MULTIGRID METHOD FOR INTEGRAL EQUATIONS AND AUTOMATIC PROGRAMS

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SUMMARY

Several iterative algorithms based on multigrid methods are introduced for solving linear Fredholm integral equations of the second kind. Automatic programs based on these algorithms are introduced using Simpson's rule and the piecewise Gaussian rule for the numerical integration.

INTRODUCTION

Several multigrid iterative methods based on the Nyström method are applied for the fast solution of the large dense systems of equations that arise from the discretization of Fredholm integral equations of the second kind. We will consider the linear Fredholm integral equation of the second kind,

$$\lambda x(s) - \int_D k(s,t)x(t)dt = y(s), \quad s \in D \tag{1}$$

with D a bounded close domain, and y$\in X$ where X is the underlying Banach space. Necessary assumptions are

(i) $k(s,t)$ is such that the associated integral operator K is compact from X into X
(ii) λ is not an eigenvalue of K and $\lambda \neq 0$

The Nyström method for solving (1) uses some type of numerical integration to obtain the approximating equation

$$\lambda x_l(s) - \sum_{j=1}^{n_l} \alpha_j(s)x_l(t_j) = y(s), \quad s \in D \tag{2}$$

the nodes $t_1, t_2, ..., t_{n_l}$ are in D, and $x_l(t) \doteq x(t)$. The weights $\alpha_j(s)$ can be defined in a variety of ways, depending on the smoothness and form of the kernel function. If $k(s,t)$ and $x(t)$ are reasonably smooth, usually $\alpha_j(s) = w_j k(s,t_j)$, where

$$\int_D f(t)dt \approx \sum_{j=1}^{n} w_j f(t_j) \tag{3}$$

is a numerical integration formula. Let the numerical integration operator K_l be defined by

$$K_l x(s) = \sum_{j=1}^{n_l} w_j k(s, t_j) x(t_j), \qquad s \in D \tag{4}$$

Using (2) and (4), (1) approximated by the linear system

$$\lambda x_l(t_i) - \sum_{j=1}^{n_l} w_j k(t_i, t_j) x_l(t_j) = y(t_i) \tag{5}$$

We will denote (1) and (5) symbolically as

$$(\lambda - K)x = y \tag{6}$$

and

$$(\lambda - K_l)x_l = y \tag{7}$$

respectively. Our discussion is based on the convergence of a sequence of approximations to the unique solution of (1).

In finding numerical solutions for equations (1), the system (5) is too large to be solved directly. The purpose of this paper is to consider some iterative variants of (4). The basic assumptions needed in our algorithms are given in section 2. In section 3, linear iterative algorithms are given based on Simpson's rule and piecewise Gaussian quadrature rule for the numerical integraion formulae. And in the section 4, we include numerical examples.

BASIC ASSUMPTIONS

The methods will be defined and discussed using the abstract formulation of Anselone [1] and Atkinson [3], [4] for families of collectively compact operators.

Let $X_l, l = 0, 1, 2...$,be finite-dimensional subspaces of the Banach space X and let $P_l, l = 0, 1, 2, ...$, be a bounded projection operator from X onto X_l. We need the following assumptions for $\{X_l\}$ and $\{P_l\}$

(A1) $X_0 \subset X_1 \subset .... \subset X_l ... \subset X$
(A2) $\lim_{l \to \infty} \|f - P_l f\| = 0$ for all f $\in X$

The sequence $\{X_l\}$ is thought as being associated with a sequence of decreasing meshsizes $\{h_l\}$ with $\lim_{l\to\infty} h_l = 0$. Corresponding with this sequence $\{h_l\}$,we approximate K by a sequence of operators $\{K_l\}, K_l : X \longrightarrow X$. In multigrid iteration, the subscript l is called "level".The hypotheses on $\{K_l : l \geq 1\}$ and K are as follows.

