FAS Multigrid Calculations of Three Dimensional Flow Using Non-staggered Grids

D. Matović[1], A. Pollard,
H. A. Becker and E. W. Grandmaison

Centre for Advanced Gas Combustion Technology,
Departments of Mechanical and Chemical Engineering,
Queen's University, Kingston, Ontario K7L 3N6, Canada

Abstract

Grid staggering is a well known remedy for the problem of velocity/pressure coupling in incompressible flow calculations. Numerous inconveniences occur, however, when staggered grids are implemented, particularly when a general-purpose code, capable of handling irregular three-dimensional domains, is sought. In several non-staggered grid numerical procedures proposed in the literature, the velocity/pressure coupling is achieved by either pressure or velocity (momentum) averaging. This approach is not convenient for simultaneous (block) solvers that are preferred when using multigrid methods. A new method is introduced in this paper that is based upon non-staggered grid formulation with a set of virtual cell face velocities used for pressure/velocity coupling. Instead of pressure or velocity averaging, a momentum balance at the cell face is used as a link between the momentum and mass balance constraints. The numerical stencil is limited to 9 nodes (in 2D) or 27 nodes (in 3D) both during the smoothing and inter-grid transfer, which is a convenient feature when a block point solver is applied. The results for a lid-driven cavity and a cube in a lid-driven cavity are presented and compared to staggered grid calculations using the same multigrid algorithm. The method is shown to be stable and produce a smooth (wiggle-free) pressure field.

[1]Ph.D. Student

1 Introduction

Multigrid methods are used in a number of applications in fluid dynamics, usually by applying the Full Approximation Scheme [1]. Incompressible flow calculations usually employ a staggered grid because of its strong coupling between the pressure and the velocity field (e.g. [2]. For complex geometries, however, as well as for calculations in non-orthogonal coordinates, the use of a staggered grid is a serious obstacle to efficient and well structured computer coding [3]. Additional complexities arise when a block-solver is used; for example, variables cannot be easily grouped into cell-bound blocks due to different node count. Some authors resort to asymmetric nodal clusters [5] while others update a symmetric block of variables around the cell centre node thereby updating face velocities twice in each relaxation sweep [5, 6]. Various levels of decoupled relaxation are also common. These include distributive relaxation, where all momentum equations are solved together and the pressure field is solved separately [1, 7], and sequential schemes that update variables throughout the flow field one by one [8, 9]. Some comparative studies of block *versus* sequential relaxation give no clear preference [10, 11]. There is a greater consensus that grid staggering is a necessary burden, particularly in the context of multigrid methods ([12, 13, 14, 15, 16] and even [1]). Comparison studies of staggered and non-staggered methods are sometimes conflicting in their assessment of the accuracy and stability of any given method. While some authors demonstrate that non-staggered methods match the staggered ones using both criteria ([13, 16, 17]), others question it ([18]). Despite this, the majority of finite volume incompressible calculations use staggered grids. The main reason may be that existing non-staggered grids *increase* rather than lessen the complexity of the staggered grid calculations. For example, the method of Rhie and Chow [19] (adopted by [13, 14, 15]) requires that both the nodal and cell face velocities are stored. Moreover, in a multigrid context, both the nodal and the face velocities need to be restricted [8], requiring even more computational work. Also, the computational cluster extends beyond either 9 or 27 point stencil in two- or three-dimensional formulations respectively for the first order discretisation and even more if the higher order methods are used.

The considerations mentioned above motivated the present contribution for a method suitable for block solvers on an irregular three-dimensional domain using a non-staggered grid.

In this paper a brief description of the multigrid procedure based on a new non-staggered method is given in section 2 and the multigrid implementation in section 3; the test cases and results are presented in section 4, followed by the discussion section where relative merits of the method are assessed.

