x_s)-F(x_r)$ is distributed as a beta variate of the first kind with parameters depending only on the difference $(s-r)$.

Using equation (54), equation (53) becomes

$$Pr\{y \ge P\} = \int_P^1 \frac{y^{s-r-1}(1-y)^{n-s+r}dy}{B(s-r,n-s+r+1)} = \gamma \ . \tag{55}$$

Equation (55) is rewritten in terms of the incomplete beta function as

$$Pr\{F(x_s)-F(x_r) \ge P\} = 1-I_p(s-r,n-s+r+1) = \gamma \ . \tag{56}$$

Equation (56) provides the nonparametric tolerance interval (x_r, x_s) and related parameters such as the minimum proportion P of $F(x)$, the probability γ, the sample size n and the order statistics' positions r and s.

Equation (56) is equivalent to another equation, namely,

$$\gamma \ge \sum_{i=0}^{r+m-1} \binom{n}{i} (1-p)^i p^{n-i} \ , \tag{57}$$

where the sample size n depends on the solution for both types of one-sided tolerance limits and for the two-sided tolerance limits. $\binom{n}{i}$ is the binomial coefficient, defined as

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \ .$$

The data for nonparametric tolerance limits computed from equation (56) are tabulated in table 12. The numerical data have shown that the sample number of at least 230 items would be required if the acceptance criteria without any failure are satisfied for 99-percent proportion at 90-percent confidence level. Should a single failure occur, then a sample of 388 items would be obtained. The sample of at least 3,000 items is based on achievement of 23 failures.

Table 12. Nonparametric tolerance limits.

α	P	No Failure		One Failure		Two Failures		Three Failures		(r–1) Failures	
		r	n	r	n	r	n	r	n	r	n
0.50	0.50	1	2	2	4	3	6	4	8	1,500	3,000
	0.75	1	3	2	7	3	11	4	15	751	3,000
	0.90	1	7	2	17	3	27	4	37	300	3,000
	0.95	1	14	2	34	3	54	4	74	151	3,000
	0.99	1	69	2	168	3	268	4	367	31	3,000
0.75	0.50	1	3	2	5	3	7	4	10	1,481	3,000
	0.75	1	5	2	10	3	15	4	20	734	3,000
	0.90	1	14	2	27	3	39	4	51	289	3,000
	0.95	1	28	2	54	3	78	4	102	143	3,000
	0.99	1	138	2	269	3	392	4	510	27	3,000
0.90	0.50	1	4	2	7	3	9	4	12	1,463	3,000
	0.75	1	9	2	15	3	20	4	25	720	3,000
	0.90	1	22	2	38	3	52	4	65	280	3,000
	0.95	1	45	2	77	3	105	4	132	135	3,000
	0.99	1	230	2	388	3	531	4	667	24	3,000
0.95	0.50	1	5	2	8	3	11	4	14	1,453	3,000
	0.75	1	11	2	18	3	24	4	29	711	3,000
	0.90	1	29	2	46	3	61	4	76	274	3,000
	0.95	1	59	2	94	3	124	4	153	131	3,000
	0.99	1	299	2	473	3	628	4	773	22	3,000
0.99	0.50	1	7	2	11	3	14	4	17	1.433	3,000
	0.75	1	17	2	24	3	31	4	37	695	3,000
	0.90	1	44	2	64	3	81	4	97	263	3,000
	0.95	1	90	2	130	3	165	4	198	124	3,000
	0.99	1	459	2	662	3	838	4	1,001	19	3,000

Notes: α = confidence level
P = proportion
Failure to meet acceptance criteria.

CONCLUSIONS

The four computer codes have been written to implement the algorithms to calculate the parameters for the exact tolerance factor k's for derivation of the one- and two-sided statistical tolerance limits, using the new theory. One resulting conclusion from this study has shown that the computer codes can perform faster and more efficiently. The exact data, based on normal distributions, have demonstrated that the approximation is somewhat satisfactory for even small sample sizes.

