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Proof Test of the Computer Program BUCKY for Plasticity Problems

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Abstract

A theoretical equation describing the elastic-plastic deformation of a cantilever beam subject to a constant pressure is developed. The theoretical result is compared numerically to the computer program BUCKY for the case of an elastic-perfectly plastic specimen. It is shown that the theoretical and numerical results compare favorably in the plastic range. Comparisons are made to another research code to further validate the BUCKY results. This paper serves as a quality test for the computer program BUCKY developed at NASA Johnson Space Center.

Introduction

The ability to predict a structure's response to loading is a necessity in the field of structural analysis. Too often, a structure quickly becomes too complicated to analyze by hand calculations. In addition, when advanced topics such as material nonlinearity are introduced, the analysis becomes virtually impossible to perform by hand. In such cases, computer models must be developed to estimate the response of the structure.

Plastic analysis of structures has been incorporated into several finite element packages. However, nearly all current finite element programs are based on the h-version of the finite element method; only a few research codes based on the p-version of the finite element method are known to exist. The computer program BUCKY is a multi-element p-version finite element program that has the ability to perform elastic-plastic studies of structures.

The assumptions used in the development of BUCKY are few; the structure in question must be a two-dimensional plate structure in a state of plane stress. In addition, the material properties must be such that the plate is isotropic in nature and has work-hardening abilities. Proportional loading and no unloading are further assumptions of the computer method.

Using the tangent stiffness method, the p-version of the finite element is ideally suited for problems of plasticity. It will be shown that the cantilever beam example is effectively described with only two finite elements.

Problem Statement

We wish to find the exact theoretical deflection for a fixed cantilever beam of length L and thickness h subject to a uniform statically applied load q_0 over the beam span, as shown in Figure 1. As the beam is loaded, the extreme fibers ($z = \pm h/2$) experience higher stresses than the rest of the beam. Thus, these areas will yield before the remainder of the beam. The beam is assumed to have elastic-perfectly plastic material behavior, as shown in Figure 2.

With increasing load q_0, the beam yields further until eventually the collapse load is reached, at which point the deflections at the beam tip are completely unbounded and a plastic hinge forms at the base of the beam. We want to determine the deflection of the beam anywhere in the beam and for any load from the fully elastic state to the collapse state.

Fully Elastic Solution

The problem of an elastic beam under a distributed load is straightforward. The differential equation describing a uniform beam under a distributed load $q(x)$ is given by

$$EI\frac{d^4w}{dx^4} = q(x). \tag{1}$$

The solution to (1) is simply a matter of integrating (1) four times, yielding four constants of integration. Realizing that the beam has four boundary conditions (two on each side), these four constants can be uniquely determined and a unique solution can be found. In our example, the boundary conditions are

$$w(0) = 0, \qquad w'(0) = 0, \qquad M(L) = 0, \qquad \text{and} \qquad V(L) = 0, \tag{2}$$

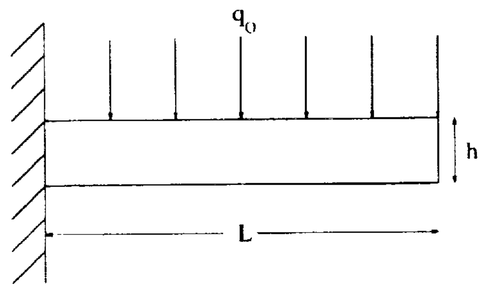

Figure 1. **Beam model.**

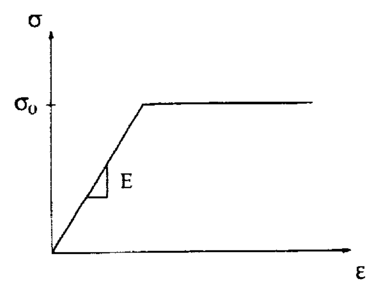

Figure 2. **Stress-strain curve for an elastic-perfectly plastic material.**

where

$$M(x) = EI\frac{d^2w}{dx^2} \qquad \text{and} \qquad V(x) = EI\frac{d^3w}{dx^3}. \tag{3}$$

