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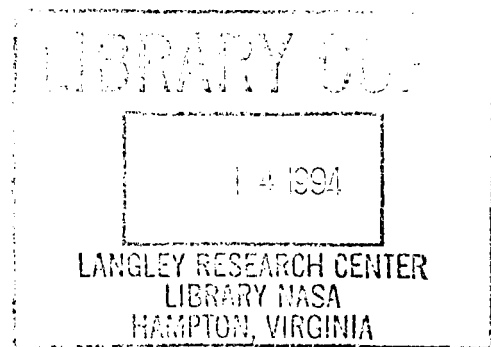
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## Abstract

Rational Bézier and B-spline representations of circles have been heavily publicized. However, all the literature assumes the rational Bézier segments in the homogeneous space are both planar and (equivalent to) quadratic. This creates the illusion that circles can only be achieved by planar and quadratic curves.

In this paper we show circles that are formed by higher order rational Bézier curves which are nonplanar in the homogeneous space. We also investigate the problem of whether it is possible to represent a complete circle with one Bézier curve. In addition, some other interesting properties of cubic Bézier arcs are discussed.

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# 1 Introduction

One of the most (if not the most) important reasons to use rational Béziers and B-splines for shape representation is to represent circles exactly. Quadratic is the lowest degree necessary to represent a circle with rational B-spline curves, and indeed, is the best representation for most applications. However, the requirement of multiple internal knots for the quadratic B-spline curves to represent a circle may create difficulties in some applications.

An excellent review on B-spline circles can be found in [1]. Other more recent work on this subject can be found in [2, 3]. All the literature implicitly assumes that the curve segments are planar in the homogeneous space and are quadratic, or equivalent to quadratic. Some of the theorems stated in the papers are true only with this implicit assumption. In this paper we show that there are more circles/circular arcs in higher order curves and in nonplanar cases.

# 2 Preliminaries

A rational Bézier curve in 2D is a vector valued rational function obtained by a perspective projection of a nonrational Bézier curve in the homogeneous space (3D). We let  $H$  denote a perspective map from the origin, in 3D, onto the plane  $w = 1$ . We have,

$$C(t) = (X(t), Y(t)) = H\{\tilde{C}(t)\} = H\{\tilde{X}(t), \tilde{Y}(t), W(t)\} \quad (1)$$

$$= H\{\sum_{i=0}^n B_i^n(t) \tilde{\mathbf{P}}_i\}, \quad (2)$$

$$= \sum_{i=0}^n B_i^n(t) w_i \frac{\mathbf{P}_i}{\sum_{j=0}^n B_j^n(t) w_j}, \quad (3)$$

where  $\tilde{\mathbf{P}}_i = (\tilde{x}_i, \tilde{y}_i, w_i) = (w_i x_i, w_i y_i, w_i)$ ,  $i = 0, \dots, n$ , are the homogeneous control points,  $B_i^n(t)$  are the  $n$ -th degree Bernstein functions,  $w_i$  are the

weights, and  $\mathbf{P}_i = H\{\tilde{\mathbf{P}}_i\} = (x_i, y_i)$  are the 2D control points.

It is easier to discuss the rational curves in the homogeneous space. For a circle, the nonrational homogeneous curves  $\tilde{C}(t)$  lie on a cone in the homogeneous space (Figure 1). This can easily be seen from the circle equation:

$$\left(\frac{\tilde{X}(t)}{\tilde{W}(t)}\right)^2 + \left(\frac{\tilde{Y}(t)}{\tilde{W}(t)}\right)^2 = 1 \quad (4)$$

$$\Rightarrow (\tilde{X}(t))^2 + (\tilde{Y}(t))^2 - (\tilde{W}(t))^2 = 0. \quad (5)$$

The above equations are those of a circle centered at the origin and with unit radius. The last equation is that of a curve on a cone. The perspective map  $H$  projects the curve onto the base circle of the cone, forming the circle  $C(t)$ .

B-spline circles can be constructed by piecing together Bézier curves going around the cone. To investigate B-spline circles of higher order, we need to study the possible Bézier curves on the cone. We assume our circles/circular arcs are centered at the origin and of unit radius. This confines the cone of our discussion to the one shown in Figure 1, but does not sacrifice the generality of our results.

