MULTIGRID AND CYCLIC REDUCTION APPLIED TO THE HELMHOLTZ EQUATION

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ABSTRACT

We consider the Helmholtz equation with a discontinuous complex parameter and inhomogeneous Dirichlet boundary conditions in a rectangular domain. A variant of the direct method of cyclic reduction is employed to facilitate the design of improved multigrid components, resulting in the method of CR-MG. We demonstrate the improved convergence properties of this method.

1 INTRODUCTION

Microwave heating of foods has revolutionised the food processing industry. Effective and efficient microwave heating depends very much on a detailed knowledge and understanding of the dielectric properties of the food to be processed. This need has given rise to extensive research into the dielectric properties of materials; see, for example, Tinga and Nelson [1].

Microwave heating can be compared to heating by alternating current. The electric field of alternating current changes direction approximately 100 times each second, whereas the microwave field changes direction approximately 5 billion times each second. The heating effect is accomplished by energy transfer to dipoles, most commonly water. Most foods contain between 50 and 90 percent water. By attempting to follow the very rapidly changing microwave electric field, the molecular vibrations of the dipoles are strengthened, thus increasing the temperature of the water and hence the food.

The scalar potential ϕ associated with the microwave field satisfies the wave equation

$$\nabla^2\phi - \epsilon\mu\frac{\partial^2\phi}{\partial t^2} = 0, \tag{1}$$

which is derived from Maxwell's equations of electromagnetics. The parameter ϵ describes the permittivity of the medium and the parameter μ the permeability. However, the radiation field in a microwave oven varies harmonically in time, and so we look for a solution of equation (1) in the form

$$\phi(\mathrm{x}\,,\,t) = e^{i\omega t}\,u(\mathrm{x}),$$

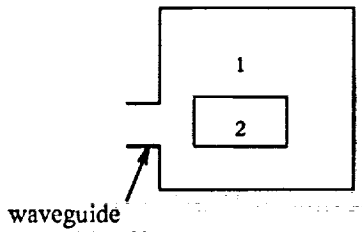

Figure 1: A two-dimensional model of a microwave oven.

where u is a time-independent scalar potential function and ω is the frequency of the microwave radiation. By substituting this expression into equation (1), we see that u satisfies the Helmholtz equation

$$\nabla^2 u + \delta u = 0,$$

where $\delta := \epsilon\mu\omega^2$. In general, ϵ and μ are complex numbers, with real parts related to a material's ability to store electrical and magnetic energy respectively, and imaginary parts related to a material's ability to dissipate electrical and magnetic energy respectively. However, the permeability of biological materials is close to that of free space, i.e. $\mu \approx \mu_0 = 4\pi \times 10^{-7}$ Hm^{-1}. Hence, since most domestic microwave ovens operate at a frequency of 2450 MHz, we can calculate δ for any given permittivity ϵ.

The oven is represented schematically (in two dimensions) by the rectangular domain depicted in Fig. 1. Region 1 corresponds to free space and so $\epsilon = \epsilon_0 = \frac{1}{36\pi} \times 10^{-9}$ Fm^{-1} and δ is a real constant in this region. Region 2 corresponds to the heated material and so δ is a complex constant in this region. Energy is fed into the system by a magnetron via the waveguide. Hence, in this paper, we consider the solution of the Helmholtz equation with a discontinuous complex parameter and inhomogeneous Dirichlet boundary conditions in a rectangular domain.

We close this section with a plan of the paper. In section 2 we describe the mathematical problem and discuss the smoothing abilities of two multigrid smoothers. In section 3 we describe the technique of approximate cyclic reduction and show how this can be used to design improved multigrid components. Numerical results are presented in section 4 and some concluding remarks are made in section 5.