2 The new non-staggered method

A transport equation for a general set of transported variables $\mathbf{u}$ in the volume Ω bounded by a boundary S can be expressed in an integral form suitable for finite volume formulation

$$\frac{\partial}{\partial t}\rho \mathbf{u} \mathrm{d}\Omega + \oint_S \left[\rho \mathbf{u} u_i - \Gamma_{\mathbf{u}} \left(\frac{\partial \mathbf{u}}{\partial x_i}\right)\right] n_i \mathrm{d}S = \mathbf{F}_{\mathbf{u}}, \tag{1}$$

where ρ is the density, u_i is the velocity component in the x_i direction and n_i is the component of unit normal to the boundary S. When (1) is applied to the momentum balance of a viscous incompressible fluid, the set $\mathbf{u}$ is a velocity vector $\mathbf{u} = \{u_j, j = 1, ..., d\}$ (d being the problem dimension), with the corresponding diffusion coefficient $\Gamma = \mu$ and the source terms

$$F_{u_j} = -\int_\Omega \left(\frac{\partial p}{\partial x_j}\right) \mathrm{d}\Omega + \oint_S \left[\mu \left(\frac{\partial u_i}{\partial x_j}\right) - \frac{2}{3}\delta_{ij}\mu \left(\frac{\partial u_k}{\partial x_k}\right)\right] n_i \mathrm{d}S \tag{2}$$

in the absence of external volume forces. The extension to other transportable properties (such as enthalpy, mass fraction, etc.) is straightforward by augmenting the vector $\mathbf{u}$ to include new variables and defining appropriate source terms and the diffusion coefficients. In the following presentation a three-dimensional implementation will be used.

The momentum equations are discretised using the hybrid (central/upwind) difference scheme [20] although higher order schemes can be employed[1]. The pressure field is resolved by means of mass conservation for the control volume around the node in a symmetric block manner as used by Vanka [5] for the staggered grid, although the extension to the line block around the node in a symmetric block manner as used by Vanka [5] for the staggered grid, although the extension to the line block formulation is

[1]For a multigrid implementation of a second order upwind scheme on a staggered grid see e.g. [21].

straightforward. The estimation of the face velocities that are substituted into the mass conservation equation is obtained by discretizing the momentum equation over a half-length volume around each cell face, directly involving the nodal pressure and velocities, while the lateral velocities are obtained by averaging values from the nearby nodes (see Fig. 1). More details on the coefficient generation are given in [22].

The principle of discretizing the face velocity using a half-size cell is applied also by Schneider and Raw [3], although in their method the coefficients of the face velocity are treated implicitly by incorporating them into the nodal velocity coefficient matrix. To ensure positive definiteness of the nodal velocities coefficient matrix, Schneider and Raw had to truncate the momentum equation applied to the face velocities [3, 16]. In the present method, the face velocities are explicitly expressed in terms of the surrounding nodal values and used in the continuity equation for the pressure correction calculation. The implication of this step is that the family of face velocities satisfies both the momentum and the mass conservation exactly at the positions where the convection velocities in a general transport equation are required. On the other hand, as an average of the (tentative) nodal velocities they are readily available without requirement for a permanent storage.

The boundaries of the flow field are coincident with the cell faces, enabling the definition of a set of boundary nodes there. This practice ensures consistency between the global mass balance of the whole calculation domain and the local mass balance of each cell, but calls for special treatment if Neumann boundary conditions are to be used. If the zero-gradient condition for the normal velocity is discretised in a usual way

$$\frac{\partial(u_i n_i)}{\partial(x_i n_i)} = \frac{(u_i n_i)_b - (u_i n_i)_{inn}}{(x_i n_i)_b - (x_i n_i)_{inn}}, \tag{3}$$

where subscripts b and inn denote boundary and the first inner node, respectively, the flow rate through the boundary will be linked to the velocity that does not belong to the mass preserving field, resulting in poor overall mass conservation. The correct way to implement Neumann boundary condition in this case is to use the face velocity. This way, the local and global mass balance become fully compatible. There is no need for any special treatment of the Dirichlet boundary conditions where the face velocities coincide with the boundary and are assumed known.

3 The multi-grid implementation

In the multigrid context, the nonlinear equation (1) can be expressed as

$$\mathcal{L}(\mathbf{u}) = \mathbf{F} \tag{4}$$

by grouping all terms that will result in a coefficient matrix (within a Newton iteration cycle) into the operator $\mathcal{L}$ and the remainder into the source term $\mathbf{F}$ as in [13, 23]. A more common practice of expressing Eqn. (4) as homogeneous (by absorbing $\mathbf{F}$ into $\mathcal{L}(\mathbf{u})$) [1, 24] is found by the present authors to be somewhat confusing, especially when defining residual transfer to the coarse grid.

The discretised (sparse, positive definite) Eqn. 4 for the grid l is linearised by a Newton iteration [24]

$$L^l \mathbf{u}^l = \mathbf{F}^l \tag{5}$$

and relaxed by a block Gauss-Seidel method.

