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A NONCONFORMING MULTIGRID METHOD USING CONFORMING SUBSPACES*

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SUMMARY

For second-order elliptic boundary value problems, we develop a nonconforming multigrid method using the coarser-grid correction on the conforming finite element subspaces. The convergence proof with an arbitrary number of smoothing steps for $\mathcal{V}$-cycle is presented.

1. INTRODUCTION

Let Ω be a convex polygon in $\mathbf{R}^2$. Let $f \in L^2(\Omega)$, $\alpha \in C^1(\bar{\Omega})$ and $\beta \in C^0(\bar{\Omega})$. We assume there exists α_0 such that $\alpha \geq \alpha_0 > 0$ and $\beta \geq 0$. In this paper we discuss convergence properties of the multigrid method for solving the Dirichlet problem

$$
\begin{aligned}
-\nabla \cdot (\alpha \nabla u) + \beta u &= f \quad \text{in} \quad \Omega, & (1)\\
u &= 0 \quad \text{on} \quad \partial\Omega, & (2)
\end{aligned}
$$

using $P1$ nonconforming finite elements(see [5, 6]).

The prototype of the multigrid convergence theory is that

> For some number of smoothing steps the multigrid process is a contraction for some norm. Moreover, the contraction number is independent of the mesh size h.

This was proved for conforming multigrid methods by Bank and Dupont[1]. Braess and Hackbusch[2] and Hackbusch[8] proved this for the $\mathcal{V}$ cycle with one smoothing step. For the nonconforming multigrid method, this was proved by Braess and Verfürth[3] and Brenner[4] for the $\mathcal{W}$-cycle under the condition that each iteration step contains many smoothing steps.

The method presented in this paper consists of a smoothing step on the nonconforming finite element space of the finest-grid and correction step which is obtained by the conforming multigrid

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method on the conforming finite element subspaces of coarser-grids. The *standard nonconforming multigrid* which was proved by Brenner in [4] is based on smoothings and correction on the nonconforming finite element spaces. The important difference is that $V_{k-1} \not\subseteq V_k$ and $W_{k-1} \subseteq V_k$, where V_k and W_k are the nonconforming and conforming finite element spaces on mesh level k, respectively. Hence we can simply use the natural injection for the intergrid transfer of grid functions and this intergrid transfer operator preserves the energy norm. Moreover, the error of the coarser-grid correction is orthogonal to W_{k-1}. Owing to these, the standard proof of convergence in [2] for the $\mathcal{V}$-cycle of one smoothing step of the conforming multigrid method carries over directly. In [3] Braess and Verfürth added the step length parameter in the correction step of the standard nonconforming multigrid algorithm to improve the convergence. They proved the convergence of two-level case of this *modified standard nonconforming multigrid* with one smoothing step. The rate of convergence of their algorithm should be better than or at least equal to that of the standard nonconforming multigrid method but it needs more cost for each iteration. While Brenner proved the convergence of the standard nonconforming multigrid algorithm only for the $\mathcal{W}$-cycle it is convergent for the $\mathcal{V}$ cycle with one smoothing step in real computation. Also the modified standard nonconforming multigrid algorithm converges for the $\mathcal{V}$ cycle with one smoothing step in real computation. Our multigrid method is easier to implement and more effective because it needs fewer computations and communications in a parallel sense. These computations were done in CM-5 Vector Units[†].

This paper is organized as follows. In Section 2 we discuss the fundamental estimates from the theory of finite elements and the intergrid transfer operator. The multigrid algorithm is discussed in Section 3. Section 4 contains the contracting properties of the k-level iteration. In the last section we compare the computational results of three algorithms.

2. THE FINITE ELEMENT SPACES

The variational formulation for (1) and (2) is defined as follows: Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = F(v) \quad \forall v \in H_0^1(\Omega)\,,$$

where

$$a(u,v) = \int_\Omega (\alpha \nabla u \cdot \nabla v + \beta u v) \quad \text{and} \quad F(v) = \int_\Omega f v\,.$$

Here, $H_0^1(\Omega)$ denotes the usual Sobolev space (see [5]).

