Theoretical and Simulated Performance for a Novel Frequency Estimation Technique

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ABSTRACT

A low complexity, open-loop, discrete-time, delay-multiply-average (DMA) technique for estimating the frequency offset for digitally modulated MPSK signals is investigated. A nonlinearity is used to remove the MPSK modulation and generate the carrier component to be extracted. Theoretical and simulated performance results are presented and compared to the Cramer-Rao lower bound (CRLB) for the variance of the frequency estimation error. For all signal-to-noise ratios (SNRs) above threshold, it is shown that the CRLB can essentially be achieved with linear complexity.

INTRODUCTION

Most conventional burst transmission systems with frequency uncertainty provide a preamble of unmodulated carrier and/or a carrier modulated with a known symbol pattern, for initial frequency estimation and synchronization purposes. There are also many other applications where it is desirable to estimate the frequency error from a modulated signal with unknown data. In either case, it is desirable to have a fast, efficient, and accurate frequency estimation algorithm, both for initial acquisition and tracking purposes.

In this paper, a low complexity, open-loop, discrete-time, delay-multiply-average (DMA) approach to estimating the frequency offset for digitally modulated signals is investigated. M-ary phase shift keyed (MPSK) signaling formats are considered. An M-power-type nonlinearity can be used to generate a carrier component when the data symbols are unknown. The special case of pure carrier and/or known symbols is included by setting M=1. Performance is theoretically approximated and compared to the Cramer-Rao lower bound (CRLB) for the variance of the frequency estimation error. Simulated performance is also presented and compared to the theoretical approximations and bounds. It is shown that, when optimum delays are employed, performance is within about 0.5 dB of the CRLB for all signal-to-noise ratios (SNRs) above threshold. A simple extension to the DMA algorithm, which approximates true maximum-likelihood (ML) estimation, is also examined. With the ML extension, the CRLB is essentially achieved for all SNRs above threshold.

Previously known open-loop techniques which provide performance close to the CRLB typically involve some form of fast Fourier transform (FFT) processing [1]. The complexity of FFT based algorithms is order $KL \log_2(KL)$ where K is the observation time in samples and L is the zero-stuffing factor required to obtain the desired frequency resolution using an FFT of size KL. Small L values of 2 or 4 are usually recommended when the FFT is used only for a coarse search [1]. To approach the CRLB, additional processing is required to perform a fine search for the peak of the likelihood function. The complexity of the DMA based algorithm presented in [2] is order KB where B is the number of DMA branches employed. The number of branches required depends on the desired threshold SNR, but can typically be made fewer than $\log_2(K)$ for many applications. For example, 3 branches were found to be sufficient for the MSAT application described in [3], with K=100. This paper presents a modified version of the basic DMA algorithm described in [2] and a simple ML extension. In addition to providing improved performance, the complexities of the new DMA algorithm and its ML extension are both of order K.

FREQUENCY ESTIMATION

Single Branch DMA Approach

Figure 1 shows an open-loop frequency phasor estimator, based on the DMA approach. The sampled (discrete) complex baseband received signal, $\{r_k\}$, is modeled as

$$\begin{aligned} r_k &= A a_k \exp(j\omega k) + w_k \\ &= A a_k W^k + w_k \end{aligned} \tag{1}$$

where the complex phasor, W, is defined as

$$W = \exp(j\omega), \tag{2}$$

A is the signal's complex amplitude, a_k represents the MPSK modulation data symbols, given by

$$a_k = \exp(j2\pi m/M), \quad m \in \{0,\ldots,M-1\}, \tag{3}$$

ω is the frequency offset measured in radians per sample or symbol period, T, and w_k is additive noise.

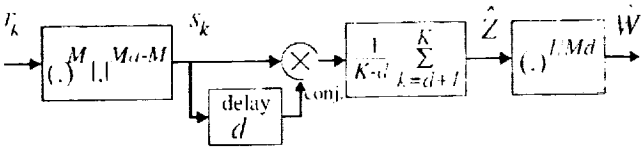

Fig. 1: Single branch DMA frequency estimator.