The updates of the variable set

$$\mathbf{u}' = \alpha(\mathrm{diag}(L))^{-1}\mathbf{R}^l \tag{6}$$

are expressed in terms of the residual $\mathbf{R}^l = \mathbf{F}^l - L^l\mathbf{u}^l$, the inverse of the coefficient matrix diagonal $(\mathrm{diag}(L))^{-1}$ and the underrelaxation coefficient α. Variables at the node i, j, k are then updated by $\mathbf{u}_{i,j,k} = \mathbf{u}_{i,j,k} + \mathbf{u'}_{i,j,k}$.

Restriction is accomplished by grouping a cluster of eight cells into one. This leads to the following operator

$$\begin{aligned}\phi_{I,J,K} = \frac{1}{8}(&\phi_{i,j,k} + \phi_{i+1,j,k} + \phi_{i,j+1,k} + \phi_{i,j,k+1} + \\ &\phi_{i+1,j+1,k} + \phi_{i,j+1,k+1} + \phi_{i+1,j,k+1} + \phi_{i+1,j+1,k+1}),\end{aligned} \tag{7}$$

where $I = 2i - 2, \ldots$ The same operator is applied both to variable and residual restriction. After both the variables and residuals are transferred to the next coarser grid $(l-1)$, Eqn. (5) is approximated as

$$L^{l-1}(\mathbf{u}^{l-1}) = \overline{\mathbf{F}}^{l-1}, \tag{8}$$

where

$$\overline{\mathbf{F}}^{l-1} = \mathbf{F}^{l-1} - (\mathbf{F}_0^{l-1} - L_0^{l-1}\mathbf{u}_0^{l-1}) + \mathcal{R}_l^{l-1}\mathbf{R}^l \tag{9}$$

is the equivalent source term on the coarse grid. The restriction at Neumann boundaries is carried out using a divided form of the boundary conditions [1]. For the velocity component perpendicular to the boundary, an additional correction is made to preserve the mass flow rate through the boundary.

Prolongation is carried out by tri-linear interpolation using a seven point stencil, shown here for one cell and with injection only:

$$\phi_{i,j,k} = \frac{1}{6}(3\phi_{I,J,K} + \phi_{I-1,J,K} + \phi_{I,J-1,K} + \phi_{I,J,K-1}) \qquad (10)$$

with $i = (I+2)/2, \ldots$ The injection upon the first visit to the fine grid (FMG cycling is assumed) and the fine grid correction are done as

$$\mathbf{u}^l_{\text{first}} = \mathcal{P}^l_{l-1}\mathbf{u}_{l-1} \quad \text{or} \quad \mathbf{u}^l_{\text{new}} = \mathbf{u}^l_{\text{old}} + \mathcal{P}^l_{l-1}(\mathbf{u}_{l-1} - \mathbf{u}_{0,l-1}). \qquad (11)$$

4 Test cases

The non-staggered method presented in this paper is compared with the staggered three-dimensional calculations employing the block symmetric Gauss-Seidel algorithm of Vanka [5]. For both methods the coding and data structures are of the same style.

The flow in a three-dimensional cavity with a moving top is used as a first test case. The residual norm history is shown in Fig. 2. The rate of convergence obtained when calculating on a staggered grid is comparable with the results of Vanka [10] where 12 work units (w.u.) were necessary for a two orders of magnitude residual reduction. In our calculations 14 w.u. was necessary for the staggered grid calculation and 18 w.u. for the non-staggered calculations.[1] However, the change in slope of the non-staggered residual may indicate that the full potential of multigrid acceleration has yet to be achieved.

In a second test case, a cube is inserted in a cavity (Fig. 3), forcing the flow to negotiate this asymmetric three-dimensional obstacle, partly by the velocity magnitude change, partly by flow separation. It is believed that this flow geometry serves as a good test of the pressure/velocity coupling

[1] Or 23 w.u. for the same residual decrease; however, this is more arbitrary, because of the much lower initial residual at the finest grid.

because the major force behind the flow adjustment is the pressure field. The residual history, (Fig. 4) indicates very similar convergence rates for the staggered and non-staggered calculations. The resulting flow field in a symmetry plane (Fig. 5) indicates well resolved separation bubbles around the cube corners.