2 MATHEMATICAL PROBLEM

Consider the complex two-dimensional Dirichlet boundary value problem

$$\begin{aligned} \nabla^2 u + \delta u = 0 \quad &\text{in} \quad \Omega = \Omega_1 \cup \Omega_2 \\ \text{s.t.} \quad u = g \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{2a}$$

with data

$$\delta = \begin{cases} \delta_1 & \text{in subdomain } \Omega_1 \\ \delta_2 & \text{in subdomain } \Omega_2 \end{cases}, \tag{2b}$$

where Ω_1 and Ω_2 are rectangular subdomains of Ω (as in Fig. 1).

Operator Definitions

Before attempting to solve this problem by the multigrid method, we need to carefully consider the definitions of the discretisation, restriction and prolongation operators. In [2], De Zeeuw considers the solution of general linear second order elliptic partial differential equations over similar domains. He notes that the rate of convergence of standard multigrid methods often deteriorates when the coefficients in the differential equation are discontinuous; he proposes matrix-dependent grid transfer operators to overcome these difficulties. However, in our case, the discontinuity occurs only in the coefficient of u (viz. δ), and not of ∇u. Hence we proceed in the following way to define operator $\mathcal{P} = \mathcal{P}(\delta)$ in the domain Ω, where $\mathcal{P}$ can be taken to represent the discretisation, restriction or prolongation operator. Firstly, if δ takes value δ_i in subdomain Ω_i $(i = 1, 2)$, then we set the value of δ on the interior boundary between Ω_1 and Ω_2 to $\delta_3 := \frac{1}{2}(\delta_1 + \delta_2)$. Secondly, $\mathcal{P}$ is defined piecewise by

$$\mathcal{P} = \begin{cases} \mathcal{P}(\delta_1) & \text{in } \Omega_1 \\ \mathcal{P}(\delta_2) & \text{in } \Omega_2 \\ \mathcal{P}(\delta_3) & \text{on } \partial\Omega_2 \end{cases} . \tag{3}$$

In practice, this definition of $\mathcal{P}$, for discontinuous δ, does not seem to impair the convergence of the multigrid algorithm for relevant values of δ.

Equivalent System of Real Equations

Consider the discrete analogue of problem (2). Suppose $\delta = \alpha + i\beta \in \mathbb{C}$ and $g \in \mathbb{R}$. Using a central difference discretisation on a mesh of $n \times n$ intervals, the matrix of the discrete system $A\mathbf{u} = \mathbf{f}$ is represented in stencil notation by

$$A \sim \frac{1}{h^2} \begin{bmatrix} & 1 & \\ 1 & p & 1 \\ & 1 & \end{bmatrix}, \tag{4}$$

where $A \in \mathbb{C}^{(n-1)^2} \times \mathbb{C}^{(n-1)^2}$, $h := \frac{1}{n}$ and $p := \delta h^2 - 4 = (\alpha h^2 - 4) + i\beta h^2$. Hence, while most linear systems which arise in practice have real coefficient matrices, the discretisation of this problem yields a complex linear system. Further applications which give rise to complex linear systems include discretisations of the time-dependent Schrödinger equation, inverse scattering problems and underwater acoustics.

A popular approach for solving complex linear systems is to solve the equivalent real linear systems for the real and imaginary parts of $\mathbf{u}$. However, the following remarks, due to Freund [3], cast doubt on this approach. Suppose that A is a general complex $m \times m$ matrix. By taking real and imaginary parts, we can rewrite the complex system as the real linear $2m \times 2m$ system

$$B\bar{\mathbf{u}} = \begin{pmatrix} \mathcal{R}e\, A & \mathcal{I}m\, A \\ \mathcal{I}m\, A & -\mathcal{R}e\, A \end{pmatrix} \begin{pmatrix} \mathcal{R}e\, \mathbf{u} \\ -\mathcal{I}m\, \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathcal{R}e\, \mathbf{f} \\ \mathcal{I}m\, \mathbf{f} \end{pmatrix}.$$

It can then be shown that B has eigenspectrum

$$\sigma(B) = \{\lambda \in \mathbb{C} | \, \lambda^2 \in \sigma(\bar{A}A)\},$$

which means that $\sigma(B)$ is symmetric with respect to the real and imaginary axes and hence the eigenvalues *always* embrace the origin. Now if A is complex symmetric (as is the case with (4)), then B is a real symmetric matrix with real eigenvalues symmetrically distributed about the origin, i.e. B is symmetric indefinite. Therefore the equivalent real system is often harder to solve than the original complex one.