(A3) K and $K_l, l \geq 1$ are linear operators on the Banach space X into X.
(A4) $K_l x \to Kx$ as $n \to \infty$, for all $x \in X$.
(A4) $\{K_l\}$ is a collectively compact family of operators.

The following is a consequence of the assumptions (A3) - (A5):

Lemma 1 *Assume (A3) - (A5). Then with n defined as in (3)*
(i) K is compact
(ii) $\|(K - K_l)K\|$ *and* $\|(K - K_l)K_l\|$ *converge to zero as* $n \to \infty$
(iii) If $a_l = \sup_{m\geq l} \sup_{n\geq 1} \|(K - K_m)K_n\|$, *then* $\lim_{l\to\infty} a_l = 0$

Proof. See Atkinson [4].

Lemma 2 *If* $(\lambda - K)^{-1}$ *exists, then*
$(\lambda - K_l)^{-1}$ *exists for sufficiently large l, say* $N(\lambda)$, *and is uniformly bounded by* $c_2(\lambda)$ *and*

$$\|x - x_l\| \leq c_2(\lambda)\,\|Kx - K_l x\|\,, \qquad l \geq N(\lambda)$$

where $x_l \equiv (\lambda - K_l)^{-1} y$

Proof. See Atkinson [4].

This shows $x_l \to x$ and gives a rate of convergence.

LINEAR ITERATIVE METHODS

Multigrid Methods

Assume that $x_{l,0}$ denotes a approximate solution of (7) with residual

$$d_l = y_l - (\lambda - K_l)x_{l,0} \tag{8}$$

Then improve on the accuracy by writing

$$x_{l,1} = x_{l,0} + \delta_l \tag{9}$$

where the correction δ_l satisfies the residual correction equation

$$(\lambda - K_l)\delta_l = d_l \tag{10}$$

In general, the correction term δ_l will be small, and it is unnecessary to solve the residual correction equation (10) exactly. Thus we may write

$$\delta_l = B_l d_l \tag{11}$$

where B_l denotes a bounded linear operator approximating $(\lambda - K_l)^{-1}$. By (??) and (9) together with (11), we obtain

$$x_{l,1} = [\lambda - B_l(\lambda - K_l)]x_{l,0} + B_l y_l \tag{12}$$

as the new approximate solution to (7). The equation (11) can be represented well by means of coarser grid functions

$$(\lambda - K_{l-1})\delta_{l-1} = d_{l-1} \tag{13}$$

where $d_{l-1}$is chosen reasonably and depends linearly on d_l. If $r : X_l \longrightarrow X_{l-1}$ is the restriction mapping, then

$$d_{l-1} = r d_l \tag{14}$$

Having defined $d_{l-1}$by (14), δ_{l-1} is obtained using (11) at level $l-1$. Having obtained δ_{l-1}which is defined only on the coarse grid level, we need to interpolate this coarse-grid function by

$$\tilde{\delta}_l = p\delta_{l-1} \tag{15}$$

where p describes the prolongation of a coarse grid function to a fine grid function.

We note here that the choice of the prolongation p in (15) must be small enough to satisfy

$$\|I - pr\| < C\ h_l^\tau \tag{16}$$

where the consistency order τ depends on the discretization. (e.g. on the order of the quadrature formula). For the restriction operator r, we will consider both trivial injection and Nystrom type restriction.

Our automatic algorithm is based on the following multigrid iteration which is given as a recursive procedure.

Multigrid iteration for solving $(\lambda - K_l)x_l = y$

Procedure Multigrid (l, x_l, y) (17)

if $l = 0$ then

solve $x_0 = (\lambda - K_l)^{-1}y$

otherwise

$\bar{x}_l = \frac{1}{\lambda}[K_l x_l + y]$
$d_l = (I - K_l)\bar{x}_l - y$
$d_{l-1} = r d_l$
repeat the Procedure Multigrid with $(l-1, \delta_{l-1}, d_{l-1})$
$x_l^{new} = \bar{x}_l - p\delta_{l-1}$

We now give some basic results of the multigrid algorithm (17) that are used in our automatic algorithm.