5 Discussion and conclusions

The new method of incompressible flow calculation using non-staggered grids and its multigrid implementation are examined for suitability in a complex flow field geometry. The presence of two sets of velocity values, both of which satisfy the (discrete form of the) governing equations increases the overall level of accuracy for a given grid size, although this remains to be quantified.

In the numerical experiments performed so far the method proved to be stable, without any tendency to produce an oscillating pressure field, which is a common feature of some non-staggered methods [18]. The method permits discretization on a trivially coarse grid (with a single node in the interior), which is very convenient in a FAS multigrid implementation, because it allows the coarse grid to contain the lowest number of nodes. Thus significantly coarser grids can be used in complex geometries. For example, in the case of a cube in a cavity (see the previous section) the coarsest grid (6x6x6 nodes) has only one control volume located between the cube and the cavity wall at one side. If the calculation method required two nodes at minimum, the overall node count at the coarse grid would increase eight times, thereby substantially increasing the work needed to obtain exact solution at the coarsest grid.

Various tests performed so far always produced smooth solutions both in velocity and pressure, which indicate a high ellipticity measure of the proposed method. The analytical evaluation of the ellipticity measure remains to be carried out (following e.g. [25, 16]).

The amount of computational work of the proposed method is slightly larger that of the Rhie and Chow [19] method. It is comparable to the work in the SCGS method of Vanka, requiring the same amount of work to calculate face velocities and pressure coefficients and, in addition, the calculation of the nodal velocity coefficients, i.e. approximately 25% increase in two-dimensional and 14% in three-dimensional calculations. This

overhead exists only for the simplest flow problem because any additional variable that is solved permits nodal velocity coefficients to be reused (with proper scaling of the diffusion part).

References

[1] A. Brandt. Guide to multigrid development. In *Multigrid Methods: Proceedings,Köln-Porz, 1981*, pages 220–312. Springer-Verlag, Berlin, 1982.

[2 B. Favini and G. Guj. MG techniques for staggered differences. In Holstein D.J., Paddon H., editor, *Multigrid Methods for Integral and Differential Equations*, pages 253–262, Oxford, 1985. Clarendon Press.

[3] G.E. Schneider. A novel colloccated finite difference procedure for the numerical computation of fluid flow. In *Proceedings of 4th AIAA/ASME Joint Thermophysics and Heat Transfer Conference.* AIAA-86-1330, 1986.

[4] B. R. Hutchinson, P. F. Galpin, and G. D. Raithby. Application of additive correction multigrid to the coupled fluid flow equations. *Numerical Heat Transfer*, 13:133–147, 1988.

[5] S. P. Vanka. Block-implicit multigrid solution of Navier-Stokes equations in primitive variables. *Journal of Computational Physics*, 65:138–158, 1986.

[6] M. C. Thompson and J. H. Ferziger. An adaptive multigrid solution technique for the incompressible navier-stokes equations. *Journal of Computational Physics*, 83:94–121, 1989.

[7] L. Fuchs and H.-S. Zhao. Solution of three-dimensional viscous incompressible flows by a multi-grid method. *International Journal for Numerical Methods in Fluids*, 4:539–555, 1984.

[8] M. Hortmann, M. Peric, and G. Scheuerer. Finite volume multigrid prediction of laminar natural convection: Bench-mark solutions. *Journal for Numerical Methods in Fluids*, 11:189–207, 1990.

[9] P. A. Rubini, H. A. Becker, E. W. Grandmaison, A. Pollard, A. Sobiesiak, and C. Thurgood. Multi-grid acceleration of three dimensional turbulent variable density flows. *Numerical Heat Transfer, B: Fundamentals*, pages 163–177, 1992.

[10] S. P. Vanka. Fast numerical computation of viscous flow in a cube. *Numerical Heat Transfer, Part B: Fundamentals*, 20:255–261, 1991.

[11] P. H. Gaskell, A. K. C. Lau, and N. G. Wright. Comparison of two solution strategies for use with higher-order discretization schemes in fluid flow simulation. *International Journal for Numerical Methods in Fluids*, 8:1203–1215, 1988.

[12] C. Hirsch. *Numerical Computation of Internal and Eternal Flows*, volume 1: Fundamentals of Numerical Discretization. John Wiley & Sons, New York, 1988.

[13] M. Perić, R. Kessler, and G. Scheuerer. Comparison of finite-volume numerical methods with staggered and collocated grids. *Computers and Fluids*, 16:389–403, 1988.