The sample SNR at the receiver input is defined as

$$\gamma = \frac{P_r}{P_w} = \frac{|A|^2}{\mathrm{E}\left[|w_k|^2\right]} \tag{4}$$

where E[.] denotes the expected value operator. For mathematical convenience, and without any loss in generality, it is assumed that $|A|=1$, so that $P_r=1$ and $P_w=1/\gamma$.

The received signal is first passed through a generalized M-power-type nonlinearity to remove the MPSK modulation. The nonlinearity is generalized in the sense that the phase is multiplied by M but the amplitude can be raised to a different power, namely M_a. From (1), the signal at the output of the nonlinearity is given by

$$s_k = r_k^M |r_k|^{M_a - M} = A^{M_a} W^{kM} + n_k \tag{5}$$

This nonlinearity is equivalent to that introduced in [4] for carrier phase estimation. The noise term, n_k, is quite complicated in general. Although simulation results are presented for different values of M_a and M, the theoretical approximations are restricted to the case of $M_a=M$. With this restriction, n_k is given by

$$n_k = \sum_{m=1}^{M} \binom{M}{m} w_k^m A^{M-m} W^{k(M-m)} \tag{6}$$

The objective is to obtain an estimate of W, since this phasor contains the phase rotation over a single sample period due to the frequency offset, ω. Multiplying the received signal samples, $\{r_k\}$, by the sequence $\{W^{-k}\}$ would remove the frequency offset. An estimate of

$$Z = W^{Md} \tag{7}$$

is obtained first, and is given by

$$\hat{Z} = \frac{1}{K-d} \sum_{k=d+1}^{K} s_k s_{k-d}^* \tag{8}$$

where K is the number of samples used in the measurement, and d is the delay in sample periods. The estimate of W is then given by

$$\hat{W} = [\hat{Z}]^{1/Md} \tag{9}$$

In the absence of noise and possible phase ambiguities associated with multiple complex roots, it is clear that $\hat{Z} = Z$ and $\hat{W} = W$.

Multiple Branch DMA Approach

There is a fundamental phase ambiguity problem associated with all frequency estimators of this type. Without a previous estimate for guidance, the maximum resolvable frequency offset is less than $1/(2TMd)$ Hz. The larger the delay, the more potential phase ambiguities. The phase ambiguity problem results from not knowing which of the Md complex roots to choose. In most cases the ambiguity can be resolved by employing a ball-park estimate to guide the selection of the appropriate complex root. Given a previous estimate, obtained using delay d_{b-1}, a new estimate, using delay $d_b > d_{b-1}$, can be obtained as follows

$$\hat{W}_b = \hat{W}_{b-1} \left[\frac{\hat{Z}_b}{\hat{W}_{b-1}^{Md_b}} \right]^{1/Md_b} \tag{10}$$

If the delays are selected such that

$$d_b = p_b\, d_{b-1} \qquad , b = 2 \ldots B \tag{11}$$

where p_b is an integer greater than or equal to 2, then (10) is equivalent to

$$\hat{W}_b = \hat{W}_{b-1} \left[\frac{\hat{Z}_b}{\hat{Z}_{b-1}^{p_b}} \right]^{1/Md_b} \tag{12}$$

If the root operation in (10) or (12) always takes the principle root and the phase difference between the current and previous estimate is within π/Md_b, which is the maximum resolvable phase difference with delay d_b, then the overall result corresponds to the correct root and the phase ambiguity is resolved. If the previous phase error is too large to resolve the phase ambiguity, then the incorrect root which is closest to the previous estimate will be selected. Equations (10) and (12) are clearly equivalent to (9) if the appropriate root is selected.

The new DMA based algorithm is depicted in Figure 2. The approach is similar to that given in [2], in that multiple DMA branches are used to resolve potential phase ambiguities as the branch delays increase. The method shown for resolving phase ambiguities is that of (12). This method can be used because the delays are specifically chosen to be increasing powers of 2, resulting in $p_b=2$ for each branch. The major difference between the DMA approach of Figure 2 and the DMA approach of [2] is the rotate-add-decimate (RAD) operation, which is performed repeatedly on the signal, s_k, at the output of the nonlinearity.