APPENDIX A

Derivation of Chi Distribution

Chi-square probability function (tail area) is constructed in the form of

$$Q[w(z)] = \frac{1}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \int_{t=vx^2}^{\infty} t^{\frac{v}{2}-1} e^{-\frac{t}{2}} dt \ , \tag{A1}$$

where v denotes degrees of freedom and $\Gamma(\cdot)$ is the gamma function

$$\left(\int_0^{\infty} t^{z-1} e^{-t} dt\right).$$

Let $t = vx^2$ and $dt = 2vxdx$.

Substituting the preceding equations into equation (A1) gives

$$Q[w(z)] = \frac{1}{2^{\frac{v}{2}}\Gamma\left(\frac{v}{2}\right)} \int_{x=vw^2}^{\infty} (vx^2)^{\frac{v}{2}-1} e^{-\frac{vx^2}{2}} 2vxdx \ . \tag{A2}$$

After manipulation, equation (A2) becomes

$$Q[w(z)] = \frac{2\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{x=vw^2}^{\infty} x^{v-1} e^{-\frac{vx^2}{2}} dx \ . \tag{A3}$$

For the reason of symmetry based on addition of integration area, the final expression is given by

$$Q[w(z)] = \frac{4\left(\frac{v}{2}\right)^{\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)} \int_{x=vw^2}^{\infty} x^{v-1} e^{-\frac{vx^2}{2}} dx \ . \tag{A4}$$

Equation (A4) is the chi distribution.

APPENDIX B1

One-Sided K_p

Standardized normal random variable exceeded with probability p

Proportion	K_p
0.99999	4.2648907939
0.99996832876	4.0000000000
0.9999	3.7190164855
0.999	3.0902323062
0.998650102	3.0000000000
0.9975	2.8070337684
0.996	2.6520698079
0.99	2.3263478743
0.97724987	2.0000000000
0.975	1.9599640498
0.96	1.7506860747
0.95	1.6448536278
0.935	1.5141018878
0.90	1.2815515655
0.85	1.0364333895
0.84134477	1.0000000000
0.80	0.8416212336
0.75	0.6744897502
0.70	0.5244005127
0.65	0.3853204664
0.60	0.2533471031
0.55	0.1256613469
0.50	0

APPENDIX B2

Two-Sided K_p

Standardized normal random variable exceeded with probability p

Proportion	K_p
0.99999	4.4171734135
0.99993665752	4.0000000000
0.9999	3.8905918864
0.999	3.2905267315
0.997300204	3.0000000000
0.99	2.5758293036
0.975	2.2414027281
0.95449974	2.0000000000
0.95	1.9599640498
0.90	1.6448536278
0.85	1.4395314711
0.80	1.2815515655
0.75	1.1503493804
0.70	1.0364333895
0.6826894925	1.0000000000
0.65	0.9345892911
0.60	0.8416212336
0.55	0.7554150264
0.50	0.6744897502
0.45	0.5977601260
0.40	0.5244005127
0.35	0.4537621902
0.30	0.3853204664
0.25	0.3186393640
0.20	0.2533471031
0.15	0.1891184263
0.10	0.1256613469
0.05	0.0627067779
0	0