Given a circular arc, there are an infinite number of homogeneous curves  $\tilde{C}(t)$  that can be mapped to the arc. For example, given the Bézier control points  $\mathbf{P}_i$  of an arc, all the homogeneous curves that have the same shape invariance [4] and that are from degree elevation/reduction [5] are mapped to the same arc. As an example, the shape invariance formula for cubic curves is

$$\frac{w_1^2}{w_0 w_2} = c_1, \frac{w_2^2}{w_1 w_3} = c_2. \quad (6)$$

Two cubic curves with the same  $c_1$  and  $c_2$  are of the same shape and mapped to the same circular arc. Geometrically, the effect of adjusting the weights is to move the control points  $\tilde{\mathbf{P}}_i$  along the line connecting  $\mathbf{P}_i$  and the origin (Figure 1). Through the shape invariance, we can fix the weights of the

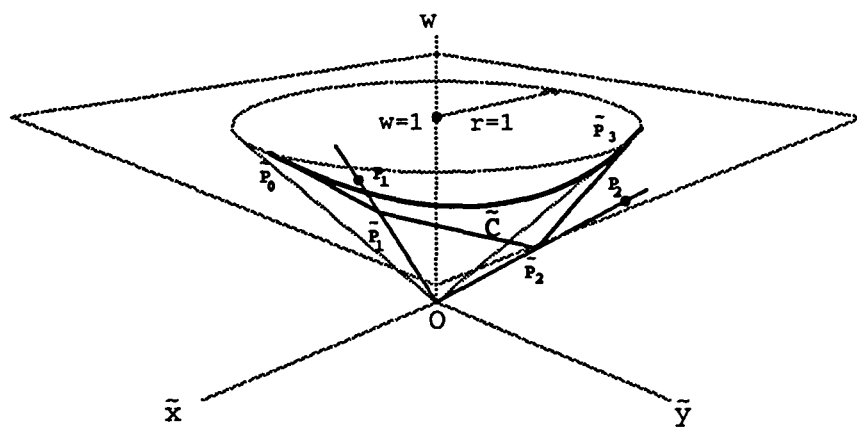


Figure 1: The Bézier curves for an arc lie on a cone in the homogeneous space.

control points at the two ends to unity. In 3D this means the end control points are on the base circle of the cone. Given a curve, after we have fixed the end weights, the position of the control points  $\tilde{\mathbf{P}}_i$  are fixed. We say all the curves obtained through degree elevation/reduction, from weight adjustment, and from affine transformation are equivalent. Our primary interest is in curves that are in different equivalent classes. Therefore, we assume the first and the last control points stay on the base circle of the cone.

### 3 A Bézier Curve as A Full Circle

The fact that a rational quadratic Bézier curve can not represent a full circle (even with negative weights) is evident from previous literature. The following is a simple proof, included for completeness.

All nonrational quadratic Bézier curves are parabolae, and all the parabolae on a cone are made by cutting the cone with planes parallel to the rulings. In Figure 2, we can push the cutting plane as close to the ruling ( $R$  in the figure) as possible, and hence make  $C(t)$  as close to a full circle as possible; but the cutting plane can not contain the ruling. When that happens,  $\tilde{C}(t)$  becomes a double line and  $C(t)$  is a point. All the quadratic Bézier arcs have the following control points:  $\tilde{\mathbf{P}}_0 = (\cos(\vartheta), -\sin(\vartheta), 1)$ ,  $\tilde{\mathbf{P}}_1 = (1, 0, \cos(\vartheta))$ ,  $\tilde{\mathbf{P}}_2 = (\cos(\vartheta), \sin(\vartheta), 1)$ , where  $\vartheta$  is one-half of the sweeping angle of the arc. If no negative weights are allowed ( $\cos(\vartheta) > 0$ ),  $C(t)$  is less than 180 degrees. When the middle weight is zero ( $\vartheta = 90$  degrees),  $C(t)$  is exactly 180 degrees.

The next natural question is: how about cubic curves? If quadratic curves can not represent a full circle, can cubic curves do it? Unfortunately, the answer is negative. We provide the following simple proof.

