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ABSTRACT

We consider preconditioning methods to accelerate convergence to a steady state for the incompressible fluid dynamic equations. The analysis relies on the inviscid equations. The preconditioning consists of a matrix multiplying the time derivatives. Thus the steady state of the preconditioned system is the same as the steady state of the original system. We compare our method to other types of pseudo-compressibility. For finite difference methods preconditioning can change and improve the steady state solutions. An application to viscous flow around a cascade with a non-periodic mesh is presented.

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1 Introduction

One way to solve the steady state incompressible equations is to march the time dependent equations until a steady state is reached. Since the transient is not of any interest one can use acceleration techniques which destroy the time accuracy but enables one to reach the steady state faster. Such methods can be considered as preconditionings to accelerate the convergence to a steady state. For the incompresible equations the continuity equation does not contain any time derivatives. To overcome this difficulty, Chorin [3] added an artificial time derivative of the pressure to the continuity equation together with a multiplicative variable, β . With this artificial term the resultant scheme is a symmetric hyperbolic system for the inviscid terms. Thus, the system is well posed and and numerical method for hyperbolic systems can be used to advance this system in time.. The free parameter β is then chosen to reach the steady state quickly. Later Turkel ([8], [9], [10]) extended this concept by adding a pressure time derivative also to the momentum equations. The resulting system after preconditioning is no longer symmetric but can be symmetrized by a change of variables.

Thus, we will consider systems of the form

$$w_t + f_x + g_y = 0.$$

This system is written in conservation form though for some applications this is not necessary. Our analysis will be based on the linearized equations so the conservation form does not appear in the analysis though it does appear in the final numerical approximation. This system is now replaced by

$$\mathbf{P}^{-1} w_t + f_x + g_y = 0, \tag{1}$$

or in linearized form

$$\mathbf{P}^{-1} w_t + A w_x + B w_y = 0, \tag{2}$$

with A and B constant matrices.

For this system to be equivalent to the original system, in the steady state, we demand that $\mathbf{P}^{-1}$ have an inverse. This only need be true in the flow regime under consideration. We shall see later that frequently $\mathbf{P}$ is singular at stagnation points. Thus, we will temporarily consider strictly flows without a stagnation point. We also assume that the Jacobian matrices $A = \frac{\partial f}{\partial w}$ and $B = \frac{\partial g}{\partial w}$ are simultaneously symmetrizable. In terms of the 'symmetrizing' variables we also demand that $\mathbf{P}$ be positive definite. We shall show later, in detail, that it does not matter which set of dependent variables are used to develop the preconditioner. One can transform between any two sets of variables. Thus, when we are finished we will analyze a system which is similar to (2), where the matrices A and B are symmetric and $\mathbf{P}$ is both symmetric and positive definite. Such systems are known as symmetric hyperbolic systems. One can then multiply this system by w and integrate by parts to get estimates for the integral of w_t^2, i.e. energy estimates. These estimates can then be used to show that the system is well posed . We stress that if $\mathbf{P}$ is not positive then we may change the physics of the problem. For example, if $\mathbf{P} = -I$ then we have reversed the time direction and must therefore change all the boundary conditions. Keeping the right signs for the eigenvalues is a necessary but not sufficient condition for well-posedness.

With this assumption the steady state solutions of the two systems are the same. Assuming that the steady state has a unique solution, it does not matter which system we march to a steady state. We shall later see that for the finite difference approximations the steady state solutions are not necessarily the same and usually the preconditioned system leads to a better behaved steady state.

2 Incompressible Equations

Consider the incompressible inviscid equations in primitive variables.

$$\begin{aligned} u_x + v_y &= 0 \\ u_t + uu_x + vu_y + p_x &= 0 \\ v_t + uv_x + vv_y + p_y &= 0 \end{aligned}$$

We generalize Chorin's pseudo-compressibility method [3]. Using the preconditioning suggested in [8] (with $\alpha = 1$) we have

$$\begin{aligned} \frac{1}{\beta^2}p_t + u_x + v_y &= 0 \\ \frac{u}{\beta^2}p_t + u_t + uu_x + vu_y + p_x &= 0 \\ \frac{v}{\beta^2}p_t + v_t + uv_x + vv_y + p_y &= 0 \end{aligned} \tag{3}$$

or in conservation form

$$\begin{aligned} \frac{1}{\beta^2}p_t + u_x + v_y &= 0 \\ \frac{2u}{\beta^2}p_t + u_t + (u^2 + p)_x + (uv)_y &= 0 \\ \frac{2v}{\beta^2}p_t + v_t + (uv)_x + (v^2 + p)_y &= 0 \end{aligned}$$