Smoothing Analysis

Multigrid smoothing methods are usually basic iterative methods, the properties of which are well understood. As the name suggests, the function of a multigrid smoothing method is to reduce the rough (high frequency) components of the error as efficiently as possible. This is basically a local task and so the smoothing efficiency of a method can be analysed by local Fourier mode analysis [4], neglecting interactions with boundaries. The smooth (low frequency) error components are reduced on the coarser grids. There is a natural distinction between high and low frequencies depending on the type of grid coarsening chosen. Essentially, the low frequencies are those which are visible on the coarser grids. In principle, smoothing methods need not be convergent (see [5], chpt 7), although in practice most are.

Consider the discrete analogue of problem (1), $A\mathbf{u} = \mathbf{f}$, defined on a mesh of $n \times n$ intervals. Basic iterative methods are based on a matrix splitting $A = M - N$ and are defined by

$$M\mathbf{u}^{(m+1)} = N\mathbf{u}^{(m)} + \mathbf{f}.$$

The algebraic error arising from the iterative solution of this system of equations is defined by $\mathbf{e}^{(m)} := \mathbf{u}^{(m)} - \mathbf{u}$ and satisfies the equation $M\mathbf{e}^{(m+1)} = N\mathbf{e}^{(m)}$. Denoting the stencils of M and N by $[M]$ and $[N]$ respectively, this equation can be rewritten in stencil form as $[M]\, e_{jk}^{(m+1)} = [N]\, e_{jk}^{(m)}$. Now if we define $\mathbf{e}^{(m+1)} := \lambda\, \mathbf{e}^{(m)}$ and note that the algebraic error can be represented as a combination of local Fourier modes

$$e_{jk}^{(m)} = \lambda^m\, e^{i(j\theta + k\phi)}, \; (\theta\,,\, \phi) \in \Theta := \{(\tfrac{2\pi p}{n}\,,\, \tfrac{2\pi q}{n}) : -\tfrac{n}{2} + 1 \leq p,\, q \leq \tfrac{n}{2}\},$$

then by substituting this into the stencil representation of the error recurrence we define the *error amplification factor*

$$\lambda(\theta\,,\, \phi) \;:=\; \frac{[N]\, e^{i(j\theta + k\phi)}}{[M]\, e^{i(j\theta + k\phi)}}.$$

The error amplification factor is the factor by which the amplitude of the $(\theta\,,\, \phi)$ Fourier mode is multiplied as a result of a single smoothing iteration. Now in the case of standard grid coarsening, the sets of smooth and rough frequencies are defined by

$$\Theta_s := \Theta \cap (-\tfrac{\pi}{2}\,,\, \tfrac{\pi}{2})^2,$$

$$\Theta_r := \Theta \backslash \Theta_s.$$

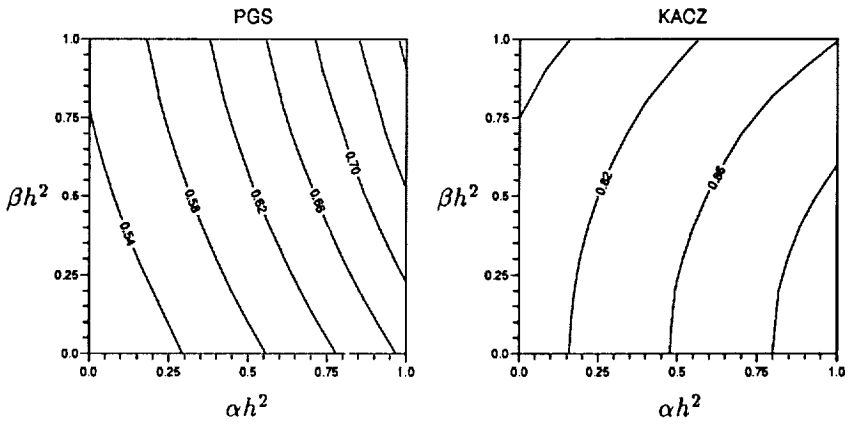

Figure 2: Fourier smoothing factors ρ^D for PGS and KACZ.

