19940018762

N94-23235

1993 NASA/ASEE SUMMER FACULTY FELLOWSHIP PROGRAM

JOHN F. KENNEDY SPACE CENTER
UNIVERSITY OF CENTRAL FLORIDA

S5-37

197191

p. 20

MASS PROPERTIES MEASUREMENT SYSTEM DYNAMICS

PREPARED BY:	Dr. Keith L. Doty
ACADEMIC RANK:	Professor
UNIVERSITY AND DEPARTMENT:	University of Florida Department of Electrical Engineering
NASA/KSC	
DIVISION:	Mechanical Engineering
BRANCH:	Special Projects
NASA COLLEAGUE:	Bill Jones Todd Graham

DATE:	August 7, 1993
CONTRACT NUMBER:	University of Central Florida NASA-NGT-60002 Supplement: 11

TABLE OF CONTENTS

ACKNOWLEDGMENTS

The author gratefully acknowledges the support of the University of Central Florida faculty and staff along with the I-NET and NASA staffs whose diligence and kindness make the NASA Faculty Fellowship program at the Kennedy Space Center so efficient and productive. Particular thanks to Bill Jones and Todd Graham of NASA; Kedron Wolcott and Rich Bennett of I-NET; and Ray Hosler and Kari Stiles of the University of Central Florida. A special thanks to Willis Crumpler of NASA who proposed the original concept of the MPMS. Finally, I would like to thank NASA Faculty Fellows Robert Byers and Narasimha Malladi for stimulating discussions.

ABSTRACT

The MPMS mechanism possess two revolute degrees-of-freedom and allows the user to measure the mass, center of gravity, and the inertia tensor of an unknown mass. This paper develops the dynamics of the Mass Properties Measurement System (MPMS) from the Lagrangian approach to illustrate the dependency of the motion on the unknown parameters.

1. INTRODUCTION

The Mass Properties Measurement System (MPMS), illustrated in Figure 1.1, consists of a serial kinematic mechanism with two intersecting revolute axes, $\mathbf{z}_1$ and $\mathbf{z}_2$, that intersect with fixed angle α_1. The joint angles and rates denoted by θ_1, θ_2 and $\dot{\theta}_1$, $\dot{\theta}_2$, respectively, turn about the respective joint axes.

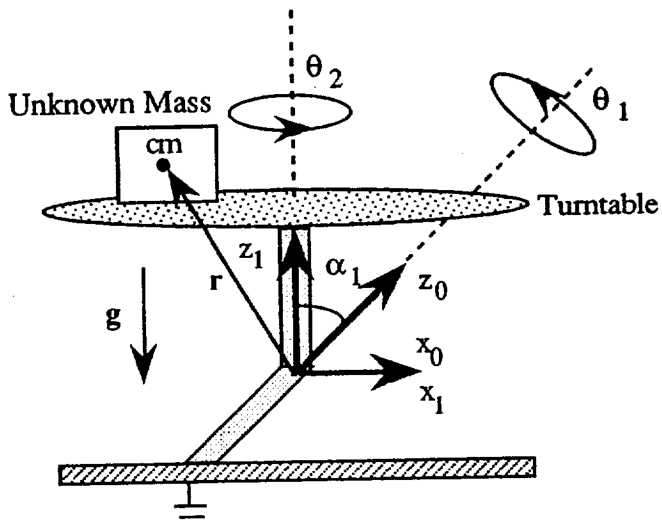

Figure 1.1 Mass Properties Measurement System (MPMS).

The function of the MPMS is to measure the mass, center-of-gravity and second-order mass moments of an unknown mass placed on the turntable. The mass and center-of-gravity can be determined from static measurements. The inertia tensor must be determined from the dynamics.

The structure of the Mass Properties Measurement System (MPMS) lends itself to a straightforward dynamics analysis using the Lagrangian approach. The dynamics analysis assumes α_1 as a parameter.

The symbolic software package *Mathematica* was used to produced and verify the equations presented here. In the analysis to follow, the standard Denevit-Hartenberg kinematic parameters of the MPMS were employed.

2. MPMS KINEMATICS

Table 2.1 lists the Denevit-Hartenberg kinematic parameters for the MPMS mechanism.

Table 2.1 : Kinematic Parameters for the MPMS Mechanism

Joint	d	θ	a	α
1 r	0	θ_1	0	α_1
2 r	0	θ_2	0	0°

From the DH-parameters of the MPMS mechanism listed in Table 2.1, the two link transforms compute to

$$\mathbf{L}_1 = \begin{pmatrix} & {}^0\mathbf{R}_1 & & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{L}_2 = \begin{pmatrix} & {}^1\mathbf{R}_2 & & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2-1a}$$

$$\text{where } {}^0\mathbf{R}_1 = \begin{pmatrix} c_1 & -\tau_1 s_1 & \sigma_1 s_1 \\ s_1 & \tau_1 c_1 & -\sigma_1 c_1 \\ 0 & \sigma_1 & \tau_1 \end{pmatrix} \quad {}^1\mathbf{R}_2 = \begin{pmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2-1b}$$

and $c_i := \cos(\theta_i)$, $s_i := \sin(\theta_i)$, $\tau_i := \cos(\alpha_i)$ and $\sigma_i := \sin(\alpha_i)$. The forward kinematics transform of the MPMS equals