For the cubic  $H(\tilde{C}(t))$  to be a circle,  $\tilde{\mathbf{P}}_3$  must be on the line  $\overline{\tilde{\mathbf{P}}_0\mathbf{O}}$ , and we can assume both  $\tilde{\mathbf{P}}_0$  and  $\tilde{\mathbf{P}}_3$  are at the same point in 3D. Since  $\tilde{C}(t)$  is on the cone, both the tangents of  $\tilde{C}(t)$  at  $\tilde{\mathbf{P}}_0$  and at  $\tilde{\mathbf{P}}_3$  must lie on the tangent

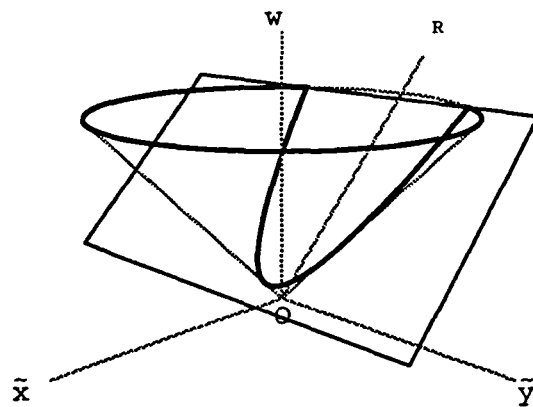


Figure 2: A single quadratic Bézier curve can not be a full circle.



plane of the cone at  $\tilde{\mathbf{P}}_0$ . By the tangent property of Bézier curves,  $\tilde{\mathbf{P}}_1$  is on the tangent line of the curve at  $\tilde{\mathbf{P}}_0$  and  $\tilde{\mathbf{P}}_2$  is on the tangent line of the curve at  $\tilde{\mathbf{P}}_3$ . From above, we have that all the four control points are on the tangent plane of the cone at  $\tilde{\mathbf{P}}_0$ ; hence by the convex hull property of Bézier curves, the curve  $\tilde{C}(t)$  is on the tangent plane. We know that the tangent plane only intersects the cone at a line.  $C(t)$  is a point.

How about quartic curves? What are the quartic curves that are circles? To find all the quartic circles, we substitute the equation for the homogeneous Bézier curves into the equation of the cone (Equation 5). After equating the coefficients of  $B_i^8(t)$ ,  $i = 0, \dots, 8$ , to zero, we obtain nine equations.

Without loss of generality, we assume  $\tilde{\mathbf{P}}_0 = \tilde{\mathbf{P}}_4 = (1, 0, 1)$ . Since  $\tilde{\mathbf{P}}_1$  and  $\tilde{\mathbf{P}}_3$  are on the tangent plane at  $\tilde{\mathbf{P}}_0$ , we also have  $\tilde{x}_1 = w_1$  and  $\tilde{x}_3 = w_3$ . With these known conditions, the nine equations reduce to five. With some simple algebraic manipulation, we have the following equations:

$$\tilde{y}_3 = -\tilde{y}_1 \quad (7)$$

$$\tilde{x}_3 = -\tilde{x}_1 \quad (8)$$

$$3\tilde{x}_2 + 4\tilde{y}_1^2 - 3w_2 = 0 \quad (9)$$

$$\tilde{x}_1\tilde{x}_2 + \tilde{y}_1\tilde{y}_2 - \tilde{x}_1w_2 = 0 \quad (10)$$

$$9\tilde{x}_2^2 - 8\tilde{y}_1^2 + 9\tilde{y}_2^2 - 9w_2^2 = 0 \quad (11)$$

The last three equations have two sets of nontrivial solutions:

$$\tilde{y}_1 = \alpha, \tilde{x}_2 = -\frac{3w_2 - 4\tilde{x}_1^2 + 2}{3}, \tilde{y}_2 = \frac{4}{3}\tilde{x}_1\alpha; \quad (12)$$

$$\tilde{y}_1 = -\alpha, \tilde{x}_2 = -\frac{3w_2 - 4\tilde{x}_1^2 + 2}{3}, \tilde{y}_2 = -\frac{4}{3}\tilde{x}_1\alpha, \quad (13)$$

where  $\alpha = \sqrt{\frac{3w_2}{2} - \tilde{x}_1^2 + \frac{1}{2}}$ . We have the following control points for the quartic circles:

$$\tilde{\mathbf{P}}_0 = (1, 0, 1) \quad (14)$$

$$\tilde{\mathbf{P}}_1 = (\tilde{x}_1, \pm\alpha, \tilde{x}_1) \quad (15)$$

$$\tilde{\mathbf{P}}_2 = \left(-\frac{3w_2 - 4\tilde{x}_1^2 + 2}{3}, \pm\frac{4}{3}\tilde{x}_1\alpha, w_2\right) \quad (16)$$

$$\tilde{\mathbf{P}}_3 = (-\tilde{x}_1, \mp\alpha, -\tilde{x}_1) \quad (17)$$

$$\tilde{\mathbf{P}}_4 = (1, 0, 1) \quad (18)$$

In order for  $\alpha$  to be a real number, we must have

$$w_2 > -\frac{1}{3}, \text{ and} \quad (19)$$

$$-\sqrt{\frac{3w_2 + 1}{2}} < \tilde{x}_1 < \sqrt{\frac{3w_2 + 1}{2}}. \quad (20)$$

Note that when  $w_2 = -\frac{1}{3}$  or  $\tilde{x}_1 = \pm\sqrt{\frac{3w_2+1}{2}}$ ,  $C(t)$  is not a circle.