Hence the *Fourier smoothing factor*, which is the worst factor by which all high frequency error components are reduced per iteration, is defined by

$$\rho := \max_{(\theta,\phi)\in\Theta_r} |\lambda(\theta, \phi)|.$$

Note however that this definition of the smoothing factor is only valid for boundary conditions of harmonic type. The influence of Dirichlet boundary conditions can be taken into account heuristically (see [6] and [7], for example) in the following way. The error at the boundary is always zero and so we define a new set of rough frequencies as

$$\Theta_r^D := \Theta_r \cap \{(\theta, \phi) \in \Theta : \theta \neq 0 \text{ and/or } \phi \neq 0\}.$$

The corresponding Fourier smoothing factor is defined by

$$\rho^D := \max_{(\theta,\phi)\in\Theta_r^D} |\lambda(\theta, \phi)|.$$

This is a mesh-dependent definition. A mesh-independent definition, introduced by Brandt [4], is obtained by replacing the discrete set Θ with a continuous analogue, but this is more difficult to compute numerically and gives less realistic results in cases where the type of boundary condition has much influence.

There are many possibilities for the choice of smoothing method (see [7], for example), but for brevity we consider only two, point Gauß-Seidel iteration (PGS) and Kaczmarz iteration (KACZ). The latter of these two methods, dating back to 1937 [8], is considered here because, when applied to the complex linear system $A\mathbf{u} = \mathbf{f}$, the method converges for all distributions $\sigma(A)$ of eigenvalues of A. The reason for this is that solving the system $A\mathbf{u} = \mathbf{f}$ using KACZ is equivalent to solving the system $AA^H\mathbf{v} = \mathbf{f}$ with $\mathbf{u} = A^H\mathbf{v}$ (i.e. *postconditioning*) using PGS, and the matrix AA^H is Hermitian positive definite, thus guaranteeing convergence. Applying the smoothing analysis to stencil (4), the error amplification factors for PGS and KACZ are

$$\lambda_{\text{PGS}} = -\frac{e^{i\theta} + e^{i\phi}}{p + e^{-i\theta} + e^{-i\phi}},$$

$$\lambda_{\text{KACZ}} = -\frac{(e^{i\theta}+e^{i\phi})(e^{i\theta}+e^{i\phi}+p+\overline{p})+2e^{i(-\theta+\phi)}}{4+p\overline{p}+(e^{-i\theta}+e^{-i\phi})(e^{-i\theta}+e^{-i\phi}+p+\overline{p})+2e^{i(\theta-\phi)}},$$

for some $p = (\alpha h^2 - 4) + i\beta h^2$ and $(\theta\,,\,\phi) \in \Theta$. Fig. 2 displays contour plots of ρ^D_{PGS} and ρ^D_{KACZ} plotted as functions of αh^2 and βh^2. For fixed values of h and $\alpha = \mathcal{R}e\,\delta$, as $\beta = \mathcal{I}m\,\delta$ increases, ρ^D_{PGS} increases and ρ^D_{KACZ} decreases. Hence we might expect the multigrid convergence rate to improve slowly with a KACZ smoother and deteriorate more rapidly with a PGS smoother as β increases. This is borne out in practice. Finally, as a rule of thumb, a *good* smoothing method has a smoothing factor $\rho^D < \frac{1}{2}$. In this sense, neither of the two methods considered here is a good smoothing method for problem (2).

3 CYCLIC REDUCTION AND MULTIGRID

Cyclic reduction (CR) is a *direct* method of solution for tridiagonal and block-tridiagonal systems of linear algebraic equations [9], [10]. For tridiagonal systems which represent approximations to 1-D second order ordinary differential equations, CR is as efficient as multigrid (MG). For problems in higher dimensions CR becomes too computationally expensive due to fill-in within the blocks. However, the *design* of MG methods in higher dimensions can be facilitated by drawing comparisons between MG and CR (see Shaw [11]).

