A THEOREM REGARDING ROOTS OF THE ZERO-ORDER BESSEL FUNCTION OF THE FIRST KIND

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ABSTRACT

This paper investigates a problem on the steady-state, conduction-convection heat transfer process in cylindrical porous heat exchangers. The governing partial differential equations for the system are obtained using the energy conservation law. Solution of these equations and the concept of enthalpy lead to a new approach to prove a theorem that the sum of inverse squares of all the positive roots of the zero order Bessel function of the first kind equals to one-forth. As a corollary, it is shown that the sum of one over pth power ($p \geq 2$) of the roots converges to some constant.

NOMENCLATURE

W   specific mass flow rate , $Kg/m^2.s$

C_p specific heat of fluid at constant pressure , $J/Kg.^{\circ}C$

h   convective heat transfer coefficient , $W/m^2.^{\circ}C$

R   radius of cylindrical board , m

r   radial coordinate , m

z   axial coordinate , m

T_s solid temperature , $^{\circ}C$

T_{s0} solid temperature at circumference , $^{\circ}C$

T_f fluid temperature , $^{\circ}C$

T_{f0} inlet fluid temperature , $^{\circ}C$

A WC_pR/λ_r , dimensionless parameter

B $\beta hR/WC_p$, dimensionless parameter

t $(T_s - T_{f0})/(T_{s0} - T_{f0})$, dimensionless temperature of solid media

T	$(T_f - T_{f0})/(T_{s0} - T_{f0})$, dimensionless temperature of fluid
x	z/R , dimensionless axial coordinate
y	r/R , dimensionless radial coordinate

Greek Letters

β	specific area of heat transfer , m^2/m^3
γ	λ_z/λ_r , ratio of the axial and radial thermal conductivity
λ_z	axial thermal conductivity of solid , $W/m.^{\circ}C$
λ_r	radial thermal conductivity of solid , $W/m.^{\circ}C$

INTRODUCTION

Porous heat exchangers play a significant role in many engineering applications, such as cryogenics, thermal storage systems, and chemical reactors. Such systems lead to a set of Partial Differential Equations (PDEs) with a strong coupling between equations for the solid and the fluid phases. Analytical schemes to solve a general class of problems in this area include the method of separation of variables (refs. 1 and 2), Riemann method (ref. 3), orthogonal collocation techniques (ref. 4) and collocation-perturbation scheme for packed beds (ref. 5). Siegwarth and Radebough (ref. 6) present a numerical technique to solve the above problems with variable physical properties. Lin, Guo, and Wang (ref. 7) present a combined orthogonal collocation-perturbation method to solve the temperature field in a cylindrical porous heat exchanger.

In this paper, the physical problem presented in (Ref. 7) is reconsidered. Using the approach of separation of variables, the PDEs associated with the problem are reduced to two Boundary Value Problems (BVPs) of Ordinary Differential Equations (ODEs). Solution of the resulting BVPs leads to a new method to prove a theorem regarding roots of the zero-order Bessel function of the first kind. As a corollary, it is shown that the sum of one over pth power ($p \geq 2$) of the roots converges to some constant.

The theory of Bessel functions and related functions is well established and the main results are summarized in textbooks and mathematical manuals (refs. 8

to 10). The properties of Bessel functions concerning operations of differentiation and integration with respect to their order have also been studied by Apelblat and Kravitsky (ref. 11). However, little is known about the convergence and summation of series for roots of Bessel functions due to no explicit expressions for the roots except for $\pm 1/2$ order Bessel functions. Although in 1874 Rayleigh derived a general form of the theorem discussed here by applying an Euler's formula (ref. 8), his approach is suitable for even power series only. Furthermore, he did not study the convergence of such series for a general case. Therefore, research is needed to further understand the properties of Bessel functions. The new method to prove the theorem stated above is based on the mathematical model of a cylindrical porous heat exchanger. A brief description of this model is presented next.

MATHEMATICAL MODELING

The model considered here is a semi-infinite cylindrical porous heat exchanger as shown in (fig. 1). Let r and z be the radial and the axial coordinates, and R be the radius of the cylinder. A fluid flows through the porous media in axial direction from left to right. The inlet temperature of the fluid is T_{f0} which can be higher or lower than the board circumference temperature T_{so}. Let W, C_p, and h be the specific mass flow rate, the specific heat of fluid at constant pressure, and the convective heat transfer coefficient respectively.

Porous materials used in such applications exhibit anisotropic behavior. It is assumed that the thermal conductivity of the solid is symmetric about the axis of the cylinder. Let β be the specific area of heat transfer , k_z and k_r be the axial and the radial thermal conductivities of the solid, and γ be the ratio of k_z and k_r.