We can also write (3) in matrix form using

$$\mathbf{P}^{-1} = \begin{pmatrix} 1/\beta^2 & 0 & 0 \\ u/\beta^2 & 1 & 0 \\ v/\beta^2 & 0 & 1 \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} \beta^2 & 0 & 0 \\ -u & 1 & 0 \\ -v & 0 & 1 \end{pmatrix}$$

Multiplying by $\mathbf{P}$ we rewrite this as

$$w_t + \mathbf{P}Aw_x + \mathbf{P}Bw_y = 0.$$

Let

$$D = \omega_1 A + \omega_2 B \qquad -1 \leq \omega_1, \omega_2 \leq 1$$

where ω_1, ω_2 are the Fourier transform variables in the x and y directions respectively. The speeds of the waves are now governed by the roots of $det(\lambda I - \mathbf{P}A\omega_1 - \mathbf{P}B\omega_2) = 0$ or equivalently $det(\lambda \mathbf{P}^{-1} - A\omega_1 - B\omega_2) = 0$. Let

$$q = u\omega_1 + v\omega_2.$$

Then the eigenvalues of **PD** are

$$\begin{aligned} d_0 &= q \\ d_\pm &= \pm\beta \end{aligned} \tag{4}$$

and so the 'acoustic' speed is isotropic.

The spatial derivatives involve symmetric matrices, i.e. D is a symmetric matrix but **P** is not symmetric. Thus, while the original system was symmetric hyperbolic the preconditioned system is no longer symmetric. In [8] it is shown that as long as

$$\beta^2 > (u^2 + v^2)$$

the equations can be symmetrized. On the other hand the eigenvalues are most equalized if $\beta^2 = (u^2 + v^2)$ [8]. So, we wish to choose β^2 slightly larger than $u^2 + v^2$. However, numerous calculations verify, that in general, a constant β is the best for the convergence rate. The reasons for this are not clear.

We wish to stress that β has the dimensions of a speed. Therefore, β cannot be a universal constant. There are papers that claim that $\beta = 1$ or $\beta = 2.5$ are optimal. Such claims cannot be true in general. It is simple to see that if one nondimensionalizes the equation then β gets divided by a reference velocity. Hence, the optimal 'constant' β depends on the dimensionalization of the problem and in particular depends on the inflow conditions. In many calculations the inflow mass flux is equal to 1 or alternatively $p + (u^2 + v^2)/2 = 1$. Such conditions will give an optimal β close to one.

We next define the Bernoulli function

$$H = p + (u^2 + v^2)/2.$$

Bernoulli's theorem states that when the flow is steady and inviscid then H is constant along streamlines. We now multiply the second equation of (3) by u and the third equation of (3) by v and add these two equations. If $\beta^2 = u^2 + v^2$, the result is

$$H_t + uH_x + vH_y = 0. \tag{5}$$

Thus, by altering the time dependence of the equations we have constructed a new equation in which H is convected along streamlines. Furthermore, if H is a uniform constant both initially and at inflow then H will remain constant for all time. On the numerical level this will usually not be true because of the introduction of an artificial viscosity or because of upwinding. For viscous flow, (5) is replaced by

$$H_t + uH_x + vH_y = \frac{1}{Re}(u\Delta u + v\Delta v)$$

We note that these relationships for H follow from the momentum equations and do not depend on the form of the continuity equation. Hence, we consider the following generalization of (3)

$$\begin{aligned} \frac{1}{\beta^2}p_t + aH_t + u_x + v_y &= 0 \\ \frac{\alpha u}{\beta^2}p_t + u_t + uu_x + vu_y + p_x &= 0 \\ \frac{\alpha v}{\beta^2}p_t + v_t + uv_x + vv_y + p_y &= 0 \end{aligned} \tag{6}$$

where, a,α and β are free parameters. When $\omega_1^2 + \omega_2^2 = 1$ the eigenvalues of **PD** are

$$\omega_1 u + \omega_2 v, \qquad \frac{s \pm \sqrt{s^2 + 4\beta^2 d}}{2d}$$

where

$$q^2 = u^2 + v^2, \quad d = 1 + a - a\frac{q^2}{\beta^2}, \quad s = (1-\alpha)(\omega_1 u + \omega_2 v).$$