Approximate Cyclic Reduction

Consider the system of equations $L\mathbf{u} = \mathbf{f}$. If $\mathbf{v}$ is an approximation to the true solution $\mathbf{u}$, then we define the error vector as $\mathbf{e} := \mathbf{u} - \mathbf{v}$ and the residual vector as $\mathbf{r} := \mathbf{f} - L\mathbf{v} = L\mathbf{e}$. Then assuming that the error vector $\mathbf{e}$ is sufficiently smooth (a condition normally guaranteed by a few applications of a smoother in a MG algorithm), the fill-in can be minimised by making accurate Taylor expansion approximations of the outlying errors. This method is known as *approximate cyclic reduction* (ACR) [12].

Now consider a two-grid method applied to a two-dimensional Toeplitz system. Suppose the two grids have mesh sizes h and $H = 2h$ and the fine grid matrix has stencil

$$L_h \sim \begin{bmatrix} & b & \\ b & a & b \\ & b & \end{bmatrix},$$

where a and b are scalars. Given an initial approximation $\mathbf{v}$ to $\mathbf{u}$, we want to solve the equation $L_h\mathbf{e} = \mathbf{r}$ for $\mathbf{e}$ to obtain an improved approximation $\mathbf{v} + \mathbf{e}$. The method of ACR approaches this problem as follows.

Eliminate the outlying errors in the stencil using neighbouring equations to give

$$\tilde{L}_h \sim \begin{bmatrix} & & b^2 & & \\ & 2b^2 & 0 & 2b^2 & \\ b^2 & 0 & 4b^2 - a^2 & 0 & b^2 \\ & 2b^2 & 0 & 2b^2 & \\ & & b^2 & & \end{bmatrix}.$$

This first step of CR has destroyed the band structure of the original five-point operator. Further steps of CR would introduce more fill-in, resulting in a relatively inefficient process. Instead, assuming the errors are sufficiently smooth, approximate the errors at the NW, NE, SW and SE positions (in compass point notation) using accurate Taylor series expansions. This defines the ACR-modified coarse grid matrix, which has stencil

$$L_H \sim \sigma \begin{bmatrix} & & 2b^2 & & \\ & & 0 & & \\ 2b^2 & 0 & 8b^2 - a^2 & 0 & 2b^2 \\ & & 0 & & \\ & & 2b^2 & & \end{bmatrix},$$

where σ is an arbitrary scaling parameter. From the above information, the definition of restriction from the fine grid to the coarse grid can also be gleaned. The ACR-modified restriction operator has stencil

$$R_h^H \sim \sigma \begin{bmatrix} & b & \\ b & -a & b \\ & b & \end{bmatrix}. \tag{5}$$

For theoretical considerations it is very convenient to choose restriction and prolongation operators which satisfy the relation $P_H^h = R_h^{H*}$, where R_h^{H*} is the adjoint operator of R_h^H with respect to a suitably defined scalar product. However, the adjoint of the five-point restriction operator (5) is not a reasonable prolongation (see [13], p. 78). Alternative definitions of the prolongation operator are discussed in the following subsection.

ACR and the Helmholtz Equation

Consider a two-grid method, with mesh sizes h and $H = 2h$, applied to the fine grid Helmholtz differential operator $\mathcal{L}_h u := \nabla^2 u + \delta u$. Using a central difference discretisation on a mesh of $n \times n$ intervals, the fine grid matrix has stencil

$$L_h \sim \frac{1}{h^2} \begin{bmatrix} & 1 & \\ 1 & p & 1 \\ & 1 & \end{bmatrix},$$

where $h := \frac{1}{n}$ and $p := \delta h^2 - 4$. Hence $a = \frac{p}{h^2}$ and $b = \frac{1}{h^2}$. Now if we choose $\sigma = \frac{h^2}{8}$, then the ACR-modified coarse grid matrix and restriction operator have stencils

$$L_H \sim \frac{1}{(2h)^2} \begin{bmatrix} & & 1 & & \\ & & 0 & & \\ 1 & 0 & 4 - \frac{1}{2}p^2 & 0 & 1 \\ & & 0 & & \\ & & 1 & & \end{bmatrix},$$

$$R_h^H \sim \frac{1}{8} \begin{bmatrix} & 1 & \\ 1 & -p & 1 \\ & 1 & \end{bmatrix}.$$