In the derivation to follow, we assume that: 1) the physical properties and convective heat transfer coefficient between porous substance and fluid are constants, 2) the dimensions of the porosity and solid particles are very small compared to the overall dimension of the heat exchanger, and therefore, the porous material can be treated as continuous media, 3) the fluid thermal conductivities are negligible compared to the solid thermal conductivities, and 4) the inner wall temperature is kept constant, and there are no thermal resistances between wall

and porous media or fluid. Finally, for simplicity, it is considered that $T_{s0} > T_{f0}$. However, the formulation presented here equally holds for $T_{s0} < T_{f0}$.

The differential equations of the system may be obtained by applying the energy conservation law to a micro unit volume (ref. 7). Nondimensional forms of these equations and the boundary conditions are given as follows:

Solid Phase :

$$\frac{\partial^2 t}{\partial y^2} + \frac{1}{y}\frac{\partial t}{\partial y} + \gamma\frac{\partial^2 t}{\partial x^2} = A\frac{\partial T}{\partial x} \tag{1}$$

Fluid Phase :

$$\frac{\partial T}{\partial x} = B(t - T) \tag{2}$$

Boundary Conditions :

$$
\begin{cases}
t = 1, & y = 1 & (3a) \\
T = 0, & x = 0 & (3b) \\
t = 1, & x \to +\infty & (3c)
\end{cases}
$$

where $t(= (T_s - T_{f0})/(T_{s0} - T_{f0}))$ and $T(= (T_f - T_{f0})/(T_{s0} - T_{f0}))$ are dimensionless temperatures of the solid and the fluid, $x(= z/R)$ and $y(= r/R)$ are dimensionless radial and axial coordinates, and $A(= WC_pR/k_r)$ and $B(= \beta hR/WC_p)$ are dimensionless parameters. Here T_s and T_f are the solid and the fluid temperature distributions. It should be noted that an additional boundary condition is required to completely define the problem. However, it is not needed here and therefore it is not considered.

Equations (1) to (3) can be used to find the temperature distributions of the solid and the fluid phases. It will be shown that these equations can also be used to prove the following theorem:

Theorem: Let α_n denote the nth positive root of the zero-order Bessel function of the first kind, then

$$\sum_{n=1}^{+\infty} \frac{1}{\alpha_n^2} = \frac{1}{2^2} \tag{4}$$

Corollary: For $p \geq 2$, series $\sum_{n=1}^{+\infty} \frac{1}{\alpha_n^p}$ converges to a constant $C(p)$ which depends on p.

Proof: In order to prove the above theorem, we begin by eliminating T from Eqs. (1) and (2). This leads to

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 t}{\partial y^2} + \frac{1}{y}\frac{\partial t}{\partial y} + \gamma\frac{\partial^2 t}{\partial x^2}\right) = AB\frac{\partial t}{\partial x} - B\left(\frac{\partial^2 t}{\partial y^2} + \frac{1}{y}\frac{\partial t}{\partial y} + \gamma\frac{\partial^2 t}{\partial x^2}\right) \qquad (5)$$

Let

$$t = 1 - XY \qquad (6)$$

where X and Y are functions of x and y respectively. Using Eqs. (3) and (5) and the method of separation of variables, we obtain the following two equivalent boundary value problems of ordinary differential equation:

$$\begin{cases} Y'' + \frac{1}{y}Y' - \lambda Y = 0 & (7a) \\ Y = 0, \quad y = 1 & (7b) \\ |Y| < \infty, \quad y = 0 & (7c) \end{cases}$$

and

$$\begin{cases} \gamma X''' + \gamma BX'' + (\lambda - AB)X' + \lambda BX = 0 & (8a) \\ X = 0, \quad x \to +\infty & (8b) \end{cases}$$

where a prime($'$) on X (Y) represents the derivative with respect to x (y), and λ is a separation constant. Equation (7c) suggests that Y is bounded at its center. Equation (7) can be brought to standard Bessel equation form by a simple linear transformation. This is a general Sturm - Liouville eigenvalue problem (ref. 14). It has nontrivial solutions only when $\lambda_n = -\alpha_n^2(\alpha_n \neq 0)$. Using the properties of Bessel functions, the solution of Eq. (7) for any α_n may be written as

$$Y_n = c_n J_0(\alpha_n y) \qquad (9)$$

where $J_0(x)$ is the zero order *Bessel* function of the first kind, α_n is the nth positive root of $J_0(x)$, and c_n is an undetermined constant.