$${}^0\mathbf{T}_2 = \mathbf{L}_1\,\mathbf{L}_2 \tag{2-2}$$

which computes to

$${}^0\mathbf{T}_2 = \begin{pmatrix} & {}^0\mathbf{R}_2 & & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2-3}$$

where

$$
{}^{0}\mathbf{R}_2 = \begin{pmatrix} c_1\, c_2 - \tau_1\, s_1\, s_2 & -c_1\, s_2 - \tau_1\, s_1\, c_2 & \sigma_1\, s_1 \\ s_1\, c_2 + \tau_1\, c_1\, s_2 & -s_1\, s_2 + \tau_1\, c_1\, c_2 & -\sigma_1\, c_1 \\ \sigma_1\, s_2 & \sigma_1\, c_2 & \tau_1 \end{pmatrix} \tag{2-4}
$$

The forward analysis presented here serves as reference. The rotation part indicates how to change frame F_2 vector representations into frame F_0 representations. The dynamics analysis presented later will make use of the forward kinematics ${}^{0}\mathbf{T}_2$.

3. MPMS END FRAME JACOBIAN

The Jacobian of the MPMS relates the joint-rates $\dot{\mathbf{q}} = [\,\dot{\theta}_1 \quad \dot{\theta}_2]^{\tau}$ to the frame-velocity $\mathbf{V} = [\mathbf{v}^{\tau} \;\; \omega^{\tau}\,]^{\tau}$ of the end-frame,

$$
\mathbf{V} = \mathbf{J}\,\dot{\mathbf{q}}. \tag{3-1}
$$

The Jacobian of the MPMS computes [1] [2] to

$$
{}^{2,0}\mathbf{J}_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \sigma_1\, s_2 & 0 \\ \sigma_1\, c_2 & 0 \\ \tau_1 & 1 \end{pmatrix} \tag{3-2}
$$

The leading superscript *2* means that this Jacobian is expressed in frame F_2 while the *0* indicates the motion the end-frame, designated by the second subscript, is relative to the base frame F_0 of the MPMS. The first subscript indicates the frame origin at which the linear velocity is measured.

For convenience, we write

$$
{}^{2,0}\mathbf{J}_{2,2} = \begin{pmatrix} {}^{2,0}\mathbf{J}_{v2,2} \\ {}^{2,0}\mathbf{J}_{\omega,2} \end{pmatrix}, \text{ where } {}^{2,0}\mathbf{J}_{v2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } {}^{2,0}\mathbf{J}_{\omega,2} = \begin{pmatrix} \sigma_1 s_2 & 0 \\ \sigma_1 c_2 & 0 \\ \tau_1 & 1 \end{pmatrix} \tag{3-3}
$$

4. MPMS Dynamics

The Lagrangian approach to dynamics of the mechanism requires the calculation of the kinetic and potential energy of the moving masses.

Potential Energy Terms

Assume the center-of-mass vectors for the first and second links equal $\mathbf{r}_1^*$ and $\mathbf{r}_2^*$. These vectors are constants in their own frames,

$$^1\mathbf{r}_1 = \begin{pmatrix} r_{x1} \\ r_{y1} \\ r_{z1} \end{pmatrix} \quad \text{and} \quad ^2\mathbf{r}_2 = \begin{pmatrix} r_{x2} \\ r_{y2} \\ r_{z2} \end{pmatrix} \tag{4-1}$$

The gravitational field vector in Figure 1.1 equals

$$^0\mathbf{g} = g_c \begin{pmatrix} 0 \\ \sigma_1 \\ -\tau_1 \end{pmatrix} \tag{4-2}$$

and the potential energy of a mass m at position $\mathbf{p}$ equals

$$P = -m\, \mathbf{g}^\tau\, \mathbf{p} \tag{4-3}$$

hence, the potential energies P_1 and P_2 of the first and second link equal

$$P_1 = -m_1\, \mathbf{g}^\tau\, \mathbf{r}_1^* = -m_1\, {}^0\mathbf{g}^\tau\, {}^0\mathbf{R}_1\, {}^1\mathbf{r}_1^* \tag{4-4a}$$

and

$$P_2 = -m_2\, \mathbf{g}^\tau\, \mathbf{r}_2^* = -m_2\, {}^0\mathbf{g}^\tau\, {}^0\mathbf{R}_2\, {}^2\mathbf{r}_2^*. \tag{4-4b}$$

The torques associated with changes in the potential energy of the mechanism links equal $\tau_{pe1} = \frac{\partial P_1}{\partial \theta_1} + \frac{\partial P_2}{\partial \theta_1}$ and $\tau_{pe2} = \frac{\partial P_1}{\partial \theta_2} + \frac{\partial P_2}{\partial \theta_2} = \frac{\partial P_2}{\partial \theta_2}$. These torques compute to

$$\tau_{pe1} = m_1\,\sigma_1\,g_c\{r_{x1}\,c_1 + r_{y1}\,\tau_1\,s_1 + r_{z1}\,\sigma_1\,s_1\} + m_2\,\sigma_1\,g_c\{r_{x2}\,(c_1\,c_2 - \tau_1\,s_1\,s_2) + r_{y2}\,(-\tau_1\,c_2\,s_1 - c_1\,s_2) + r_{z2}\,\sigma_1\,s_1\} \tag{4-4a}$$

$$\tau_{pe2} = m_2\,\sigma_1\,g_c\{r_{x2}(-c_2\,\tau_1 + \tau_1\,c_1\,c_2 - s_1\,s_2) + r_{y2}\,(\tau_1\,s_2 - c_2\,s_1 - \tau_1\,c_1\,s_2)\} \tag{4-4b}$$

Equations (4-4) will provide the means for measuring the mass and center-of-gravity for each link.