Therefore the analogous coarse grid Helmholtz differential operator is defined as $\mathcal{L}_H u := \nabla^2 u + \delta(1 - \frac{\delta H^2}{32})u$, i.e. ACR suggests solving the Helmholtz equation with a different value of δ on the coarse grid in order to stabilise the MG process. For positive real values of δ for which L_h is indefinite, this corresponds to solving the Helmholtz equation with a *smaller* value of δ on the coarse grid, thus reducing the indefiniteness of L_H. There are various ways to define the prolongation operator. Possibilities include seven-point and nine-point prolongation [14]. However, a more effective definition of the prolongation operator for this interface problem is

$$P_H^h \sim \frac{1}{2p^2}\begin{bmatrix} 4 & -4p & 4 \\ -4p & 3p^2 & -4p \\ 4 & -4p & 4 \end{bmatrix},$$

which is derived from the tensor product of the one-dimensional ACR-modified prolongation operator. To extend these ideas to an m-grid process, where h_i is the mesh size of grid Ω_i and $h_{i+1} = 2h_i$, we proceed as follows.

Define $\delta_1 := \delta$ and $\delta_k := \delta_{k-1}(1 - \frac{\delta_{k-1}h_k^2}{32}) := \delta_{k-1}\, c_k$ $(2 \leq k \leq m)$ and $p_k := \delta_k h_k^2 - 4$. Then the differential operator on grid Ω_k is defined as

$$\mathcal{L}_k u := \nabla^2 u + \delta_k u,$$

for $1 \leq k \leq m$, provided $\sigma = \frac{h_k^2}{8}$. Therefore, the ACR-modified definitions of the matrix of the discrete system on grid Ω_k and the restriction and prolongation operators have stencils

$$L_k \sim \frac{1}{h_k^2}\begin{bmatrix} & 1 & \\ 1 & p_k & 1 \\ & 1 & \end{bmatrix},$$

$$R_k^{k+1} \sim \frac{1}{8}\begin{bmatrix} & 1 & \\ 1 & -p_k & 1 \\ & 1 & \end{bmatrix},$$

$$P_{k+1}^k \sim \frac{1}{2p_k^2}\begin{bmatrix} 4 & -4p_k & 4 \\ -4p_k & 3p_k^2 & -4p_k \\ 4 & -4p_k & 4 \end{bmatrix}$$

respectively. We call this ACR-modified multigrid process CR-MG. Note that the CR-MG restriction operator is similar to the operator naturally suggested by the principle of *total reduction* (see [15] and [16], for example). Further, for Laplace's equation (i.e. $\delta = 0$), $p_k = -4$ and the CR-MG restriction operator corresponds to half weighting.

4 NUMERICAL RESULTS

Consider the complex two-dimensional Dirichlet boundary value problem

$$\nabla^2 u + \delta u = 0 \quad \text{in} \quad \Omega = \Omega_1 \cup \Omega_2 \ : \text{unit square}$$
$$\text{s.t.} \quad u = g \quad \text{on} \quad \partial\Omega,$$

with data

$$\delta = \begin{cases} 30 + 10i & \text{in } \Omega_2 : \frac{3}{8} < x, y < \frac{5}{8} \\ 1 & \text{in } \Omega_1 : \Omega \backslash \overline{\Omega}_2 \end{cases},$$

$$g = \begin{cases} \sin(4y - \frac{3}{2})\pi & \text{on } x = 0, \frac{3}{8} < y < \frac{5}{8} \\ 0 & \text{elsewhere on } \partial\Omega \end{cases}.$$

For convenience, we consider a domain Ω consisting of two concentric squares. The value of δ in Ω_2 is a typical value calculated from the data in [1]. In the following experiment we assess the efficiency of the CR-MG algorithm, as described in the preceding section, and compare it with standard MG using full weighting restriction and nine-point prolongation.