With this set of control points, it is easy to check that  $W(t) > 0$ , for  $t \in [0, 1]$ . That is, points on the circle can be computed safely. However, from Equations 15 and 17,  $w_1 = -w_3$ ; it is impossible to have a quartic Bézier circle with positive weights.

If we require the curve to be symmetric with respect to the plane  $y = 0$ , we have  $\tilde{\mathbf{P}}_0 = \tilde{\mathbf{P}}_4 = (1, 0, 1)$  and

$$\tilde{\mathbf{P}}_1 = (0, \pm\beta, 0) \quad (21)$$

$$\tilde{\mathbf{P}}_2 = \left(-\frac{2}{3} - w_2, 0, w_2\right) \quad (22)$$

$$\tilde{\mathbf{P}}_3 = (0, \mp\beta, 0) \quad (23)$$

$$(24)$$

where  $\beta = \sqrt{\frac{1}{2} + \frac{3}{2}w_2}$ . In particular, when  $w_2 = \frac{1}{3}$ , we have

$$\tilde{\mathbf{P}}_0 = (1, 0, 1), \tilde{\mathbf{P}}_1 = (0, 1, 0), \tilde{\mathbf{P}}_2 = \left(-1, 0, \frac{1}{3}\right), \quad (25)$$

$$\tilde{\mathbf{P}}_3 = (0, -1, 0), \tilde{\mathbf{P}}_4 = (1, 0, 1). \quad (26)$$

Figure 3 shows this curve. This curve can also be obtained by squaring the quadratic semicircle with control points<sup>1</sup>

$$\tilde{\mathbf{P}}_0 = (1, 0, 1), \tilde{\mathbf{P}}_1 = (0, 1, 0), \tilde{\mathbf{P}}_2 = (-1, 0, 1). \quad (27)$$

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<sup>1</sup>This fact was first pointed out to the author by Tim Strotman at SDRC.

Now we have a Bézier curve (a rational polynomial piece) as a circle. Unfortunately, all the quartic Bézier circles are with zero or negative weights. This is highly undesirable in many applications. The next natural question is: is it possible to have a quintic Bézier circle with positive weights?

This question can be answered easily by elevating the degree of the curves in Equations 14-18. The conditions for the degree-elevated curves to have positive weights can be easily determined. For example, the curve in Equations 25-26 satisfies the condition, and all the weights of the degree-elevated curve are positive.

## 4 Cubic Semicircles

From the previous section we know that quadratic Bézier curves can not represent a semicircle with positive weights. In this section we study cubic Bézier curves as a semicircle.

We begin by substituting the homogeneous cubic Bézier curve equation into the equation of the cone (Equation 5). After equating the coefficients of  $B_i^6(t)$ ,  $i = 0, \dots, 6$ , to zero, we obtain seven equations.

Again we assume  $\tilde{\mathbf{P}}_0 = (1, 0, 1)$ ,  $\tilde{\mathbf{P}}_3 = (\cos(\theta), \sin(\theta), 1)$ , where  $\theta$  is the angle of the circular arc, and  $\theta = \pi$  for a semicircle. Since  $\tilde{\mathbf{P}}_1$  and  $\tilde{\mathbf{P}}_2$  must lie on the tangent plane at  $\tilde{\mathbf{P}}_0$ , we have

$$w_1 = \tilde{x}_1 \tag{28}$$

and

$$w_2 = \cos(\theta)\tilde{x}_2 + \sin(\theta)\tilde{y}_2. \tag{29}$$

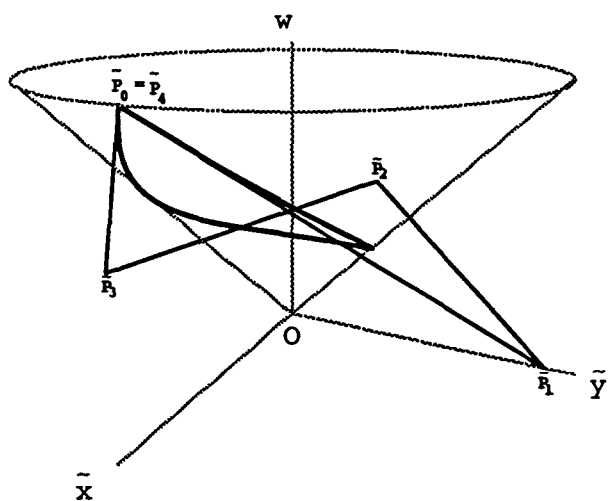


Figure 3: A quartic Bézier curve as a circle.