APPENDIX C1

Simulation Code, One-Sided *k*

```
'K1-FACTOR
DEFDBL A-Z
INPUT"SAMPLE SIZE=";NS
INPUT "PROPORTION";P
INPUT "CONFIDENCE=";CL
START=TIMER
PI=3.141592654#
'INVERSE NORMAL
Q=1-P:T=SQR(-2*LOG(Q))
AO=2.30753:A1=.27061:B1=.99229:B2=.04481
NU=A0+A1*T:DE=1+B1*T+B2*T*T
X=T-NU/DE
L0: Z=1/SQR(2*PI)*EXP(-X*X/2):IF X>2 GOTO L3
V=25-13*X*X
FOR N=11 TO 0 STEP-1
U=(2*N+1)+(-1)^(N+1)*(N+1)*X*X/V
V=U:NEXT N
F=.5-Z*X/V
W=Q-F:GOTO L2
L3:V=X+30
FOR N=29 TO 1 STEP-1
U=X+N/V
V=U :NEXT N
F=Z/V :W=Q-F :GOTO L2
L2:L=L+1
R=X:X=X-W/Z
E=ABS(R-X)
IF E>.00001 GOTO L0
'END OF INVERSE NORMAL

'CALCULATION OF FACTORIAL
N=NS:NU=N-1
MT=INT(NU/2):UT=NU-2*MT
GT=1
FOR I=1 TO MT-1+UT
KT=I
IF UT=0 GOTO L1
KT=I-.5
L1:GT=GT*KT
NEXT I
GT=GT*(1+UT*(SQR(PI)-1))
GF=GT*2^(NU/2-1)
'END OF FACTORIAL
```

```
'SECANT METHOD
KP=X:J=1:K=KP
K0=K:GOSUB INTEGRATION:SF0=SF
K=K*(1+.0001):K1=K:GOSUB INTEGRATION:SF1=SF
BEGIN:K=K1-SF1*(K1-K0)/(SF1-SF0)
IF ABS((K1-K)/K1)<.000001 GOTO RESULT
J=J+1:K0=K1:K1=K:SF0=SF1
GOSUB INTEGRATION:SF1=SF:GOTO BEGIN
RESULT:FINISH=TIMER
BEEP:BEEP
PRINT K
PRINT "K";"(";J;") =";USING"##.####";K
PRINT "TIME=";FINISH-START;"SECONDS"
'END OF SECANT METHOD
WHILE INKEY$="":WEND

'SIMPSON
INTEGRATION:L1=0:L2=10
IF N>40 THEN L2=20
DL=KP*SQR(N):TP=K*SQR(N)
Y=NU/2
M=2:E=0:H=(L2-L1)/2
X=L1:GOSUB FUNCTION
Y0=Y:X=L2:GOSUB FUNCTION
YN=Y:X=L1+H:GOSUB FUNCTION
U=Y:S=(Y0+YN+4*U)*H/3
START:M=2*M
D=S:H=H/2:E=E+U:U=0
FOR I-1 TO M/2
X=L1+H*(2*I-1):GOSUB FUNCTION
U=U+Y
NEXT I
S=(Y0+YN+4*U+2*E)*H/3
IF ABS((S-D)/D)>.00001# GOTO START
SF=S/GF-CL
RETURN
'END OF SIMPSON

FUNCTION: Z=TP*X/SQR(NU)-DL
T0=Z:G0=1/SQR(2*PI)*EXP(-Z*Z/2)
A1=.3193815:A2=-.3565638:A3=1.781478:A4=-1.821256:A5=1.330274
IF Z<0 THEN T0=-Z
W=1/(1+.231649*T0)
P1=((((A5*W+A4)*W+A3)*W+A2)*W+A1)*W
PH=1-G0*P1
IF Z<0 THEN PH=1-PH
Y=PH*X^(NU-1)*EXP(-X*X/2)
RETURN
```