Hence, the 'acoustic' eigenvalue is isotropic if $\alpha = 1$. Furthermore, d = 1 if either a = 0 or $\beta^2 = u^2 + v^2$. For $a = 0$ we recover our original scheme. For $a = -1$ the time derivative of the pressure no longer appears in the continuity equation. For general α, β we have

$$\mathbf{P}^{-1} = \frac{1}{\beta^2}\begin{pmatrix} (a+1) & au & av \\ \alpha u & \beta^2 & 0 \\ \alpha v & 0 & \beta^2 \end{pmatrix},$$

$$\mathbf{P} = \frac{1}{1 + a - a\alpha\frac{u^2+v^2}{\beta^2}} \cdot \begin{pmatrix} \beta^2 & -au & -av \\ -\alpha u & 1 + a - \frac{a\alpha v^2}{\beta^2} & \frac{a\alpha uv}{\beta^2} \\ -\alpha v & \frac{a\alpha uv}{\beta^2} & 1 + a - \frac{a\alpha u^2}{\beta^2} \end{pmatrix}$$

If we write the equation in conservation form (1) we have

$$\mathbf{P}^{-1}_{conservative} = \frac{1}{\beta^2}\begin{pmatrix} (a+1) & au & av \\ (1+a+\alpha)u & \beta^2 + au^2 & auv \\ (1+a+\alpha)v & auv & \beta^2 + av^2 \end{pmatrix},$$

$$\mathbf{P}_{conservative} = \frac{1}{1 + a - a\alpha\frac{u^2+v^2}{\beta^2}} \cdot \begin{pmatrix} \beta^2 + a(u^2+v^2) & -au & -av \\ -(1+a+\alpha)u & 1 + a - \frac{a\alpha v^2}{\beta^2} & \frac{a\alpha uv}{\beta^2} \\ -(1+a+\alpha)v & \frac{a\alpha uv}{\beta^2} & 1 + a - \frac{a\alpha u^2}{\beta^2} \end{pmatrix}$$

In [9] an analogy to the symmetric preconditioning of van Leer, Lee and Roe was constructed for the incompressible equations. If we choose $a = 1$ **P** is symmetric. If we also choose $\beta^2 = u^2 + v^2$ then we get the preconditioning of van Leer et.al. .

$$\mathbf{P} = \begin{pmatrix} u^2+v^2 & -u & -v \\ -u & 1+\frac{u^2}{u^2+v^2} & \frac{uv}{u^2+v^2} \\ -v & \frac{uv}{u^2+v^2} & 1+\frac{v^2}{u^2+v^2} \end{pmatrix}$$

These examples show that preconditioning is not unique. If fact, since the determinant of the transpose of a matrix is equal to the determinant of the original matrix it follows that the transpose of **P** is also a preconditioner with the same eigenvalues for the preconditioned system. These various systems will have the same eigenvalues but different eigenvectors for the preconditioned system. Numerous calculations show that the system given by **P** in (3) is more robust and converges faster than with the transpose preconditioner. This shows that it is not sufficient to consider just the eigenvalues but that the eigenvectors are also of importance. The eigenvectors are given in ([10]).

We next consider the preconditioner considered by de Jouette, Viviand et. al. ([4]). Define:

$$\begin{aligned} q &= uu_x + vu_y + p_x - (\tau_{xx} + \tau_{xy}) \\ r &= uv_x + vv_y + p_y - (\tau_{xy} + \tau_{yy}) \\ s &= uq + vr \\ U^2 &= u^2 + v^2 \end{aligned}$$

Then they consider the following extension of the incompressible Navier-Stokes equations.

$$\begin{aligned} p_t + d_V(u_x + v_y) + \alpha_V s = 0 \\ u_t + a_V q + \beta_V u(u_x + v_y) + e_V us = 0 \\ v_t + a_V r + \beta_V v(u_x + v_y) + e_V vs = 0 \end{aligned} \tag{7}$$

In the steady state $q = r = s = 0$ and $u_x + v_y = 0$ and so we recover the usual incompressible equations. a_V,d_V,e_V,α_V,β_V are free parameters that satisfy the following conditions

$$\begin{aligned} \alpha_V \beta_V = d_V e_V \\ a_V \geq 0 \qquad d_V \geq 0 \\ (d_V + \alpha_V U^2)(d_V + \beta_V U^2) \geq 0 \end{aligned}$$

In addition, in order for the speed of the convective wave to remain unchanged we add the condition $a_V = 1$. From the momentum equations we obtain

$$s = -\frac{[uu_t + vv_t + \beta_V U^2(u_x + v_y)]}{1 + e_V U^2}$$

Hence we can rewrite (8) as

$$\begin{aligned} \frac{1 + e_V U^2}{d} p_t - \frac{\alpha_v}{d_V}(uu_t + vv_t) & \\ + u_x + v_y =& \ 0 \\ -\frac{\beta_V u}{d} p_t + u_t + q =& \ 0 \\ -\frac{\beta_V v}{d} p_t + v_t + r =& \ 0 \end{aligned}$$