To simplify the description of the technique, the observation time in samples is restricted to be

$$K = 3 \times 2^{B-2}, \qquad B \geq 2, \tag{13}$$

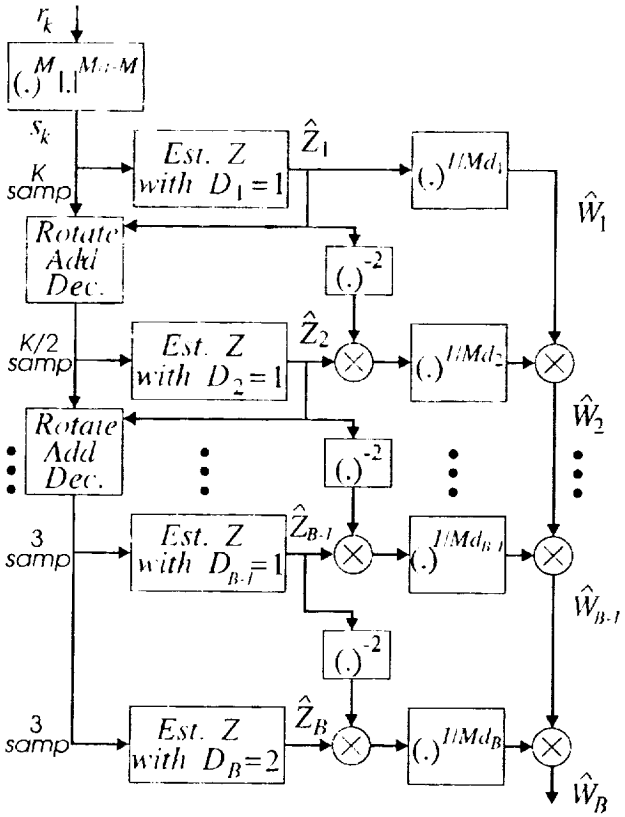

Fig. 2: Bank of B frequency estimators with rotate-add-decimate (RAD) processing.

where it is assumed that at least the bottom 2 branches shown in Figure 2 are employed. More general values of K can be accommodated, but the values of K given in (13) are the most convenient. The desired delays in original samples for the B branches are

$$d_b = 2^{b-1}, \qquad b = 1 \ldots B \tag{14}$$

The RAD operation always decimates by 2. Thus the corresponding delays in decimated samples for the B branches are given by

$$\begin{aligned} D_b &= 1, & b &= 1 \ldots B-1 \\ &= 2, & b &= B \end{aligned} \tag{15}$$

Only 3 samples are processed in the final 2 branches and the RAD operation is not used between the last 2 branches. This is why a delay of 2 samples is used in the final branch. In [2], it is shown that the optimum delay for the final branch is 2/3 the number of samples.

The idea behind the RAD operation is to pseudo-coherently combine sample pairs to improve the sample SNR by approximately 3 dB, while simultaneously lowering the complexity by reducing the number of samples to be processed later. The RAD operation performed after the b-th branch is given by

$$s_{k,b+1} = s_{2k-1,b} + \hat{z}_b^* s_{2k,b}, \; k = 1 \ldots K_{b+1}, \; b = 1 \ldots B-2 \tag{16}$$

where

$$\hat{z}_b = \hat{Z}_b / \left| \hat{Z}_b \right|, \qquad b = 1 \ldots B-2 \tag{17}$$

is the unit amplitude rotation factor applied after the b-th branch, and

$$\begin{aligned} K_b &= 3 \times 2^{B-b-1}, & b &= 1 \ldots B-1 \\ &= 3, & b &= B \end{aligned} \tag{18}$$

is the number of decimated samples used to estimate Z in the b-th branch. The RAD operation performed after the b-th branch requires only $K_b/2$ multiplies and adds. The RAD operation removes the estimated frequency error from the input signal in a pairwise fashion, enabling approximate coherent combining. The estimated frequency error is not completely removed, as this would require about $2K_b$ multiplies. The RAD operation also has an interesting frequency domain interpretation. It is equivalent to performing down-conversion, low-pass filtering with a 100% roll-off root-raised-cosine (RRC) filter, decimating-by-2, followed by upconversion or reintroduction of the frequency error. After decimation, the actual frequency error may lie within one of the aliased spectra. The processing used to select the correct root is equivalently selecting the appropriate aliased spectrum.