With these conditions and some algebraic manipulation, the seven equations become three:

$$2(1 - \cos(\theta))\tilde{x}_2 + 3\tilde{y}_1^2 - 2\sin(\theta)\tilde{y}_2 = 0 \quad (30)$$

$$9(1 - \cos(\theta))\tilde{x}_1\tilde{x}_2 + 9\tilde{y}_1\tilde{y}_2 - 9\sin(\theta)\tilde{x}_1\tilde{y}_2 + \cos(\theta) - 1 = 0 \quad (31)$$

$$\begin{aligned} 3\sin^2(\theta)\tilde{x}_2^2 - 2(1 - \cos(\theta))\tilde{x}_1 + 3\cos^2(\theta)\tilde{y}_2^2 + 2\sin(\theta)\tilde{y}_1 \\ - 6\cos(\theta)\sin(\theta)\tilde{x}_2\tilde{y}_2 = 0 \end{aligned} \quad (32)$$

Note that all the cubic Bézier arcs with angles  $\theta$  and  $(2\pi - \theta)$  satisfy the above equations.

By substituting  $\theta = \pi$  into the above equations, we obtain the following set of solution:

$$\tilde{\mathbf{P}}_0 = (1, 0, 1), \tilde{\mathbf{P}}_1 = \left(\frac{1}{3\alpha^2}, \frac{2\alpha}{3}, \frac{1}{3\alpha^2}\right), \quad (33)$$

$$\tilde{\mathbf{P}}_2 = \left(-\frac{\alpha^2}{3}, \frac{2}{3\alpha}, \frac{\alpha^2}{3}\right), \tilde{\mathbf{P}}_3 = (-1, 0, 1) \quad (34)$$

where  $\alpha = 3\tilde{y}_1/2$ . The 2D control points are  $\mathbf{P}_0 = (1, 0)$ ,  $\mathbf{P}_1 = (1, 2\alpha^3)$ ,  $\mathbf{P}_2 = (-1, 2/\alpha^3)$ ,  $\mathbf{P}_3 = (-1, 0)$ . Several interesting properties of the curves are discussed below.

First we provide a geometric method for constructing an arbitrary rational cubic Bézier curve forming a semicircle. As shown in Figure 4, we draw two lines  $x = 1$  and  $x = -1$ . After that, we draw an arbitrary tangent line,  $L$ , to the semicircle. Let the tangent point be  $q$ .  $L$  intersects the vertical lines at  $y_r$  and  $y_l$ . We find the points,  $(1, 2y_r)$  and  $(-1, 2y_l)$ , on the lines. These two points, along with  $(1, 0)$  and  $(-1, 0)$ , are the 2D control points of the cubic semicircle. The weights are

$$w_1 = \frac{1}{3(y_r)^{2/3}}, w_2 = \frac{1}{3(y_l)^{2/3}}. \quad (35)$$

To prove the correctness of the construction, we observe that  $y_1y_2 = 4$  (from Equations 33 and 34). The fact that  $y_ly_r = 1$  for any line tangent to the

semicircle can be proved by simple algebra. The rest of the construction follows directly from Equations 33 and 34. The angle  $\phi$  (Figure 4) at the tangent point is related to  $\alpha$  by  $\tan(\phi) = (2\alpha^3)/(1 - \alpha^6)$ .

We check the parametrization of the semicircles by examining the points at  $t = 1/2$ . It is easy to calculate that  $C(1/2) = (2\alpha)/(1 - \alpha^2)$ . The point is shown in Figure 4 (as an '\*'), along with the line tangent to the semicircle at this point. This tangent line intersects  $x = 1$  and  $x = -1$  at  $y = \alpha$  and  $y = 1/\alpha$  respectively. From the construction we have that for  $\alpha < 1$ ,  $C(1/2)$  lies between point  $(0, 1)$  and  $q$ . As  $q$  moves toward  $(1, 0)$  (with decreasing  $\alpha$ ),  $C(1/2)$  follows, producing increasingly skewed parametrization. An interesting case is when  $q$  is at  $(0, 1)$ . In this case  $\alpha = 1$ , and the curve is symmetric with respect to the  $x = 0$  line. The control points of this curve are  $\tilde{\mathbf{P}}_0 = (1, 0, 1)$ ,  $\tilde{\mathbf{P}}_1 = (1/3, 2/3, 1/3)$ ,  $\tilde{\mathbf{P}}_2 = (-1/3, 2/3, 1/3)$ ,  $\tilde{\mathbf{P}}_3 = (-1, 0, 1)$ , which can also be obtained by elevating the degree of the quadratic semicircle [1].