Let ζ_k be the contraction number of the multigrid iteration employed at level k

$$\left\| x_k^{j+1} - x_k \right\| \leq \zeta_k \left\| x_k^j - x_k \right\| \tag{18}$$

Then it is known that $\{\zeta_k\}$ are uniformly bounded by some $\zeta < 1$.

Let

$$\zeta := \max_{1 \leq k \leq l} \zeta_k \tag{19}$$

where l is the maximum level in (17). The relative discretization error, the difference between x_k and x_{k-1}, is often estimated by

$$\begin{gathered} \|\tilde{p} x_{k-1} - x_k\| \leq C_1 h_k^\tau \\ for \;\; 1 \leq k \leq l \end{gathered} \tag{20}$$

where $\tilde{p}$ is a prolongation operator and τ is the consistency order.

Theorem 3 *Assume (20) and*

$$C_2 \zeta^i < 1 \tag{21}$$

with

$$C_2 := \max_{1 \leq k \leq l} \left[\frac{h_{k-1}}{h_k} \right]^\tau$$

then the i th iteration of the multigrid procedure (17) at level k results in $\tilde{x}_k$ and satisfies the error estimate

$$\begin{gathered} \|\tilde{x}_k - x_k\| \leq C_3 C_1 {h_k}^\tau \\ for \;\; 0 \leq k \leq l \end{gathered} \tag{22}$$

where

$$C_3 = \frac{\zeta^i}{1 - C_2 \zeta^i} \tag{23}$$

Proof. See Hackbush [11].

Theorem 4 *Assume the validity of (22) and suppose* $\frac{h_{k-1}}{h_k} \leq \frac{1}{2}$ *then the i th iteration of the multigrid procedure (17) at level k results in* $\tilde{x}_k$ *satisfies the error estimate*

$$\|\tilde{x}_k - x_k\| \leq C_4 \|x_k - x\| \tag{24}$$

where

$$C_4 = \frac{(2^\tau - 1)\zeta^i}{1 - C_2\zeta^i} \tag{25}$$

Proof. See Hackbush [11].

Automatic Algorithms

The automatic algorithm ζ_k in (18) is used to estimate the iteration error. Then together with the discretization error the global error in the solution is estimated. Often ζ_k is estimated by

$$\zeta_k \doteq \frac{\left\|x_k^{j+1} - x_k^j\right\|}{\left\|x_k^j - x_k^{j-1}\right\|} \tag{26}$$

Then

$$\left\|x_k - x_k^{j+1}\right\| \doteq \frac{\zeta_k}{1 - \zeta_k} \left\|x_k^{j+1} - x_k^j\right\| \tag{27}$$

is used to estimate the iteration error. Thus at any level, a minimum of two iteration is required to estimate the iteration error. However, (24) together with (25) can be used to estimate ζ using

$$C_4 \doteq \frac{\textit{iteration error}}{\textit{discretization error}} \tag{28}$$

and it will enable us to estimate (27) with only one iteration.

Our first algorithm is based on Simpson's rule with double the node points as the level increases, i.e. dimension of the linear system at a level l is $2^{l+1} + 1$. In this case we have $C_2 = 16$ in (21). Thus by the condition (21), if $\zeta < \frac{1}{16}$ the estimates in (22) holds with i=1, i.e. only one multigrid iteration per level. The result is computational savings. As the level increases the amount of computation increases, so that there is a significant time savings in performing only one iteration as the dimension of the linear system being solved becomes larger. Moreover ζ_k in (18) goes to zero as the level k increases, which means that after a certain level k, ζ_k becomes so small that the iteration error becomes much less significant than the discretization error, hence more accurate estimation of it is not needed. Thus one iteration is sufficient at this stage.

The second algorithm is based on the piecewise Gaussian quadrature rule for the numerical integration scheme. We adapt the iteration error estimation scheme discussed earlier.