APPENDIX C2

Simulation Code, Two-Sided *k*

```
'K2-FACTOR
START:
DEFDBL A-Z
PI=3.141592654#

INPUT"SAMPLE SIZE=";N
INPUT"PROPORTION=";P
INPUT"CONFIDENCE=";CL
START=TIMER

NU=N-1:X=(NU+2)/2
DEF FNHTN(X)=(EXP(X)-EXP(-X))/(EXP(X)+EXP(-X))
CALL GAMMA
CALL NORMIN
PRINT"KP=";KP
'SECANT METHOD
K=KP:J=1
K0=K:GOSUB INTEGRATION:SF0=SF
K=K*(1+.0001):K1=K:GOSUB INTEGRATION:SF1=SF
BEGIN:K=K1-SF1*(K1-K0)/(SF1-SF0)
IF ABS((K1-K)/K1)<.000001 GOTO RESULT
J=J+1:K0=K1:K1=K:SF0=SF1
GOSUB INTEGRATION:SF1=SF:GOTO BEGIN
RESULT:FINISH=TIMER
BEEP:BEEP
PRINT K
PRINT "K";"(";J;") =";USING"##.####";K
PRINT "TIME=";FINISH-START;"SECONDS"
'END OF SECANT METHOD
WHILE INKEY$="":WEND
GOTO START

'START SIMPSON
INTEGRATION:
A=0:B=5:M=32
H=(B-A)/2/M
FE=0:FU=0
X=A:W=KP/K:UC=K/SQR(N):HC=H/K/SQR(N)
CALL FUNCTION:FA=F
FOR I-1 TO 2*M-1
CALL RUNGE
X=A+I*H
CALL FUNCTION
U=I MOD 2
IF U=0 THEN FE=FE+F ELSE FU=FU+F
```

```
NEXT I
CALL RUNGE
X=B:CALL FUNCTION:FB=F
S=H*(FA+2*FE+4*FU+FB)/3
SF=2*S-CL
RETURN

SUB RUNGE STATIC
SHARED HC,UC,W,H,X
'RUNGE-KUTTA/3
X1=X:W1=W:U=X*W*UC:K1=HC*FNHTN(U)
X=X1+H/2:W=W1+K1/2:U=X*W*UC:K2=HC*FNHTN(U)
X=X1+H:W=W1-K1+2*K2:U=X*W*UC:K3=HC*FNHTN(U)
W=W1+(K1+4*K2+K3)/6
END SUB

SUB FUNCTION STATIC
SHARED PI,NU, GNU,F,W,X
'CHI-SQUARE
WL=NU*W*W
C=(WL/2)^(NU/2)*EXP(-WL/2)/GNU
Q=1:J=0:S=0
LINE2: J=J+1:D=NU+2*J
R=WL/D:Q=Q*R:S=S+Q
IF Q>.000001 GOTO LINE2
P=(1+S)*C
'END CHI-SQUARE
F=(1-P)*EXP(-X*X/2)/SQR(2*PI)
END SUB

SUB GAMMA STATIC
SHARED X,PI,GNU
Z=X+4:IF X>4 THEN Z=X
G=SQR(2*PI/Z)*EXP(Z*LOG(Z)-Z+1/12/Z-1/360/Z^3+1/1260/Z^5-1/1680/Z^7+1  /1188
/Z^9)
IF X>4 GOTO L1
G=G/X/(X+1)/(X+2)/(X+3)
L1:GNU=G
END SUB

SUB NORMIN STATIC
SHARED KP,P,PI
Q=(1-P)/2:T=SQR(-2*LOG(Q))
A0=2.30753:A1=.27061:B1=.99229:B2=.04481
NU=A0+A1*T:DE=1+B1*T+B2*T*T
X=T-NU/DE
```

```
LO: Z=1/SQR(2*PI)*EXP(-X*X/2):IF X>2 GOTO L3
V=25-13*X*X
FOR N=11 TO 0 STEP-1
U=(2*N+1)+(-1)^(N+1)*(N+1)*X*X/V
V=U:NEXT N
F=.5-Z*X/V
W=Q-F:GOTO L2
L3:V=X+30
FOR N=29 TO 1 STEP -1
U=X+N/V
V=U :NEXT N
F=Z/V :W=Q-F :GOTO L2
L2:L=L+1
R=X:X=X-W/Z
E=ABS(R-X)
IF E>.000001 GOTO LO
KP=X
END SUB
```

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APPROVAL

STATISTICAL COMPUTATION OF TOLERANCE LIMITS

By J.T. Wheeler

The information in this report has been reviewed for technical content. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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Director, Structures and Dynamics Laboratory

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Based on a new theory, two computer codes have been developed specifically to calculate the exact statistical tolerance limits for normal distributions within unknown means and variances for the one-sided and two-sided cases for the tolerance factor, k. The quantity k is defined equivalently in terms of the noncentral t-distribution by the probability equation. Two of the four mathematical methods employ the theory developed for the numerical simulation. Several algorithms for numerically integrating and iteratively root-solving the working equations are written to augment the program simulation. The program codes generate some tables of k's associated with the varying values of the proportion and sample size for each given probability to show accuracy obtained for small sample sizes.

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