Measuring MPMS Link Mass and Link Center-of-Gravity

By measuring the balancing torque on joint one and setting the joint angles at different angles one can determine the required information about the first-order mass moments of each link. This information will allow us to subtract the gravity torque terms due to the mass of the MPMS when we desire to measure the mass properties of the unknown mass.

Experiment 4.1 $\theta_1 = 0, \theta_2 = 0,$

$$\tau_{x1} := m_1\,\sigma_1\,g_c\,r_{x1} + m_2\,\sigma_1\,g_c\,r_{x2} \tag{4-5}$$

Experiment 4.2 $\theta_1 = 0, \theta_2 = \pi/2,$

$$\tau_{x2} := m_1\,\sigma_1\,g_c r_{x1} - m_2\,\sigma_1\,g_c\,r_{y2} \tag{4-6}$$

From these two measurements one computes

$$m_2\,r_{y2} = \frac{\tau_{x1} - \tau_{x2}}{2\,\sigma_1\,g_c} \quad \text{and} \quad m_2\,r_{x1} = \frac{\tau_{x1} + \tau_{x2}}{2\,\sigma_1\,g_c} \tag{4-7}$$

Experiment 4.3 $\theta_1 = \pi/2, \theta_2 = 0,$

$$\tau_{x3} := m_1\,\sigma_1\,g_c(\,r_{y1}\,\tau_1 + r_{z1}\,\sigma_1) + m_2\,\sigma_1\,g_c(-\,r_{y2}\,\tau_1 + r_{z2}\,\sigma_1) \tag{4-8}$$

Experiment 4.4 $\theta_1 = \pi/2, \theta_2 = \pi/2,$

$$\tau_{x4} := m_1\,\sigma_1\,g_c(r_{y1}\,\tau_1 + r_{z1}\,\sigma_1) + m_2\,\sigma_1\,g_c(-\,\tau_1\,r_{x2} + r_{z2}\,\sigma_1) \tag{4-9}$$

From *Experiments 3 and 4* one can find

$$m_2\, r_{x2} = \frac{\tau_{x3} - \tau_{x4}}{\sigma_1\, \tau_1\, g_c} + m_2\, r_{y2} \tag{4-10}$$

Relation (4-10) does not apply when $\tau_1 = 0$, i.e., when $\alpha_1 = \pi/2$.

From (4-7) and (4-10) we can develop a linear relation in the other unknowns,

$$\tau'_{pe1} := \frac{1}{s_1}\left\{\frac{\tau_{pe1}}{\sigma_1\, g_c} - m_1 r_{x1}\, c_1 - m_2\, r_{x2}\,(c_1\, c_2 - \tau_1\, s_1\, s_2)\right.$$

$$\left. + m_2\, r_{y2}\,(-\tau_1\, c_2\, s_1 - c_1\, s_2)\right\} = m_1\, \{r_{y1}\, \tau_1 + r_{z1}\, \sigma_1\} + m_2\, r_{z2}\, \sigma_1 \tag{4-11}$$

Since the unknowns in (4-11) have fixed coefficients, those unknowns cannot be resolved further.

In summary, (4-7), (4-10) and (4-11) provide the necessary information for determining the MPMS gravity terms. The actual mass values and center-of-gravity terms do not need to be determined completely.

MPMS Kinetic Energy Terms

The total kinetic energy K of the MPMS motion equals $K = K_1 + K_2$, where K_i, $i = 1,2$, equals the kinetic energy of link L_i defined with respect to the origin define in Figure 1.1. The joint torques τ_{ke1} and τ_{ke2} required to generate the motion, assuming a conservative system, equal, according to Lagrange,

$$\tau_{kei} = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{\theta}_i}\right] - \frac{\partial K}{\partial \theta_i}, \quad i = 1,2. \tag{4-12}$$

The torque terms associated with each kinetic energy

$$\tau_{keji} = \frac{d}{dt}\left[\frac{\partial K_j}{\partial \dot{\theta}_i}\right] - \frac{\partial K_j}{\partial \theta_i}, \quad j = 1,2;\ i = 1,2. \tag{4-13}$$

sum to produce the total torque for that joint,

$$\tau_{kei} = \tau_{ke1i} + \tau_{ke2i} \tag{4-14}$$

Since the origin does not translate, the kinetic energy for both links of the MPMS equals the rotational kinetic energy,

$$K_i = \frac{1}{2}\, \omega_i <\bullet> \mathbf{I}_i\, \omega_i, \quad i = 1,2, \tag{4-15}$$

where ω_i equals the angular velocity and $\mathbf{I}_i$ equals the standard inertia matrix for link L_i. In frame F_i the matrix $\mathbf{I}_i$ has constant terms.