The majority of the processing is that required to compute the Z estimates for each branch. The total number of complex multiplies and adds is

$$\begin{aligned} \#\,mult &= 3K - B - 4 \\ \#\,adds &= 3K - 2B - 4 \end{aligned} \tag{19}$$

which indicates a complexity of only order K.

Maximum Likelihood Extension

Consider the pure tone case with M_a=M=1 so that s_k=r_k as defined in (1) and (5). The additive noise is assumed to be white and Gaussian with n_k=w_k. The maximum likelihood (ML) frequency estimator finds the frequency $\hat{\omega}_{ML} = u$ which maximizes the function

$$f(u) = |S(u)|^2 = S(u)\, S^*(u) \tag{20}$$

where

$$S(u) = \sum_{k=1}^{K} s_k \, U^{-k} \tag{21}$$

is the Fourier transform of $\{s_k\}$ with U defined as

$$U = \exp(ju) \tag{22}$$

Newton's method can be used to find the maximum of $f(u)$ by finding the zero-crossing of the first derivative of $f(u)$, provided the initial guess is close to the peak of the main lobe of $f(u)$. A good initial guess is given by the frequency estimate $\hat{\omega}_B = phase(\hat{W}_B)$ from the final branch of the DMA based estimator of Figure 2. The simulation results show that there is little to be gained by using more than a single step of Newton's method. Thus, an approximate ML

extension to the DMA based frequency estimator of Figure 2 is given by

$$\hat{\omega}_{ML} \approx \hat{\omega}_B - \frac{f'(\hat{\omega}_B)}{f''(\hat{\omega}_B)} \tag{23}$$

The first and second derivatives of $f(u)$ are given by

$$f'(u) = 2\,\mathrm{Re}[S'(u)\,S^*(u)]$$
$$f''(u) = 2|S'(u)|^2 + 2\,\mathrm{Re}[S''(u)\,S^*(u)] \tag{24}$$

where the n-th derivative of $S(u)$ is given by

$$S^{(n)}(u) = \frac{d^n}{du^n} S(u) = \sum_{k=1}^{K} (-jk)^n\, s_k\, U^{-k} \tag{25}$$

Combining the above results to further simplify (23) gives

$$\hat{\omega}_{ML} \approx \hat{\omega}_B + \frac{\mathrm{Im}\left[S_0\, S_1^*\right]}{|S_1|^2 - \mathrm{Re}\left[S_0\, S_2^*\right]} \tag{26}$$

where the 3 sums, S_0, S_1 and S_2 are defined as

$$S_n = \sum_{k=1}^{K} k^n\, s_k\, \hat{w}_B^{-k}, \qquad n = 0,1,2 \tag{27}$$

with the definition that

$$\hat{w}_B = \hat{W}_B \Big/ \left|\hat{W}_B\right| = \exp(j\hat{\omega}_B) \tag{28}$$

With a few further minor manipulations to the sums in (27), it can be shown that the total number of multiplies and adds required to implement the ML extension is upper bounded by

$$\#\,mult = \#\,adds = 2.5K \tag{29}$$

Thus, the complexity of the ML extension is also order K. The ML extension can also be applied to one of the sets of K_b decimated samples. With this slight modification, the complexity of the ML extension can be reduced even further to that of a constant. It is shown in the next section that the performance penalty with this modification is very small.

THEORETICAL ANALYSIS

For the theoretical results which follow it is assumed that the noise samples, $\{w_k\}$, are Gaussian and uncorrelated, that M_a=M in the nonlinearity, and that all potential phase ambiguities are correctly resolved. An approximation for the variance of the frequency estimator shown in Figure 1, measured in (radians/T)^2, was derived in [2]. The approximation is most accurate for high SNRs and/or long observation times, when the true angular variance of $\hat{W}$ is small. The result is

$$V(K,d,N) = \frac{\min[d,K-d]N}{(K-d)^2 d^2 M^2} + \frac{N^2}{2(K-d)d^2M^2} \quad (rad/T)^2 \tag{30}$$

where

$$N = \sum_{m=1}^{M} \binom{M}{m}^2 m!\,\gamma^{-m} \tag{31}$$

is the power of the noise terms defined in (6).