Integrating (1) twice with $q(x) = q_0$ and applying the boundary conditions at $x = L$, we find

$$M(x) = q_0 L^2 \left(\frac{x^2}{2L^2} - \frac{x}{L} + \frac{1}{2}\right). \tag{4}$$

Integrating (4) twice more and applying the fixed ended conditions yields

$$w(x) = \frac{q_0 x^2 L^2}{EI}\left(\frac{x^2}{24L^2} - \frac{x}{6L} + \frac{1}{4}\right) \tag{5}$$

for the displacement of the beam. Equation (5) provides the beam displacement anywhere in the beam under the assumption that the beam is comprised of a linearly elastic material.

Moment Due to a Partially Plastic Cross Section

In this section, we compute the moment that is the result of a partially plastic cross section. As the base of the beam goes from completely elastic to partially and subsequently fully plastic, the contribution of the stresses on the moment changes. We can quantify this easily by writing the stress distribution in a partially plastic section, as shown in Figure 3. The stress distribution shown in Figure 3 can be described analytically by

$$\sigma = \begin{cases} -\sigma_0, & h/2 \geq y \geq \eta, \\ -\sigma_0 y/\eta, & \eta \geq y \geq -\eta, \\ \sigma_0, & -\eta \geq y \geq -h/2. \end{cases} \tag{6}$$

The moment due to this distribution is

$$M_p(x) = 2b\int_0^{h/2}(-\sigma)y\,dy = 2b\int_0^{\eta}\sigma_0\frac{y^2}{\eta}dy + 2b\int_{\eta}^{h/2}\sigma_0 y\,dy \tag{7}$$

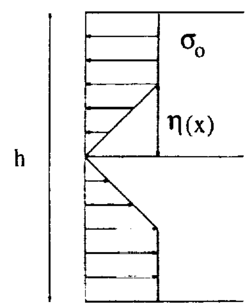

Figure 3. **Elastic and plastic zones in a partially plastic cross section.**

or

$$M_p(x) = \frac{\sigma_0}{3} b \left(\frac{3h^2}{4} - \eta^2 \right), \tag{8}$$

where η is a function of x, and b is the width of the beam.

From equation (8) and Figure 3 it is seen that if $\eta = h/2$, the moment at that cross section has achieved the yield moment M_y, or

$$M_y = \frac{\sigma_0 b h^2}{6}. \tag{9}$$

Conversely, if the plastic region governed by $\eta(x)$ takes the value of zero, then the cross section has completely collapsed (section is fully plastic), and the moment is

$$M_c = \frac{\sigma_0 b h^2}{4}. \tag{10}$$

In the elastic-plastic region, the moment at any cross section is bounded by equations (9) and (10).

Displacements of an Elastic-Plastic Cantilever Beam

In this section we use the results of the last two sections to derive the expression for the beam displacements anywhere in the beam. To find the displacements, we match the moments of the elastic and elastic-plastic sections and integrate the resulting differential equation. The integration leads to an expression for the beam displacements.

Recalling the moment for the fully elastic case and equating it to the moment due to a partially plastic cross section, we see that

$$q_0 L^2 \left(\frac{x^2}{2L^2} - \frac{x}{L} + \frac{1}{2} \right) = \frac{\sigma_0}{3} b \left(\frac{3h^2}{4} - \eta^2 \right). \tag{11}$$

Solving for the variable η, which represents the size of the elastic core in an otherwise plastic region, we find

$$\eta = \left[\frac{3h^2}{4} - \frac{3q_0 L^2}{\sigma_0 b} \left(\frac{x^2}{2L^2} - \frac{x}{L} + \frac{1}{2} \right) \right]^{1/2}. \tag{12}$$

From (12), we can predict the extent of the plastic zone anywhere in the beam.