Comparing this with (6) we see that the two approaches are identical if

$$
\begin{aligned}
d_V &= \frac{\beta^4}{(a+1)\beta^2 - a\alpha U^2} \\
\alpha_V &= -\frac{a}{\beta^2} d_V \\
\beta_V &= -\frac{\alpha}{\beta^2} d_V \\
e_V &= \frac{a\alpha}{\beta^4} d_V
\end{aligned}
$$

Choosing $\alpha = 1$ and $\beta = U^2$ we get the standard preconditioning (3). The Viviand parameters become $\alpha_V = -a$, $\beta_V = -1$, $a_V = 1$, $d_V = U^2$, $e_V = a$. Then $a = 0$ gives the Turkel preconditioner and $a = 1$ gives the van-Leer (symmetric) preconditioner.

3 Difference equations

Until now the entire analysis has been based on the partial differential equation. We now make some remarks on important points for any numerical approximation of this system. When using a scheme based on a Riemann solver this solver should be for the preconditioned system and not the original scheme. When using a central difference schemes there is a need to add an artificial viscosity. Accuracy is improved for low Mach number flows if the preconditioner is applied only to the physical convective and viscous terms but not to the artificial viscosity. The use of a matrix artificial dissipation ([7]) should be based on the preconditioned equations as for Riemann solvers. difference scheme. Hence, both for upwind and central difference schemes the Riemann solver or artificial viscosity should be based on $\mathbf{P}^{-1}|\mathbf{P}A|$ and not $|A|$ i.e. in one dimension solve $w_t + \mathbf{P}f_x = \mathbf{P}(\mathbf{P}^{-1}|\mathbf{P}A|w_x)_x$. When using characteristics for extrapolation at the boundaries it should be based on the characteristics of the modified system and not the physical system. Preconditioning is even more important when using multigrid than with an explicit scheme. With the original system, the stiffness of the eigenvalues greatly affects the smoothing rates of the slow components and so slows down the multigrid method, [6]. We conclude that the steady state solution of the preconditioned system may be different from that of the physical system. Thus, on the finite difference level the preconditioning can improve the accuracy as well as the convergence rate.

We next consider adding artificial viscosity to the system (3). We first rewrite this system eliminating p_t from the velocity equations. This gives

$$
\begin{aligned}
p_t + \beta^2(u_x + v_y) &= 0 \\
u_t + p_x + vu_y - uv_y &= 0 \\
v_t + uv_x - vu_x + p_y &= 0
\end{aligned}
$$

or in matrix form

$$
\begin{pmatrix} p \\ u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & \beta^2 & 0 \\ 1 & 0 & 0 \\ 0 & -v & u \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & v & -u \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix}_y = 0 \qquad (8)
$$

or

$$w_t + \mathbf{P}Aw_x + \mathbf{P}Bw_y = 0.$$

We next consider the use of a matrix valued viscosity. Let $D = \omega_1 A + \omega_2 B$ with $\omega_1^2 + \omega_2^2 = 1$. The non-preconditioned matrix viscosity is given by $|A|$ in the x direction and $|B|$ in the y direction. Then

$$|D| = \frac{1}{(\lambda_2 - \lambda_3)\, q^2} \begin{pmatrix} 2q^2 & q\,R^2 & q\,R\,S \\ q\,R^2 & \left({\lambda_2}^2 + {\lambda_3}^2\right) R^2 + (\lambda_2 - \lambda_3)\, S^2\, |R| & R\,S\,\left({\lambda_2}^2 + {\lambda_3}^2 - (\lambda_2 - \lambda_3)\,|R|\right) \\ q\,R\,S & R\,S\,\left({\lambda_2}^2 + {\lambda_3}^2 - (\lambda_2 - \lambda_3)\,|R|\right) & \left({\lambda_2}^2 + {\lambda_3}^2\right) S^2 + (\lambda_2 - \lambda_3)\, R^2\, |R| \end{pmatrix}.$$

with

$$\lambda_2 = \frac{R + \sqrt{R^2 + 4}}{2}, \quad \lambda_3 = \frac{R - \sqrt{R^2+4}}{2}$$
$$R = u\omega_1 + v\omega_2, \quad S = u\omega_2 - v\omega_2$$