Let $\{\mathcal{T}^k\}$, $k \geq 1$, be a family of triangulations of Ω, where $\mathcal{T}^{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}^k$. Let $h_k := \max_{T \in \mathcal{T}^k} \operatorname{diam} T$, then $h_k = 2h_{k+1}$. Throughout this paper, C denotes the positive constant independent of k which may vary from occurence to occurence even in the proof of the same theorem.

[†]These results are based upon a test version of the software where the emphasis was on providing functionality and the tools necessary to begin testing the CM5 with vector units. This software release has not had the benefit of optimization or performance tuning and, consequently, is not necessarily representative of the performance of the full version of this software.

It is worth pointing out the motivation of the nonconforming finite elements. In the stationary Stokes problem for an incompressible viscous fluid, it is realized that a major difficulty exists in the numerical treatment of the incompressibility condition. Crouzeix and Raviart in [6] advocated the method that the incompressibility condition is approximated. They have found it very convenient to use nonconforming finite elements for this purpose. By Uzawa's method the Stokes equation is reduced to a sequence of Dirichlet problems for the operator $-\Delta$. Thus we shall develop a nonconforming multigrid method for solving (1) and (2).

Now let's define the nonconforming finite element space

$$V_k := \{v : v|_T \text{ is linear for all } T \in \mathcal{T}^k,\ v \text{ is continuous at the midpoints of the edges and } v = 0 \text{ at the mid points on } \partial\Omega\}.$$

Note that functions in V_k are not continuous.

We also use a conforming finite element space for our multigrid method *NC-CMG*. Define

$$W_k := \{w : w|_T \text{ is linear for all } T \in \mathcal{T}^k,\ \ w \text{ is continuous on } \Omega \text{ and } w|_{\partial\Omega} = 0\}.$$

The space V_k will be used in the finest-grid space and W_k in the coarser-grid spaces to obtain *NC-CMG*. Observe that $W_k = V_k \cap H_0^1(\Omega) = V_k \cap V_{k+1}$.

For each k, define (on $V_k + H_0^1(\Omega)$)

$$a_k(u,v) := \sum_{T\in\mathcal{T}^k} \int_T (\alpha\nabla u\cdot\nabla v + \beta uv)$$

and the energy norm induced by a_k

$$\|u\|_k := \sqrt{a_k(u,u)}.$$

The bilinear form $a_k(\cdot,\cdot)$ is symmetric and positive definite on V_k. Moreover, we have the inverse estimate[4]

$$\|u\|_k \le C h_k^{-1}\|u\|_{L^2} \quad \forall u \in V_k. \tag{3}$$

We also note that if $u, v \in H_0^1(\Omega)$, then $a_k(u,v) = a(u,v)$.

We now recall some fundamental estimates from the theory of finite elements.

Since $f \in L^2(\Omega)$, elliptic regularity implies that $u \in H^2(\Omega)$(see [7]). For the same f, let $u_k \in V_k$ satisfy

$$a_k(u_k, v) = \int_\Omega fv \quad \forall v \in V_k$$

and let $\tilde{u}_k \in W_k$ satisfy

$$a_k(\tilde{u}_k, v) = \int_\Omega fv \quad \forall v \in W_k.$$

Since V_k satisfies the patch test(see [11]), we have the following estimate for the discretization error:

$$\|u - u_k\|_{L^2} + h_k\|u - u_k\|_k \le C h_k^2 \|u\|_{H^2} \tag{4}$$

(see [6]). The estimate for the conforming descretization error is, of course, well known(see [5]):

$$\|u - \tilde{u}_k\|_{L^2} + h_k\|u - \tilde{u}_k\|_k \le Ch_k^2\|u\|_{H^2}. \tag{5}$$