Next, we write that the strain due to bending of the plate is given by the expression

$$\epsilon = z \frac{d^2 w}{dx^2} = \frac{\sigma}{E}. \tag{13}$$

In regions of the beam where the cross section is partially plastic, equation (13) becomes

$$\frac{d^2 w_p}{dx^2} = \frac{\sigma_0}{Ez} = \frac{\sigma_0}{E\eta}. \tag{14}$$

Equation (14) provides a differential equation for the curvature of the beam in the partially plastic zone. The parameter η in (14) is given by equation (12). Thus, the differential equation to solve is

$$\frac{d^2 w_p}{dx^2} = \frac{\sigma_0}{E}\left[\frac{3h^2}{4} - \frac{3q_0 L^2}{\sigma_0 b}\left(\frac{x^2}{2L^2} - \frac{x}{L} + \frac{1}{2}\right)\right]^{-1/2} \tag{15}$$

subject to the boundary conditions

$$w_p(0) = 0 \qquad \text{and} \qquad w_p'(0) = 0. \tag{16}$$

The solution to (15) is not trivial, but it can be found analytically. Integrating (15) twice subject to the boundary conditions (16) yields

$$\begin{aligned} w_p(x) \quad = \quad & \frac{\sqrt{3}\sigma_0^2 bh}{3Eq_0}\left\{\frac{L}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\left(\frac{x}{L}-1\right)\sin^{-1}\left[\frac{L}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\left(\frac{x}{L}-1\right)\right] + \sqrt{1 - \frac{2q_0 L^2}{\sigma_0 h^2 b}\left(\frac{x}{L}-1\right)^2}\right. \\ & \left. + \frac{x}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\sin^{-1}\left(\frac{L}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\right) - \frac{L}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\sin^{-1}\left(\frac{L}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\right) - \left(1 - \frac{2q_0 L^2}{\sigma_0 b h^2}\right)^{1/2}\right\}. \end{aligned} \tag{17}$$

Equation (17) can only be used where the beam is partially (or fully) plastic. Recall that the elastic core in the plastic zone is denoted by the variable η. When $\eta = h/2$, the beam is entirely elastic. From equation (12), we see that this transition occurs at the location

$$\frac{x^*}{L} = 1 - \frac{h}{L}\sqrt{\frac{\sigma_0 b}{3q_0}}. \tag{18}$$

As long as the beam position is less than x^*, equation (17) can be used to compute the beam deflection.

Equation (17) provides the exact solution to the differential equation (15) subject to the boundary conditions (16) in the partially plastic zone. However, the entire beam is not in a plastic state; the majority of the beam is in a fully elastic state. For this region, we apply to the problem the traditional beam solution used earlier. If we compute the differential equation for a beam with no moment or shear at its end, we find

$$w'(x) = \frac{q_0 x L^2}{Ebh^3}\left(2\frac{x^2}{L^2} - 6\frac{x}{L} + 6\right) + C_3 \tag{19}$$

and

$$w(x) = \frac{q_0 x^2 L^2}{Ebh^3}\left(\frac{x^2}{2L^2} - 2\frac{x}{L} + 3\right) + C_3 x + C_4. \tag{20}$$

To find the constants of integration C_3 and C_4, we must apply the condition that the slope and displacement of the beam at the transition point x^* are equal in the partially plastic and fully elastic zones. That is, we compute the displacement in the plastic zone from (17) at the point x^* and set that equal to the displacement given by (20) at x^*. Additionally, we equate the slope of equation (19) to that of the derivative of (17) at x^*. The two conditions, then, yield expressions for the integration constants, such that

$$C_3 = -\frac{q_0 x^* L^2}{Ebh^3}\left(2\frac{x^{*2}}{L^2} - 6\frac{x^*}{L} + 6\right) + w_p'(x^*) \tag{21}$$

and

$$C_4 = -\frac{q_0 x^{*2} L^2}{Ebh^3}\left(\frac{x^{*2}}{2L^2} - 2\frac{x^*}{L} + 3\right) - C_3 x^* + w_p(x^*) \tag{22}$$

where

$$w_p'(x) = \frac{\sqrt{3}\sigma_0^2 bh}{3Eq_0}\left\{\frac{1}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\sin^{-1}\left[\frac{L}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\left(\frac{x}{L}-1\right)\right] + \frac{1}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\sin^{-1}\left(\frac{L}{h}\sqrt{\frac{2q_0}{\sigma_0 b}}\right)\right\}. \tag{23}$$

To compute the displacements in the beam, we use equation (17) in the region $[0, x^*]$ and (20) in the region $[x^*, L]$.