For simplicity we use $h_l = \frac{b-a}{2^l}$ for $l = 1, 2, ...$ This means that we reduce the length of each subinterval by half as the level increases. Suppose at some level l, we have a partition

$$Q_l = \{a = q_0 < q_1 < .... < q_{m_l} = b\} \tag{29}$$

with

$$q_i = a + i * h_l \quad for \;\; i = 0, 1, 2, ..., m_l$$

and $m_l = 2^l$:=number of subintervals, for $l = 0, 1, 2, ...$

Then

$$\int_a^b f(t)dt \doteq \sum_{i=1}^{m_l} h_i \sum_{j=1}^{p} \hat{w}_j \; f(q_{i-1} + h_i \hat{t}_j) \tag{30}$$

where

$$\int_0^1 f(t)dt \doteq \sum_{j=1}^{p} \hat{w}_j \; f(\hat{t}_j) \tag{31}$$

is the Gaussian quadrature rule on [0,1] with p node points.

Unlike Simpson's rule, we do not have nested node points. In the following algorithm, both restriction and prolongation are done with Nyström type interpolation.

Procedure Multigrid with piecewise Gaussian (l, x_l, y) (32)

if $l = 0$ then

solve $x_0 = (\lambda - K_0)^{-1} y$

otherwise

$\bar{x}_l = \frac{1}{\lambda}[K_l x_l + y]$

$d_l = (\lambda - K_l)\bar{x}_l - y = K_l x_l - K_l \bar{x}_l$

$d_{l-1} = r(K_l x_l - K_l \bar{x}_l)$

repeat the Procedure Multigrid with $(l - 1, \delta_{l-1}, d_{l-1})$

$x_l^{new} = \bar{x}_l - p\delta_{l-1}$

Nyström type interpolations as in the procedure (32) are costly. Each interpolation involves $O(n_l^2)$ multiplications at each level. However this can be improved as suggested in our conclusion later.

The following theorem which is due to Atkinson-Potra [7] gives the theoretical iterative rate of convergence for piecewise Gaussian quadrature with Nyström type interpolation. We will assume

that the kernel $k(s,t)$ belongs to the class $G(\alpha,\gamma)$. This means that the kernel $k(s,t)$ has the following properties:

(G1) Define

$$\Psi_1 = \{(s,t) \mid a \le s \le t \le b\}$$
$$\Psi_2 = \{(s,t) \mid a \le t \le s \le b\}$$

Then there are functions $k_i \in C^{\alpha}(\Psi_i), i = 1,2$ with

$$k(s,t) = k_1(s,t), \quad (s,t) \in \Psi_1, \quad t \neq s$$
$$k(s,t) = k_2(s,t), \quad (s,t) \in \Psi_2$$

(G2) If $\gamma \ge 0$, then $k(s,t) \in C^{\gamma}([a,b] \times [a,b])$. If $\gamma = -1$, then the kernel $k(s,t)$ may have a discontinuity of the first kind along the line $t = s$

Theorem 5 *Assume that $k(s,t) \in G(\alpha,\gamma)$. Then solve the Nyström equation*

$$x_l(s) = \sum_{j=1}^{N} w_j k(s,t_j) x_l(t_j) + y(s) \tag{33}$$

using piecewise Gaussian quadrature rule with p node points in subintervals by first obtainning $x_l(t_1), ..., x_l(t_N)$ as a solution of the linear system

$$x_l(t_i) = \sum_{j=1}^{N} w_j k(t_i,t_j) x_l(t_j) + y(t_i) \tag{34}$$

then using (33) as an iterpolation formula gives an error estimate

$$\|x = x_l\| = O(h_l{}^w) \tag{35}$$

where $w = \min\{\alpha, 2p, \gamma + 2\}$.

Proof. See Atkinson-Potra [7] for the case p=r+1.

Finally to determine i, the needed number of iteration at any level l, use (24) and (25) with $\tau = 2p$,hence $C_2 = 2^{2p}$.