[14] S. Majumdar. Role of underrelaxation in momentum interpolation for calculation of flow with nonstaggered grids. *Numerical Heat Transfer*, 13:125–132, 1988.

[15] P. Coelho, J. C. F. Pereira, and M. G. Carvalho. Calculation of laminar recirculating flows using a local non-staggered grid refinement system. *International Journal for Numerical Methods in Fluids*, 12:535–557, 1991.

[16] W. W. Armfield. Finite difference solutions of the navier-stokes equations on staggered and non-staggered grids. *Computers and Fluids*, 20(1):1–17, 1991.

[17] T. M. Shih and A. L. Ren. Primitive-variable formulations using nonstaggered grids. *Numerical Heat Transfer*, 7:413–428, 1984.

[18] T. M. Shih, C. H. Tan, and B. C. Hwang. Effects of grid staggering on numerical schemes. *International Journal for Numerical Methods in Fluids*, 9:413–428, 1989.

[19] C. M. Rhie and W. L. Chow. A numerical study of the turbulent flow past an isolated airfoil with trailing edge separation. AIAA-82-0998, 1982.

[20] D. B. Spalding. A novel finite-difference formulation for differential expressions involving both first and second derivatives. *International Journal of Numerical Methods in Engineering*, 4:551–559, 1972.

[21] D. Xiao, H. A. Becker, E. W. Grandmaison, and A. Pollard. The calculation of a jet in a crossflow using embedded/overlapping grid techniques, multi-grid acceleration and a modified QUICK differencing scheme. *Submitted to: International Journal of Computational Fluid Dynamics*, 1993.

[22] D. Matovic and A. Pollard. Evaluation of a new block non-staggered calculation method for an incompressible flow and its multigrid acceleration. Submitted for publication.

[23] W. Shyy and C-S. Sun. Development of a pressure-correction/staggered-grid based multigrid solver for incompressible recirculating flows. *Computers and Fluids*, 22:51–76, 1993.

[24] W. Hackbush. *Multi-Grid Methods and Applications.* Springer-Verlag, Berlin, 1985.

[25] A. Brandt and N. Dinar. Multigrid solutions to elliptic flow problems. In *Numerical Methods for Partial Differential Equations*, pages 53–147. Academic Press, New York, 1979.

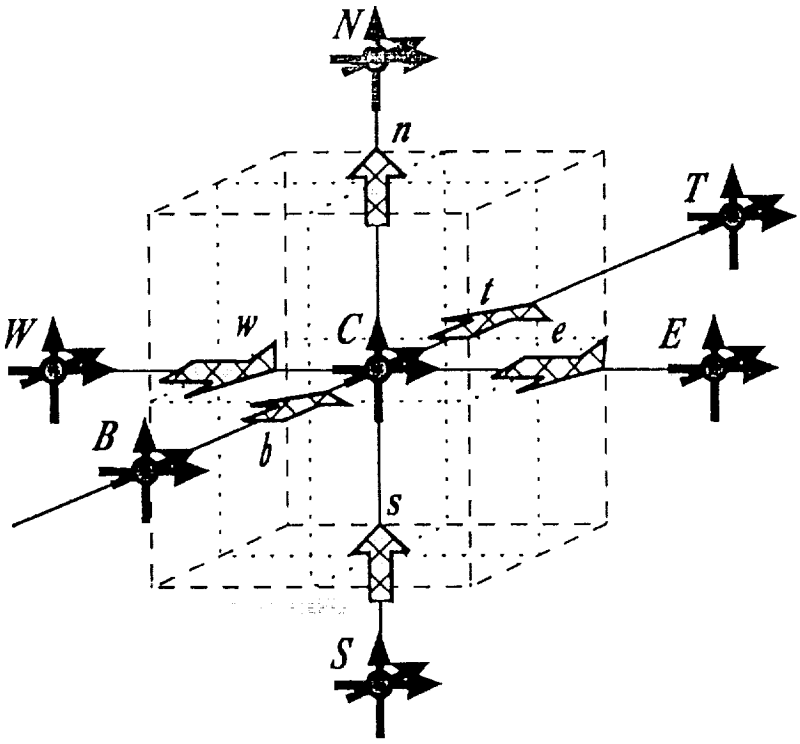

Figure 1: The layout of a non-staggered grid. Only the nodal variables require storage.