From (3-1) and (3-3),

$$^{1,0}\omega_1 = \dot{\theta}_1 \begin{pmatrix} 0 \\ \sigma_1 \\ \tau_1 \end{pmatrix}, \quad ^{2,0}\omega_2 = {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \tag{4-16}$$

The kinetic energy of the first link

$$K_1 = \frac{1}{2}\ {}^{1,0}\omega_1 <\bullet> {}^{1}\mathbf{I}_1\ {}^{1,0}\omega_1 \tag{4-17}$$

does not depend on either of the joint variables nor $\dot{\theta}_2$, hence,

$$\tau_{ke11} := \frac{d}{dt}\left[\frac{\partial K_1}{\partial \dot{\theta}_1}\right] - \frac{\partial K_1}{\partial \theta_1} = \ddot{\theta}_1\,(0 \quad \sigma_1 \quad \tau_1)^{\tau}\ {}^{1}\mathbf{I}_1 \begin{pmatrix} 0 \\ \sigma_1 \\ \tau_1 \end{pmatrix} \tag{4-18}$$

$$\tau_{ke12} := \frac{d}{dt}\left[\frac{\partial K_1}{\partial \dot{\theta}_2}\right] - \frac{\partial K_1}{\partial \theta_2} = 0 \tag{4-19}$$

The kinetic energy of the second link equals

$$K_2 = \frac{1}{2} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} <\bullet> {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\ {}^{2}\mathbf{I}_2\ {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}, \tag{4-20}$$

and does not depend upon θ_1. Hence, the torque contribution by joint one to the motion of the second link equals

$$\tau_{ke21} := \frac{d}{dt}\left[\frac{\partial K_2}{\partial \dot{\theta}_1}\right] - \frac{\partial K_2}{\partial \theta_1} = \frac{d}{dt}\left[\frac{\partial K_2}{\partial \dot{\theta}_1}\right] = {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} <\bullet> \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} +$$

$${}^{2,0}\dot{\mathbf{J}}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} <\bullet> \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} + {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\dot{\mathbf{J}}_{\omega,2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} <\bullet> \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}. \qquad (4\text{-}21)$$

Similarly, the torque contribution by joint two to the motion of link two equals

$$\tau_{ke22} := \frac{d}{dt}\left[\frac{\partial K_2}{\partial \dot{\theta}_2}\right] - \frac{\partial K_2}{\partial \theta_2} = {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} <\bullet> \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} +$$

$${}^{2,0}\dot{\mathbf{J}}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} <\bullet> \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} + {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\dot{\mathbf{J}}_{\omega,2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} <\bullet> \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

$$- \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} <\bullet> \left\{ \frac{\partial\, {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}}{\partial \theta_2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\mathbf{J}_{\omega,2} + {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, \frac{\partial\, {}^{2,0}\mathbf{J}_{\omega,2}}{\partial \theta_2} \right\} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \qquad (4\text{-}22)$$

From (3-3) compute

$${}^{2,0}\dot{\mathbf{J}}_{\omega,2} = \sigma_1 \dot{\theta}_2 \begin{pmatrix} c_2 & 0 \\ -s_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \frac{\partial\, {}^{2,0}\mathbf{J}_{\omega,2}}{\partial \theta_2} = \sigma_1 \begin{pmatrix} c_2 & 0 \\ -s_2 & 0 \\ 0 & 0 \end{pmatrix} \qquad (4\text{-}23)$$

and deduce

$${}^{2,0}\dot{\mathbf{J}}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\dot{\mathbf{J}}_{\omega,2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (4\text{-}24)$$

$${}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\, {}^{2}\mathbf{I}_2\, {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sigma_1 (s_2 I_{xz2} + c_2 I_{yz2}) \\ I_{zz2} \end{pmatrix}, \qquad (4\text{-}25)$$

Relations (4-24) and (4-25) allows us to simplify (4-22), and, coupled with (4-19) yields the joint-two, kinetic energy derived torque τ_{ke2},

$$\tau_{ke2} = \tau_{ke22} + \tau_{ke12} = {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\ {}^{2}\mathbf{I}_2\ {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix}0\\1\end{pmatrix} <\bullet> \begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} +$$

$$- \begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix} <\bullet> \{ \frac{\partial\ {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}}{\partial \theta_2}\ {}^{2}\mathbf{I}_2\ {}^{2,0}\mathbf{J}_{\omega,2} + {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\ {}^{2}\mathbf{I}_2 \frac{\partial^{2,0}\mathbf{J}_{\omega,2}}{\partial \theta_2} \} \begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix} \quad (4\text{-}26)$$