Automatic Implementation

Our automatic implementation is divided into two stages based on the results from the iteration method. In stage 1, $(\lambda - K_m)x_m = y$ is solved directly, and then an attempt is made to solve $(\lambda - K_l)x_l = y \quad for\ l > m$, iteratively. If the rate of convergence is sufficiently rapid then the stage 2 is entered. Otherwise m is replaced by l and the stage 1 is repeated. In stage 2, the value of m will serve as the coarsest grid level in the multigrid procedure (17) and solve $(\lambda - K_l)x_l = y$ iteratively until termination of the algorithm. The iteration procedure attempts to use the minimum number of iterates such that once the iterative solutions satisfy a certain criteria

we will try to estimate the rate of convergence asymptotically, which enables the estimation of the rate of convergence with only one iteration per level. As shown in our numerical examples, this scheme results in computational savings at finer grid levels.

The initial guess for an iteration of the higher level is the interpolation of the solution of the preceding level which may have been obtained either directly or iteratively. The error $\|x - x_m\|$ and $\|x - x_l\|$ in stages 1 and 2, respectively, are monitored continuously, regardless of whether the iteration method is being used or not. Thus the multigrid iteration may not have been invoked successfully before the attainment of an answer within the desired error tolerance.

In order to estimate the global error in the current solution, we need to monitor the discretization error and the iteration error. For the iteration error estimation, (27) is used with estimated ζ in place of ζ_k. In stage 1, a test is made to determine whether the speed of convergence is sufficient to enter stage 2. If

$$\zeta \leq [\text{Ratio}]^{1/2} \tag{36}$$

then the speed of convergence is adequate for stage 2. This requirment will usually insure that only two iterates are needed to be calculated in stage 2 at any given level. The number Ratio is the theoretical rate at which the error in x_l should decrease when l is increased to the next level. In our case, since we are doubling the node points as the level increases, Ratio $= \left(\frac{1}{2}\right)^{\tau}$ with $\tau = 4$ for Simpson's rule and $\tau = 2p$ for p points piecewise Gaussian quadrature in each subinterval.

For the discretization error estimation, we compute the rate at which the error is decreasing for the current level. For each computed level l,

$$\text{NumDE} := \|x_l - x_{l-1}\| \tag{37}$$

and let DenDE be the previous value of NumDE, if any. Then the rate is computed using

$$\text{DE} := \frac{\text{NumDE}}{\text{DenDE}} \tag{38}$$

Using this value of DE, we estimate the error $x - x_l$,

$$\text{Error} := \left[\frac{\text{DE}}{1 - \text{DE}}\right] \text{NumDE} \tag{39}$$

which is a standard error estimate for sequences which are converging geometrically with a rate DE. Having estimated Error as in (39), we use the final test

$$\text{Error} \leq \epsilon \tag{40}$$

with ϵ a desired error tolerance supplied by the user.

To ensure that only needed accuracy in x_l is computed, we want to test

$$\text{iteration error} \leq \text{quadrature error} \tag{41}$$

This is done by

$$\left\|x_l^{(2)} - x_l^{(1)}\right\| \leq \left(\frac{1-\zeta}{\zeta}\right)\left(\frac{\mathrm{DE}}{1-\mathrm{DE}}\right)\left\|x_l^{(2)} - x_l^{(0)}\right\| \tag{42}$$

The test (42) is obtained by using (41) and the approximations

$$\|x - x_l\| \doteq \left(\frac{\mathrm{DE}}{1-\mathrm{DE}}\right)\left\|x_l^{(2)} - x_l^{(0)}\right\| \tag{43}$$

$$\left\|x - x_l^{(2)}\right\| \doteq \left(\frac{\zeta}{1-\zeta}\right)\left\|x_l^{(2)} - x_l^{(1)}\right\| \tag{44}$$

If the test (42) is not satisfied, then the new iterate is calculated, and (42) is tested again. Once an iterate is acceptable according to (42), we check for accuracy in the most recently computed iterate using (39) and (40).