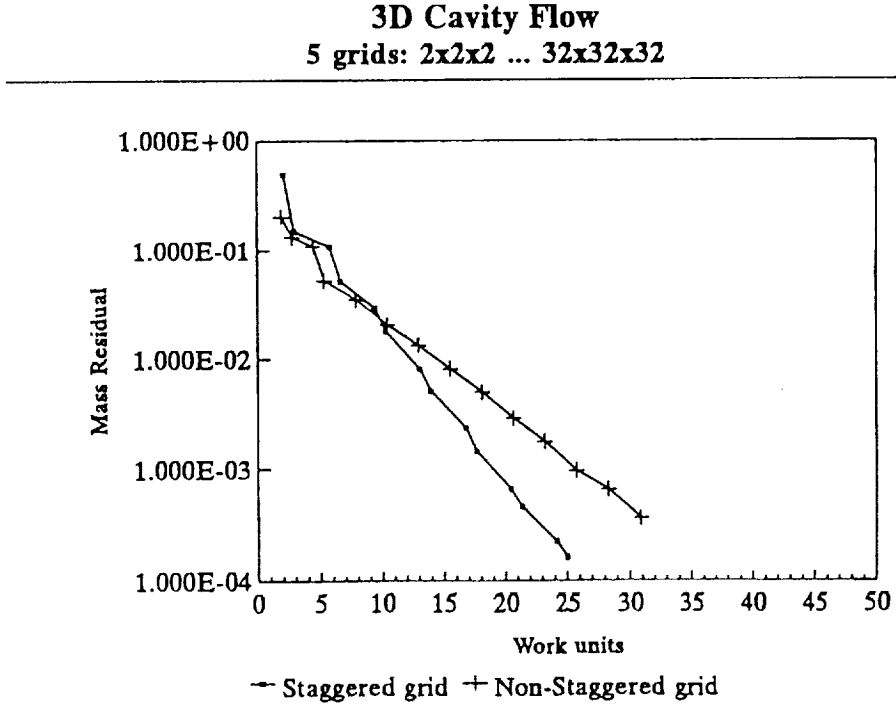

Figure 2: Mass residual history for a lid-driven cavity flow. Re = 400.

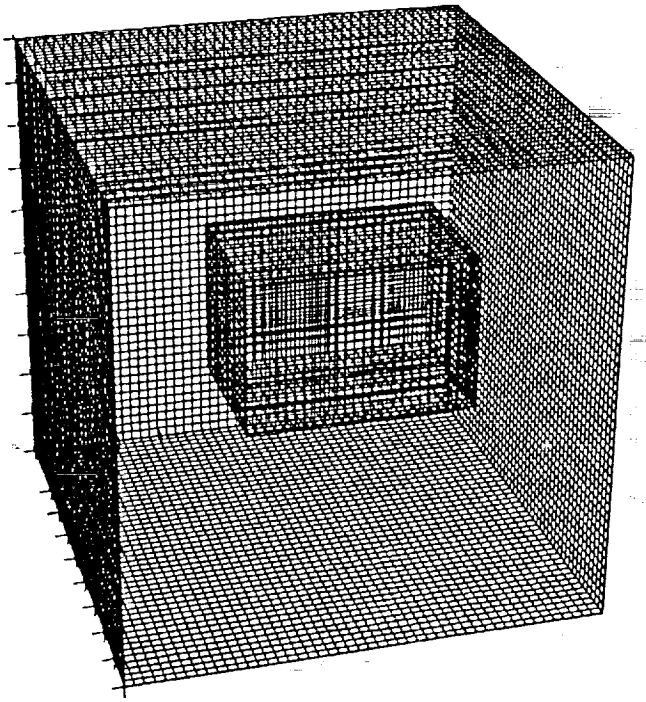

Figure 3: The grid for a flow around the cube in a lid-driven cavity.