The problem is discretised piecewise according to (3) and (4), using central differences on a 65×65 grid. A four-grid method is employed, with standard grid coarsening. This ensures good resolution of the inner subdomain Ω_2 on the coarsest grid. The multigrid schedule used is the V-cycle with two pre-smoothing and two post-smoothing iterations, and LU decomposition with partial pivoting is used to solve the defect equation exactly on the coarsest grid. The initial estimate is taken to be the zero vector and convergence is measured by $\log_{10} ||r||_2$, where r is the residual vector and $||\cdot||_2$ is the usual Euclidean norm.

With convergence set to a tolerance of

$$\log_{10} ||r||_2 < -9,$$

the convergence times of MG and CR-MG with PGS and KACZ smoothers were measured and the results are displayed in Table 1. All convergence times were measured in seconds on a Sun SPARC-

Table 1: CPU Convergence Times

time (s)	PGS	KACZ
MG	22.8	191.6
CR-MG	18.5	155.9

workstation. We immediately notice that both MG and CR-MG converge much more rapidly with a PGS smoother than with a KACZ smoother. This is not unexpected, considering the smoothing properties of these two iterative methods. Further, KACZ is a more computationally intensive smoother than PGS, having a 13-point stencil as compared to the 5-point stencil of PGS.

However, most importantly, we find that with both smoothers the rate of convergence of CR-MG is significantly faster than that of MG. In fact, with both smoothers CR-MG provides a 19 percent saving in CPU time over MG. This is a significant saving, especially for larger problems. The rates of convergence of MG and CR-MG with a PGS smoother are compared graphically in Fig. 3. Both plots are approximately straight lines, a consequence of the grid-independent convergence of the multigrid method.

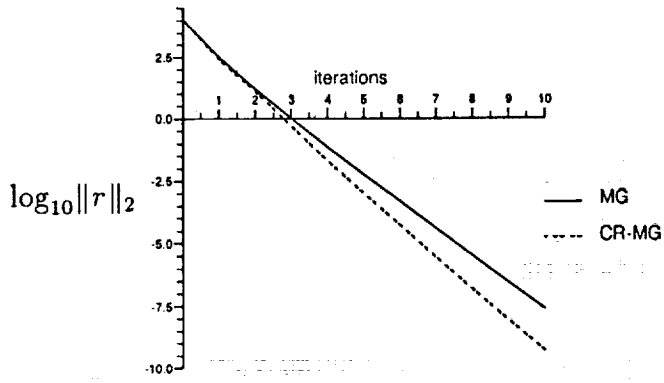

Figure 3: Convergence of MG and CR-MG with a PGS smoother.

5 CONCLUDING REMARKS

In this paper, attention has been focussed on improving the design of the standard multigrid method with respect to a particular problem, namely the complex-valued microwave oven problem. By drawing a comparison with the direct method of cyclic reduction, improved discretisation, restriction and prolongation operators have been designed, resulting in savings of up to 19 percent in CPU time used.

Only two smoothing methods have been considered here, point Gauß-Seidel and Kaczmarz. However, there are many more robust smoothers, such as alternating damped Jacobi, alternating symmetric line Gauß-Seidel and incomplete LU decomposition. These methods, and many more, have been summarised and analysed in detail in [7]. Improvements in the convergence properties of the modified multigrid method (CR-MG) will almost certainly be realised by using such smoothers.

Finally, attention in this paper has been restricted to the microwave oven problem, although the ideas presented here can be extended to other problems. For example, in [11], these ideas were applied to the convection-diffusion equation and it was shown that approximate cyclic reduction can be used to define the ideal quantity of coarse grid artificial viscosity and the direction in which it lies.

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