For the preconditioned artificial viscosity we consider instead $\mathbf{P}^{-1}|\mathbf{PA}|$ and $\mathbf{P}^{-1}|\mathbf{PB}|$ (see [7]). We consider the case $\alpha = 1$ with β and a arbitrary. Then

$$\mathbf{P}^{-1}|\mathbf{P}D| = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}.$$

with

$$V_{11} = \frac{1+a}{\beta\sqrt{d}} + \frac{a\,S^2\,q^2\,X}{\beta^2}$$

$$V_{12} = a\left[\frac{u}{\beta\sqrt{d}} + S^2\,u\,X + \frac{R\,S\,(\beta^2 - q^2)\,v\,X}{\beta^2}\right]$$

$$V_{21} = \frac{u}{\beta\sqrt{d}} + S^2\,u\,X + \frac{R\,S\,(\beta^2 - aq^2)\,v\,X}{\beta^2}$$

$$V_{13} = a\left[\frac{v}{\beta\sqrt{d}} + S^2\,v\,X - \frac{R\,S\,(\beta^2 - q^2)\,u\,X}{\beta^2}\right]$$

$$V_{31} = \frac{v}{\beta\sqrt{d}} + S^2\,v\,X - \frac{R\,S\,(\beta^2 - aq^2)\,u\,X}{\beta^2}$$

$$V_{22} = \frac{\beta}{\sqrt{d}} + \frac{\beta^2\,S^2\,u^2\,X}{q^2} - R\,S\left(1 + a - \frac{2\,\beta^2}{q^2}\right) u\,v\,X + \frac{R^2\,(\beta^2 - d\,q^2)\,v^2\,X}{q^2}$$

$$V_{33} = \frac{\beta}{\sqrt{d}} + \frac{\beta^2\,S^2\,v^2\,X}{q^2} + R\,S\left(1 + a - \frac{2\,\beta^2}{q^2}\right) u\,v\,X + \frac{R^2\,(\beta^2 - d\,q^2)\,u^2\,X}{q^2}$$

$$V_{23} = \frac{(\beta^2\, S\, v + R\, u\, (q^2 - \beta^2))\ (\beta^2\, S\, u + R\, v\, (\beta^2 - a\, q^2))\ X}{\beta^2\, q^2}$$

$$V_{32} = \frac{(\beta^2\, S\, v + R\, u\, (aq^2 - \beta^2))\ (\beta^2\, S\, u + R\, v\, (\beta^2 - q^2))\ X}{\beta^2\, q^2}$$

where

$$R = \frac{u\omega_1 + v\omega_2}{q} \qquad S = \frac{u\omega_2 - v\omega_1}{q}$$

$$d = 1 + a - a\frac{q^2}{\beta^2} \qquad q^2 = u^2 + v^2$$

$$X = \frac{|Rq|\sqrt{d} - \beta}{d\ Z}$$

$$Z = \frac{\beta^4 S^2 - (\beta^2 - q^2)\,(\beta^2 - aq^2)\,R^2}{\beta^2}$$

By inspection the matrix is symmetric when $a = 1$. For the special case $a = 1$ and $\beta^2 = u^2 + v^2$ the formulas simplify and we get

$$V_{11} = \frac{R+1}{q}$$

$$V_{21} = V_{12} = R\frac{u}{q}$$

$$V_{31} = V_{13} = R\frac{v}{q}$$

$$V_{22} = q + \frac{u^2(R-1)}{q} = \frac{v^2 + Ru^2}{q}$$

$$V_{33} = q + \frac{v^2(R-1)}{q} = \frac{u^2 + rv^2}{q}$$

$$V_{23} = V_{32} = \frac{uv(R-1)}{q}$$

For the equations in conservation form we multiply the continuity equation by u and add to the x velocity equation. We also multiply the continuity equation by v and add to the y velocity equation.

4 Computational Results

We now present a calculation for two dimensional flow around an cascade to demonstrate the previous theory. The discretization is based on the multistage time method coupled with a central difference approximation as described in ([5], [7]). The basic scheme is accelerated by using a local time step, residual smoothing and multigrid. This code was further developed to consider cascade configurations in which the grid is not necessarily continous across the wake ([1], [2]). We compute the flow about a NACA0012 with periodic external boundaries. The flow is turbulent and we use a Baldwin-Lomax turbulence model, with $Re = 500,000$, $Pr = 0.7$, $Pr_t = 0.9$ At inflow the angle of attack is specified as well as the Bernoulli constant, $p + \frac{u^2+v^2}{2} = 1$. The mesh is 192×32 and is shown in figure 1. We use a four stage Runge-Kutta method as a smoother for a full multigrid iteration. We choose $a = 0$ and $\beta^2 = max(K(u^2 + v^2), \beta_m)$ with $K = 1.1, \beta_m = 0.4$, see (6). In figure 2 we plot the convergence rate for different values of α. We see that the fastest convergence occurs when $\alpha = 1$ followed by $\alpha = 0$ and finally $\alpha = -1$. We also considered viscous flow about a VKI cascade (figure 3). In this case the convergence of all the methods slowed down. $\alpha = 1$ was still the most efficient method but the differences were less dramatic than in the previous case. In other cases in was necessary to choose β almost constant. The symmetric preconditioner, $a = 1$ was more robust but not faster than $a = 0$.