Let the trial solution of Eq. (8) be given as

$$X = a'e^{\beta x} \qquad (10)$$

Substitution of Eq. (10) into Eq. (8) leads to the following characteristic equation,

$$\gamma\beta^3 + \gamma B\beta^2 - (\alpha_n^2 + AB)\beta - \alpha_n^2 B = 0 \qquad (11)$$

Equation (11) is cubic in β. The discriminant Δ of this equation is given as

$$\Delta = -\frac{1}{4\gamma^3}\{4\alpha_n^2(B^2\gamma - \alpha_n^2)^2 + 18AB^2\alpha_n^2\gamma + AB^3(AB + 2\alpha_n^2)+$$

$$4AB(A^2B^2 + 3\alpha_n^4 + 3\alpha_n^2 AB)\} \qquad (12)$$

Equation (12) suggests that $\Delta < 0$. Thus, using the theory of cubic equations, it follows that all roots of Eq. (11) are real (ref. 12). Let these roots be given as β_{n1}, β_{n2}, and β_{n3}. From Eq. (11), it follows that

$$\beta_{n1} + \beta_{n2} + \beta_{n3} = -B < 0 \qquad (13)$$

and

$$\beta_{n1} \cdot \beta_{n2} \cdot \beta_{n3} = \alpha_n^2 B/\gamma > 0 \qquad (14)$$

Equation (13) suggests that atleast one of the roots is negative and Eq. (14) indicates that negative roots appear in pair. Thus, it is concluded that Eq. (11) has two negative roots and one positive root for each α_n. From physical consideration (or Eq. (8b)), the positive root is disregarded. The general solution for X may now be written as

$$X_n = \sum_{i=1}^{2} a'_{ni}e^{\beta_{ni}x} \qquad (15)$$

Substituting Eqs. (9) and (15) into Eq. (6), we get

$$t = 1 - \sum_{n=1}^{\infty} J_0(\alpha_n y) \cdot \sum_{i=1}^{2} a_{ni}e^{\beta_{ni}x} \qquad (16)$$

where $a_{ni} = c_n a'_{ni}$. From Eqs. (1), (2) and (16), we obtain

$$T = 1 - \frac{1}{AB} \cdot \sum_{n=1}^{\infty} J_0(\alpha_n y) \cdot \sum_{i=1}^{2} a_{ni}e^{\beta_{ni}x} \cdot (\alpha_n^2 - \gamma\beta_{ni}^2 + AB) \qquad (17)$$

Differentiation of Eq. (17) with respect to x yields,

$$\frac{\partial T}{\partial x} = -\frac{1}{AB} \cdot \sum_{n=1}^{\infty} J_0(\alpha_n y) \cdot \sum_{i=1}^{2} a_{ni}\beta_{ni} \cdot (\alpha_n^2 - \gamma\beta_{ni}^2 + AB)e^{\beta_{ni}x} \qquad (18)$$

Using Eq. (17), the inlet condition (Eq. (3b)), and the property of orthogonality of $J_0(\alpha_n y)$ in the interval $[0, 1]$, we derive

$$\sum_{i=1}^{2} a_{ni}(\alpha_n^2 - \gamma\beta_{ni}^2 + AB) = \frac{2AB}{\alpha_n \cdot J_1(\alpha_n)} \tag{19}$$

where $J_1(x)$ is a *Bessel* function of the first kind of order one.

ENTHALPY RELATIONS

Some enthalpy relations help prove the above theorem. Let $H(x)$ and $H(x + dx)$ be quantities of enthalpy carried by the fluid across the axial planes at x and $x + dx$ respectively (Figure 1). Expression for $H(x)$ is given as

$$H(x) = WC_p \cdot \int_0^R \{rdr \cdot \int_0^{2\pi} d\theta \cdot [T_{f0} + T(T_{s0} - T_{f0})]\} \tag{20}$$

A closed form expression for $H(x)$ can be obtained by substituting Eq. (17) into Eq. (20), and integrating the resulting equation. Expanding $H(x+dx)$ in a Taylor's series, and neglecting the second and the higher order terms, we obtain

$$dH(x) = H(x + dx) - H(x) = \frac{\partial H(x)}{\partial x} \cdot dx \tag{21}$$

where $dH(x)$ is the enthalpy variation of the fluid passing through the control volume between x and $x + dx$ (Figure 1). The total change in enthalpy of the fluid is obtained by integrating Eq. (21) over the length of the heat exchanger. Thus,

$$\Delta H_0^{+\infty} = \int_0^{+\infty} dH(x) = \int_0^{+\infty} \frac{\partial H(x)}{\partial x} \cdot dx \tag{22}$$

Using Eqs. (18), (20), and (22), we obtain

$$\Delta H_0^{+\infty} = \frac{2\pi R^2 WC_p(T_{s0} - T_{f0})}{AB} \cdot \sum_{n=1}^{\infty} \{\frac{J_1(\alpha_n)}{\alpha_n} \cdot \sum_{i=1}^{2} a_{ni}(\alpha_n^2 - \gamma\beta_{ni}^2 + AB)\} \tag{23}$$