## 5 General Bézier Cubic Arcs

In this section we discuss some properties of cubic Bézier arcs. In particular we ask the question: what is the largest angle achievable by a cubic Bézier curve without negative weights?

For an arc of angle  $2\vartheta$  we can write the control points of the cubic Bézier curve as:  $\tilde{\mathbf{P}}_0 = (\cos(\vartheta), -\sin(\vartheta), 1)$ ,  $\tilde{\mathbf{P}}_1 = (\tilde{x}_1, \tilde{y}_1, w_1)$ ,  $\tilde{\mathbf{P}}_2 = (\tilde{x}_2, \tilde{y}_2, w_2)$ ,  $\tilde{\mathbf{P}}_3 = (\cos(\vartheta), \sin(\vartheta), 1)$ . There is one and only one curve whose control points are symmetric with respect to the plane  $\tilde{y} = 0$ , i.e., a curve with  $\tilde{x}_1 = \tilde{x}_2$ ,  $\tilde{y}_1 = -\tilde{y}_2$ , and  $w_1 = w_2$ . The cubic curve with this property arises from the degree elevation of the quadratic Bézier arc with the same angle. The correctness of the statement is obvious, and its proof is omitted here.

Curves with positive weights are particularly useful in computer applications. As pointed out previously, the largest angle of an arc achievable by a

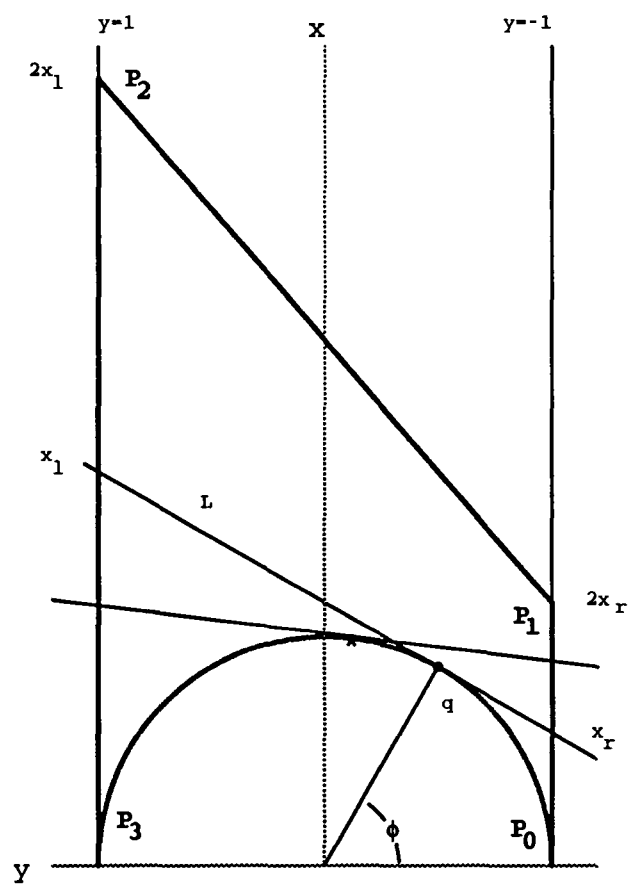


Figure 4: Geometric construction of a cubic Bézier semicircle.

quadratic Bézier curve without negative weights is 180 degrees. A natural question to ask is: what is the largest angle achievable by a cubic Bézier curve without negative weights?