From the spectral theory, there exist eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n_k}$ and eigenfunctions $\psi_1, \psi_2, \ldots, \psi_{n_k} \in V_k$, $(\psi_i, \psi_j)_{L^2} = \delta_{ij}$ (= the Kronecker delta), such that $a_k(\psi_i, v) = \lambda_i(\psi_i, v)_{L^2}$ for all $v \in V_k$. From the inverse estimate (3), there exists $C > 0$ such that

$$\lambda_i \le Ch_k^{-2}. \tag{6}$$

The same results hold for the conforming finite element spaces. The norm $|\!|\!|v|\!|\!|_{s,k}$ is defined (see [1]) as follows:

$$|\!|\!|v|\!|\!|_{s,k} := \left(\sum_{i=1}^{n_k} \lambda_i^s \nu_i^2\right)^{1/2} \quad \text{where} \quad v = \sum_{i=1}^{n_k} \nu_i\psi_i \in V_k. \tag{7}$$

Moreover,

$$|\!|\!|v|\!|\!|_{0,k} = \|v\|_{L^2} \quad \text{and} \quad |\!|\!|v|\!|\!|_{1,k} = \|v\|_k. \tag{8}$$

And, the Cauchy-Schwarz inequality implies

$$|a_k(v, w)| \le |\!|\!|v|\!|\!|_{1+t,k}|\!|\!|w|\!|\!|_{1-t,k}$$

for any $t \in \mathbf{R}$ and $v, w \in V_k$.

For $v \in V_{k-1}$ the intergrid transfer operator $I_{k-1}^k : V_{k-1} \to V_k$ is defined as follows. Let p be a midpoint of a side of a triangle in $\mathcal{T}^k$. If p lies in the interior of a triangle in $\mathcal{T}^{k-1}$, then we define

$$(I_{k-1}^k v)(p) := v(p).$$

Otherwise, if p lies on the common edge of two adjacent triangles T_1 and T_2 in $\mathcal{T}^{k-1}$, then we define

$$(I_{k-1}^k v)(p) := \frac{1}{2}[v|_{T_1}(p) + v|_{T_2}(p)].$$

From the definition of I_{k-1}^k, it is clear that

$$I_{k-1}^k v = v \quad \forall v \in W_{k-1} = V_k \cap V_{k-1} \subseteq H_0^1(\Omega).$$

In other words, $I_{k-1}^k|_{W_{k-1}}$ is just the natural injection.

Now we are ready to state an approximation property.

Lemma 1 *Given $u \in V_k$ let $u^\star \in W_{k-1}$ be the solution of*

$$a_k(u - u^\star, v) = 0 \quad \forall v \in W_{k-1}.$$

Then

$$|\!|\!|u - u^\star|\!|\!|_{1,k} \le Ch_k|\!|\!|u|\!|\!|_{2,k}.$$

Proof. Let $g \in V_k$ satisfy

$$(g, v) = a_k(u, v) \quad \forall v \in V_k.$$

Then

$$\forall v \in W_{k-1}, \quad a_k(u^\star, v) = a_k(u, v) = (g, v).$$

Now let $w \in H_0^1(\Omega)$ be the solution of the Dirichlet problem

$$\begin{aligned} -\nabla \cdot (\alpha \nabla w) + \beta w &= g \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then by elliptic regularity $\|w\|_{H^2} \leq C\|g\|_{L^2}$. It follows from the discretization error estimates (4) and (5) that

$$\begin{aligned} \|u - u^\star\|_{L^2} &\leq \|u - w\|_{L^2} + \|w - u^\star\|_{L^2} && (9) \\ &\leq Ch^2 \|w\|_{H^2} && (10) \\ &\leq Ch^2 \|g\|_{L^2}. && (11) \end{aligned}$$

But

$$\|g\|_{L^2}^2 = (g, g) = a_k(u, g) \leq |\!|\!| u |\!|\!|_{2,k} \|g\|_{L^2}.$$

Therefore,

$$\|g\|_{L^2} \leq |\!|\!| u |\!|\!|_{2,k}.$$

Combining inverse estimate (3) and (11), we obtain

$$|\!|\!| u - u^\star |\!|\!|_{1,k} \leq \frac{C}{h} \|u - u^\star\|_{L^2} \leq Ch |\!|\!| u |\!|\!|_{2,k}. \quad \square$$

3. THE MULTIGRID ALGORITHM

Now, we consider a decreasing sequence of mesh size h_k:

$$h_0 > h_1 > \cdots > h_k > \cdots > h_{k_{\max}}.$$

We first describe the k-level iteration scheme of the conforming multigrid algorithm. The k-level iteration with initial guess z_0 yields $CMG(k, z_0, G)$ as a conforming approximate solution to the following problem.