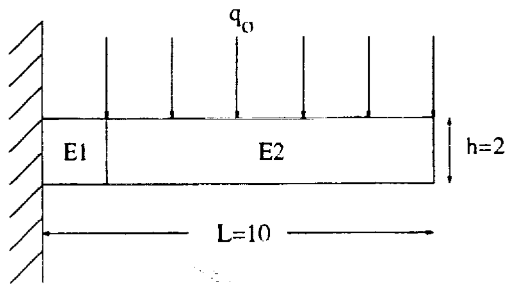

Figure 4. **Elastic-plastic cantilever beam model.**

BUCKY Finite Element Model

To perform the plastic analysis numerically, the computer program BUCKY was used with its elastic-plastic analysis capabilities. Utilizing the tangent stiffness method of plasticity analysis, BUCKY is a p-version finite element method program capable of accurately predicting the response of simple structures to applied loads. The p-version finite element method is a highly accurate method that exploits the qualities of higher order smooth functions to describe large gradients. Traditional h-finite elements have a general lack of detail in this regard unless very fine meshing is employed.

The finite element model used to verify the plastic analysis of BUCKY is shown in Figure 4. Note that only two elements are employed in the solution of the problem. Obviously the p-method has the tremendous advantage of simplified modeling. The element labeled $E1$ has a size of the same order of the beam thickness. The small size of the element allows us to accurately trace the elastic-plastic boundary. The second element, $E2$, is much larger since we expect it to remain elastic. Even if this element were to become partially plastic, BUCKY would be able to effectively show this.

The physical parameters of the beam in Figure 4 are such that the length $L = 10$, the thickness $h = 2$, and the width $w = 1$. In addition, the beam material has some Young's modulus $E10E6$ associated with it, as well as a yield strength of $\sigma_0 = 50000$. Finally, since the material is elastic-perfectly plastic, the tangent modulus E_T is identically zero.

The load q_0 of Figure 4 is varied so that the beam can be compared to theoretical predictions at a variety of states. One caveat to the BUCKY analysis is that BUCKY is a two-dimensional finite element code, whereas the theory used to predict the beam deflection above is based upon a reduction from the full three-dimensional equations of elasticity to one dimension. The comparisons, however, will be shown to be close.

Comparison to Theoretical Results

There are four comparison points given by the loads $q_0 = 667$, $q_0 = 800$, $q_0 = 900$, and $q_0 = 1000$. For the beam under consideration, the load $q_0 = 667$ corresponds to the distributed load at which the beam first experiences a plastic effect. The first occurrence of a plastic component of stress is at the top and bottom outer fibers at the wall. The load $q_0 = 1000$ represents the theoretical collapse load of the beam. That is, at this load, the plate forms a fully plastic core and plastic hinge at the beam-wall interface. This gives unbounded deflections and the stiffness in some points of the beam quickly goes to zero. Numerically, this means singular element stiffness matrices, which is an undesirable effect.

At the moment the beam first reaches a yield point, the applied load takes the value $q_0 = 667$. At this point, the BUCKY analysis compares well to another benchmark, the PEGASYS finite element package. PEGASYS is a p-version finite element program based on secant modulus principles. However, both programs deviate from the elastic-plastic solution and fully elastic solution. Figure 5 shows the four deflections based on theoretical predictions for elastic, elastic-plastic, and the two computer programs. The deflections throughout the beam are shown.