From the proof in Section 3, we know that a cubic Bézier curve can not form a full circle. In the following we prove that a cubic Bézier curve can form an arc near 360 degrees only if some of its weights are negative. We start by assuming  $\tilde{\mathbf{P}}_0 = (1, 0, 1)$ ,  $\tilde{\mathbf{P}}_3 = (\cos(\theta), \sin(\theta), 1)$ . When  $\theta$  approaches 360 degrees, we have

$$\cos(\theta) \simeq (1 - \delta^2), \sin(\theta) \simeq -\delta, \quad (36)$$

where  $\delta = (2\pi - \theta)$  is an arbitrarily small number. From Equations 28, 29, and 30, we have  $\tilde{x}_0 = 1$ ,  $\tilde{x}_1 = w_1$ ,  $\tilde{x}_2 = w_2 - \frac{3}{2}\tilde{y}_1^2$ ,  $\tilde{x}_3 \simeq 1 - \delta^2/2$  (see Appendix). In the Appendix we prove that  $\tilde{y}_1$  is linearly proportional to  $\delta$  and that when  $w_2$  is less than one,  $w_1$  is negative. If all the weights are positive, i.e.,  $w_1 > 0$  and  $w_2 > 1$ , then, all the  $\tilde{x}_i$ 's are positive when  $\delta$  is small. By the convex hull property of Bézier curves, a curve lies on the  $x > 0$  half-space if the  $x$ -components of all its control points are positive. Therefore, the curve can not be mapped to an arc greater than 180 degrees. We conclude that a cubic Bézier curve forming an arc near 360 degrees must have negative weights.

The largest angle achievable by a cubic Bézier curve with non-negative weights and symmetric control points is 240 degrees. It is achieved by elevating the degree of a quadratic Bézier arc with the same angle. The author has not been able to prove that 240-degree is the largest angle achievable by all cubic Bézier curves; however the author conjectures that this is true.

## 6 Conclusion

We investigated some interesting properties of Bézier curves as circles/circular arcs. Most of the properties are derived from the fact that all the rational curves forming an arc lie on a cone in the homogeneous space. Given an angle, Equations 30-32 define all the possible cubic Bézier curves mapping



to the arc with that angle (with the assumption on the positions of  $\tilde{\mathbf{P}}_0$  and  $\tilde{\mathbf{P}}_3$ ).

There is one degree of freedom in Equations 30-32. In the Appendix, we show a way to simplify the equations. However, in order to solve the equations, a complex cubic equation (Equation 45) has to be solved. In Equation 45, the degree of freedom is chosen to be  $w_2$ . For each value of  $w_2$ , there may be up to three solutions to Equations 30-32, depending on the determinant of Equation 45. Methods like Cardan's Solution or Trigonometric Solution [6] can be employed to solve the equation. However, to find an explicit solution as a function of the angle (like the one for quadratic curves) is difficult; especially, some of the solutions may represent degenerated cases or solutions for the complementary arc. Even when the solutions are obtained, the geometric interpretation of the solutions is quite illusive, except for certain special cases.

In this paper we show that cubic Bézier arcs are much richer in variety than the quadratic ones. How to take advantage of this variety to create cubic B-spline circular arcs or circles with better properties, e.g., higher continuity, better parametrization, should be a topic of future research.

## Appendix

In this appendix, we show that when the angle  $\theta$  of a cubic arc is near  $2\pi$ ,  $\tilde{y}_1$  is linearly proportional to  $\delta = (2\pi - \theta)$ . In the latter half we prove that when  $w_2 < 1$ ,  $w_1 < 0$ .

We start by substituting Equation 29 into Equations 30-32. From Equation 30, we have

$$\tilde{x}_2 = w_2 - \frac{3}{2}\tilde{y}_1^2. \quad (37)$$

From Equation 32, we have

$$\tilde{x}_1 = \frac{1}{2(1 - \cos(\theta))}$$

$$\left[ 3 \left( w_2 - \frac{3}{2} \tilde{y}_1^2 \right)^2 + 3 \left( \frac{3\cos(\theta)}{2\sin(\theta)} \tilde{y}_1^2 + \frac{(1 - \cos(\theta))}{\sin(\theta)} w_2 \right)^2 - 3w_2^2 \right]. \quad (38)$$

Substituting  $\tilde{x}_2$  into Equation 37, we obtain

$$\tilde{y}_2 = \frac{3\cos(\theta)}{2\sin(\theta)} \tilde{y}_1^2 + \frac{(1 - \cos(\theta))}{\sin(\theta)} w_2. \quad (39)$$