The frequency estimate variance for each of the branches shown in Figure 2 can be approximated by

$$V_b' = (K_b/K)^2\, V(K_b, D_b, N_b), \qquad b = 1 \ldots B \tag{32}$$

where K, D_b, and K_b are as defined in (13), (15), and (18), respectively. The scale factor in(32) is required to convert from decimated sample periods back to original sample periods, T, to preserve the units of (radians/T)^2. The N_b term represents the effective noise power at the input to the b-th branch. For SNRs above threshold, the frequency estimation error remaining after each branch is typically well within the 3 dB bandwidth of the 100% roll-off RRC filter used in the following RAD operation. Since this filter cuts the noise power in half each time it is applied, a good approximation for N_b, for SNRs above threshold, is

$$N_b = (K_b/K)\,N, \qquad b = 1 \ldots B \tag{33}$$

where K and K_b are again given by (13) and (18). For the final branch in Figure 2, the approximation becomes

$$\begin{aligned} V_B' &= (K_B/K)^2\, V(K_B, D_B, N_B) \\ &= (3/K)^2\, V(3, 2, 3N/K) \\ &= \frac{27N}{4K^3M^2}\left[1 + \frac{3N}{2K}\right] \quad (rad/T)^2 \end{aligned} \tag{34}$$

For high SNRs (or for all SNRs with M=1), N can be approximated by the first term in (31), which gives

$$N(\gamma >> 1) \approx M^2\gamma^{-1} \tag{35}$$

With this approximation

$$V_B(\gamma >> 1) \approx \frac{27}{4K^3\gamma} \quad (rad/T)^2 \tag{36}$$

Note that the variance at high SNRs is not a function of M. For low SNRs the extra noise terms become more significant and performance does depend on M. However, for the new DMA frequency estimator with RAD, the last noise term in (34) is reduced by an additional factor of K^{-1}, which is not present for the frequency estimator presented in [2]. At low SNRs, where large values of K are typically required, this improvement can be very significant.

The Cramer-Rao lower bound (CRLB) on the variance of any discrete-time frequency estimator is given by [2, 5]

$$\mathrm{CRLB}(K,\gamma) = \frac{6}{K(K^2-1)\gamma} \quad (rad/T)^2 \tag{37}$$

Comparing this with (36), the degradation in dB relative to the CRLB for the frequency estimator of Figure 2, at high SNRs, is given by

$$\mathrm{Deg}(\gamma >> 1) \approx 10\log\left(\frac{9(K^2-1)}{8K^2}\right) \quad \mathrm{dB} \tag{38}$$

For large observation times, K>>1, the degradation from the CRLB is approximately 10log(9/8)=0.5 dB. Note that there is no degradation from the CRLB with K=3. The

simulation results show that the performance of the new DMA frequency estimator with RAD remains very close to the CRLB for all SNRs above threshold.

The CRLB, as given in (37), applies to the original K received samples, $\{r_k\}$, and is valid for the MPSK signal model used with any value for M. For the pure carrier case, without a nonlinearity (i.e. M_a=M=1), a CRLB can also be derived for each set of K_b decimated samples at the input to the b-th branch of Figure 2. The result is

$$CRLB_b = (K_b/K)^2\, CRLB(K_b,\gamma_b), \qquad b=1\ldots B \tag{39}$$

where K and K_b are defined in (13) and (18). The scale factor in (39) is required to convert from decimated sample periods to original sample periods, T, to preserve the units of (radians/T)^2. The γ_b term represents the sample SNR at the input to the b-th branch. Using the same arguments as for (33), a good approximation for γ_b, for all SNRs above threshold, is

$$\gamma_b = (K/K_b)\,\gamma, \qquad b=1\ldots B \tag{40}$$

where K and K_b are again given by (13) and (18). Simplifying (39) further gives

$$CRLB_b = \frac{6}{K\left(K^2-(K/K_b)^2\right)\gamma}, \qquad b=1\ldots B \tag{41}$$