Find $z \in W_k$ such that $a_k(z, v) = G(v) \quad \forall v \in W_k$, where $G \in W_k'$.

Here, W_k' is the dual space of W_k. For $k = 1$, $CMG(1, z_0, G)$ is the solution obtained from a direct method. For $k > 1$, $CMG(k, z_0, G) = z_m + I_{k-1}^k q$, where the approximation $z_m \in W_k$ is constructed recursively from the initial guess z_0 and the equations

$$z_i = z_{i-1} + \frac{1}{\Lambda_k}(G - A_k z_{i-1}), \quad 1 \leq i \leq m.$$

Here, Λ_k is greater than or equal to the largest eigenvalue of A_k which is the stiffness matrix of a_k in the conforming finite element space W_k, and m is an integer to be determined later. The coarser-grid correction $q \in W_{k-1}$ is obtained by applying the $(k-1)$-level iteration 1 time. In other words, it is the $\mathcal{V}$-cycle multigrid method. More precisely,

$$q = CMG(k-1, 0, \bar{G})$$

where $\bar{G} \in W'_{k-1}$ is defined by $\bar{G}(v) := G(I^k_{k-1}v) - a_k(z_m, I^k_{k-1}v)$ for all $v \in W_{k-1}$.

The nonconforming multigrid algorithm of this paper is as follows: The $k_{\max}$-level iteration with initial guess z_0 yields $NC\text{-}CMG(k_{\max}, z_0, F)$ as a nonconforming approximate solution to the following problem.

Find $z \in V_{k_{\max}}$ such that

$$a_{k_{\max}}(z, v) = F(v) = \int_\Omega fv \quad \forall v \in V_{k_{\max}}. \tag{12}$$

For $k_{\max} = 1$, $NC\text{-}CMG(1, z_0, F)$ is the solution obtained from a direct method.

For $k_{\max} > 1$,

Smoothing Step: the approximation $z_m \in V_k$ is constructed recursively from the initial guess z_0 and the equations

$$z_i = z_{i-1} + \frac{1}{\Lambda_{k_{\max}}}(F - A_{k_{\max}} z_{i-1}), \quad 1 \le i \le m. \tag{13}$$

Here, $\Lambda_{k_{\max}}$ is greater than or equal to the largest eigenvalue of $A_{k_{\max}}$ which is the stiffness matrix of $a_{k_{\max}}$ in the nonconforming finite element space $V_{k_{\max}}$.

Correction Step: The coarser-grid correction $q \in W_{k-1}$ is obtained by applying the $(k_{\max}-1)$-level conforming iteration 1 time. More precisely,

$$q = CMG(k_{\max}-1, 0, \bar{F})$$

where $\bar{F} \in W'_{k_{\max}-1}$ is defined by $\bar{F}(v) := F(I^k_{k-1}v) - a_k(z_m, I^k_{k-1}v)$ for all $v \in W_{k-1}$.

Put

$$NC\text{-}CMG(k_{\max}, z_0, F) = z_m + I^{k_{\max}}_{k_{\max}-1} q.$$

4. ESTIMATE OF CONVERGENCE RATE

Now, we can proceed with the well-known analysis of the conforming multigrid method in [2].

Define the linear mapping $J : V_k \to V_k$ by

$$Jw = \sum_i \nu_i \left(1 - \frac{\lambda_i}{\lambda_{\max}}\right) \psi_i \quad \text{for } w = \sum_i \nu_i \psi_i \,.$$

Here λ_i's are the eigenvalues of a_k. The smoothing step (13) amplifies the error $e_i = z - z_i$ by J, i.e., $e_i = Je_{i-1}$. Note that J is a self adjoint and semidefinite operator with respect to the energy norm.