NUMERICAL EXAMPLES

The integral equation

$$x(s) - \lambda \int_a^b k(s,t)x(t)dt = y(s), \qquad a \leq s \leq b \tag{45}$$

is solved with the kernel

$$k(s,t) = \cos(\pi st)$$

on [0,1]. A variety of parameters λ that are close to the dominant characteristic values (the reciprocals of eigenvalues) are considered, as the equation becomes more difficult to solve as λ approaches characteristic values. The dominant characteristic value that we use in our example is 1.4278. The right hand function y(s) is so chosen that

$$x(s) = e^x \cos(7s), \quad 0 \leq s \leq 1 \tag{46}$$

Table I. The First Algorithm

				Dimension (Level)	
λ	Desired	Estimated	Actual	Coarsest	Finest
1.00	1.0E-6	6.82E-7	6.76E-7	3 (0)	65 (5)
1.40	1.0E-4	1.62E-5	1.60E-5	5 (1)	65 (5)
1.43	1.0E-4	1.31E-5	1.31E-5	5 (1)	129 (6)

In Table I, the Estimated column is computed using (39). As λ approaches the characteristic value of 1.4278, both the coarsest grid level and the finest grid level were increased. In Table II, we give the iterative rate of convergence at each level, and the number of iterations performed at each level is also given in parentheses. As noted in section 3, only one iteration is needed as the level increases. Whenever only one iteration is performed at any given level, the iterative rate of convergence is the maximum contraction number ζ in (19) estimated using (24) and (25).

Table II. Iterative Rate of Convergence of The First Algorithm

		Level		
λ	Desired	1	2	3
1.00	1.0E-6	2.10E-2 (2)	5.14E-2 (1)	2.03E-3 (1)
1.40	1.0E-4	2.10E-1 (2)	5.31E-2 (2)	7.57E-3 (2)
1.43	1.0E-4	-	1.44E-1 (2)	1.44E-2 (2)
		4	5	6
1.00	1.0E-6	3.40E-3 (1)	3.80E-3 (1)	-
1.40	1.0E-4	5.93E-2 (1)	3.79E-3 (1)	-
1.43	1.0E-4	4.40E-2 (1)	3.75E-3 (1)	3.89E-3 (1)

For the second algorithm, the coarsest level corresponds to two subintervals. In order to give a reasonable comparison with the first algorithm, we first give the results with 2 node points in each subinterval. Thus the quadrature order coincides with that of the first algorithm.

Table III. The Second Algorithm with p=2

				Dimension (Level)	
λ	Desired	Estimated	Actual	Coarsest	Finest
1.00	1.0E-6	6.82E-7	6.76E-7	4 (0)	64 (5)
1.40	1.0E-4	1.62E-5	1.60E-5	4 (0)	64 (5)
1.43	1.0E-5	8.74E-6	8.72E-6	4 (0)	128 (6)

In the next table, we have results from the second algorithm with more node points on each subinterval. To show the superiority of the Gaussian quadrature rule, we give results for a smaller desired error for $\lambda = 1.40$ and $\lambda = 1.43$.

Table IV. The Second Algorithm with p=3,4

					Dimension (Level)	
λ	p	Desired	Estimated	Actual	Coarsest	Finest
1.40	3	1.0E-8	1.95E-9	1.93E-9	6 (0)	96 (4)
1.43	3	1.0E-8	3.97E-10	3.96E-10	6 (0)	192 (5)
1.43	4	1.0E-8	6.52E-10	6.28E-10	8 (0)	64 (3)

Table V. Iterative Rate of Convergence of The Second Algoritm with p=3, 4

			Level		
λ	p	Desired	1	2	3
1.40	3	1.0E-8	1.09E-4 (2)	1.43E-6 (1)	8.84E-4 (1)
1.43	3	1.0E-8	1.39E-3 (2)	1.04E-2 (1)	2.10E-4 (1)
1.43	4	1.0E-8	9.35E-6 (2)	2.55E-3 (1)	1.30E-5 (1)
				4	5
1.40	3	1.0E-8		9.90E-4 (1)	-
1.43	3	1.0E-8		2.36E-4 (1)	2.42E-4 (1)
1.43	4	1.0E-8		-	-