The degradation in the CRLB, measured in dB for the b-th branch, where K_b decimated samples are used instead of the original K samples, is given by

$$Deg_b = 10\log\left[\frac{CRLB_b}{CRLB}\right] = 10\log\left[\frac{K_b^2(1-K^{-2})}{K_b^2-1}\right]\ dB, \qquad b=1\ldots B \tag{42}$$

For K>>1, the degradation is approximately given by

$$Deg_b(K>>1) \approx 10\log\left[\frac{K_b^2}{K_b^2-1}\right]\ dB, \quad b=1\ldots B \tag{43}$$

Representative examples of the degradations in the CRLB for K_b=3, 6 and 12 are 0.51, 0.12 and 0.03 dB, respectively. The degradation in the CRLB is clearly negligible for $K_b \geq 12$. Note that the ML extension described earlier can be applied to any set of K_b decimated samples (e.g. K_b=12), and not just to the initial set of K samples. Thus, for large values of K, the complexity of the ML extension can be reduced to a fixed constant, independent of K, with negligible degradation in performance. Thus, the complexity of the complete frequency estimator with the ML extension remains approximately 3K.

EXAMPLE PERFORMANCE RESULTS

The simulated performance results are presented in terms of measured root-mean-squared (RMS) frequency error in (cycles/T) versus sample SNR, γ, in dB. An observation time of K=48 samples was used, and 5000 independent trials were simulated for each SNR. Figure 3 shows the results for the case of pure carrier with no nonlinearity (M_a=M=1). Three sets of simulation results are shown. The first set, with d=1, is for the single branch estimator of Figure 1 or the first branch in Figure 2. The second set, with d_B=32, is for the final branch of the new DMA estimator of Figure 2. The third set is for the ML extension applied to the original K=48 samples. The performance is essentially the same for a decimated set of 12 or more samples. Also shown, for comparison, are the corresponding theoretical approximations and the CRLB. It is observed that the theoretical approximations are quite accurate for all SNRs above threshold. With the ML extension, the CRLB is essentially achieved for all SNRs above threshold. The threshold SNR is observed to be about 0 dB for this case.

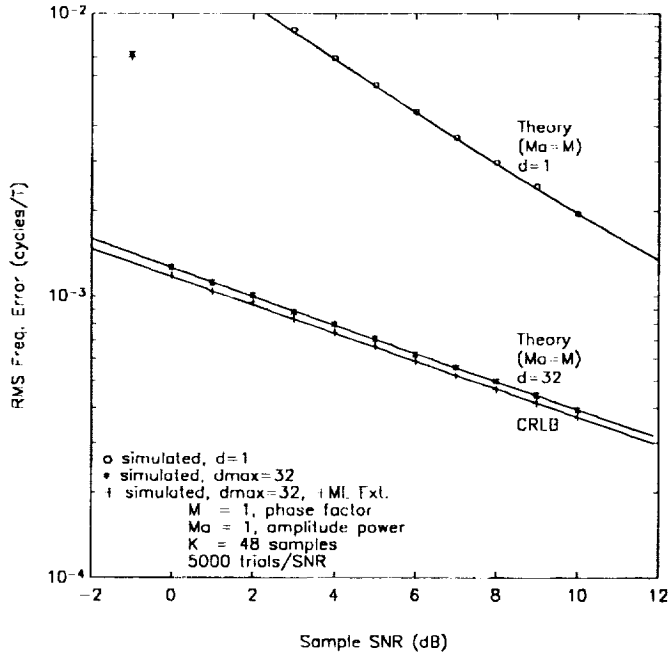

Figure 3: RMS frequency error versus sample SNR, γ, for pure carrier (M_a=M=1, dmax=d_B).

Figures 4 and 5 show simulation results for BPSK and QPSK signaling, respectively. For the simulated BPSK results in Figure 4, M=2 and M_a=1. For the simulated QPSK results in Figure 5, M=4 and M_a=1. Not shown are the simulation results with M_a=M, but they closely match

the theoretical approximations for all SNRs above threshold. The simulation results with M_a=1 are clearly better than the theoretical approximations with M_a=M. Note that the simulated performance of the DMA estimator with RAD remains within about 0.5 dB of the CRLB for all SNRs above threshold, and that the CRLB is essentially achieved with the ML extension. As expected, the threshold SNRs are much higher with M>1. Longer observation times are required to provide lower thresholds.