The joint-one, kinetic energy derived torque τ_{ke1} equals the sum of (4-18) and (4-21),

$$\tau_{ke1} = \tau_{ke11} + \tau_{ke21} = \ddot{\theta}_1\ I_{T1} + {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\ {}^{2}\mathbf{I}_2\ {}^{2,0}\mathbf{J}_{\omega,2} \begin{pmatrix}1\\0\end{pmatrix} <\bullet> \begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix}$$

$$+ \{ {}^{2,0}\dot{\mathbf{J}}^{\tau}_{\omega 2}\ {}^{2}\mathbf{I}_2\ {}^{2,0}\mathbf{J}_{\omega,2} + {}^{2,0}\mathbf{J}^{\tau}_{\omega 2}\ {}^{2}\mathbf{I}_2\ {}^{2,0}\dot{\mathbf{J}}_{\omega,2} \} \begin{pmatrix}1\\0\end{pmatrix} <\bullet> \begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}. \quad (4\text{-}27)$$

where

$$I_{T1} := (0 \quad \sigma_1 \quad \tau_1)^{\tau}\ {}^{1}\mathbf{I}_1 \begin{pmatrix}0\\ \sigma_1\\ \tau_1\end{pmatrix} \quad (4\text{-}28)$$

cannot be resolved further by any joint measurements

Using *Mathematica* to expand (4-27) and (4-26) yields

$$\tau_{ke1} = \ddot{\theta}_1 I_{T1} + I_{zz}\tau_1(\ddot{\theta}_2 + \ddot{\theta}_1\tau_1) + \sigma_1\{ I_{yy} c_2 \sigma_1 (\ddot{\theta}_1 c_2 - 2\dot{\theta}_1\dot{\theta}_2 s_2)$$

$$+ I_{xx}\sigma_1 s_2 (2\dot{\theta}_1\dot{\theta}_2 c_2 + \ddot{\theta}_1 s_2)$$

$$+ I_{yz}(-\ddot{\theta}_2 c_2 - 2\ddot{\theta}_1 c_2\tau_1 + \dot{\theta}_2^2 s_2 + 2\dot{\theta}_1\dot{\theta}_2\tau_1 s_2)$$

$$+ I_{xz}(-\dot{\theta}_2^2 c_2 - 2\dot{\theta}_1\dot{\theta}_2 c_2\tau_1 - \ddot{\theta}_2 s_2 - 2\ddot{\theta}_1\tau_1 s_2)$$

$$- I_{xy}\sigma_1(2\dot{\theta}_1\dot{\theta}_2 c_{2\theta 2} + \ddot{\theta}_1 s_{2\theta 2})\ \} \qquad (4\text{-}29)$$

and

$$\tau_{ke2} = I_{zz}(\ddot{\theta}_2 + \ddot{\theta}_1\tau_1) + \sigma_1\{ I_{xy}\dot{\theta}_1^2 c_{2\theta 2}\sigma_1 + I_{xz}(\dot{\theta}_1^2 c_2\tau_1 - \ddot{\theta}_1 s_2)$$

$$+ I_{yz}(-\ddot{\theta}_1 c_2 - \dot{\theta}_1^2\tau_1 s_2) - \tfrac{1}{2} I_{xx}\dot{\theta}_1^2\sigma_1 s_{2\theta 2} + \tfrac{1}{2} I_{yy}\dot{\theta}_1^2\sigma_1 s_{2\theta 2}\ \} \qquad (4\text{-}30)$$

Equations (4-29) and (4-30) yield the essential relations for determining the unknown inertia matrix $\mathbf{I}_2$ and the term I_{T1} from torque, velocity and acceleration measurements.

5. Dynamics of the Unknown Mass

The functional objective of the MPMS is to measure the unknown mass properties of an object placed on the table of link two (Figure 1.1). The unknown mass is rigidly attached to link two during the measurements. Conceptually, this makes the unknown object a part of the second link, hence, the combined inertia tensor $\mathbf{I}_2'$ equals

$$\mathbf{I}_2' = \mathbf{I}_2 + \mathbf{I}_u \qquad (5\text{-}1)$$

where ${}^2\mathbf{I}_u$ is the inertia matrix of the unknown mass described about the common origin of frame F_0 and F_1. The dynamics analysis in Section 4 applies to this problem with $\mathbf{I}_2$ replaced by $\mathbf{I}_2'$. Thus, the torques derived from the kinetic energy of the unknown mass alone have the form of $\tau_{ke1} - \ddot{\theta}_1 I_{T1}$ in (4-29) and τ_{ke2} in (4-30) with the inertia terms replaced by those of the unknown mass,

$$\tau_{keu1} := I_{zzu} \tau_1 (\ddot{\theta}_2 + \ddot{\theta}_1 \tau_1) + \sigma_1 \{ I_{yyu} c_2 \sigma_1 (\ddot{\theta}_1 c_2 - 2 \dot{\theta}_1 \dot{\theta}_2 s_2)$$

$$+ I_{xxu} \sigma_1 s_2 (2 \dot{\theta}_1 \dot{\theta}_2 c_2 + \ddot{\theta}_1 s_2)$$

$$+ I_{yzu} (-\ddot{\theta}_2 c_2 - 2 \ddot{\theta}_1 c_2 \tau_1 + \dot{\theta}_2^2 s_2 + 2 \dot{\theta}_1 \dot{\theta}_2 \tau_1 s_2)$$

$$+ I_{xzu} (-\dot{\theta}_2^2 c_2 - 2 \dot{\theta}_1 \dot{\theta}_2 c_2 \tau_1 - \ddot{\theta}_2 s_2 - 2 \ddot{\theta}_1 \tau_1 s_2)$$

$$- I_{xyu} \sigma_1 (2 \dot{\theta}_1 \dot{\theta}_2 c_{2\theta 2} + \ddot{\theta}_1 s_{2\theta 2}) \} \qquad (5\text{-}2)$$

and

$$\tau_{ke2} = I_{zzu} (\ddot{\theta}_2 + \ddot{\theta}_1 \tau_1) + \sigma_1 \{ I_{xyu} \dot{\theta}_1^2 c_{2\theta 2} \sigma_1 + I_{xzu} (\dot{\theta}_1^2 c_2 \tau_1 - \ddot{\theta}_1 s_2)$$

$$+ I_{yzu} (-\ddot{\theta}_1 c_2 - \dot{\theta}_1^2 \tau_1 s_2) - \tfrac{1}{2} I_{xxu} \dot{\theta}_1^2 \sigma_1 s_{2\theta 2} + \tfrac{1}{2} I_{yyu} \dot{\theta}_1^2 \sigma_1 s_{2\theta 2} \} \qquad (5\text{-}3)$$