$\Delta H_0^{+\infty}$ can also be evaluated directly by subtracting the enthalpy of the fluid at the inlet ($= H(0)$) from that at the outlet ($= H(\infty)$). Expressions for $H(0)$ and $H(\infty)$ are given as

$$H(0) = WC_p\pi R^2 \cdot T_{f0}$$

and

$$H(\infty) = WC_p\pi R^2 \cdot T_{s0}$$

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Hence,

$$\Delta H_0^{+\infty} = WC_p\pi R^2 \cdot (T_{s0} - T_{f0}) \tag{24}$$

Comparing Eqs. (23) and (24), we obtain

$$\frac{2}{AB} \cdot \sum_{n=1}^{\infty} \frac{J_1(\alpha_n)}{\alpha_n} \cdot \{\sum_{i=1}^{2} a_{ni}(\alpha_n^2 - \gamma\beta_{ni}^2 + AB)\} = 1 \tag{25}$$

Finally, substituting Eq. (19) into Eq. (25) and simplifying, we obtain

$$\sum_{n=1}^{+\infty} \frac{1}{\alpha_n^2} = \frac{1}{2^2}$$

This proves the theorem. Since function $J_0(y)$ is symmetrical, it follows that

$$\sum_{n=-\infty}^{+\infty} \frac{1}{\alpha_n^2} = \frac{1}{2} \tag{26}$$

In order to prove the corollary, observe that all positive roots of $J_0(x)$ are greater than 1 (ref. 8). This implies that for $p \geq 2$

$$\frac{1}{\alpha_n^p} \leq \frac{1}{\alpha_n^2} \tag{27}$$

Equations (4) and (27) suggest that the series

$$C_n(p) = \sum_{i=1}^{n} \frac{1}{\alpha_i^p}$$

increases monotonically and it is bounded. Thus, by monotone convergence theorem (ref. 13), it follows that the series $C_n(p)$ ($n \rightarrow \infty$) is convergent.

CONCLUSION

A mathematical model for a steady-state conduction-convection heat transfer processes in a cylindrical porous heat exchanger has been investigated. It has been shown that the equations resulting in this model may be used to prove a theorem and a corollary regarding roots of the zero-order Bessel function of the first kind.

Research reported in this paper provides an alternate approach to prove a theorem in this area. The advantages of this physical approach are that it enriches physical understanding of a theorem, and that we may avoid difficulties emerging from rigid mathematical arguments.

ACKNOWLEDGMENT

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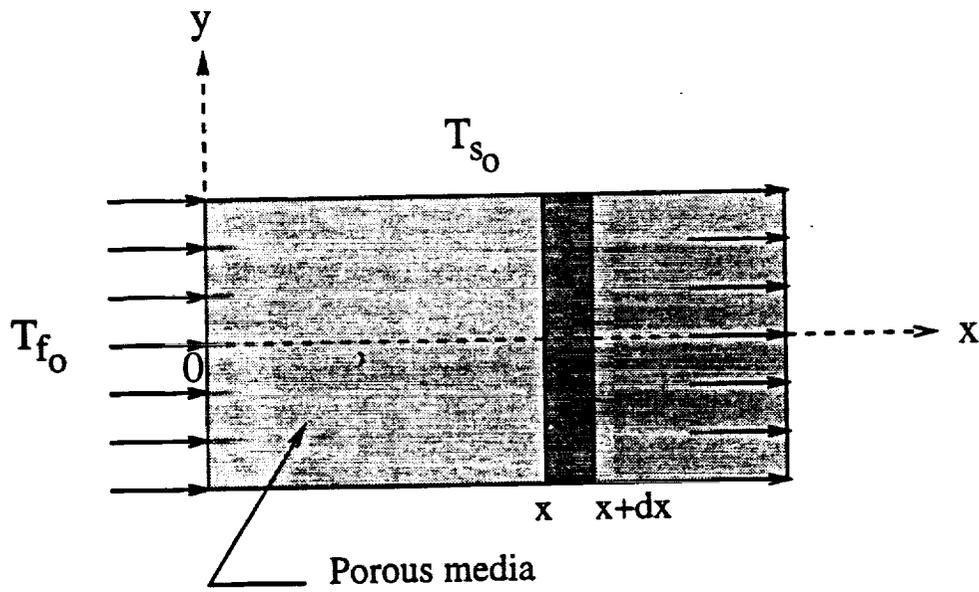14. Haberman, R.: *Elementary Applied Partial Differential Equations*. pp. 134-135, pp. 216-217, Prentice - Hall, 1987.

Figure 1. Porous heat exchanger model