5 Conclusions

A three parameter preconditioning matrix has been introduced for the incompressible inviscid equations. This is equivalent to the pseudo- compressibility methods considered by de Jouette et. al. When $\alpha = 1$ the 'acoustic' speeds are symmetric. Furthermore, one can choose the parameter a so that the preconditioning matrix is symmetric. For the inviscid case considered computed a considerable increase in the convergence rate was achieved.

In addition the incompressible equations offer a theoretical advantage over the compressible equations for the theoretical study of preconditioning methods. This is because of the simpler nature of the equations and the fact that the original method of Chorin is already symmetric. Nevertheless, a central difference scheme coupled with a Runge-Kutta time advancement suffers from lack of robustness. In particular β needs to be bounded away from zero at a relatively high level for many of the cases. Using the symmetric preconditioner $a = \alpha = 1$ yields a more robust scheme though it does not seem to converge faster than the nonsymmetric preconditioner. Furthermore, changes of the physical inflow boundary condition can greatly affect the choice of the optimal α and β. The major increases in the convergence rate are for the Euler equations. For the Navier-Stokes equations it is necessary to reformulate the preconditioning matrix to account for the viscous effecrs.

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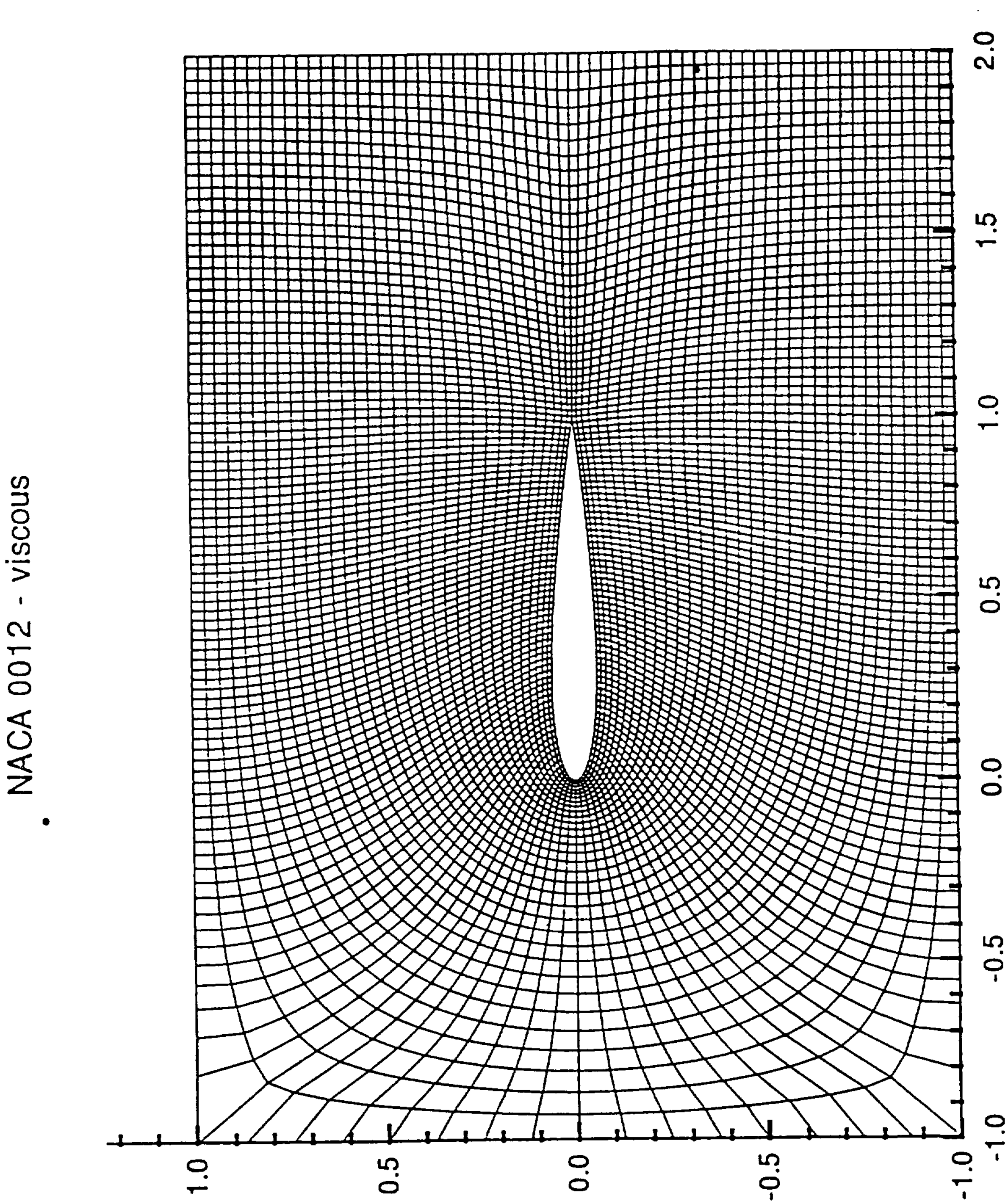
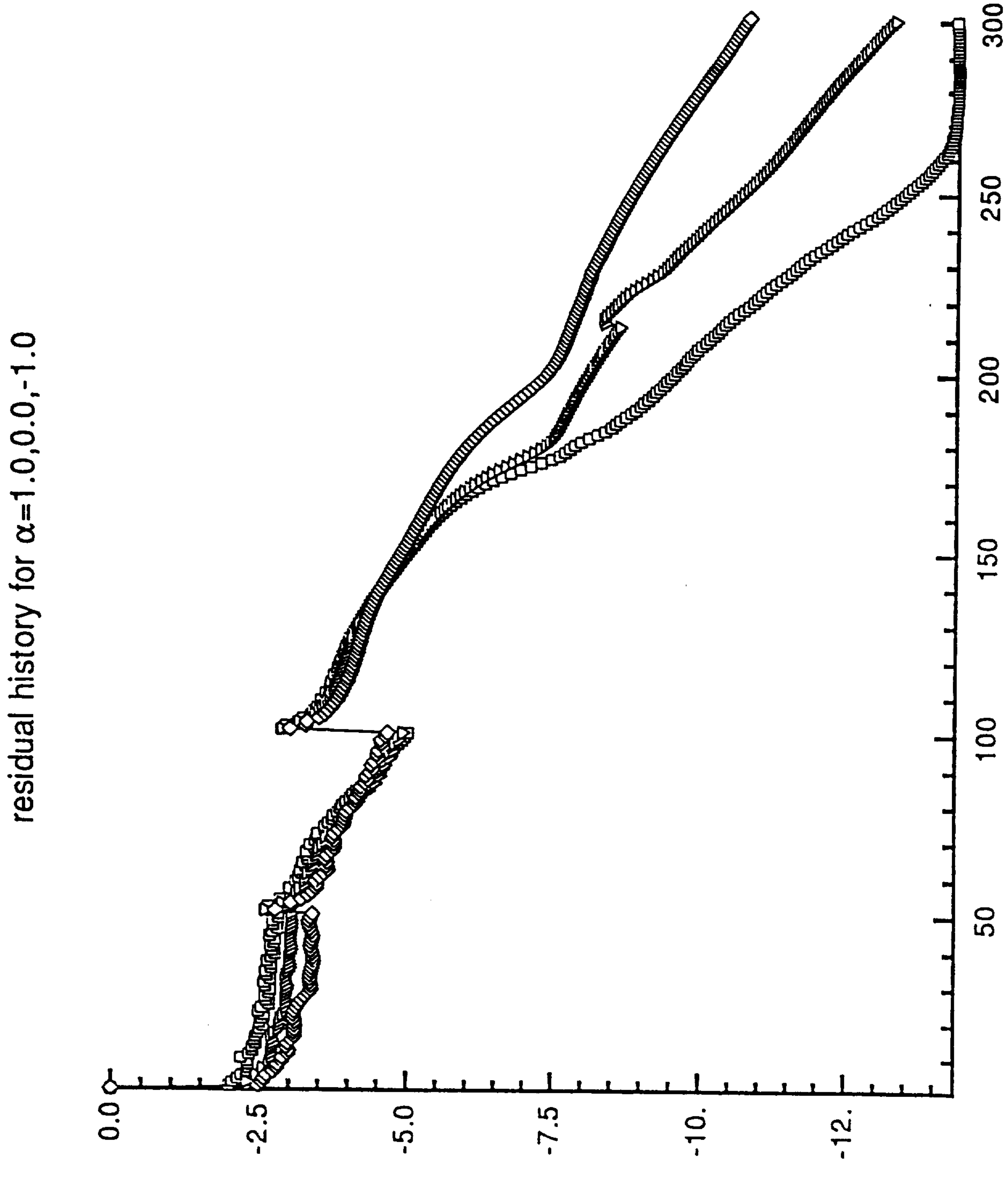
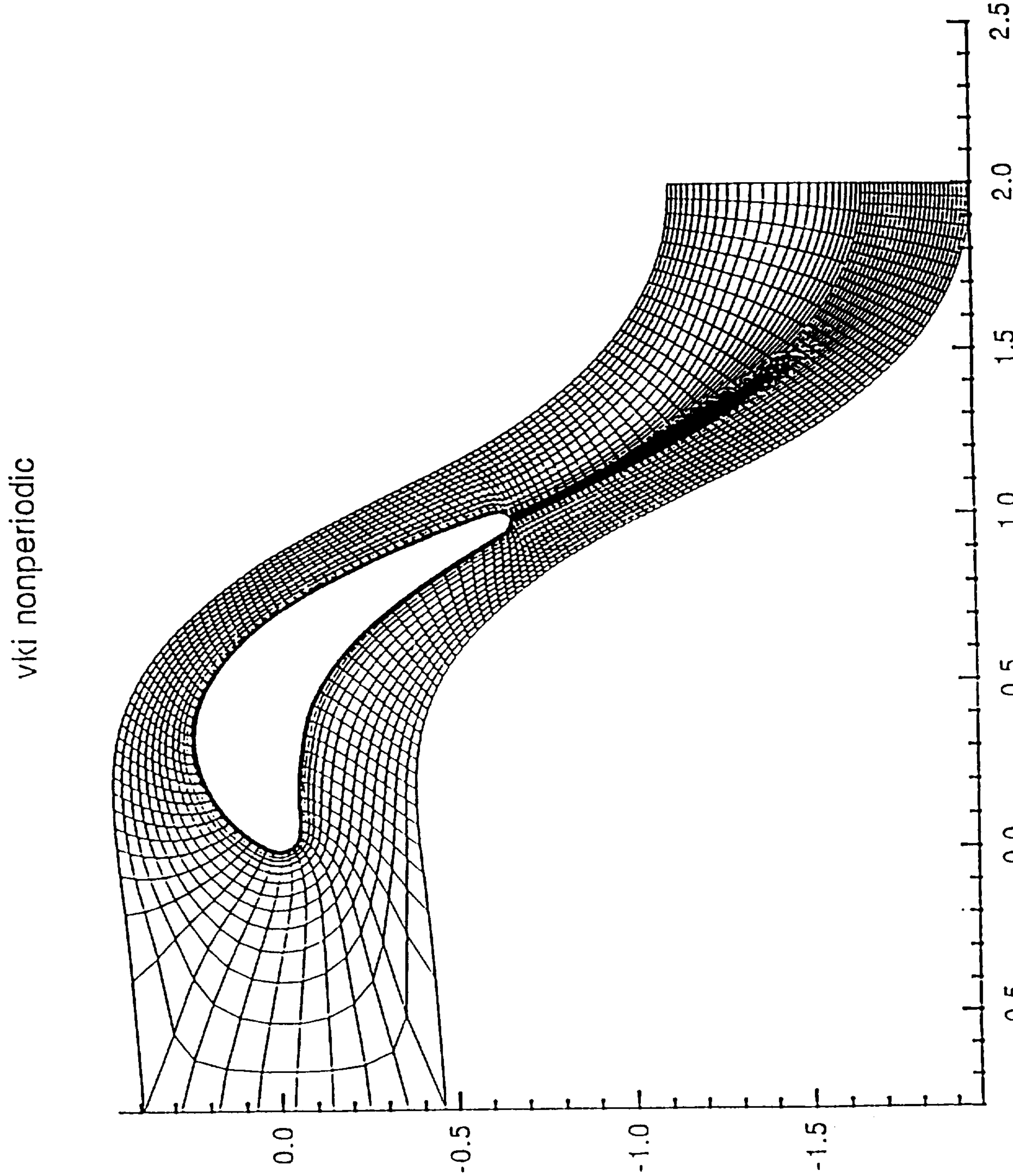


1. Mesh for turbulent flow around a NACA0012 cascade.



2. Convergence rate for turbulent flow around the cascade of fig. 1.



3. Mesh for turbulent flow around a VKI cascade.

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We consider preconditioning methods to accelerate convergence to a steady state for the incompressible fluid dynamic equations. The analysis relies on the inviscid equations. The preconditioning consists of a matrix multiplying the time derivatives. Thus the steady state of the preconditioned system is the same as the steady state of the original system. We compare our method to other types of pseudo-compressibility. For finite difference methods preconditioning can change and improve the steady state solutions. An application to viscous flow around a cascade with a non-periodic mesh is presented.

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