Define the weaker seminorm

$$|w|^2 := \sum_i \lambda_i \left(1 - \frac{\lambda_i}{\lambda_{\max}}\right) \nu_i^2 \quad \text{for } w = \sum_i \nu_i \psi_i \,.$$

From (7) and (8) we know $\|w\|_k^2 = \sum \lambda_i \nu_i^2$ and $|w| \le \|w\|_k$. Define the ratio

$$\rho(w) := \begin{cases} |w|^2/\|w\|_k^2 & \text{if } w \neq 0\,, \\ 0 & \text{if } w = 0 \end{cases} \,.$$

It can be regarded as a measure for the smoothness of $w \in V_k$ because for a smooth function the coefficient ν_i for small λ_i's dominate and $|w| \approx \|w\|_k$.

Lemma 2 *Given $w \in V_k$ put $\rho = \rho(J^m w)$. Then*

$$\|J^m w\|_k \le \rho^m \|w\|_k \,.$$

Proof. Similar to the proof of Lemma 4.3. in [2]. □

Let $\bar{q}\,(\in W_{k-1})$ be the exact coarser-grid correction i.e.

$$a_{k-1}(\bar{q},\, v) = F(v) - a_k(z_m,\, v) \quad \forall v \in W_{k-1} \,.$$

Define

$$Qe_m := e_m - \bar{q}$$

Then Q is the a_k-orthogonal projector from V_k into $W_{k-1}^{\perp}$. Note that $\bar{q}$ is a_k-orthogonal projection of e_m into W_{k-1}.

Lemma 3 *Given $w \in V_k$ we have*

$$|\!|\!| Qw |\!|\!|_{1,k} \le \min\left\{1, C\sqrt{1-\rho(w)}\right\} |\!|\!| w |\!|\!|_{1,k} \,.$$

Proof. For $w = \sum \nu_i \psi_i$, we have

$$\begin{aligned} |\!|\!| w |\!|\!|_{1,k}^2 - |w|^2 &= \sum_i \lambda_i \nu_i^2 - \sum_i \lambda_i \left(1 - \frac{\lambda_i}{\lambda_{\max}}\right) \nu_i^2 \\ &= \frac{1}{\lambda_{\max}} \sum_i \lambda_i^2 \nu_i^2 = \frac{1}{\lambda_{\max}} |\!|\!| w |\!|\!|_{2,k}^2 . \end{aligned}$$

It follows from Lemma 1 that $|\!|\!| Qw |\!|\!|_{1,k} \leq Ch |\!|\!| w |\!|\!|_{2,k}$. This and the estimate (6) for $\lambda_{\max}$ imply

$$\begin{aligned} |\!|\!| Qw |\!|\!|_{1,k}^2 &\leq Ch^2 \lambda_{\max} (|\!|\!| w |\!|\!|_{1,k}^2 - |w|^2) \\ &\leq C(|\!|\!| w |\!|\!|_{1,k}^2 - |w|^2) \\ &= C(1 - \rho(w)) |\!|\!| w |\!|\!|_{1,k}^2 . \end{aligned}$$

Moreover, since Q is an orthogonal projector, we have

$$|\!|\!| Qw |\!|\!|_{1,k}^2 \leq \min \left\{ 1, C\sqrt{1 - \rho(w)} \right\} |\!|\!| w |\!|\!|_{1,k}^2 . \quad \square$$

We are now (as in [10]) in a position to define three multigrid iterative schemes for the solution of (12).

1. the symmetric scheme $NC\text{-}CMGV_k$: symmetric smoothing *NC-CMG* scheme
2. the coarse-to-fine cycle $NC\text{-}CMG/_k$: postsmoothing *NC-CMG* scheme
3. the fine-to-coarse cycle: $NC\text{-}CMG\backslash_k$: our *NC-CMG* scheme.