Equations (5-2) and (5-3) yield the essential relations for determining the unknown inertia matrix $\mathbf{I}_u$ from torque, velocity and acceleration measurements.

6. Measuring the Unknown Mass

Since the torque equations are essentially the same for the inertia terms of link two of the MPMS and the unknown mass, only the experimental technique for the latter will be developed here.

The first question to resolve:

Will measurements of the torque, angular position, velocity and acceleration of joint one provide sufficient data to compute all the inertia parameters of the unknown mass?

The answer will to this question is *No*. To verify this claim, note that, potentially, all six independent parameters in $\mathbf{I}_u$ might be determined from (5-2) given that

Experiment 6.1: $\dot{\theta}_2 = 0$, $\ddot{\theta}_2 = 0$,

$$\tau_{keu1} := \ddot{\theta}_1 \{ I_{zzu}\, \tau_1^2 + \sigma_1^2 \{ I_{yyu}\, c_2^2 + I_{xxu}\, s_2^2 - I_{xyu}\, s_{2\theta 2} \} - 2\, \sigma_1\, \tau_1 \{ I_{yzu}\, c_2 + I_{xzu}\, s_2 \} \}$$

which, in inner product form, equals,

$$\tau_{keu1} = \ddot{\theta}_1\, \mathbf{k} <\bullet> \mathbf{I}_v \tag{6-1}$$

where $\mathbf{k}$ is the *coefficient vector* and $\mathbf{I}_v$ a six-vector of the independent parameters in $\mathbf{I}_u$:

$$\mathbf{k} := \begin{pmatrix} \tau_1^2 \\ \sigma_1^2 c_2^2 \\ \sigma_1^2 s_2^2 \\ -\sigma_1^2 s_{2\theta 2} \\ -2\,\sigma_1\,\tau_1\, c_2 \\ -2\,\sigma_1\,\tau_1 s_2 \end{pmatrix} \text{ and } \mathbf{I}_v := \begin{pmatrix} I_{zzu} \\ I_{yyu} \\ I_{xxu} \\ I_{xyu} \\ I_{yzu} \\ I_{xzu} \end{pmatrix} \tag{6-2}$$

If a vector can be found orthogonal to the coefficient vector for all choices of MPMS configurations, then, six independent coefficient vectors $\mathbf{k}$ cannot be obtained and, therefore, six independent equations in the six unknowns in $\mathbf{I}_v$ cannot be obtained. The vector $\mathbf{x} := [\sigma_1^2 \;\; -\tau_1^2 \;\; -\tau_1^2 \;\; 0 \;\; 0 \;\; 0]^\tau \neq \mathbf{0}$ is such a vector: $\mathbf{k} <\bullet> \mathbf{x} = \mathbf{0}$ for all possible $\mathbf{k}$. This implies no non-singular measurement matrix $\mathbf{M}$ can be formed from the set of possible $\mathbf{k}$ that will yield six independent equations. Since $\mathbf{M}\,\mathbf{x} = \mathbf{0}$ for any $\mathbf{M}$ whose rows are constructed from different $\mathbf{k}$'s, the matrix $\mathbf{M}$ possesses a non-zero vector in its null space and, therefore, must be singular. Note that this observation holds for any choice of angle α_1 between the two joint axes of the mechanism.

Observe that $\mathbf{x}$ cannot be solution to any inertia problem because it forces at least one of the diagonal terms of $\mathbf{I}_u$ negative, an impossible situation physically.

Substitute $\mathbf{x} = \mathbf{I}_v$, this is not a physically possible $\mathbf{I}_v$, into the general expression for τ_{keu1} and find that $\tau_{keu1} = I_{zzu}\,\sigma_1\,\tau_1\,\ddot{\theta}_2$ which proves that, minimally, one must measure $\ddot{\theta}_2$ on the second joint.