As the load increases to $q_0 = 800$, the extent of the plastic zone increases somewhat. Figure 6 shows that the beam tip deflection increases slightly for an increase in the load q_0. Whereas in the

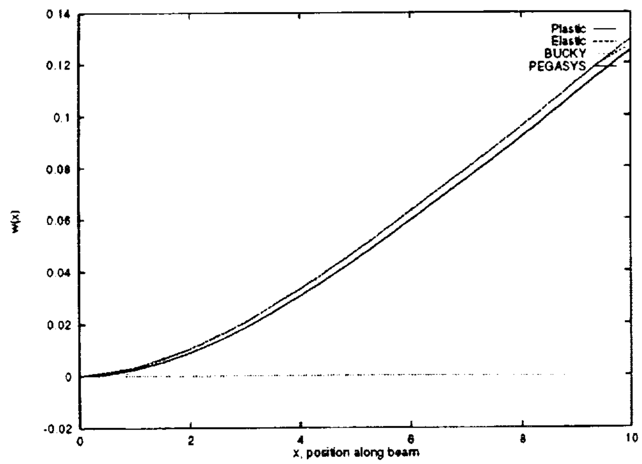

Figure 5. **Deflections of an elastic-perfectly plastic cantilever for q_0=667.**

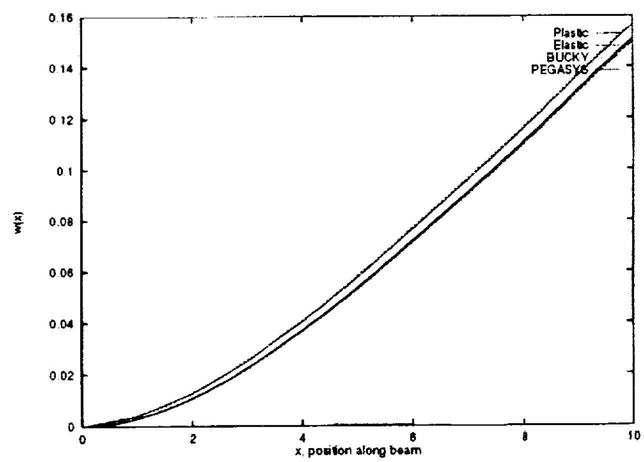

Figure 6. **Deflections of an elastic-perfectly plastic cantilever for q_0=800.**

first case where the theoretical predictions for the tip deflection were nearly equal for the elastic and plastic analysis, we now see in Figure 6 that the elastic solution underpredicts the deformation. This is because a linear elastic anaylsis cannot model material nonlinearities. We also see in Figure 6 that the two computer codes again give similar results.

As the load is increased further to $q_0 = 900$, BUCKY predicts the highest tip deflection. The deformations for this case are shown in Figure 7. Finally, in Figure 8, the deflections in the beam at full collapse are shown. The load at full collapse is $q_0 = 1000$. At this point, we see a close agreement between the theoretical predictions and the computer analysis. We also see that the PEGASYS code underpredicts the deformation slightly. Of course, the elastic solution trails well below the elastic-plastic solutions. As the material becomes plastic, the modulus decreases rapidly, resulting in larger deflections.

Finally, in Figure 9, we show the elastic-plastic boundary at the theoretical collapse load $q_0 = 1000$. The boundary is given by the light band on the left side of the beam. The plastic region is the darker area on the left side of the beam. Notice that the cross section at the wall is not completely plastic. Because of the manner in which the stress updates are generated at the element Gauss points, BUCKY can only compute an approximate stress at points deviating from the Gauss points. The stresses are found by determining an equation over the element which describes the stresses exactly at the Gauss points. All other points within the element have potential error. However, even though the stress distribution at the wall is not in a completely plastic state, the stress distribution throughout the beam does have the expected shape.