In particular, we have[10]

$$\begin{aligned} \|NC\text{-}CMG/_k\|_k &= \|NC\text{-}CMG\backslash_k\|_k , \\ \|NC\text{-}CMGV_k\|_k &= \|NC\text{-}CMG\backslash_k\|_k^2 . \end{aligned}$$

The symmetrical method *NC-CMGV* enables us to use estimates with respect to the energy norm and to apply a duality argument.

Lemma 4 *The multigrid algorithm $NC\text{-}CMGV_k$ has a convergence factor*

$$\|NC\text{-}CMGV_k\|_k \leq \max_{0 \leq \rho \leq 1} \rho^{2m} \{\epsilon + (1 - \epsilon) \min(1, C[1 - \rho])\} , \tag{14}$$

with respect to the energy norm. ϵ is the error in $(k-1)$-level $CMGV_{k-1}$ and the constant C is independent of k and m.

We note that the right-hand side of (14) is a monotone function of ϵ due to the cut-off induced by the min-operation which is contained in the expression.

Proof.

$$z_{m+1} = z_m + \bar{q} + \epsilon w' \quad (\text{i.e. } \|q - \bar{q}\|_k \le \epsilon \|\bar{q}\|_k)$$

with some $w' \in W_{k-1}$. Hence the error is

$$e_{m+1} = e_m - \bar{q} - \epsilon w' = Qe_m - \epsilon w'.$$

Since Qe_m is orthogonal to W_{k-1} and $w' \in W_{k-1}$, we get

$$\begin{aligned} \|Qe_m - w'\|_k^2 &= \|Qe_m\|_k^2 + \|w'\|_k^2 \le \|Qe_m\|_k^2 + \|\bar{q}\|_k && (15) \\ &\le \|Qe_m\|_k^2 + \|(I-Q)e_m\|_k^2 = \|e_m\|_k^2. && (16) \end{aligned}$$

In order to estimate the final error $e_{2m+1} = J^m e_{m+1}$, we use a duality argument: $\|e_{2m+1}\|_k = \sup_{\hat{w}} a(\hat{w}, e_{2m+1})/\|w\|_k$. Note that (16) , $Q^2 = Q$ and Cauchy-Schwarz's inequality imply

$$\begin{aligned} a_k(\hat{w}, e_{2m+1}) &= a_k(\hat{w}, J^m(Qe_m - \epsilon w')) \\ &= a_k(J^m\hat{w}, (1-\epsilon)Q^2 e_m + \epsilon(Qe_m - w')) \\ &\le (1-\epsilon)a_k(J^m\hat{w}, Q^2 e_m) + \epsilon\|J^m\hat{w}\|_k\|e_m\|_k \\ &\le (1-\epsilon)\|QJ^m\hat{w}\|_k\|QJ^m e_0\|_k + \epsilon\|J^m\hat{w}\|_k\|J^m e_0\|_k \\ &\le [(1-\epsilon)\|QJ^m\hat{w}\|_k^2 + \epsilon\|J^m\hat{w}\|_k^2]^{1/2}[(1-\epsilon)\|QJ^m e_0\|_k^2 + \epsilon\|J^m e_0\|_k^2]^{1/2}. \end{aligned}$$

Given $w \in V_k$ by the Lemmas 2 and 3 it follows that

$$(1-\epsilon)\|QJ^m w\|_k^2 + \epsilon\|J^m w\|_k^2 \le \rho^{2m}\{\epsilon + (1-\epsilon)\min(1, C[1-\rho])\}\|w\|_k^2,$$

where $\rho = \rho(J^m w)$. Hence

$$\|e_{2m+1}\|_k \le \max_{0\le\rho\le1} \rho^{2m}\{\epsilon + (1-\epsilon)\min(1, C[1-\rho])\}\|e_0\|_k. \quad \square$$