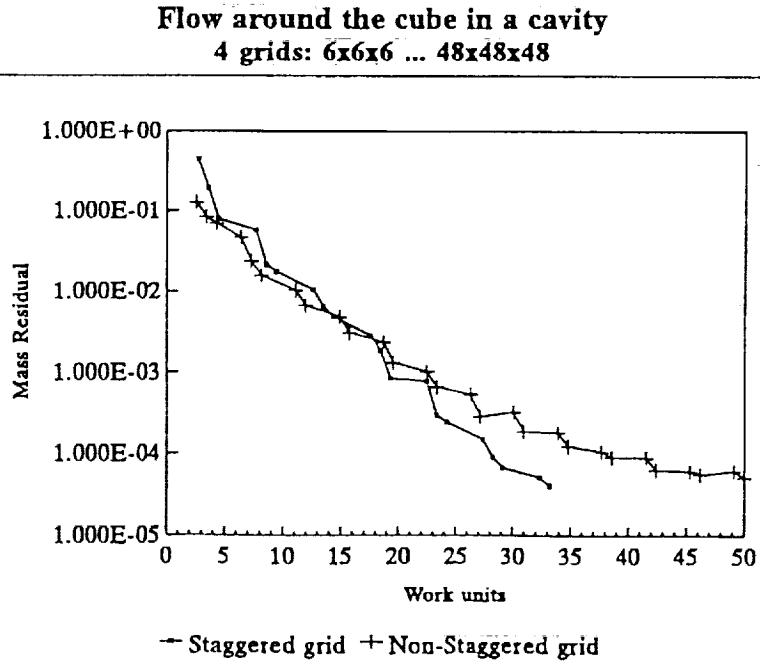

Figure 4: Mass residual history for the flow around the cube in a lid-driven cavity. Cavity Re=400.

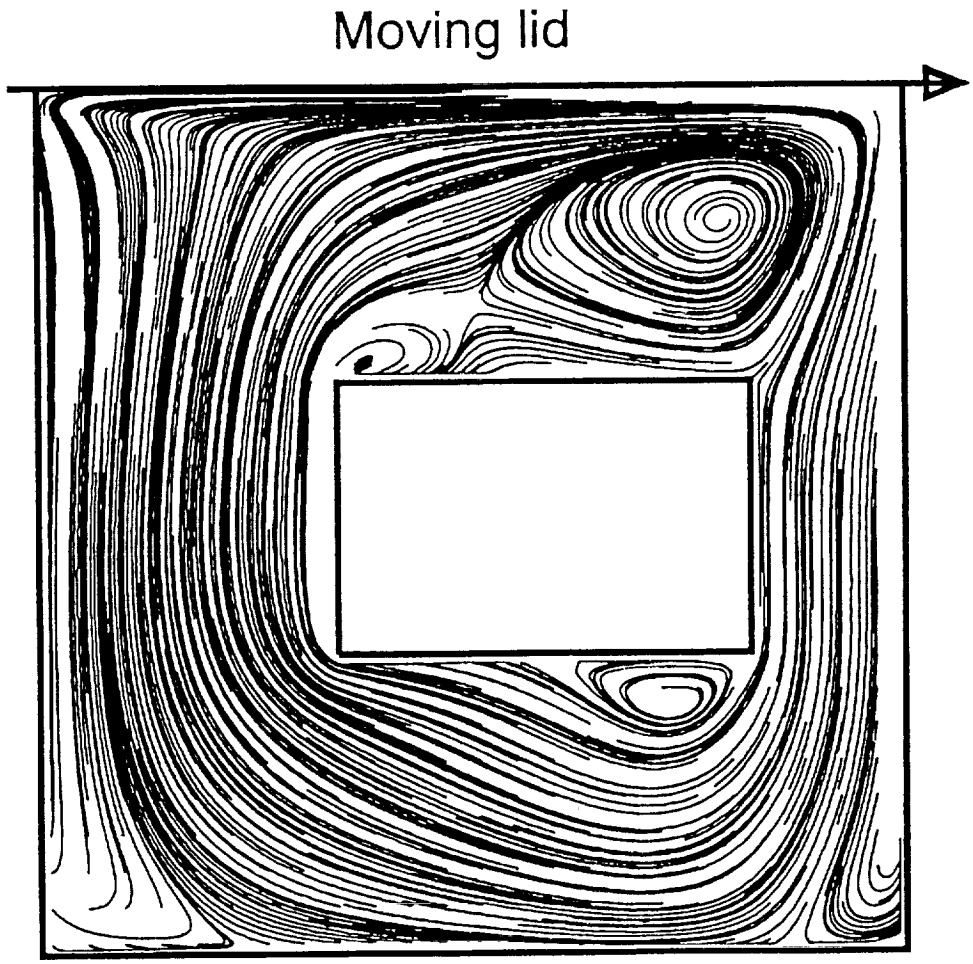

Figure 5: Flow around a cube in a lid-driven cavity: particle traces in a symmetry plane.