Experiment 6.2 $\dot{\theta}_1 = 0$, $\ddot{\theta}_1 = 0$,

$$\tau_{keu1} := I_{zzu}\,\tau_1\,\ddot{\theta}_2 + \sigma_1\,I_{yzu}\,(-\ddot{\theta}_2\,c_2 + \dot{\theta}_2^{\,2}\,s_2) + \sigma_1\,I_{xzu}\,(-\dot{\theta}_2^{\,2}\,c_2 - \ddot{\theta}_2\,s_2) \tag{6-3}$$

hence,

$$\tau_{keu1} = \mathbf{k} <\bullet> \mathbf{I}_s \tag{6-4}$$

where $\mathbf{k}$ is the *coefficient vector* and $\mathbf{I}_s$ a three-vector :

$$\mathbf{k} := \begin{pmatrix} \tau_1\,\ddot{\theta}_2 \\ \sigma_1\,(-\ddot{\theta}_2\,c_2 + \dot{\theta}_2^{\,2}\,s_2) \\ \sigma_1\,(-\dot{\theta}_2^{\,2}\,c_2 - \ddot{\theta}_2\,s_2) \end{pmatrix} \text{ and } \mathbf{I}_s := \begin{pmatrix} I_{zzu} \\ I_{yzu} \\ I_{xzu} \end{pmatrix} \tag{6-5}$$

By measuring the torque on joint one and the angular position, velocity and acceleration on joint two, one can measure three independent equations (6-4) to compute I_{zzu} , I_{yzu} , I_{xzu}. For example, if $\theta_2 = \frac{\pi\,t^2}{4}$, $\dot{\theta}_2 = \frac{\pi\,t}{2}$, $\ddot{\theta}_2 = \frac{\pi\,t}{2}$, $0 \le t \le \sqrt{2}$, and $\alpha_1 = \tan^{-1}[\sqrt{2}]$, the measurement values of (6-3) for t=0, t = 1, t = $\sqrt{2}$ lead to the measurement matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{k}^\tau(t=0) \\ \mathbf{k}^\tau(t=1) \\ \mathbf{k}^\tau(t=\sqrt{2}) \end{pmatrix} = \begin{pmatrix} 0.9069 & 0 & -1.28255 \\ 0.9069 & -2.33145 & 0.517655 \\ 0.9069 & -1.28255 & 4.02925 \end{pmatrix}$$

whose determinant equals det[M] = -9.13734, proving that **M** non-singular and that $\mathbf{M}\mathbf{I}_s = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}$ can be solved for $\mathbf{I}_s$. With $\mathbf{I}_s$ known, the remaining inertia parameters can be determined from the measurements in Experiment 6.1.

***Conclusion:** If one measures torques and angular acceleration on joint one, then the angular position, velocity and acceleration on joint two must be measured (or calculated) in order to obtain all the inertia parameters of the unknown mass.*

7. PROPOSED DESIGN FOR MPMS

Based upon the analysis in this paper, we note that

1. All the mass properties of the MPMS and the unknown mass itself cannot be determined by measurements performed only on joint axis one quantities.

Discussions with Kedron Wolcott on the construction of the current prototype indicates

2. Construction of the bearings for the MPMS for a twist $0 < \alpha_1 < 90°$ present significant cost and design penalties.

A talk with Richard Bennett indicates that

3. The current design occupies a large volume due to the twist $\alpha_1 \approx 54°$ between the two axes.

4. The current design is not easily scalable to handle larger loads.

Based upon the four considerations above, the author proposes that the two joint axes of the MPMS be oriented at right angles, $\alpha_1 = 90°$, as shown in Figure 7.1. This design eliminates the problems mentioned in items 2,3, and 4, but will require a torque sensor on the second joint. The following analysis demonstrates the latter observation.

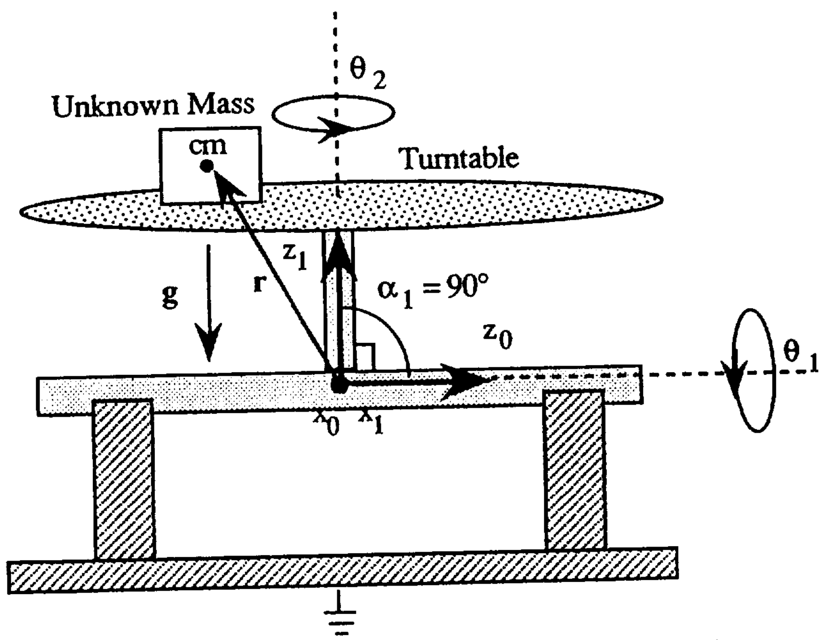

Figure 7.1 Proposed MPMS.