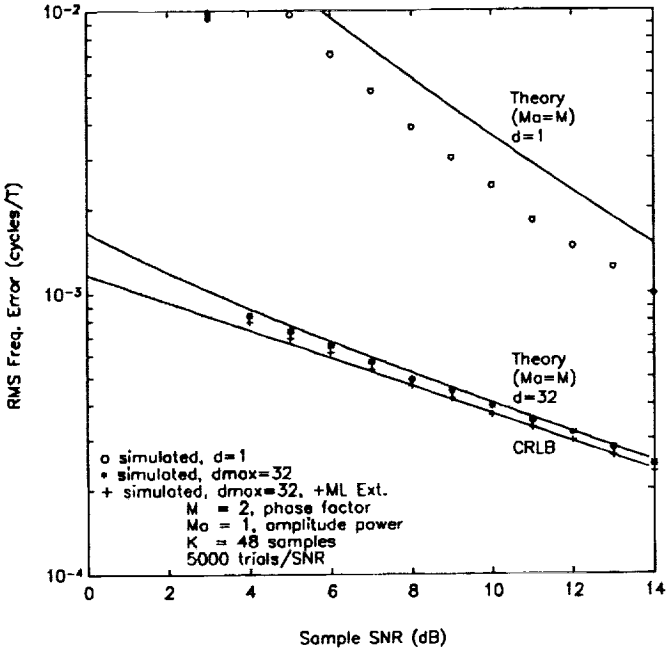

Figure 4: RMS frequency error versus sample SNR, γ, for BPSK signaling (M=2, M_a=1, dmax=d_B).

CONCLUSIONS

A low-complexity, open-loop, discrete-time, delay-multiply-average (DMA) approach to estimating the frequency offsets for MPSK modulated signals was investigated. A simple maximum likelihood (ML) extension was also considered. Theoretical and simulated performance results were presented and compared to the Cramer-Rao lower bound (CRLB) for the variance of the frequency estimation error. It was shown that the frequency estimate variance can be improved by orders of magnitude over that obtained with a delay of d=1. Without the ML extension, performance is typically within about 0.5 dB of the CRLB, for all SNRs above threshold. With the ML extension, the CRLB is essentially achieved. The complexity of the new DMA algorithm, with or without the ML extension, is approximately $3K$, where K is the observation time in samples.

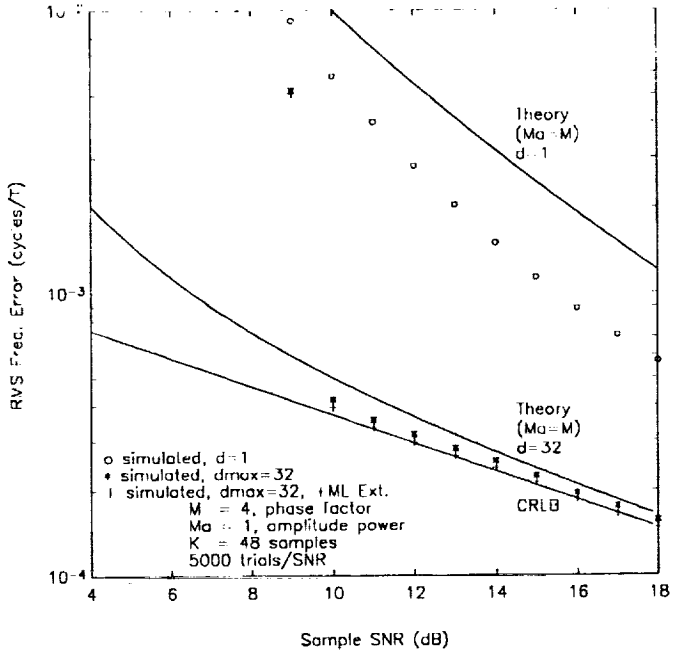

Figure 5: RMS frequency error versus sample SNR, γ, for QPSK signaling (M=4, M_a=1, dmax=d_B).

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