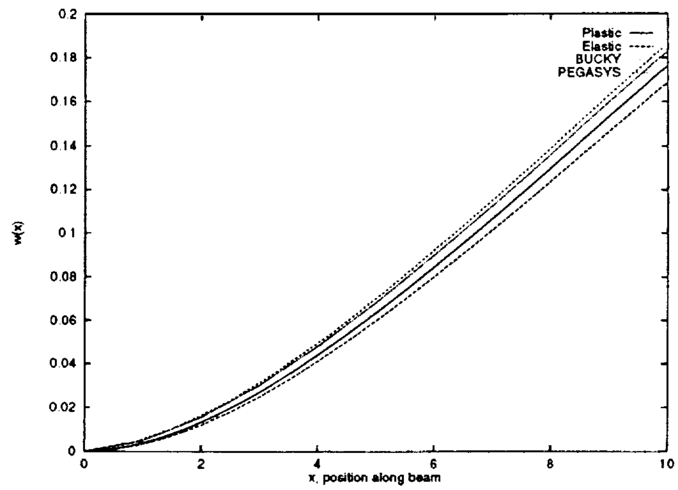

Figure 7. **Deflections of an elastic-perfectly plastic cantilever for q_0=900.**

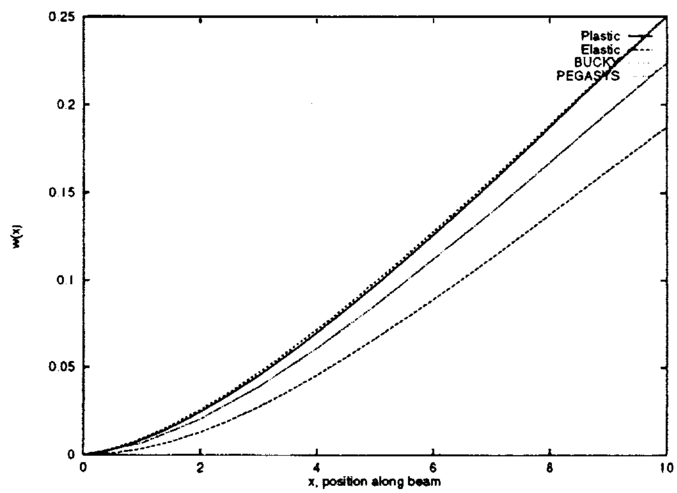

Figure 8. **Deflections of an elastic-perfectly plastic cantilever for q_0=1000.**

Conclusions

As was seen from the example of the cantilever beam, the theoretical solution and the computer solution agree closely. At most, the tip deflections disagree by no more than a few percent. This error is remarkable considering that only two elements were used in the finite element analysis. Had more elements been utilized, the number of degrees of freedom would have increased and the error would have decreased further. Indeed, the BUCKY model could have been extended to four elements (two element by two element mesh) to trace the plastic boundary better.

From the example problem, it has been shown that BUCKY is an effective tool for the analysis of structures involving material nonlinearities. The p-version of the finite element method has proven its value with linear static analysis, and the same exceptional behavior is exhibited in nonlinear problems as well. By contrast, the convergence properties of traditional h-version finite elements yield poorer results for the same number of equations used in the BUCKY analysis.

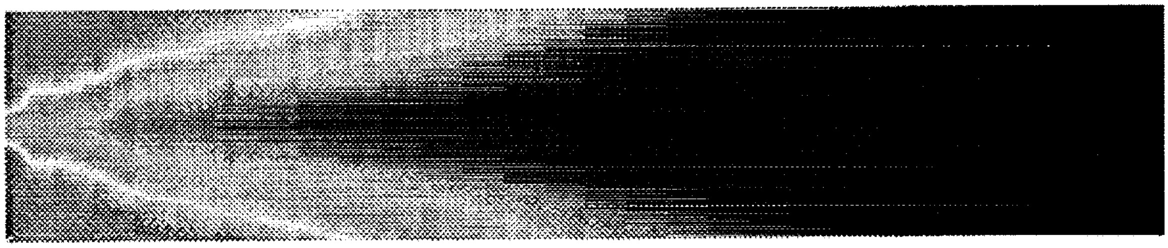

Figure 9. **Von Mises stress distribution at theoretical collapse load.**

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