Theorem 5 *If* $\|CMG\backslash_{k-1}\|_{k-1} \le \delta^{1/2}$ *where* $\frac{C}{C+2m} \le \delta < 1$, *then*

$$\|NC\text{-}CMG\backslash_k\|_k \le \delta^{1/2}.$$

Proof. We conclude from Lemma 4,

$$\|NC\text{-}CMGV_k\|_k = \max_{0\le\rho\le1} \rho^{2m}\{\delta + (1-\delta)\min(1, C[1-\rho])\},$$

because $\|CMGV_{k-1}\|_{k-1} = \|CMG\backslash_{k-1}\|_{k-1}^2 \le \delta$. Maximum δ is attained at $\rho = 1$ when $\delta \ge \frac{C}{C+2m}$.

$$\|NC\text{-}CMG\backslash_k\|_k = \|NC\text{-}CMGV_k\|_k^{1/2} \le \delta^{1/2}. \quad \square$$

Table I: Number of Grid = 8 i.e. h = 1/8

	S-NCMG		*M-NCMG*		*NC-CMG*	
smoothing	iter	time(sec)	iter	time(sec)	iter	time(sec)
1	4	.909	3	.788	3	.233
2	3	.689	2	.523	2	.156
3	2	.471	2	.540	2	.170
4	2	.483	2	.549	2	.177

Since the conforming multigrid method with the $\mathcal{V}$-cycle and arbitrary smoothing step is convergent we can choose δ such that $1 \geq \delta \geq \frac{C}{C+2m}$ and $\|CMG\backslash_{k-1}\|_{k-1} \leq \delta^{1/2}$.

5. EXPERIMENTAL RESULTS

We implement the standard nonconforming multigrid algorithm *S-NCMG* in [4], the modified standard nonconforming multigrid algorithm *M-NCMG* in [3] and *NC-CMG* with the $\mathcal{V}$-cycle for the Laplace's equation

$$\begin{aligned} -\Delta u &= -1 \quad \text{in} \quad \Omega = \text{unit square}, \\ u &= 0 \quad \text{on} \quad \partial\Omega . \end{aligned}$$

Let $\{\phi_1^k, \ldots, \phi_{n_k}^k\}$ be the basis of V_k such that each ϕ_j^k equals 1 at exactly one midpoint and equals 0 at all other midpoints. The stiffness matrix representing $a_k(\cdot,\cdot)$ with respect to this basis of nonconforming space has at most five entries per row. In the conforming case, the stiffness matrix has again at most five entries per row. Therefore z_m can be obtained from z_0 by iterating a sparse band matrix. We use the Gershgorin theorem in order to get the bounds of the maximum eigenvalues. These are the rough bounds so that the convergence rate is not optimal, but there is a trade-off because finding the exact maximum eigenvalue costs more. Note that the matrix for I_{k-1}^k has again at most five entries per row.

We take an initial guess $z_0 = 0$. The programs execute the multigrid iterations until the discrete energy norm of the real error is below the tolerance 1/(number of basis) for various mesh size and the number of smoothing. The real solution comes from the SSOR preconditioning conjugate gradient method for the five point finite difference scheme in which the difference of two consecutive solutions is less than the tolerance 10^{-9} in the descrete l_2 sense. The experiments reported here were run in double-precision arithmetic on CM-5 Vector Units which has 32K processors.

There are many ways to measure the performance of a parallel algorithm running on a parallel processor(see [9]). The most important and commonly used metric is the elapsed cpu time to run a job on a given machine even though it depends on how to optimize the program. We used the power method to get the rate of convergence. In the Table V–VIII the rate of convergence of *S-NCMG* and *M-NCMG* is slightly smaller or larger than the rate of convergence of *NC-CMG*.