CONCLUSION

The piecewise Gaussian rule is superior to Simpson's rule. However, as pointed out in section 3, restrictions and prolongations are done with Nyström type interpolation. And it involves $O(n_l^2)$ multiplications at each level l without counting kernel evaluations. It appears that these operations cause the bottleneck of our algorithms. We are in the process of applying the idea suggested by Achi Brandt in [9] to our current algorithms which will reduce the operation count by far. Our preliminary results appear to be promising, and progress is being made in developing them further.

REFERENCES

1. Anselone, P.M.: *Collectively Compact Operator Approximation Theory*, Prentice-Hall, 1971.

2. Atkinson, K.E.: The Numerical Solution of Fredholm Integral Equations of the Second Kind. SIAM J. Numer. Anal., Vol. 4, 1967, pp. 337-348.

3. Atkinson, K.E.: Iterative Variants of the Nyström Method for the Numerical Solution of Integral Equations. Numer. Math., Vol. 22, 1973, pp. 17-31.

4. Atkinson, K.E.: *A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind*, SIAM, 1976.

5. Atkinson, K.E.; and Potra, F.A.: Projection and Iterative Projection Methods for Nonlinear Integral Equations, SIAM J. Numer. Anal., Vol. 20, 1987, pp. 1352-1373.

6. Atkinson, K.E.; and Potra, F.A.: The Discrete Galerkin Method for Nonlinear Integral Equations. J. Intgeral Eqns. and Appl., Vol. 1, no. 1, 1988, pp. 17-54.

7. Atkinson, K.E.; and Potra, F.A.: On the Discrete Galerkin Method for Fredholm Integral Equations of the Second Kind, IMA J. Numer. Anal., Vol. 9, 1989, pp. 385-403.

8. Brandt, A.: Multi-level Adaptive Solutions to Boundary-value Problems. Math. Comp., Vol. 31, 1977, pp. 333-390.

9. Brandt, A.: Multilevel Computations of Integral Transforms and Particle Interaction with Oscillatory Kernels. Comp. Phy. Comm., Vol. 65, 1991, pp. 24-38.

10. Chatelin, F.; and Lebbar, R.: Superconvergence Results for the Iterated Projection Method Applied to Fredholm Integral Equations of the Second Kind and the Corresponding Eigenvalue Problems, J. Int. Eqns. Vol. 6, 1984, pp. 71-91.

11. Hackbush, W.: *Multigrid Methods and Applications*, Springer-Verlag, 1985.

12. Hemker, P.W.; and Schippers, H.: Multiple Grid Method for the Solution of Fredholm Integral Equations of the Second Kind. Math. Comp., Vol. 36, 1981, pp. 215-232.

13. Hashimoto, M.: A Method of Solving Large Matrix Equations Reduced From Fredholm Integral Equations of the Second Kind. J. Assoc. Comp. Mach., Vol. 17, 1970, pp. 629-636.

14. Kantorovich, L.; and Akilov, G.: *Functional Analysis in Normed Spaces*, Pergamon Press, 1964.

15. Kress, R.: *Linear Integral Equations*, Springer-Verlag, Applied Math Sciences 82, 1989.

16. Schippers, H.: The Automatic Solution of Fredholm Equations of the Second Kind. Report NW 99/80, Mathematisch Centrum, Amsterdam 1980.

17. Schippers, H.: Application of Multigrid Methods for Integral Equations to Two Problems from Fluid Dynamics. J. of Comp. Physics, Vol 48, 1982, pp. 441-461.

18. Stetter, H.J.: The Defect Correction Principle and Discretization Methods, Numer. Math., Vol. 29, 1978, pp. 425-443.