After eliminating  $\tilde{x}_1, \tilde{x}_2$  and  $\tilde{y}_2$  from Equation 31 (with Equations 37-39), we obtain

$$\frac{27\tilde{y}_1^2}{4(1-\cos(\theta))} \left[ \frac{27}{4} \frac{\tilde{y}_1^4}{\sin(\theta)^2} + \frac{9(\cos(\theta)-1)}{\sin(\theta)^2} \tilde{y}_1^2 w_2 + \frac{3(1-\cos(\theta))^2}{\sin(\theta)^2} w_2^2 + 2\sin(\theta) \tilde{y}_1 \right]$$

$$- \frac{27\cos(\theta)}{2\sin(\theta)} \tilde{y}_1^3 - \frac{9(1-\cos(\theta))}{\sin(\theta)} \tilde{y}_1 w_2 - (\cos(\theta) - 1) = 0 \quad (40)$$

$$\Rightarrow \left( \frac{\gamma^3}{8\sin(\theta)(1-\cos(\theta))} + 1 \right) \left( \frac{\gamma^3}{2\sin(\theta)} - \frac{3(1-\cos(\theta))\gamma w_2}{\sin(\theta)} \right)$$

$$- \frac{3\alpha w_2}{4\sin(\theta)} \left( \frac{\gamma^3}{2\sin(\theta)} - \frac{3\gamma(1-\cos(\theta))w_2}{\sin(\theta)} \right) - (\cos(\theta) - 1) = 0 \quad (41)$$

$$\Rightarrow \mu(\mu + 4(1 - \cos(\theta))) + 4(\cos(\theta) - 1)^2 = 0 \quad (42)$$

$$\Rightarrow \mu = 2(\cos(\theta) - 1) \quad (43)$$

where  $\gamma = 3\tilde{y}_1$ ,

$$\mu = \frac{\gamma^3}{2\sin(\theta)} - \frac{3\gamma(1 - \cos(\theta))w_2}{\sin(\theta)}. \quad (44)$$

From Equations 43 and 44 we obtain a cubic equation for  $\gamma$ ,

$$\gamma^3 - 6(1 - \cos(\theta))w_2\gamma + 4\sin(\theta)(1 - \cos(\theta)) = 0. \quad (45)$$

The equation of  $\tilde{x}_1$  (Equation 38) can be simplified with Equation 45:

$$\tilde{x}_1 = -\frac{9}{4\sin(\theta)^2}w_2\tilde{y}_1^2 + \frac{(1+2\cos(\theta))}{2\sin(\theta)}\tilde{y}_1 + \frac{3}{2(1+\cos(\theta))}w_2^2. \quad (46)$$

When  $\theta$  is near  $2\pi$  and  $\delta$  is small,  $\sin(\theta) \simeq -\delta$  and  $\cos(\theta) \simeq (1 - \delta^2/2)$ . The equation for  $\tilde{x}_1$  becomes:

$$\tilde{x}_1 = -\frac{9}{4\delta^2}\tilde{y}_1^2w_2 - \frac{3}{2\delta}\tilde{y}_1 + \frac{3}{4}w_2^2 \quad (47)$$

The cubic equation for  $\gamma$  becomes:

$$\gamma^3 - 3\delta^2w_2\gamma - 2\delta^3 = 0. \quad (48)$$

The determinant of this cubic equation is  $-108Q$ , where

$$Q = \delta^6(1 - w_2^3). \quad (49)$$

Equation 48 has one real root, multiple roots (real), or three different real roots, if  $w_2 < 1$ ,  $w_2 = 1$ , or  $w_2 > 1$ , respectively. For each of the cases, the solutions for  $\gamma$  can be calculated with either Cardan's Solution, when  $w_2 \leq 1$ , or Trigonometric Solution, when  $w_2 > 1$ . In all the cases,  $\gamma$  is linearly proportional to  $\delta$ , and so is  $\tilde{y}_1$ . In the following, we demonstrate the case for  $w_2 < 1$ .

When  $w_2 < 1$ , the only real root in the equation is

$$\gamma = \delta \left[ \left(1 + \sqrt{1 - w_2^3}\right)^{1/3} + \left(1 - \sqrt{1 - w_2^3}\right)^{1/3} \right] = \delta\zeta. \quad (50)$$

With this solution for  $\tilde{y}_1(= \gamma/3)$ ,  $\tilde{x}_1$  in Equation 47 becomes

$$\tilde{x}_1 = -\frac{\zeta^2}{4}w_2 - \frac{\zeta}{2} + \frac{3}{4}w_2^2. \quad (51)$$

With some algebra, it can be proved that  $\tilde{x}_1 < 0$  for  $0 < w_2 < 1$ . Since  $w_1 = \tilde{x}_1$ , we have  $w_1 < 0$  for  $0 < w_2 < 1$ .

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