Table II: Number of Grid = 16 i.e. h = 1/16

	S-NCMG		M-NCMG		NC-CMG	
smoothing	iter	time(sec)	iter	time(sec)	iter	time(sec)
1	7	2.604	5	2.089	5	.766
2	4	1.526	3	1.187	3	.481
3	3	1.183	3	1.247	3	.512
4	3	1.212	3	1.240	2	.360

Table III: Number of Grid = 32 i.e. h = 1/32

	S-NCMG		M-NCMG		NC-CMG	
smoothing	iter	time(sec)	iter	time(sec)	iter	time(sec)
1	10	6.037	7	4.294	7	1.625
2	6	3.723	5	3.163	4	.970
3	5	3.196	4	2.573	4	1.034
4	4	2.641	3	1.975	3	.832

Table IV: Number of Grid = 64 i.e. h = 1/64

	S-NCMG		M-NCMG		NC-CMG	
smoothing	iter	time(sec)	iter	time(sec)	iter	time(sec)
1	14	16.668	10	11.879	9	2.874
2	8	9.560	7	8.396	5	1.692
3	6	7.196	5	6.059	4	1.447
4	5	6.200	4	4.987	4	1.544

Table V: Number of Grid = 8 i.e. h = 1/8

	S-NCMG	M-NCMG	NC-CMG
smoothing	rate of conv.	rate of conv.	rate of conv.
1	.903	.903	.906
2	.815	.815	.820
3	.736	.736	.742
4	.665	.665	.672

Table VI: Number of Grid = 16 i.e. h = 1/16

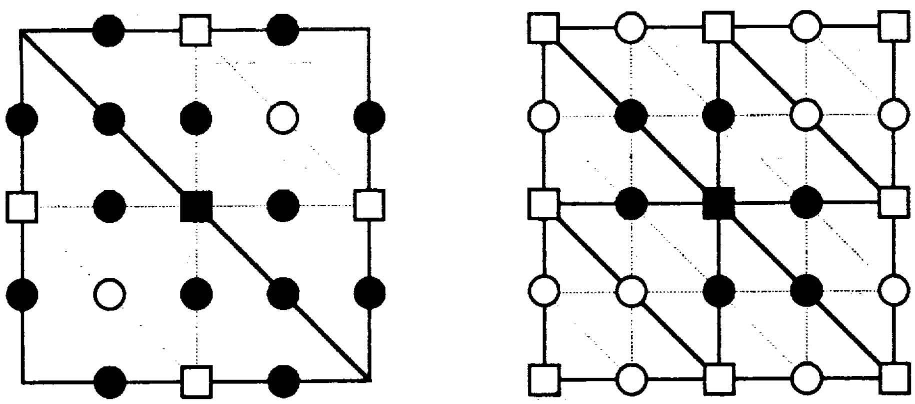
	S-NCMG	*M-NCMG*	*NC-CMG*
smoothing	rate of conv.	rate of conv.	rate of conv.
1	.904	.904	.910
2	.817	.818	.829
3	.739	.739	.754
4	.668	.669	.687

Table VII: Number of Grid = 32 i.e. h = 1/32

	S-NCMG	*M-NCMG*	*NC-CMG*
smoothing	rate of conv.	rate of conv.	rate of conv.
1	.904	.904	.911
2	.818	.818	.830
3	.740	.740	.757
4	.669	.669	.689

Table VIII: Number of Grid = 64 i.e. h = 1/64

	S-NCMG	*M-NCMG*	*NC-CMG*
smoothing	rate of conv.	rate of conv.	rate of conv.
1	.904	.904	.911
2	.939	.818	.830
3	.888	.740	.757
4	.773	.669	.690



(A) (B)

Figure 1: Nonconforming vs. conforming.

In Figure 1, (A) and (B) represent the location of the nodal basis of nonconforming finite elements and conforming finite elements, respectively. Squares represent the basis in V_{k-1} or W_{k-1} and circles represent the basis in V_k or W_k. In the correction step the centered black square is communicating with the black circles around it. Therefore *S-NCMG* and *M-NCMG* need further communications. Since the performance is determined mainly by the communication time in a massively parallel machine like CM-5, *S-NCMG* and *M-NCMG* require more cpu time than *NC-CMG*. It is shown in tables I-IV. Moreover *NC-CMG* does less computation and is easier to implement because the number of the basis of V_k is approximately three times of that of W_k and $W_{k-1} \subseteq W_k$.

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