Evaluate (5-2) and (5-3) for $\alpha_1 = 90°$,

$$\tau_{keu1} := I_{yyu}\, c_2(\ddot{\theta}_1\, c_2 - 2\, \dot{\theta}_1\, \dot{\theta}_2\, s_2) + I_{xxu}\, s_2\, (2\, \dot{\theta}_1\, \dot{\theta}_2\, c_2 + \ddot{\theta}_1\, s_2)$$

$$+ I_{yzu}\, (-\ddot{\theta}_2\, c_2 + \dot{\theta}_2^{\,2}\, s_2) - I_{xzu}\, (\dot{\theta}_2^{\,2}\, c_2 + \ddot{\theta}_2\, s_2) - I_{xyu}\, (2\, \dot{\theta}_1\, \dot{\theta}_2\, c_{2\theta 2} + \ddot{\theta}_1\, s_{2\theta_2}) \quad (7\text{-}1)$$

and

$$\tau_{ke2} = I_{zzu}\, \ddot{\theta}_2 + I_{xyu}\, \dot{\theta}_1^{\,2}\, c_{2\theta_2} - I_{xzu}\, \ddot{\theta}_1\, s_2 - I_{yzu}\, \ddot{\theta}_1\, c_2$$

$$- \tfrac{1}{2} I_{xxu}\, \dot{\theta}_1^{\,2}\, s_{2\theta_2} + \tfrac{1}{2} I_{yyu}\, \dot{\theta}_1^{\,2}\, s_{2\theta_2} \quad (7\text{-}2)$$

Observe that I_{zzu} cannot be determined from (7-1). However, if one measures τ_{ke2} and fixes $\dot{\theta}_1 = 0$, $\ddot{\theta}_1 = 0$, then (7-2) yields

$$I_{zzu} = \frac{\tau_{ke2}}{\ddot{\theta}_2} \quad \text{when } \dot{\theta}_1 = 0,\ \ddot{\theta}_1 = 0. \tag{7-3}$$

To measure the other inertia parameters set $\dot{\theta}_2 = 0$, $\ddot{\theta}_2 = 0$ in (7-1) and (7-2) to obtain

$$\tau_{keu1} := (I_{yyu}\, c_2^{\ 2} + I_{xxu}\, s_2^{\ 2} - I_{xyu}\, s_{2\theta_2})\, \ddot{\theta}_1 \tag{7-4}$$

$$(I_{xyu}\, c_{2\theta_2} - \tfrac{1}{2} I_{xxu}\, s_{2\theta_2} + \tfrac{1}{2} I_{yyu}\, s_{2\theta_2})\, \dot{\theta}_1^{\ 2} - \tau_{ke2} = (I_{xzu}\, s_2 + I_{yzu}\, c_2)\, \ddot{\theta}_1 \tag{7-5}$$

The procedure would be to use (7-4) to obtain I_{yyu}, I_{xxu} ,I_{xyu} and then substitute these values in (7-5) to obtain the remaining inertia terms I_{xzu} ,I_{yzu} .

Experiments 7.1 and 7.2 entail measuring both torques, $\dot{\theta}_1$ and $\ddot{\theta}_1$ with the second joint axis fixed, respectively, at $\theta_2 = 0, \pi/2$ and $\pi/4$. For the last angle, measurement of the torque on the second joint is not required.

Experiment 7.1 $\quad \dot{\theta}_2 = 0,\ \ddot{\theta}_2 = 0$

1. $\theta_2 = 0$, $$I_{yyu} = \frac{\tau_{ke1}}{\ddot{\theta}_1} \tag{7-6a}$$

2. $\theta_2 = \pi/2$, $$I_{xxu} = \frac{\tau_{ke1}}{\ddot{\theta}_1} \tag{7-6b}$$

3. $\theta_2 = \pi/4$, $$I_{xyu} = \frac{I_{yyu}\, c_2^{\ 2} + I_{xxu}\, s_2^{\ 2} - \tau_{ke1}}{\ddot{\theta}_1} \tag{7-6c}$$

Experiment 7.2 $\dot{\theta}_2 = 0, \ddot{\theta}_2 = 0$

$$1.\ \theta_2 = 0\ ,\ I_{yzu} = \frac{I_{xyu}\,\dot{\theta}_1^{\,2} - \tau_{ke2}}{\ddot{\theta}_1} \tag{7-6d}$$

$$2.\ \theta_2 = \pi/2\ ,\ I_{xzu} = \frac{-I_{xyu}\,\dot{\theta}_1^{\,2} - \tau_{ke2}}{\ddot{\theta}_1} \tag{7-6e}$$

Equations (7-3) and (7-6) determines the inertia tensor of the unknown mass . Further, the same measurements performed on the MPMS when it is unloaded will allow one to compute the inertia tensor for the turntable and link.

8. CONCLUSION

Theoretical analysis proves the inertia parameters of an unknown mass can be determined from the joint torques, positions, angular velocities and angular accelerations of the Mass Properties Measurement System (MPMS). In particular, the existing system requires measurement of the torque and angular acceleration on joint one and the angular position, velocity and acceleration on joint two. A proposed system, where the twist angle between the MPMS turning axes equals 90°, permits computing the inertia terms in a simple manner with the additional requirement of measuring the second joint torque for one set of measurements. The new design offers some significant advantages. It packages more compactly, allows simple mechanical scaling, and is easier and less costly to construct.

The next report will discuss actual physical measurements with the existing system to determine the precision with which the MPMS can measure the mass properties of an unknown mass.

REFERENCES

1. Doty, Keith L,1992, EEL 6667 Class Notes.

2. Doty, Keith L., 1989, "Tabulation of the Midframe Jacobian," *Int. J. of Robotics Research,* 6:3.